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Asymptotic properties of Sobolev orthogonal polynomials[1]

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Abstract

In this report we will survey some of the main ideas and tools which appeared recently in the study of the analytic properties of polynomials orthogonal with respect to inner products involving derivatives. Although some results on weak asymptotics are mentioned, the strong outer asymptotics constitutes the core of the paper. Both the discrete and the continuous cases are considered, and several open problems and conjectures are posed.

1. Introduction

More than a survey paper, pretending to mention all the contributors and all the results, this report deals with an up-to-date account on some new tools and ideas, which appeared in the study of the analytic properties of the Sobolev orthogonal polynomials. Thus, it is quite natural to start by answering the question: what are these polynomials and where is the difference between them and the standard orthogonal polynomials?

1.1. Sobolev vs. standard inner products

Let X be a vector space of complex-valued functions with an inner product $\langle\cdot,\cdot\rangle$.

Definition 1.1. The inner product $\langle\cdot,\cdot\rangle$ is called *standard* if

$$\langle xf(x), g(x)\rangle = \langle f(x), \bar{x}g(x)\rangle, \tag{1}$$

for every $f, g \in X$.

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In other words, standard means that the operator of multiplication by the variable is symmetric.

If the polynomials form a subspace of X, well-known arguments allow to establish the existence of the unique (up to normalization) sequence of orthogonal polynomials. If the given inner product is standard, the corresponding sequence of orthogonal polynomials has a very important property: the three-term recurrence. This is a connection with several fields (say, difference equations, operator theory), leading to many beautiful asymptotic results. Without recurrence, this approach is no longer valid.

This is what happens to inner products modified by terms involving derivatives. Along this presentation, we will restrict ourselves to the simplest (although not trivial) case of two terms, the second one containing derivatives.

For the time being, assume that μ_0 and μ_1 are finite Borel measures on $\mathbb{C}$, with infinitely many points of increase. For $\lambda > 0$, we define

$$(f,g)_S = \int f(z)\overline{g(z)}\,\mathrm{d}\mu_0 + \lambda \int f'(z)\overline{g'(z)}\,\mathrm{d}\mu_1. \tag{2}$$

Besides historical reasons, the explicit parameter λ is a useful tool for establishing some algebraic relations (it allows to "supress" smoothly one of the two terms when necessary).

The key fact that should be observed is that $(\cdot,\cdot)_S$ is, generally speaking, nonstandard, and thus, the classical theory of orthogonal polynomials cannot be directly applied to the Sobolev case. Probably, the first example is due to Althammer in 1962 [4]. He proved that if we take on $[-1,1]$,

$$\mathrm{d}\mu_0(x) = \mathrm{d}x \quad \text{and} \quad \mathrm{d}\mu_1(x) = \begin{cases} 10\,\mathrm{d}x, & -1 \leqslant x < 0, \\ \mathrm{d}x, & 0 \leqslant x \leqslant 1, \end{cases}$$

then the monic Sobolev polynomial of degree 2 is

$$Q_2(x) = x^2 + \tfrac{27}{35}x - \tfrac{1}{3},$$

and has a zero at $x = -1.08 \notin (-1,1)$. In fact, the existence of zeros of Sobolev orthogonal polynomials out of the support of the measures is a frequently occurring phenomenon.

1.2. Historical notes

There are two exhaustive surveys on this topic, including the early period and the maturity of the algebraic and formal theory. The first one is due to Alfaro et al. [1] and corresponds to an invited talk given at the previous Symposium on Orthogonal Polynomials; the other one was published some years later by Meijer [21]. Thus, we shall recall only the main dates and characters and, taking advantage of the fact that the mentioned surveys were written before the breakthrough in the study of the analytic properties of these polynomials, we will go straight to the topic of asymptotics.

In his pioneer work, Althammer considered the so-called Legendre–Sobolev orthogonal polynomials, corresponding to the case when both measures, μ_0 and μ_1, are the Lebesgue measure supported on $[-1,1]$. In fact, the first period in the study of the Sobolev polynomials (1962–1973) was characterized by the use of absolutely continuous measures (defined by classical weights, as a rule). The main topics of research were existence, algebraic properties and location of zeros, and the major tool was integration by parts.

Probably the first asymptotic result was given by Schäfke [26] for the derivatives of the Legendre–Sobolev orthogonal polynomials, 10 years after they were introduced by Althammer. Schäfke's result reads as

$$Q_n'(x) = nP_{n-1}(x) + \kappa_n^{1/2}\mathrm{O}(n^{-3/2}),$$

where Q_n are monic Legendre–Sobolev polynomials with Sobolev norm $\kappa_n^{1/2}$. Unfortunately, this study did not continue.

A different line of research, started about 1988, considered the so-called discrete case, when the measure corresponding to the derivatives, μ_1, is a finite (or at most, denumerable) collection of mass points. A rich algebraic theory grew and it is still evolving. It was in 1993 when Marcellán and Van Assche [16] gave the first asymptotic result. An essential paper was published in 1995 by López et al. [11], where using techniques from the analytic theory of Padé approximants they proved the asymptotics in a very general case and under mild conditions on the measures.

Nevertheless, albeit the origins, the asymptotic results in the non-discrete case (excluding the one by Schäfke) are very recent. In order to avoid unnecessary details, we will pay major attention to the case of the first derivative and two measures compactly supported on the real line; at the end more general problems will be mentioned.

1.3. Notation

Let us make some conventions concerning notation that will be used below. For two finite Borel measures μ_0 and μ_1 we collect the definitions in Table 1.

Table 1

Measure	Density	Inner prod.	MOPS	Norm^2
μ_0	$\mu_0' = \rho_0$	$\langle\cdot,\cdot\rangle_0$	P_n	$\pi_n = \langle P_n, P_n\rangle_0$
μ_1	$\mu_1' = \rho_1$	$\langle\cdot,\cdot\rangle_1$	T_n	$\tau_n = \langle T_n, T_n\rangle_1$
(μ_0, μ_1)		$(\cdot,\cdot)_S$	Q_n	$\kappa_n = (Q_n, Q_n)_S$

Although the last row depends upon the parameter λ, we assume it fixed (unless we say the contrary) and omit it from the notation.

Our goal is the study of the monic orthogonal polynomial system (MOPS) corresponding to the inner product

$$(p,q)_S = \int p\overline{q}\,\mathrm{d}\mu_0 + \lambda\int p'\overline{q'}\,\mathrm{d}\mu_1 = \langle p,q\rangle_0 + \lambda\langle p',q'\rangle_1. \tag{3}$$

We will use the following additional notation:

- Function $\Phi(z) = z + \sqrt{z^2-1}$ with $\sqrt{z^2-1} > 0$ when $z > 1$, providing the conformal mapping of $\mathbb{C}\setminus[-1,1]$ on the exterior of the unit disc. When necessary, we assume $\Phi(\pm 1) = \pm 1$.
- $\mathbb{P}_n$ will be the family of all polynomials with complex coefficients and degree $\leqslant n$, and $\mathbb{P} = \bigcup_{n\geqslant 0}\mathbb{P}_n$.

We begin with the problem of the weak asymptotics, though chronologically it was not the beginning.

2. Weak asymptotics

An essential result on the weak asymptotics of Sobolev orthogonal polynomials is contained in a paper by Gautschi and Kuijlaars [9]. The article comprises of both a numerical part and theoretical results, where a natural tool was the logarithmic potential theory (see the monograph [27] for details and necessary notions from the potential theory).

We will restrict our attention to the class **Reg** of regular measures; this is the class of the regular nth root behavior and can be characterized as follows: $\mu \in \mathbf{Reg}$ if for the sequence $p_n(z) = k_n z^n + \cdots$, $k_n > 0$, of polynomials orthonormal w.r.t. μ,

$$\lim_n k_n^{1/n} = \frac{1}{\operatorname{cap}(\operatorname{supp}\mu)}.$$

The asymptotic zero distribution of a sequence of polynomials can be described in terms of the weak convergence of measures:

$$\mu_n \xrightarrow{*} \mu,$$

if for every continuous and compactly supported on $\mathbb{C}$ function f,

$$\lim_n \int f(x)\,\mathrm{d}\mu_n(x) = \int f(x)\,\mathrm{d}\mu(x).$$

With a polynomial

$$P(x) = \text{const} \prod_{k=1}^{n} (x - z_k)$$

we associate the unit measure

$$v(P) = \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k}.$$

In particular, denote

$$v_n = v(Q_n) \quad \text{and} \quad v_n' = v(Q_n')$$

(the sequence of zeros and critical points of the Sobolev polynomials, respectively). We are interested in the (weak) limit (or at least, accumulation points) of these sequences.

In [9] Gautschi and Kuijlaars proved, among others, the following result:

Theorem 2.1. *Assume*
(a) $\Delta_0 = \operatorname{supp}(\mu_0)$ *and* $\Delta_1 = \operatorname{supp}(\mu_1)$ *are regular for the Dirichlet's problem*;
(b) $\mu_0, \mu_1 \in \mathbf{Reg}$.
Then,

$$v_n' \xrightarrow{*} \omega_{\Delta_0 \cup \Delta_1}$$

and

$$\Delta_1 \subset \Delta_0 \Rightarrow v_n \xrightarrow{*} \omega_{\Delta_0}, \quad v_n' \xrightarrow{*} \omega_{\Delta_0}.$$

Note that the sequence of derivatives $\{Q_n'\}$ has the so-called regular asymptotic zero distribution. However, this does not imply that the zeros of $\{Q_n'\}$ are all real. The authors conjectured that they accumulate at the convex hull of the union of the supports of μ_0 and μ_1.

Let us see very briefly the scheme of the proof of the first result. One of the features of the regular class **Reg** is that for $\mu \in \mathbf{Reg}$, the $L^2(\mu)$ and the sup-norms have the same nth root asymptotic behavior. In particular,

$$\lim_n \left(\frac{\|Q_n\|_{\Delta_0}}{\|Q_n\|_{L^2(\mu_0)}} \right)^{1/n} = 1,$$

and a similar identity holds for $\{Q_n'/n\}$ on Δ_1. Using the Bernstein–Walsh lemma and Cauchy's formula, the authors show that the Markov constants M_n in the inequality

$$\|Q_n'\|_{\Delta_0} \leqslant M_n \|Q_n\|_{\Delta_0}$$

have subexponential growth, that is

$$\limsup_n \left(\frac{\|Q_n'\|_{\Delta_0}}{\|Q_n\|_{\Delta_0}} \right)^{1/n} \leqslant 1.$$

This kind of reasoning allows to establish the inequality

$$\limsup_n \|Q_n'\|_{\Delta_0 \cup \Delta_1}^{1/n} \leqslant \operatorname{cap}(\Delta_0 \cup \Delta_1),$$

and the assertion follows from a result of [5].

As we will see later, for Sobolev orthogonal polynomials it is easier to say something about their derivatives than about themselves. For example, the set of accumulation points of the zeros is in general much more difficult to describe. Let $\Delta = \Delta_0 \cup \Delta_1$, $\Omega = \overline{\mathbb{C}} \setminus \Delta$ and $g_\Omega(z, \infty)$ be the Green function for Ω with a pole at ∞. Recall that

$$g_\Omega(z, \infty) = 0 \quad \text{for } z \in \Delta .$$

For $r > 0$, denote by A_r the union of components of $\{z \in \mathbb{C}:\ g_\Omega(z, \infty) < r\}$ having empty intersection with Δ_0. The set of accumulation points of the zeros is

$$A = \bigcup_{r>0} A_r.$$

Theorem 2.2 (Gautschi and Kuijlaars [9]).

$$v_n \xrightarrow{*} v, \quad n \in \Lambda \subset \mathbb{N} \ \Rightarrow\ \operatorname{supp}(v) \subset \overline{A} \cup \Delta.$$

Moreover, the balayage of v onto $K = \partial A \cup (\Delta \setminus A)$ is equal to the balayage of ω_Δ onto K.

The notion of balayage of a measure onto a compact set can be found, for example, in the monograph [27]. Note that from this theorem it does not follow that the full sequence v_n converges.

3. The discrete case

Historically, the first results concerning strong asymptotics were obtained in the discrete case. Discrete Sobolev orthogonal polynomials appeared in the works of Koekoek, Bavinck and Meijer, who were interested in the Laguerre inner product modified by derivatives evaluated at zero. Since the results were strongly tailored to the specific properties of the Laguerre weight, in 1990 Marcellán and Ronveaux [15] focused on the problem from a more general point of view. They joined forces with Alfaro and Rezola and continued this research two years later in [2].

In 1993, Marcellán and Van Assche published [16], considering the inner product of the type

$$(f,g)_S = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}\mu_0(x) + \lambda f'(c)g'(c),$$

where $c \in \mathbb{R}$, $\lambda > 0$. Their goal was to compare the Sobolev orthogonal polynomials with the standard orthogonal polynomials associated with the measure μ_0, in order to investigate how the addition of the derivatives in the inner product influences the orthogonal system. With this purpose they assumed that μ_0 is a measure for which the asymptotic behavior of the orthogonal polynomials is known; the most relevant class of this type is the Nevai's class $M(0,1)$ of orthogonal polynomials with appropriately converging recurrence coefficients.

The cornerstone of their approach was the expansion of Q_n in series of P_n, whose coefficients are asymptotically known. The authors establish that

$$\lim_{n\to\infty} \frac{Q_n(z)}{P_n(z)} = \begin{cases} 1 & \text{if } c \in \operatorname{supp}\mu_0, \\ \dfrac{(\Phi(z)-\Phi(c))^2}{2\Phi(z)(z-c)} & \text{if } c \in \mathbb{R}\setminus\operatorname{supp}\mu_0. \end{cases}$$

This shows that the situation is very similar to adding a mass point distribution to the measure μ_0 and comparing the corresponding polynomials. In particular, a zero of Q_n is attracted by c and the rest accumulate at the support of μ_0. We find reminiscences of the Nevai's monograph [24] here.

In 1995 López et al. [11] extended the above result to the inner product involving a linear differential operator, complex measures and several points in $\mathbb{C}$. In particular, for the inner product

$$(f,g)_S = \int f(x)g(x)\,\mathrm{d}\mu_0(x) + \sum_{j=1}^{m}\sum_{i=0}^{N_j} M_{j,i} f^{(i)}(c_j)\mathcal{L}_{j,i}(g;c_j),$$

where $c_j \in \mathbb{C}$, $\mathcal{L}_{j,i}(g;c_j)$ is the evaluation at $c_j \in \mathbb{C}$ of the linear differential operator $\mathcal{L}_{j,i}$ with constant coefficients acting on g, $M_{j,i} \geqslant 0$, $m, N_j > 0$, they studied the asymptotic behavior of the ratios

$$\frac{Q_n^{(\nu)}(x)}{P_n^{(\nu)}(x)}, \quad \nu \in \mathbb{Z}^+, \ \nu \text{ fixed}, \tag{4}$$

on compact subsets of de $\overline{\mathbb{C}}\setminus\operatorname{supp}\mu_0$, assuming that the complex measure μ_0 supported on $\mathbb{R}$ belongs to the generalized Nevai's class $M_{\mathbb{C}}(0,1)$. Their result confirms the parallelism between the discrete Sobolev and standard orthogonal polynomials with mass modification of the measure.

Here the key was the possibility to transform the Sobolev orthogonality in the standard quasi-orthogonality, using the following property:

$$(UV, Q_n)_S = \int V(x)Q_n(x)\,\mathrm{d}\hat{\mu}_0(x) = 0 \quad \text{for all } V \in \mathbb{P}_{n-N-1},$$

where, if N_j is the maximal order of the derivative evaluated at c_j,

$$U(x) = \prod_{j=1}^{m} (x - c_j)^{N_{j+1}} \quad \text{and} \quad \mathrm{d}\hat{\mu}_0(x) = U(x)\mathrm{d}\mu_0(x),$$

with $N = \deg U$. As a consequence, we can express the polynomial Q_n as a linear combination (with a fixed number of terms) of standard orthogonal polynomials corresponding to the modified measure $\hat{\mu}_0$:

$$Q_n(x) = \sum_{k=0}^{N} \lambda_{n,k}\hat{P}_{n-k}(x).$$

It remains to apply techniques developed in the study of convergence of Padé approximants to meromorphic functions.

The study of the discrete case continued in the works of Alfaro, Marcellán, Rezola and others and, though not concluded yet, can be considered more established in the sense of methods and approaches than the continuous case.

4. Coherence of measures

We have seen that when the operator of multiplication by a variable commutes with the inner product (we call such a product standard), we have the 3-term recurrence relation and all what it implies. If not, we still can find an operator of multiplication by a fixed polynomial symmetric with respect to the inner product (that is what happens in the discrete case) causing higher order recurrence relations appear, and the situation is still hopeful. But, generally speaking, in the continuous case such operators cannot be expected (a result from [7]). Thus, the "discrete" techniques relying on recurrence or algebraic relations do not work. If we need some, we must assume a kind of "correlation" or coherence between the measures μ_0 and μ_1. This definition appears in the work of Iserles and others [10], and can be motivated by the following considerations (see [12]).

4.1. Motivation

Using standard arguments (say expressing the monic polynomial Q_n as the ratio of two determinants, see [25]) we can see that its coefficients are rational functions in λ, with the numerator and denominator of the same degree. Thus, for $\lambda \to \infty$ there exists the "limit" polynomial, R_n. This polynomial can be used as a "bridge" between Q_n and P_n and T_n. Simple computation allows us to see that

$$R'_{n+1}(x) = (n+1)T_n(x) \quad \text{and} \quad \langle R_n, 1\rangle_0 = 0 \quad \text{for } n \geqslant 1. \tag{5}$$

If we expand the limit polynomials R_{n+1} in terms of both P_n and Q_n, we obtain that

$$R_{n+1}(x)=\sum_{j=1}^{n+1} b_{n,j}P_j(x)=\sum_{j=0}^{n+1} \beta_{n+1,j}Q_j(x), \tag{6}$$

where $\beta_{n+1,j}=\kappa_j^{-1}(R_{n+1},Q_j)_S$. Of course, it is desirable to have on both sides of this relation a fixed finite number of terms. For example, assume that the left-hand side is

$$\sum_{j=n-k}^{n+1} b_{n,j}P_j(x), \quad k\geqslant -1,\ n\geqslant k+1, \tag{7}$$

where k is a fixed integer. In particular,

$$T_n(x)=\frac{1}{n+1}\sum_{j=n-k}^{n+1} b_{n,j}P_j'(x), \quad k\geqslant -1,\ n\geqslant k+1. \tag{8}$$

Using (5) and (7) we set $\beta_{n+1,j}=0$, for $0\leqslant j\leqslant n-k-1$. Thus, (6) reads

$$\sum_{j=n-k}^{n+1} b_{n,j}P_j(x)=\sum_{j=n-k}^{n+1} \beta_{n+1,j}Q_j(x), \quad k\geqslant -1,\ n\geqslant k+1. \tag{9}$$

In this way, we are ready to introduce the following definition.

Definition 4.1. Let (μ_0,μ_1) be a pair of positive Borel measures, and $\{P_n\}_n$ and $\{T_n\}_n$ the corresponding sequences of MOP. We say that (μ_0,μ_1) constitutes a *k-coherent pair*, $k\geqslant -1$, if

$$T_n(x)=\sum_{j=n-k}^{n+1}\frac{b_{n,j}}{j}P_j'(x), \quad n\geqslant k+1,$$

with $b_{n,n+1}=1$ and $b_{n,n-k}\neq 0$.

Note that for $k=-1$ this relation is only satisfied by the sequences of the classical orthogonal polynomials (Laguerre, Jacobi, Hermite). We fix our attention on the 0 and 1-coherence.

4.2. Coherent pairs on $[-1,1]$

So, we start from

Definition 4.2. Let (μ_0,μ_1) be a pair of positive Borel measures, and $\{P_n\}_n$ and $\{T_n\}_n$ the corresponding sequences of MOP. We say that (μ_0,μ_1) constitutes a 0-*coherent* (or just *coherent*) pair, if there exist real non-zero constants (*coherence parameters*) $\sigma_1,\sigma_2,\ldots$, such that

$$T_n(x)=\frac{P_{n+1}'(x)}{n+1}-\sigma_n\frac{P_n'(x)}{n}, \quad n\geqslant 1. \tag{10}$$

In the previous discussion we have eventually established the following property (see [25]):

Proposition 4.3. *If (μ_0, μ_1) is a coherent pair of measures, then*

$$P_{n+1}(x)-\sigma_n\frac{n+1}{n}P_n(x)=Q_{n+1}(x)-\alpha_nQ_n(x), \quad n\geqslant 1, \tag{11}$$

where

$$\alpha_n=\sigma_n\frac{n+1}{n}\frac{\pi_n}{\kappa_n}\neq 0, \quad n\geqslant 1. \tag{12}$$

Taking into account (11), the following plan looks promising:
(1) Describe all the coherent pairs of measures.
(2) Compute the asymptotics of the (standard) polynomials P_n.
(3) Find the limits of σ_n and α_n.
(4) Establish the asymptotics of Q_n.
The first step was recently made by Meijer; in the paper [22] he gave the complete classification of all coherent pairs of measures. In particular, he proved that necessarily either one of the measures must be classical. Restricted to the case of the interval $[-1,1]$ we have the following coherent pairs:

Case	μ_0	μ_1
1	$(1-x)^{\alpha-1}(1+x)^{\beta-1}\mathrm{d}x$	$\frac{1}{\lvert x-\xi_2\rvert}(1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x+M\delta(\xi_2)$
2	$\lvert x-\xi_1\rvert(1-x)^{\alpha-1}(1+x)^{\beta-1}\mathrm{d}x$	$(1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x$
3	$(1+x)^{\beta-1}\mathrm{d}x+M\delta(1)$	$(1-x)^{\beta}\mathrm{d}x$
4	$(1-x)^{\alpha-1}\mathrm{d}x+M\delta(-1)$	$(1-x)^{\alpha}\mathrm{d}x$

with $\alpha,\beta>0$, $|\xi_1|>1$, $|\xi_2|\geqslant 1$, and $M\geqslant 0$.

Note that in all the cases μ_0 is the same rational modification of μ_1 plus a possible mass point on $\mathbb{R}$. Thus, the question of the relative asymptotics of T_n/P_n can be explicitly solved in terms of the Szegő's function of the ratio of weights.

Now, rewriting the coherence condition (11) as

$$\sigma_n=\frac{\frac{1}{n+1}\frac{P'_{n+1}(x)}{P_{n+1}(x)}\frac{P_{n+1}(x)}{P_n(x)}-\frac{T_n(x)}{P_n(x)}}{\frac{1}{n}\frac{P'_n(x)}{P_n(x)}}, \tag{13}$$

and using the previous remark, it is easy to establish the convergence of the sequence σ_n in all the cases and to compute the limit explicitly.

Now we turn to α_n; taking into account expression (12), it is convenient to study the behavior of the Sobolev norm of Q_n. The result, though elementary, has an independent interest.

Lemma 4.4 (Moreno Balcázar [23]). *For $n\geqslant 2$,*

$$\pi_n+\lambda n^2\tau_{n-1}\leqslant\kappa_n\leqslant\pi_n+\sigma_{n-1}^2\left(\frac{n}{n-1}\right)^2\pi_{n-1}+\lambda n^2\tau_{n-1}. \tag{14}$$

The proof is straightforward: let us consider the expression

$$\kappa_n = \langle Q_n, Q_n \rangle_0 + \lambda n^2 \langle Q_n'/n, Q_n'/n \rangle_1,$$

and use the fact that both P_n and T_{n-1} give the minimal values to the corresponding brackets in the r.h.s. On the other hand, κ_n is minimal, thus an upper bound can be obtained substituting Q_n by the limit polynomial R_n which we can express in terms of P_n.

A direct consequence of (14) is

$$\lim_{n\to\infty} \frac{\pi_n}{\kappa_n} = 0,$$

and it is immediate to obtain

Proposition 4.5. *The sequence α_n defined in* (12) *satisfies*

$$\lim_{n\to\infty} \alpha_n = 0. \tag{15}$$

Now we have all the ingredients for the proof of the main result:

Theorem 4.6 (Martínez-Finkelshtein [19]). *Let (μ_0, μ_1) be a coherent pair of measures,* $\operatorname{supp} \mu_0 = [-1, 1]$. *Then,*

$$\lim_{n\to\infty} \frac{Q_n(x)}{T_n(x)} = \frac{2}{\Phi'(x)}, \tag{16}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$.

Among the consequences of this formula we can mention that all the zeros of Q_n accumulate at $\operatorname{supp} \mu_0$, presenting the arcsin distribution, which means that Q_n has the classical weak asymptotics (compare with the results of Gautschi and Kuijlaars).

This approach can be extended to the general case of k-coherence. In fact, it has been done in [13] for particular cases of 1-coherence (the symmetric coherence) and in [12] for 2 and 3-coherence. In the latter cases we obtain pairs of measures where neither μ_0 nor μ_1 is classical (Jacobi).

Moreover, nothing prevents from using this method in the case of coherent pairs of measures with unbounded support. According to Meijer's classification, either one of them must be Laguerre (up to affine change of variable), and the other is its rational modification plus a mass point. Recently, Prof. Meijer joined forces with the Madrid-Granada tandem (Marcellán–Pérez–Piñar) and considered this problem in [14]. I will not present their formulas here in order to avoid introducing additional notation.

5. Quest for Szegő's theory

It is clear that the coherence approach does not allow to move far enough. We need some general methods of establishing asymptotics. Naturally, the classical methods of Bernstein–Szegő or Widom come to mind. Although suitable for the study of extremal polynomials in L^p norm, we will concentrate on the orthogonal polynomials.

5.1. Informal reasoning

We start with some informal reasoning in order to "guess" the answer. Recall that the MOPS $\{Q_n\}$ solves the extremal problem

$$\kappa_n = (Q_n, Q_n)_S = \min\{(P,P)_S\colon \deg P = n,\ P \text{ monic}\}. \tag{17}$$

Assume that μ_0 and μ_1 are absolutely continuous measures supported on $[-1,1]$ and $\mu_i'(x) = \rho_i(x)$ for $i = 0, 1$. Above we have defined the function $\Phi(x)$, which is analytic in $\mathbb{C}\setminus[-1,1]$ and

$$|\Phi(x)| = 1 \text{ for } x \in [-1,1] \quad \text{and} \quad \Phi(x) = 2x + \cdots,\ x \to \infty.$$

Thus,

$$4^{n-1}\kappa_n = \int_{-1}^{1} \left| \frac{2^{n-1}Q_n(x)}{\Phi^n(x)} \right|^2 \rho_0(x)\,\mathrm{d}x + \lambda \int_{-1}^{1} \left| \frac{2^{n-1}Q_n'(x)}{\Phi^{n-1}(x)} \right|^2 \rho_1(x)\,\mathrm{d}x.$$

If we denote

$$f_n(x) = \frac{2^n Q_n(x)}{\Phi^n(x)} = 1 + \mathrm{O}\left(\frac{1}{x}\right) \quad \text{and} \quad g_n(x) = \frac{(\ln f_n(x))'}{(\Phi(x))'},$$

then the previous equation can be rewritten as

$$\frac{4^{n-1}\kappa_n}{n^2} = \frac{1}{4} \int_{-1}^{1} \left| \frac{f_n(x)}{n} \right|^2 \rho_0(x)\,\mathrm{d}x + \lambda \int_{-1}^{1} \left| \frac{\Phi'(x)}{2} f_n(x) \right|^2 \left| 1 + \frac{g_n(x)}{n} \right|^2 \rho_1(x)\,\mathrm{d}x.$$

Assume that $f_n(z)$ converges to an analytic nonvanishing function $\mathscr{F}$ in $\mathbb{C}\setminus[-1,1]$, so that $g_n(z)$ is bounded. Then, if ρ_0 is "good enough",

$$\frac{4^{n-1}\kappa_n}{n^2} \to \lambda v(\rho_1) = \lambda \int_{-1}^{1} |\mathscr{D}(x)|^2 \rho_1(x)\,\mathrm{d}x,$$

where $\mathscr{D}(x)$ is the function minimizing the integral on the r.h.s. in the class of all analytic functions in $H^2(\rho_1)$ equal to 1 at infinity. Thus,

$$\mathscr{F}(x) = \frac{2}{\Phi'(x)} \mathscr{D}(x),$$

which coincides with the results for the coherent pairs obtained so far.

Without any doubt, these considerations are absolutely informal, but allow to believe that the following statement is true:

Theorem 5.1. *If ρ_0 and ρ_1 satisfy the Szegő's condition on $[-1,1]$, then*

$$\lim_{n\to\infty} \frac{Q_n(x)}{T_n(x)} = \frac{2}{\Phi'(x)},$$

locally uniformly in $\mathbb{C}\setminus[-1,1]$.

This assertion is contained in [17], but part of the ideas of the proof appeared already in [18, 23]. Let us consider the case when ρ_1 is the Jacobi weight. We take advantage of the well-known property that the derivative of a Jacobi polynomial is again a Jacobi polynomial.

5.2. Jacobi weight

Thus, assume that μ_1 is the absolutely continuous measure given by the Jacobi weight

$$\rho_1(x) = \rho^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta, \quad \alpha,\beta > 0,$$

on $[-1,1]$, and, for the time being, μ_0 is arbitrary. Then, T_n is the nth monic Jacobi polynomial, and is the solution of the extremal problem

$$\tau_n = \|T_n\|_1^2 = \min\{\|P\|_1^2: \deg P = n,\ P \text{ monic}\}. \tag{18}$$

(for the sake of brevity, we write $\|\cdot\|_i$ instead of $\|\cdot\|_{L^2(\mu_i)}$). Since we will use different parameters in the Jacobi weight, we also denote by $P_n^{(\alpha,\beta)}$ and $p_n^{(\alpha,\beta)}$ the monic and the orthonormal Jacobi polynomial (with the parameters α and β, respectively).

As usual, bounds for the Sobolev norm κ_n will be the key to the asymptotics of $\{Q_n\}$.

Theorem 5.2 (Martínez-Finkelshtein and Moreno-Balcázar [18]). *With the notation introduced above, for* $n \geqslant 1$,

$$\pi_n + \lambda n^2 \tau_{n-1} \leqslant \kappa_n \leqslant \|P_n^{(\alpha-1,\beta-1)}\|_0^2 + \lambda n^2 \tau_{n-1}. \tag{19}$$

Furthermore, if μ *satisfies the condition*

$$\|p_n^{(\alpha-1,\beta-1)}\|_0 = \mathrm{o}(n), \quad n \to \infty, \tag{20}$$

then

$$\lim_{n\to\infty} \frac{\kappa_n}{n^2 \tau_{n-1}} = \lambda. \tag{21}$$

Remark. (1) Using the well-known bound [24, Lemma 16, p. 83],

$$|p_n^{(\alpha,\beta)}(x)|^2 \leqslant C(\sqrt{1-x} + 1/n)^{-1-2\alpha}(\sqrt{1+x} + 1/n)^{-1-2\beta}, \quad x \in [-1,1],$$

several sufficient conditions for (20) can be produced. For instance, if μ_0 is an absolutely continuous measure with respect to Lebesgue measure on $[-1,1]$ and

$$\mu_0'(x) = h(x)(1-x)^{\alpha-1}(1+x)^{\beta-1}, \tag{22}$$

with

$$\int_{-1}^{1} (1-x^2)^\delta h(x) < \infty,$$

for some $\delta < 1/2$, then (20) holds. In particular, condition $\alpha,\beta \in (0,\frac{3}{2})$ guarantees (20) for every finite Borel measure μ_0 supported on $[-1,1]$. Furthermore, addition to μ_0 of a finite number of mass points on $(-1,1)$ (but not at -1 or 1) does not affect (20).

(2) We can take $h(x)=|x-\xi|$ in (22) with $\xi\in\mathbb{R}\setminus(-1,1)$. Then measures μ_0 and μ_1 constitute a coherent pair. Analogously, a symmetrically coherent pair can be obtained if $\alpha=\beta$ and we choose in (22) $h(x)=x^2+\xi^2$ with $\xi\in\mathbb{R}\setminus\{0\}$, or $h(x)=\xi^2-x^2$ with $\xi\in\mathbb{R}\setminus(-1,1)$.

In what follows we write $\mu\in S$ to denote that the finite measure μ with $\operatorname{supp}\mu=[-1,1]$ satisfies the Szegő's condition, i.e.

$$\int_{-1}^{1}\frac{\ln\mu'(x)}{\sqrt{1-x^2}}\,\mathrm{d}x > -\infty. \tag{23}$$

A finite measure μ_0 is *admissible* if $\mu_0\in S$ and (20) holds.

Now we state the asymptotic result for the monic polynomials Q_n:

Theorem 5.3 (Martínez-Finkelshtein and Moreno-Balcázar [18]). *If μ_0 is admissible, then*

$$\lim_{n\to\infty}\frac{Q_n(x)}{T_n(x)}=\frac{2}{\Phi'(x)}, \tag{24}$$

locally uniformly in $\Omega=\overline{\mathbb{C}}\setminus[-1,1]$.

Besides the strong outer asymptotics of $\{Q_n\}$, the following corollary immediately follows:

Corollary 5.4. *The zeros of Sobolev orthogonal polynomials Q_n distribute on $[-1,1]$ according to the arcsin law. All of them accumulate at $[-1,1]$, i.e.,*

$$\bigcap_{n\geqslant 1}\overline{\bigcup_{k=n}^{\infty}\{x:\ Q_k(x)=0\}}=[-1,1]. \tag{25}$$

We will outline the proof now. Recall that the following extremal problem,

$$v(\rho_1)=\inf\left\{\int_{-1}^{1}(|F_+(x)|^2+|F_-(x)|^2)\rho_1(x)\mathrm{d}x:\ F\in H^2(\rho_1), F(\infty)=1\right\}, \tag{26}$$

(where F_+ and F_- are boundary values of F on $[-1,1]$) has the unique solution (extremal function)

$$\mathscr{F}(z)=\mathscr{F}(\rho_1;z),$$

(see e.g. [29] for details). Function $\mathscr{F}$ and the extremal constant $v(\rho_1)$ are closely related to the asymptotics of the monic polynomials T_n. In fact,

$$\lim_{n\to\infty}4^n\|T_n\|^2=v(\rho_1) \tag{27}$$

and

$$\lim_{n\to\infty}2^n\frac{T_n(z)}{\Phi^n(z)}=\mathscr{F}(z), \tag{28}$$

locally uniformly in Ω.

By (28) the limit (24) is equivalent to

$$\lim_{n\to\infty} \frac{2^{n-1}Q_n(z)}{\Phi^n(z)} = \frac{\mathscr{F}(z)}{\Phi'(z)} \tag{29}$$

locally uniformly in Ω. Since $\lambda\langle Q_n', Q_n'\rangle_1 \leqslant \kappa_n$, by (19) and (27),

$$\left\langle \frac{Q_n'}{n}, \frac{Q_n'}{n} \right\rangle_1 \leqslant \tau_{n-1}\,(1+\mathrm{o}(1)). \tag{30}$$

Then, $\{Q_n'/n\}$ is an extremal sequence for the problem (18) and its asymptotic behavior is determined. Indeed, if

$$H_n(x) = 2^{n-1}\{\Phi_-^{n-1}(x)F_-(x) + \Phi_+^{n-1}(x)F_+(x)\}, \quad x \in (-1,1),$$

then standard arguments from [29] allow to prove that

$$\lim_{n\to\infty} \|H_n - 2^{n-1}Q_n'/n\|_1 = 0. \tag{31}$$

Via the reproducing property of the Szegő kernel it yields

$$\lim_{n\to\infty} \frac{2^{n-1}Q_n'}{n\Phi^{n-1}}(z) = \mathscr{F}(z), \tag{32}$$

locally uniformly in Ω. Hence, we have established the asymptotics of the derivative Q_n'; we should prove now that (32) implies (29).

Recall that

$$\kappa_n = \|Q_n\|_0^2 + \lambda n^2 \left\| \frac{Q_n'}{n} \right\|_1^2,$$

so that

$$\frac{4^{n-1}\kappa_n}{n^2} = \left\| \frac{2^{n-1}Q_n}{n\Phi^n} \right\|_0^2 + \lambda\|2^{n-1}Q_n'/n\|_1^2. \tag{33}$$

From (21), (26) and (31) we see that both the l.h.s. and the second term of the r.h.s. of (33) tend to $\lambda v(\rho_1)$. Thus,

$$\lim_{n\to\infty} \left\| \frac{2^{n-1}Q_n}{n\Phi^n} \right\|_0 = 0. \tag{34}$$

Since a function in $H^2(\rho_0)$ can be recovered from its boundary values on $[-1,1]$, we have (cf. [29, Corollary 7.4])

Lemma 5.5. *If $\rho_0 \in S$ then for any compact subset $K \subset \Omega$ there exists a constant $C = C(K)$ such that*

$$\max_{z\in K} |f(z)|^2 \leqslant C \int_{-1}^{1} \{|f_+(x)|^2 + |f_-(x)|^2\}\rho_0(x)\,\mathrm{d}x \quad \textit{for all } f \in H^2(\rho_0).$$

Applying this inequality to Q_n/Φ^n, with account of (34) we obtain that

$$\lim_{n\to\infty} \frac{2^{n-1}Q_n}{n\Phi^n}(z)=0,$$

locally uniformly in Ω. Then,

$$\lim_{n\to\infty} \left(\frac{2^{n-1}Q_n}{n\Phi^n}\right)'(z)=0,$$

also locally uniformly in Ω. The identity

$$\frac{2^{n-1}Q_n(z)}{\Phi^{n+1}(z)}\Phi'(z)=\frac{2^{n-1}Q_n'(z)}{n\Phi^n(z)}-\left(\frac{2^{n-1}Q_n(z)}{n\Phi^n(z)}\right)',$$

and (32) remains to be used to obtain (29). The theorem is proved. □

5.3. Theorem 5.1: the general case

Going through the proof above, we can see that the essential step consisted in establishing sharp bounds for κ_n, in order to assure that

$$\lim_{n\to\infty} \frac{\kappa_n}{n^2\tau_{n-1}}=\lambda. \tag{35}$$

Actually, it was sufficient to obtain a sequence of monic polynomials $\{U_n\}$ which are asymptotically extremal in the $L^2(\mu_1)$ norm, and a corresponding sequence of monic primitives V_n, satisfying

$$\frac{\langle V_n, V_n\rangle_0}{n^2\tau_{n-1}}\to 0. \tag{36}$$

In the case of Jacobi weight we took $U_n=T_n$.

A natural question that arises here is how to estimate polynomials in terms of their derivatives. In 1939, Turán [28] studied this problem and established some lower bounds for derivatives, for example in $L^\infty([-1,1])$. Nevertheless, no such inequalities are known in general weighted L^p spaces (see, e.g., [6]).

Thus, we exploit a different idea. Assume again $\mu_1\in S$ absolutely continuous on $[-1,1]$ such that

$$\frac{1}{\rho_1}\in L^1[-1,1]. \tag{37}$$

For example, the Bernstein class of weights on $[-1,1]$ satisfies this condition. As the extremal sequence for the $L^2(\mu_1)$-norm we take the MOPS T_n, and let V_n be the sequence of monic primitives normalized by $V_n(-1)=0$, that is

$$V_n(x)=n\int_{-1}^{x} T_{n-1}(t)\,\mathrm{d}t.$$

Proposition 5.6. *If* (37) *holds, then for any finite measure* μ_0 *on* $[-1,1]$ (36) *holds, i.e.,*

$$\frac{\langle V_n, V_n\rangle_0}{n^2\tau_{n-1}}\to 0.$$

Indeed, observe that condition (37) assures that

$$g(x,y)=\chi_{[-1,x]}(y)\frac{1}{\rho_1(y)}\in L^2(\mu_1),$$

for every $x\in[-1,1]$ (here χ_A means the characteristic function of the set A). Then, its Fourier coefficients w.r.t. the orthonormal sequence $t_n(x)=\tau_n^{-1/2}T_n(x)$ tends to zero. The Lebesgue's theorem of dominated convergence concludes the proof.

Corollary 5.7. *If* (37) *holds, then for any finite measure* $\mu_0\in S$ *on* $[-1,1]$,

$$\lim_{n\to\infty}\frac{Q_n(x)}{T_n(x)}=\frac{2}{\Phi'(x)}.$$

Now we can close the gap in the proof of Theorem 5.1 by showing that a weight from S can be approximated (in a suitable metric) by those satisfying (37). We refer the reader to [17] for details.

Clearly, the sufficient conditions mentioned in Theorem 5.1 are far from being necessary. For instance, they are not necessary for the measure μ_1: among the coherent pairs there are μ_1 containing mass points outside the support of the absolutely continuous component, and hence not satisfying the Szegő's condition. Moreover, an analog of Riemann–Lebesgue lemma for μ_1 is sufficient for Proposition 5.6 to be valid. Although for general orthogonal polynomial systems we cannot expect this lemma to be valid (see e.g. [8]), some milder (or different) conditions than the Szegő class could be considered.

On the other hand, having the necessary behavior of the Sobolev norms κ_n, the condition $\mu_1\in S$ was sufficient to establish strong asymptotics for the sequence of derivatives Q_n'. Assumption $\mu_0\in S$ was introduced in order to recover from here the asymptotics of Q_n. But this is not a necessary condition either; it is enough to consider the following example: take $\mu_0=\delta_0$ and μ_1 given by the Jacobi weight $\rho^{(\alpha,\beta)}$. Then easy computation shows that

$$Q_n(x)=P_n^{(\alpha-1,\beta-1)}(x)-P_n^{(\alpha-1,\beta-1)}(0),$$

and the same asymptotics (24) actually takes place.

In this sense an interesting problem would be to find an example of a pair (μ_0,μ_1) with $\operatorname{supp}\mu_0\subset[-1,1]$ and $\operatorname{supp}\mu_1=[-1,1]$, such that the asymptotic formula (24) is no longer valid.

6. Jordan curves and arcs

Observing again the scheme of the proof given above, we can see that it is tailored neither to the interval $[-1,1]$ nor to the first derivative in the Sobolev inner product. Moreover, most arguments are standard in the theory of H^2 spaces and can be easily extended to more general problems. Thus, once again we must concentrate on the norm bounds.

For instance, let us consider the problem on the unit circle. Now the measures μ_0 and μ_1 are supported on $\mathbb{T}=\{z\in\mathbb{C}\colon |z|=1\}$ and satisfy the Szegő's condition. Standard arguments with generalized Faber polynomials are suitable for establishing (35) after some "adjustment": now not only convergence of the sequence of these polynomials but of their derivatives must be controlled.

Assume that the density $\rho_1(\theta)$ on $[0,2\pi]$ is strictly >0 and satisfies the Szegő's condition

$$\int_0^{2\pi} \rho_1(\theta)\,d\theta > -\infty$$

(again we denote it $\rho_1 \in S$). Assume further that ρ_1 is such that the Szegő's function

$$D(\rho_1;z)=\exp\left\{\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\rho_1(\theta)\,d\theta\right\},\quad |z|<1,$$

has analytic continuation throughout $\mathbb{T}$ (for example, trigonometric weights satisfy this condition). Then the extremal function

$$\mathscr{F}(z)=\frac{D(\rho_1;0)}{\overline{D}(\rho_1;1/z)}$$

has the same property.

If we take the sequence of generalized Faber polynomials

$$U_n(z)=\frac{1}{2\pi i}\oint_{|t|=R}\frac{\mathscr{F}(t)t^n}{t-z}\,dt,\quad |z|<R,\ R>1,$$

then with our assumptions it is easy to prove that

$$U_n(z)=z^nF(z)(1+o(1)),\qquad U_n'(z)=nz^{n-1}F(z)(1+o(1)), \tag{38}$$

uniformly on the unit circle $\mathbb{T}$. Assuming the extra condition

$$\frac{1}{\rho_1}\in L^1(\mu_0),$$

and using the extremal property of κ_n and (38), we have that

$$\kappa_n \leqslant \lambda n^2 v(\rho_1)(1+o(1)). \tag{39}$$

Now (35) can be obtained in one step. Thus, we can state the following result:

Proposition 6.1. *Let $\mu_0,\mu_1\in S$ on $\mathbb{T}$, μ_1 is absolutely continuous with $\mu_1'(\theta)=\rho_1(\theta)>0$ and $1/\rho_1\in L^1(\mu_0)$. Assume additionally that the Szegő's function $D(\rho_1;z)$ can be analytically continued through $\mathbb{T}$. Then*

$$z^{-n}Q_n(z)\to\mathscr{F}(z),$$

or equivalently,

$$\frac{Q_n(z)}{T_n(z)} \to 1,$$

locally uniformly in $|z|>1$.

All these considerations with minor changes can be extended to sufficiently smooth Jordan curves or arcs in $\mathbb{C}$. In fact, a natural generalization of Theorem 5.1 to this case was established in [17]. Furthermore, Sobolev inner products with derivatives of higher order are also suitable for attacking with this method; see [20]. Finally, in the spirit of the work of Widom, we can consider supports with nonconnected components. Nevertheless, here the picture must be much more complicated; a slight idea can be obtained from the weak asymptotic behavior.

The approach based on the extremal property of the norm of Sobolev polynomials found application also in the discrete case. In the 1980s Nikishin, and later Kaliaguin, have studied the extension of the Szegő's method to measures with mass points outside of the support of the absolutely continuous part, producing the appropriate extremal problem. Recently, these results were applied to Sobolev orthogonality by Foulquié, Marcellán and Branquinho. Once again, following the parallel between discrete Sobolev inner product and the standard one but with addition of mass points, they reduced the first to the latter, extending asymptotics to more general configurations, like rectifiable Jordan curves on the plane, and also giving formulas valid on the support of the first measure.

7. "Balanced" Sobolev products

The asymptotic formulas obtained so far in the continuous case show that essentially only the second measure matters (corresponding to the derivatives) and the role of the first one is reduced to "not disturb". A closer look at the inner product (3) reveals that the measures μ_0 and μ_1 do not play an equivalent role: differentiation makes the leading coefficients of the polynomials involved in the second integral of (3) to be multiplied by their degrees. This effect is more notorious for larger degrees, explaining the apparent independence of the asymptotics from the measure μ_0.

These considerations motivate to "balance" the role of both terms of (3) by considering only monic polynomials. In fact, we can study the asymptotic behavior of polynomials $Q_n(x)=x^n+\cdots$ minimizing the norm

$$\|Q_n\|^2=\int Q_n^2\,\mathrm{d}\mu_0+\int\left(\frac{Q_n'}{n}\right)^2\mathrm{d}\mu_1,$$

$n\geqslant 1$. Clearly, this problem can be reduced to the study of orthogonality with respect to a "varying" Sobolev inner product

$$(p,q)_n=\int p\overline{q}\,\mathrm{d}\mu_0+\lambda_n\int p'\overline{q'}\,\mathrm{d}\mu_1,$$

where (λ_n) is a monotone decreasing sequence of real positive numbers such that

$$\lim_n \lambda_n n^2=\lambda\in[0,+\infty],\qquad \lim_n n^2(\lambda_{n-1}-\lambda_n)=0. \tag{40}$$

First results in this direction have been obtained assuming coherence of the measures μ_0 and μ_1, both supported on $[-1,1]$ (see Section 4.2 for the possible cases). Define a new measure μ^* on $[-1,1]$ by

$$d\mu^*(x) = d\mu^*(x;\lambda) = \begin{cases} d\mu_0(x) + \lambda|\Phi'(x)|^2\,d\mu_1(x) & \text{if } 0 \leqslant \lambda < +\infty, \\ |\Phi'(x)|^2\,d\mu_1(x) & \text{if } \lambda = +\infty. \end{cases} \quad (41)$$

Let $R_n(x) = x^n + \cdots$ be the sequence of monic polynomials, orthogonal on $[-1,1]$ with respect to μ^* and

$$\varrho_n(\lambda) = \|R_n\|^2_{L^2(\mu^*)} = \int_{-1}^{1} |R_n(x)|^2\,d\mu^*(x).$$

As above, denote by Q_n the monic polynomial of degree n such that

$$\kappa_n = (Q_n, Q_n)_n = \min\{(P,P)_n\colon \deg P = n,\ P \text{ monic}\}.$$

Theorem 7.1 (Alfaro et al. [3]). *Let (μ_0, μ_1) be a coherent pair of measures supported on $[-1,1]$, and the sequence $\{\lambda_n\}$ as in (40). Then,*

$$\lim_n \frac{\varrho_n(\lambda)}{\kappa_n(\lambda_n)} = 1 \quad (42)$$

and

$$\lim_n \frac{Q_n(z)}{R_n(z)} = 1, \quad (43)$$

locally uniformly in $\overline{\mathbb{C}} \setminus [-1,1]$.

In other words, the sequence $\{Q_n\}$ behaves asymptotically as the monic orthogonal polynomial sequence corresponding to the measure (41). This result should remain true without the assumption on coherence of μ_0 and μ_1.

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