JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **212,** 571–582 (1997)
ARTICLE NO. AY975547

A q^2-Analogue Operator for q^2-Analogue Fourier Analysis

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Received January 13, 1997

1. INTRODUCTION

In this paper we introduce q^2-analogue differential operators adapted to study certain q^2-analogue functions investigated by T. H. Koornwinder and R. F. Swarttouw. We shall consider the q^2-analogue trigonometric functions introduced in [10],

$$\cos(z;q^2) = {}_1\phi_1(0;q;q^2,q^2z^2) \tag{1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} z^{2k}}{(q;q)_{2k}} \tag{2}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J_{-1/2}(z;q^2) \tag{3}$$

and

$$\sin(z;q^2) = \frac{z}{1-q}\, {}_1\phi_1(0;q^3;q^2,q^2z^2) \tag{4}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} z^{2k+1}}{(q;q)_{2k+1}} \tag{5}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J_{1/2}(z;q^2), \tag{6}$$

0022-247X/97 $25.00

using the q^2-analogue Bessel function introduced by H. Exton [2], and W. Hahn [5],

$$J_\alpha(z;q^2) = \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} z^\alpha {}_1\phi_1(0;q^{2\alpha+2};q^2,q^2z^2). \tag{7}$$

Here we are using the basic notational conventions of [10, 4]. In particular, let $0 < q < 1$, define $(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j)$, $k \in \mathbf{Z}$, $(a;q)_0 = 1$, $(a;q)_\infty = \lim_{k\to\infty}(a;q)_k$, and

$$ {}_1\phi_1(0;b;q,z) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} z^k}{(b;q)_k (q;q)_k}. \tag{8}$$

(${}_1\phi_1(0;b;q;z)$ defines an entire analytic function in z provided b is outside the set $\{1, q^{-1}, q^{-2}, \dots\}$.)

A key property of $J_\alpha(z;q^2)$, $\cos(z;q^2)$, and $\sin(z;q^2)$ is that they satisfy appropriate q-analogue orthogonality relations (cf. [10]). In fact, the q^2-analogue Bessel functions and closely related variants have received much attention because of their importance in the study of q-analogues of representations of the Group of Plane Motions and of the Quantum Group of Plane Motions, q-differential equations, and other topics. In addition to the work already cited, see, e.g., Vaksman and Korogodskiĭ [12], Kalnins, Miller, and Mukherjee [6], Koelink and Swarttouw [8], Koelink [7], and Swarttouw and Meijer [11].

However, the fact that $\cos(z;q^2)$ and $\sin(z;q^2)$ have disjoint sets of eigenvalues with respect to the classical q-differential operator

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \tag{9}$$

and also with respect to $D_q \circ D_q$ has limited their consideration and discouraged efforts to construct a q-exponential built from functions defined by $\cos(z;q^2)$ and $\sin(z;q^2)$.

In this paper we consider the operator

$$\Delta_q f(z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{(q^{-1}-q)z}, \tag{10}$$

and its renormalized version

$$\partial_q = \frac{(q^{-1}-q)}{2}\Delta_q. \tag{11}$$

We will show that Δ_q (or ∂_q) is a useful q^2-analogue of the derivative in that it produces analogues of the standard differential relationships between $\cos(z; q^2)$ and $\sin(z; q^2)$. We will define a q^2-analogue exponential function in terms of these functions, study some of its properties, and use it to define and study a q^2-analogue Fourier Transform.

To conclude the introduction, we would like to discuss some differences between Δ_q and D_q. Historically, the relation between q-analogues and the classical hypergeometric functions is based on observations such as $\lim_{q \to 1}((1-q)/(1-q^k)) = 1/k$ so that, $\lim_{q \to 1}([(1-q)z]^k/(q;q)_k) = z^k/k!$. (Also see Remark 1 later in this paper.) In our context, we have the limits

$$\lim_{q \to 1} \cos\big((1-q)z; q^2\big) = \cos(z) \tag{12}$$

$$\lim_{q \to 1} \sin\big((1-q)z; q^2\big) = \sin(z). \tag{13}$$

Another classical fact is that if f is differentiable at z,

$$\lim_{q \to 1} D_q f(z) = f'(z). \tag{14}$$

We also have for f differentiable at z,

$$\lim_{q \to 1} \Delta_q f(z) = f'(z). \tag{15}$$

However, when studying expressions such as (12) and (13), frequently we are interested in limits of the form $\lim_{q \to 1} T_q f((1-q)z; q^2)$ where both the operator and its argument function change with q. Computing the classical q-derivative of the functions in (12) and (13) gives

$$D_q \sin\big((1-q)z; q^2\big) = \cos\big((1-q)z : q^2\big) \tag{16}$$

$$D_q \cos\big((1-q)z; q^2\big) = -q\sin\big((1-q)qz; q^2\big). \tag{17}$$

Although, $\lim_{q \to 1} D_q \sin((1-q)qz; q^2) = \cos(z)$ and $\lim_{q \to 1} D_q \cos((1-q)z; q^2) = -\sin(z)$, we see that the eigenvalue relationship between these functions and D_q is not analogous to the classical situation for $0 < q < 1$. On the other hand,

$$\Delta_q \sin\big((1-q)z; q^2\big) = \frac{2q}{1+q}\cos\big((1-q)z; q^2\big) \tag{18}$$

$$\Delta_q \cos\big((1-q)z; q^2\big) = -\frac{2q}{1+q}\sin\big((1-q)z; q^2\big) \tag{19}$$

giving the classical limit as $q \to 1$ and, more importantly, a useful analogue relationship for $0 < q < 1$. These relationships will allow us to define a q^2-analogue exponential which exhibits appropriate behavior under Δ_q.

In addition to studying the interaction of ∂_q and related operators with the q^2-analogue trigonometric functions and with a q^2-analogue exponential function and applications, the operator can also be used for other analogue results. For example, we have a number of ∂_q-formulas for q^2-Bessel functions of integer order n which yield classical Bessel function identities in the appropriate limit. Corresponding formulas in terms of D_q have been obtained by others, cf. [10, 8, 11] and references found in these papers.

2. THE OPERATOR ∂_q

We use the notation $f_w(z) = f(w^{-1}z)$ where w represents a complex constant. Unless otherwise specified we will always assume that functions f are defined on sets, S, which are symmetric in the sense that if $z \in S$ then $-z \in S$ and $\pm q^{\pm 1} z \in S$.

In this section n will always be integer-valued.

Our first lemma lists some useful computational properties of Δ_q and ∂_q, and reflects the sensitivity of these operators to the parity of their arguments.

LEMMA 1. (a) *If f is an odd function,* $\Delta_q f(z) = (2q/(1+q))\, D_q f(z)$.

(a′) *If f is an odd function,* $\partial_q f(z) = z^{-1}(f(z) - f(qz))$.

(b) *If f is an even function,* $\Delta_q f(z) = (2q/(1+q))\, D_q f_q(z)$.

(b′) *If f is an even function,* $\partial_q f(z) = z^{-1}(f(q^{-1}z) - f(z))$.

(c) *If f and g are both even,* $\partial_q(fg)(z) = f_q(z)\partial_q g(z) + g(z)\partial_q f(z)$.

(d) *If f and g are both odd,* $\partial_q(fg)(z) = f_q(z)\partial_q g_q(z) + g(z)\partial_q f_q(z)$.

(e) *For any complex number* α, $\alpha(\partial_q(f_\alpha)(z)) = (\partial_q f)_\alpha(z)$.

The verification of these formulas is straightforward and will be left to the reader.

We consider a context in which the behavior of ∂_q is very useful. Using the q^2-analogue trigonometric functions defined in (2) and (5) we define a q^2-analogue exponential by

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2) \tag{20}$$

$$= \sum_{k=0}^{\infty} \frac{q^{k(k+1)} z^{2k}}{(q;q)_{2k}} + \sum_{k=0}^{\infty} \frac{q^{k(k+1)} z^{2k+1}}{(q;q)_{2k+1}}. \tag{21}$$

$e(z; q^2)$ is absolutely convergent for all z in the plane, $0 < q < 1$, since both of its component functions are. $\lim_{q\uparrow 1} e((1-q)z; q^2) = e(z)$ pointwise and uniformly on compacta, because both of its component functions satisfy corresponding limits by the following remark.

Remark 1. $u > -1 \Rightarrow \lim_{q\uparrow 1} {}_1\phi_1(0; q^{u+1}; q, (1-q)^2 z) = {}_0F_1(-; u+1; -z)$, *uniformly in z in compact subsets of the plane*.

Here ${}_0F_1$ denotes the (generalized) hypergeometric series. The remark is proved in [10, p. 459].

The basic computational ∂_q formulas for these q^2-analogue functions are given by:

LEMMA 2.

$$\text{(a)}\ \partial_q \sin(z; q^2) = \cos(z; q^2). \tag{22}$$

$$\text{(b)}\ \partial_q \cos(z; q^2) = -\sin(z; q^2). \tag{23}$$

$$\text{(c)}\ \partial_q e(z; q^2) = e(z; q^2). \tag{24}$$

Proof. $\cos(z; q^2)$ is an even function, so

$$\begin{aligned}
\partial_q \cos(z; q^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} z^{2k}}{(q;q)_{2k}} \left(\frac{q^{-2k} - 1}{z} \right) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k+1)-2k} z^{2k-1}}{(q;q)_{2k}} (1 - q^{2k}) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k-1)} z^{2k-1}}{(q;q)_{2k-1}} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} q^{m(m+1)} z^{2m+1}}{(q;q)_{2m+1}} \\
&= -\sin(z; q^2).
\end{aligned}$$

Part (a) is proved similarly, and (c) follows from (a) and (b) using the linearity of ∂_q and Lemma 1(e). ∎

3. A q^2-ANALOGUE FOURIER TRANSFORM

We will use the notation for q-integrals introduced by Jackson, cf. [4],

$$\int_0^{\infty} f(t)\, d_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n$$

and

$$\int_{-\infty}^{\infty} f(t)\,d_q t = (1-q)\sum_{n=-\infty}^{\infty} \{f(q^n) + f(-q^n)\} q^n.$$

Set

$$L^p(d_q) = \left\{ f : \|f\|_p^p = \int_{-\infty}^{\infty} |f(t)|^p d_q t = (1-q)\sum_{n=-\infty}^{\infty} \{|f(q^n)|^p + |f(-q^n)|^p\} q^n < \infty \right\}$$

for $1 < p < \infty$, and set

$$L^\infty(d_q) = \left\{ f : \|f\|_\infty = \sup\left\{ |f(\pm q^k)| : k \in \mathbf{Z} \right\} < \infty \right\}.$$

The following properties can be verified by direct calculation.

LEMMA 3. *If* $\int_{-\infty}^{\infty} f(t)\,d_q t$ *exists,*

(a) *f odd implies* $\int_{-\infty}^{\infty} f(t)\,d_q t = 0$.

(b) *f even implies* $\int_{-\infty}^{\infty} f(t)\,d_q t = 2\int_0^\infty f(t)\,d_q t$.

(c) *s an integer implies* $\int_{-\infty}^{\infty} f(q^s t)\,d_q t = \int_{-\infty}^{\infty} f(t) q^{-s}\,d_q t$.

We will also use the following lemma. Let $F^*(t) \equiv \operatorname{sgn} t F(t)$.

LEMMA 4. *If* $\int_{-\infty}^{\infty} (\partial_q f)^*(t) g(t)\,d_q t$ *exists,* $\int_{-\infty}^{\infty} (\partial_q f)^*(t) g(t)\,d_q t = \int_{-\infty}^{\infty} f(t)(\partial_{q^{-1}} g)^*(t)\,d_q t$.

Proof. The verification of this results is a tedious calculation using the definitions of the q-integral given above and of ∂_q. We will give a typical term:

$$\sum_{n=-\infty}^{\infty} \frac{q^n f(q^n q) g(q^n)}{q^n} = \sum_{n=-\infty}^{\infty} f(q^{n+1}) g(q^n) = \sum_{k=-\infty}^{\infty} f(q^k) g(q^{k-1})$$
$$= \sum_{k=-\infty}^{\infty} \frac{q^k f(q^k) g(q^{k-1})}{q^k}. \qquad \blacksquare$$

We will need the q-Gamma function defined by $\Gamma_q \equiv (q;q)_\infty/(q^z;q)_\infty)(1-q)^{(1-z)}$, cf. [4]. Note that $\lim_{q\uparrow 1}\Gamma_q(z) = \Gamma(z)$, see [9].

Define the q^2-analogue Fourier Transform to be

$$\hat{f}(x;q^2) \equiv \frac{(1+q)^{1/2}}{2\Gamma_{q^2}(1/2)} \int_{-\infty}^{\infty} f(t) e(-i(1-q)tx; q^2)\,d_q t. \tag{25}$$

If we impose the condition that q be in $\{q \in (0,1) : 1 - q = q^{2m}$ for some integer $m\}$ and if we let $q \uparrow 1$ under this side condition, we obtain, formally, the classical Fourier Transform on the line. (See also the comment after the next lemma.) Therefore, in the remainder of this paper, we assume that

$$q \in \{q \in (0,1) : 1 - q = q^{2m} \text{ for some integer } m\}. \tag{26}$$

(It should be noted that if we disregard the limit as $q \uparrow 1$, we can formulate a definition of the Fourier Transform which will satisfy all the corresponding versions of the results discussed below for all $q \in (0, 1)$.)

For convenience, set

$$C = \frac{(1+q)^{1/2}}{2\Gamma_{q^2}(1/2)}. \tag{27}$$

If we show that $|e(\pm i(1-q)q^k; q^2)|$ is bounded for all integers k, then it will follow immediately that the q^2-Fourier Transform defines a bounded linear operator from $L^1(d_q)$ to $L^\infty(d_q)$. We turn to this task

We use the following results of Koornwinder and Swarttouw [10]:

LEMMA 5. (a) *For* $|z| < 1$ *and* n, m *integers*,

$$\delta_{mn} = \sum_{k=-\infty}^{\infty} z^{k+n} \frac{(z^2;q)_\infty}{(q;q)_\infty} {}_1\phi_1(0; z^2; q, q^{n+k+1})$$
$$\times z^{k+m} \frac{(z^2;q)_\infty}{(q;q)_\infty} {}_1\phi_1(0; z^2; q, q^{m+k+1}),$$

where the sum converges absolutely, and uniformly on compact subsets of the open unit disk.

(b) *For* $f \in L^2(d_q)$,

$$g(q^n) = 2C\int_0^\infty \cos((1-q)tq^n; q^2) f(t)\, d_q t$$

implies

$$f(q^k) = 2C\int_0^\infty \cos((1-q)sq^k; q^2) g(s)\, d_q s$$

or

$$g(q^n) = 2C\int_0^\infty \sin((1-q)tq^n; q^2) f(t)\, d_q t$$

implies

$$f(q^k) = 2C\int_0^\infty \sin((1-q)sq^k; q^2) g(s)\, d_q s.$$

A study of the convergence as $q \uparrow 1$ of the cosine transform defined in part (b) of this lemma can be found in [3].

We obtain some useful inequalities by combining this result with the following classical equalities of Rogers and Ramanujan [1, p. 8],

$$\sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q;q)_{2k}} = \prod_{n \in CI} \frac{1}{(1-q^n)}, \tag{28}$$

where $CI \equiv \{n \in \mathbf{Z} : n \geq 1, n \neq 0(\text{mod}10), \pm 1(\text{mod}10), \pm 8(\text{mod}20)\}$ and

$$\sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q;q)_{2k+1}} = \prod_{n \in SI} \frac{1}{(1-q^n)}, \tag{29}$$

where $SI \equiv \{n \in \mathbf{Z} : n \geq 1, n \neq 0(\text{mod}10), \pm 3(\text{mod}10), \pm 4(\text{mod}20)\}$.

LEMMA 6. $\cos((1-q)x; q^2)$, $\sin((1-q)x: q^2)$, *and* $|e(i(1-q)x; q^2)|$ *are all bounded for* $x \in \{\pm q^k : k$ *is an integer*$\}$. *In fact*

$$|\cos((1-q)x;q^2)| \leq (1+q) \prod_{n \in CI} \frac{1}{(1-q^n)} \qquad \text{and}$$

$$|\sin((1-q)x;q^2)| \leq \prod_{n \in SI} \frac{1}{(1-q^n)}.$$

Proof. Letting $m = n$ and replacing q by q^2 in part (a) of Lemma 5 gives

$$1 = \sum_{k=-\infty}^{\infty} z^{2(k+n)} \left[\frac{(z^2;q^2)_\infty}{(q^2;q^2)_\infty} {}_1\phi_1(0; z^2; q^2, q^{2(n+k+1)}) \right]^2. \tag{30}$$

Taking $z = \pm q^{1/2}$, $n = 0$ in (30) shows that

$$q^k \cos^2(q^k; q^2) = q^k \left[{}_1\phi_1(0; q; q^2, q^{2(k+1)}) \right]^2 \leq \left[\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \right]^2 \tag{31}$$

for all integers $k, 0 < q < 1$. Alternately, taking $z = q^{3/2}$, $n = 0$ in (30) gives that

$$q^k \sin^2(q^k; q^2) = q^{3k} \left[\frac{{}_1\phi_1(0; q^3; q^2, q^{2(k+1)})}{1-q} \right]^2 \leq \left[\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \right]^2 \tag{32}$$

for all integers k, $0 < q < 1$. This shows that $x^{1/2}\cos(x; q^2)$ and $x^{1/2}\sin(x; q^2)$ are bounded for $x \in \{q^k : \text{k is an integer}\}$. Thus $\cos(x; q^2)$ is bounded for $x \in \{q^k : \text{k is a non-positive integer}\}$.

Now for $|z| \le 1$, using the expansion (5) we get

$$|\sin(z;q^2)| \le \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q;q)_{2k+1}} = \prod_{n\in SI} \frac{1}{(1-q^n)}. \tag{33}$$

The last equality follows from (29). Moreover the fact that $0<q<1$ implies that $(q^2;q^2)_\infty \le \prod_{n=1}^{\infty}(1-q^{10n-8})(1-q^{10n-4})(1-q^{10n})(1-q^{20n-16})(1-q^{10n-2}) \le \prod_{n=1}^{\infty}(1-q^{10n-7})(1-q^{10n-3})(1-q^{10n})(1-q^{20n-16})(1-q^{20n-4})$ and $(q;q^2)_\infty \ge (q;q)_\infty$. Combining these inequalities,

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \le \prod_{n\in SI} \frac{1}{(1-q^n)}.$$

This, combined with (32) and (33) gives

$$|\sin(q^k;q^2)| \le \prod_{n\in SI} \frac{1}{(1-q^n)} \qquad \text{for k an integer.} \tag{34}$$

A similar argument using (28) shows

$$|\cos(q^k;q^2)| \le (1+q) \prod_{n\in CI} \frac{1}{(1-q^n)}, \qquad \text{for k an integer.} \tag{35}$$

Finally, using the evenness of $\cos(x;q^2)$, we see that it is also bounded with the same bound for $x \in \{\pm q^k : \text{k is an integer}\}$. Applying condition (26), we see that $\cos((1-q)x;q^2)$ is bounded with the above bound for $x \in \{\pm q^k : \text{k is an integer}\}$. A similar argument shows that $\sin((1-q)x;q^2)$ is bounded for $x \in \{\pm q^k : \text{k is an integer}\}$. Finally, combining the above, we see that $|e(i(1-q)x;q^2)|^2 = \cos^2((1-q)x;q^2) + \sin^2((1-q)x;q^2)$ is bounded on the same set. ∎

At this point, even though we do not have a simple addition formula for $e(z;q^2)$, we can establish several q^2-analogue Fourier transform results using essentially standard arguments. For example, it is easy to show:

COROLLARY 1. *If $f,\ g \in L^1(d_q)$, then*

(a) *$\hat{f}$ is a bounded linear operator from $L^1(d_q)$ to $L^\infty(d_q)$.*

(b) $\int_{-\infty}^{\infty} \hat{f}(t;q^2) g(t) d_q t = \int_{-\infty}^{\infty} f(t) \hat{g}(t;q^2) d_q t.$

(c) *For s an integer, $(f_{q^{-s}})^\wedge(x) = q^{-s}(\hat{f})_{q^s}(x)$.*

(d) *If $uf(u) \in L^1(d_q)$, $\partial_q \hat{f}(x;q^2) = (-i(1-q)uf(u))^\wedge(x)$.*

A complement to part (d) of the above corollary is:

COROLLARY 2. *If* $(\partial_{q^{-1}}f)^* \in L^1(d_q): (\partial_{q^{-1}}f)^{*\wedge}(x: q^2) = -i(1-q) \times (f^*)^{\wedge}(x; q^2)$.

Proof. The formula follows by expanding the left-hand side using the definition of the analogue Fourier transform and then applying Lemma 4 and using Lemmas 1e, 2, and 6.

We turn to the L^2 theory of the q^2-analogue Fourier Transform. $(L^1 \cap L^2)(d_q)$ is dense in $L^2(d_q)$. (Consider functions with finite support.) Since the q^2-analogue Fourier Transform is defined and bounded on $(L^1 \cap L^2)(d_q)$ for such functions, it defines a bounded extension to all of $L_2(d_q)$. We can use Lemmas 3 and 5 to prove an inversion theorem.

THEOREM 1. $f \in (L^1 \cap L^2)(d_q)$ *implies* $f(q^n) = C\int_{-\infty}^{\infty}\hat{f}(t; q^2)e(i(1-q)tq^n; q^2)d_q t$.

Proof. We begin by rewriting the transform pair of Lemma 3(b) as

$$f(q^n) = 2C\int_0^\infty \left[2C\int_0^\infty \cos((1-q)ts; q^2)f(t)\,d_q t\right]\cos((1-q)sq^n; q^2)\,d_q s \tag{36}$$

$$f(q^n) = 2C\int_0^\infty \left[2C\int_0^\infty \sin((1-q)ts; q^2)f(t)\,d_q t\right]\sin((1-q)sq^n; q^2)\,d_q s. \tag{37}$$

As far as possible, we will apply a standard strategy to derive Fourier Inversion. Namely, write $f = f_{ev} + f_{od}$ with f_{ev} even and f_{od} odd. Using (36) and then Lemma 3(b) and (a) we get

$$\begin{aligned} f_{ev}(q^n) &= 4C^2\int_0^\infty \cos((1-q)tq^n; q^2)\int_0^\infty f_{ev}(s)\cos((1-q)st; q^2)\,d_q s\,d_q t \\ &= 2C^2\int_0^\infty \cos((1-q)tq^n; q^2)\int_{-\infty}^\infty f_{ev}(s)\cos((1-q)st; q^2)\,d_q s\,d_q t \\ &= 2C^2\int_0^\infty \cos((1-q)tq^n; q^2)\int_{-\infty}^\infty f(s)\cos((1-q)st; q^2)\,d_q s\,d_q t. \end{aligned}$$

Similarly, we get

$$f_{od}(q^n) = 2C^2\int_0^\infty \sin((1-q)tq^n; q^2)\int_{-\infty}^\infty f(s)\sin((1-q)st; q^2)\,d_q s\,d_q t.$$

Combining these expressions and using the evenness of the integrand in the t-variable gives

$$\begin{aligned} f(q^n) &= 2C^2\int_0^\infty\int_{-\infty}^\infty \{\sin((1-q)tq^n;q^2)\sin((1-q)st;q^2) \\ &\qquad +\cos((1-q)tq^n)\cos((1-q)st;q^2)\}f(s)\,d_qs\,d_qt \\ &= C^2\int_{-\infty}^\infty\int_{-\infty}^\infty \{\sin((1-q)tq^n;q^2)\sin((1-q)st;q^2) \\ &\qquad +\cos((1-q)tq^n)\cos((1-q)st;q^2)\}f(s)\,d_qs\,d_qt. \end{aligned}$$

Since $e(ix;q^2) = \cos(x;q^2) + i\sin(x;q^2)$, we see that

$$\begin{aligned} &e(i(1-q)x;q^2)e(-i(1-q)y;q^2) \\ &\quad = \cos((1-q)x;q^2)\cos((1-q)y;q^2) \\ &\qquad + \sin((1-q)x;q^2)\sin((1-q)y;q^2) \\ &\qquad + i\big[\sin((1-q)x;q^2)\cos((1-q)y;q^2) \\ &\qquad\quad -\sin((1-q)y;q^2)\cos((1-q)x;q^2)\big]. \end{aligned}$$

Using Lemma 6 and the fact that $f \in L^1(d_q)$, apply Fubini's Theorem to show that

$$\begin{aligned} &\int_{-\infty}^\infty\int_{-\infty}^\infty \big[\sin((1-q)tq^n;q^2)\cos((1-q)st;q^2) \\ &\qquad -\sin((1-q)st;q^2)\cos((1-q)tq^n;q^2)\big]\,d_qs\,d_qt \\ &\quad = \int_{-\infty}^\infty\left\{\int_{-\infty}^\infty \big[\sin((1-q)tq^n;q^2)\cos((1-q)st;q^2)\right. \\ &\qquad \left. -\sin((1-q)st;q^2)\cos((1-q)tq^n;q^2)\big]\,d_qt\right\}f(s)\,d_qs. \end{aligned}$$

This last integral is zero by Lemma 3(a) since the integrand of the t-integral is odd in t. Applying this to the above expression for $f(q^n)$ yields

$$\begin{aligned} f(q^n) &= C\int_{-\infty}^\infty C\int_{-\infty}^\infty e(i(1-q)tq^n;q^2)e(-i(1-q)st;q^2)f(s)\,d_qs\,d_qt \\ &= C\int_{-\infty}^\infty e(i(1-q)tq^n;q^2)\left\{C\int_{-\infty}^\infty e(-i(1-q)st;q^2)f(s)\,d_qs\right\}d_qt \\ &= C\int_{-\infty}^\infty e(i(1-q)tq^n;q^2)\hat{f}(t;q^2)\,d_qt. \end{aligned}$$

∎

Now we can establish an analogue to the Plancherel Theorem. In what follows assume $x \in \{\pm q^k : \text{k is an integer}\}$. Define $\tilde{f}(x;q^2) = \hat{f}(-x;q^2)$. Theorem 1 says that $f \in (L^1 \cap L^2)$ implies $f(x) = \hat{\tilde{f}}(x)$. Also, if $\overline{f(z)}$ is the complex conjugate of $f(z)$, note that $\overline{e(ix;q^2)} = e(-ix;q^2)$. Thus $\hat{\bar{f}}(-x;q^2) = C\int_{-\infty}^{\infty}\overline{f(t)}e(i(1-q)tx;q^2)d_qt = \overline{\hat{f}\ (x;q^2)}$. For $f \in (L^1 \cap L^2)(d_q)$, say, with finite support, using Corollary 1(b), $\int_{-\infty}^{\infty} f(t)\overline{f(t)}d_qt = \int_{-\infty}^{\infty} f(t)\widetilde{\widehat{\overline{f(t)}}}d_qt = \int_{-\infty}^{\infty} f(t)\hat{\hat{\bar{f}}}(-t;q^2)d_qt = \int_{-\infty}^{\infty}\hat{f}(t;q^2)\hat{\bar{f}}(-t;q^2)d_qt = \int_{-\infty}^{\infty}\hat{f}(t;q^2)\overline{\hat{f}\ (t;q^2)}d_qt$. Since the functions with finite support are dense in $L^2(d_q)$, we get

COROLLARY 3. *$f \in L^2(d_q)$ implies $\|f\|_2 = \|\hat{f}(\cdot;q^2)\|_2$.*

REFERENCES

1. G. Andrews, “q Series Their Development and Application to Analysis, Number Theory, Combinatorics, Physics and Computer Algebra,” Amer. Math. Soc., Providence, 1986.
2. H. Exton, “Basic Hypergeometric Functions and Applications,” Ellis Horwood, Chichester, 1983.
3. M. Fichtmuller and W. Weich, The limit transition q → 1 of the q-Fourier transform *J. Math. Phys.* **37** (1996), 4683–4689.
4. G. Gasper and M. Rahman, “Basic Hypergeometric Series,” Encyclopedia Math. Appl, Vol. 35, Cambridge Univ. Press, Cambridge, 1990.
5. W. Hahn, Die mechanishce Deutung einer geometrischen Differenzengleichung, *Z. Angew. Math. Mech.* **33**. (1953), 270–272.
6. E. G. Kalnins, W. Miller and S. Mukherjee, Models of q-algebra representations: The group of plane motions, *SIAM J. Math. Anal.* **25** (1994), 513–527.
7. H. T. Koelink, The quantum group of plane motions and the Hahn-Exton q-Bessel function, *Duke Math. J.* **76**, (1994), 483–508.
8. H. T. Koelink and R. F. Swarttouw, On the zeroes of the Hahn-Exton q-Bessel function and associated q-Lommel Polynomials, *J. Math. Anal. Appl.* **186** (1994), 690–710.
9. T. H. Koornwinder, Jacobi functions as limit cases of q-ultraspherical polynomials, *J. Math. Anal. Appl.* **148** (1990), 44–54.
10. T. H. Koornwinder and R. F. Swarttouw, On q-analogues of the Fourier and Hankel transforms, *Trans. Amer. Math. Soc.* **333** (1992), 445–461.
11. R. F. Swarttouw and H. G. Meijer, A q-analogue of the Wronskian and a second solution of the Hahn-Exton q-Bessel difference equation, *Proc. Amer. Math. Soc.* **120** (1994), 855–864.
12. L. L. Vaksman and L. I. Korogodskiĭ, An algebra of bounded functions on the quantum group of the motions of the plane, and q-analogues of Bessel functions, *Soviet Math. Dokl.* **39** (1989), 173–177.