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ACADEMIC PRESS
Journal of Combinatorial Theory, Series A 102 (2003) 367–382
Journal of Combinatorial Theory Series A
http://www.elsevier.com/locate/jcta

On intervals in some posets of forests

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Received 30 October 2002

Abstract

We compute the characteristic polynomials of intervals in some posets of leaf-labeled forests of rooted binary trees.

0. Introduction

The aim of this article is to study the poset $\mathrm{For}(I)$ attached to a finite set I which was introduced in [1] in relation with a Hopf operad of forests of binary trees. The underlying set of $\mathrm{For}(I)$ is the set of leaf-labeled forests of rooted binary trees with label set I. The main result is the following theorem.

Theorem 0.1. *The characteristic polynomial of any interval in the poset* $\mathrm{For}(I)$ *has only non-negative integer roots.*

Furthermore, an explicit description of the roots is obtained for all intervals. In particular, this gives simple product expressions for all Möbius numbers. The simplest case is the interval between the minimal element E of the poset $\mathrm{For}(I)$ and a rooted binary leaf-labeled tree T on I. To each inner vertex of T, one associates the product of the number of leaves of its two subtrees. These positive integers are the roots of the characteristic polynomial of $[E, T]$. Fig. 1 displays two examples of this computation. When the tree T is a comb, the interval $[E, T]$ is isomorphic to the partition lattice and the roots are $1, 2, \ldots, n$, where $n+1$ is the cardinal of I, see the right example in Fig. 1. One recovers the well-known factorization of the characteristic polynomial of the partition lattice, by a method which differs from

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0097-3165/03/$ - see front matter
doi:10.1016/S0097-3165(03)00048-7

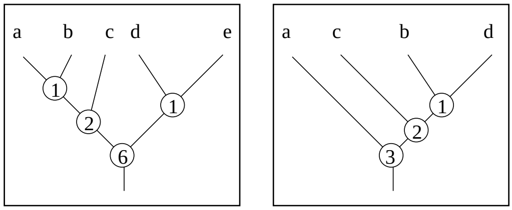

Fig. 1. Roots of characteristic polynomials.

those reviewed in [2]. The other main result is an explanation of the coincidence of some characteristic polynomials observed from the obtained description. This is shown to be a consequence of some isomorphisms between the intervals. The strategy of proof is to decompose as much as possible the intervals as products of simpler intervals. This gives a reduction to the case of some special intervals, for which another kind of decomposition can be done. The first section is devoted to general results on these posets and to the relation between combs and the partition lattice. The intervals and their decompositions are studied in the second section. The third section contains the proof that these posets are ranked by the number of inner vertices. In the fourth section, invariants of the intervals are computed, including the characteristic polynomials. The last section contains the proof of some expected isomorphisms between the intervals.

1. Definition of posets

1.1. Notations

A *tree* is a leaf-labeled rooted binary tree and a *forest* is a set of such trees. Vertices are either inner vertices (valence 3) or leaves and roots (valence 1). By convention, edges are oriented towards the root. Leaves are bijectively labeled by a finite set. Trees and forests are pictured with their roots down and their leaves up, but are not to be considered as planar. A leaf is an *ancestor* of a vertex if there is a path from the leaf to the root going through the vertex. If T_1 and T_2 are trees on I_1 and I_2, let $T_1 \vee T_2$ be the tree on $I_1 \sqcup I_2$ obtained by grafting the roots of T_1 and T_2 on a new inner vertex. If $F_1, F_2, \ldots, F_k$ are forests on $I_1, I_2, \ldots, I_k$, let $F_1 \sqcup F_2 \sqcup \cdots \sqcup F_k$ be their disjoint union. If F is the disjoint union of a forest on J and a forest on J', these restricted forests are denoted by $F[J]$ and $F[J']$. For a forest F, let $V(F)$ be the set of its inner vertices. The number of trees in a forest F on I is the difference between the cardinal of I and the cardinal of $V(F)$.

1.2. Posets of forests

Let F and F' be forests on the label set I. Then set $F \leqslant F'$ if there is a topological map from F to F' with the following properties:

1. It is increasing with respect to orientation towards the root.
2. It maps inner vertices to inner vertices injectively.
3. It restricts to the identity of I on leaves.
4. Its restriction to each tree of F is injective.

In fact, such a topological map from F to F' is determined up to isotopy by the images of the inner vertices of F. One can recover the map by joining the image of an inner vertex of F in F' with the leaves of F' which were its ancestor leaves in F. Observe that there can be different F lower than a given F' with the same image of $V(F)$ in $V(F')$.

Lemma 1.1. *Let F, F' be two distinct forests on I. If $F \leqslant F'$ then the cardinal of $V(F)$ is strictly less than the cardinal of $V(F')$.*

Proof. Assume that $F \leqslant F'$ and the cardinal of $V(F)$ is equal to that of $V(F')$. Then F and F' have the same number of trees. But each tree of F is contained in a tree of F' by connectivity. Each tree of F' contains at least one tree of F by the third condition in the definition of $\leqslant$. Therefore each tree of F is contained in exactly one tree of F'. As these two trees have the same number of vertices, they must be equal. Hence $F = F'$. □

Proposition 1.2. *The relation $\leqslant$ defines a partial order on the set* $\mathrm{For}(I)$ *of forests on I.*

Proof. Reflexivity is given by the identity map. Transitivity is easy to check for each of the four required properties. Antisymmetry is clear by Lemma 1.1. □

A counterexample, not injective on inner vertices, is given in Fig. 2 and an example in Fig. 3.

Lemma 1.3. *If T_1 and T_2 are trees on I_1 and I_2 then $T_1 \sqcup T_2 \leqslant T_1 \vee T_2$.*

Proof. Obvious. □

Lemma 1.3 implies that, for each forest F which is not a tree, there exists a forest F' with strictly less trees such that $F \leqslant F'$. Lemma 1.1 implies that trees are maximal elements. Therefore the maximal elements of the poset $\mathrm{For}(I)$ are exactly the trees. The forest without inner vertex is the unique minimal element, denoted by E. The intervals in the poset $\mathrm{For}(I)$ are not semimodular in general, as can be seen on the interval depicted in Fig. 4.

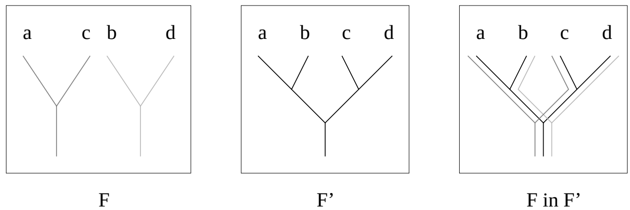

Fig. 2. A counterexample for the order relation.

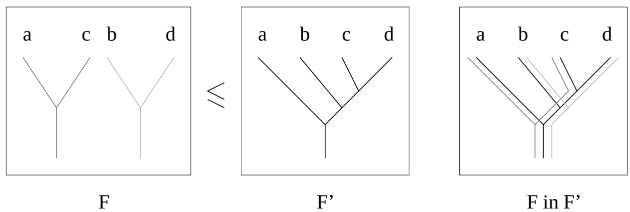

Fig. 3. An example for the order relation.

1.3. Relation to the partition lattice

A *comb* is a tree such that each inner vertex has at least one of its two subtrees reduced to an edge.

Proposition 1.4. *The interval between E and a comb C on the set I is isomorphic to the partition lattice of the set I.*

Proof. Let us remark first that a forest which is lower than a comb is necessarily composed of combs. The isomorphism ϕ is given by mapping a forest of combs to the partition of I defined on the leaves by the combs. Let J be a subset of I. Then there is exactly one comb C_J with leaf set J such that there exists an injective topological map from C_J to C which respects orientation and restricts to the identity of J on leaves.

This implies that each partition of I can in only one way be realized as the leaf set of a forest of combs which is lower than C. Hence ϕ is bijective. That the map ϕ is an isomorphism of posets follows easily from the description of the partial order, which is seen to coincide via ϕ with the refinement order on partitions. □

2. Properties of intervals

2.1. Decomposition by connected components

Let $F \leqslant F'$ be forests on I. Let $F' = T'_1, T'_2, \dots, T'_k$ seen as a set of trees T'_j on I_j with $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_k$. Then F can be uniquely decomposed as a union of forests $F_j = F[I_j]$ on I_j satisfying $F_j \leqslant T'_j$.

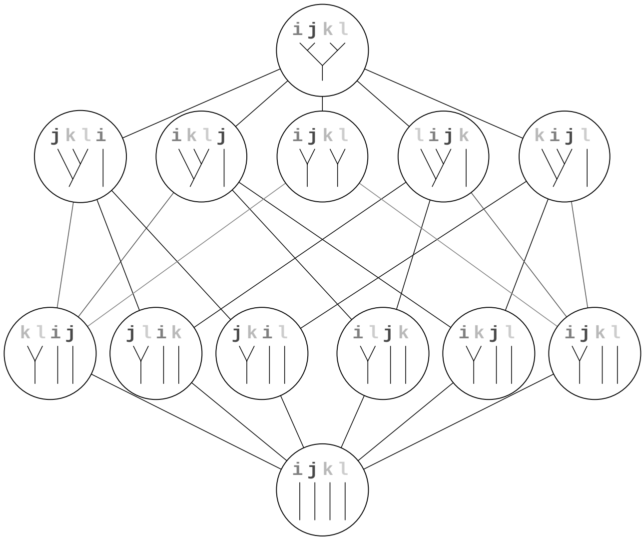

Fig. 4. An interval in the poset of forests on $\{i,j,k,\ell\}$.

Proposition 2.1. *The interval* $[F,F']$ *is isomorphic to the product of the intervals* $[F_j,T_j']$ *in* $\mathrm{For}(I_j)$ *for* $1 \leqslant j \leqslant k$.

Proof. Each element of this interval can in the same way be uniquely decomposed as a union of forests on I_j. The conditions defining the partial order then become equivalent to independent conditions on each part I_j. $\square$

One can therefore restrict attention to intervals between a forest and a tree.

2.2. Elements lower than a tree

Let T be a tree on the set I. Let us describe all elements F of $\mathrm{For}(I)$ which are lower than T. The binary tree T defines a partition $I = I_1 \sqcup I_2$ and two subtrees T_1 on I_1 and T_2 on I_2. If $F_1 \leqslant T_1$ and $F_2 \leqslant T_2$ then clearly $F_1 \sqcup F_2 \leqslant T$. Let F_1, F_2 be forests with $F_1 \leqslant T_1$ and $F_2 \leqslant T_2$. Let J_1 (resp. J_2) be a chosen part of I_1 (resp. I_2) corresponding to a chosen tree of F_1 (resp. F_2). Denote by $G(F_1,J_1,F_2,J_2)$ the forest constructed from the disjoint union of F_1 and F_2 by grafting a new inner vertex to the roots of the chosen trees. This forest satisfies $G(F_1,J_1,F_2,J_2) \leqslant T$.

Proposition 2.2. *A forest* F *lower than* T *is either the disjoint union* $F_1 \sqcup F_2$ *where* $F_1 \leqslant T_1$ *and* $F_2 \leqslant T_2$ *or is equal to* $G(F_1,J_1,F_2,J_2)$ *where* $F_1 \leqslant T_1$, $F_2 \leqslant T_2$ *and* J_1, J_2 *are parts of* I_1, I_2 *corresponding to trees of* F_1, F_2.

Proof. Two forests F_1 and F_2 can be defined as follows. Consider the inner vertices of F having only elements of I_1 as ancestors. By joining them in T to their ancestor leaves, one gets F_1 on I_1 which satisfies $F_1 \leqslant T_1$. The same construction gives F_2 on I_2 with $F_2 \leqslant T_2$.

Assume first that the image of $V(F)$ in $V(T)$ does not contain the lowest inner vertex of T. From the definition of the poset, F is in fact lower than $T_1 \sqcup T_2$ and is the disjoint union $F_1 \sqcup F_2$.

Assume now on the contrary that the image of $V(F)$ in $V(T)$ contains the lowest inner vertex of T. By injectivity on inner vertices, there exists a unique tree T' of F which has an inner vertex mapped to the lower inner vertex of T. By injectivity on trees, the tree T' can be written $T'_1 \vee T'_2$ where T'_1 (resp. T'_2) has leaf set $J_1 \subset I_1$ (resp. $J_2 \subset I_2$). The tree T'_1 (resp. T'_2) is a tree of F_1 (resp. F_2) and F is indeed equal to $G(F_1, J_1, F_2, J_2)$. $\square$

2.3. Intervals under a tree

Let T be a tree and F a forest on the set I such that $F \leqslant T$ and the image of $V(F)$ in $V(T)$ contains the lowest inner vertex of T. This implies that F can be written $G(F_1, J_1, F_2, J_2)$ as explained in the previous section.

Proposition 2.3. *The interval* $[F, T]$ *is isomorphic to the product of the intervals* $[F_1, T_1]$ *in* $\mathrm{For}(I_1)$ *and* $[F_2, T_2]$ *in* $\mathrm{For}(I_2)$.

Proof. Let F' be an element of the interval $[F, T]$. Necessarily the image of $V(F')$ contains the lowest vertex of T. Therefore one can write $F' = G(F'_1, J'_1, F'_2, J'_2)$ with $F'_1 \leqslant T_1$ and $F'_2 \leqslant T_2$. By definition of the partial order, the inequality $F \leqslant F'$ implies that $F_1 \leqslant F'_1$, $F_2 \leqslant F'_2$ and that J'_1 (resp. J'_2) must contain J_1 (resp. J_2). It follows that J'_1 and J'_2 are uniquely determined for a given F'_1 and F'_2. Therefore, any pair (F'_1, F'_2) with $F_1 \leqslant F'_1 \leqslant T_1$ and $F_2 \leqslant F'_2 \leqslant T_2$ can be uniquely extended to an element of $[F, T]$.

The elements of the interval $[F, T]$ are therefore in bijection with pairs (F'_1, F'_2) in $[F_1, T_1] \times [F_2, T_2]$.

The conditions defining the partial order do not depend on J_1 and J_2, and are mapped by the bijection to independent conditions on I_1 and I_2. Hence the bijection is an isomorphism of posets. $\square$

2.4. Special intervals

Let F be a forest and T be a tree on the set I with $F \leqslant T$. Assume that the image of $V(F)$ in $V(T)$ does not contain the lowest inner vertex of T, that is to say F is a disjoint union $F_1 \sqcup F_2$ on I_1 and I_2. The intervals of the form $[F, T]$ for such F and T are called *special intervals*.

Proposition 2.4. *There are three kinds of sub-intervals in a special interval* $[F, T]$:

1. $[F_1' \sqcup F_2', F_1'' \sqcup F_2'']$ *with* $F_1 \leqslant F_1' \leqslant F_1'' \leqslant T_1$ *and* $F_2 \leqslant F_2' \leqslant F_2'' \leqslant T_2$. *This interval is isomorphic to* $[F_1', F_1''] \times [F_2', F_2'']$.
2. $[G(F_1', J_1', F_2', J_2'), G(F_1'', J_1'', F_2'', J_2'')]$ *with* $F_1 \leqslant F_1' \leqslant F_1'' \leqslant T_1$ *and* $F_2 \leqslant F_2' \leqslant F_2'' \leqslant T_2$ *where* J_1'' *and* J_2'' *are the unique parts of* F_1'' *and* F_2'' *containing* J_1' *and* J_2'. *This interval is isomorphic to* $[F_1', F_1''] \times [F_2', F_2'']$.
3. $[F_1' \sqcup F_2', G(F_1'', J_1'', F_2'', J_2'')]$ *with* $F_1 \leqslant F_1' \leqslant F_1'' \leqslant T_1$, $F_2 \leqslant F_2' \leqslant F_2'' \leqslant T_2$, *and* J_1'', J_2'' *are arbitrary parts of* F_1'' *and* F_2''.

Proof. First, let us determine which elements F' can be lower than T and greater than F. If F' is a disjoint union $F_1' \sqcup F_2'$, then it is necessary and sufficient that $F_1 \leqslant F_1' \leqslant T_1$ and $F_2 \leqslant F_2' \leqslant T_2$. If $F' = G(F_1', J_1', F_2', J_2')$, then necessary and sufficient conditions are also that $F_1 \leqslant F_1' \leqslant T_1$ and $F_2 \leqslant F_2' \leqslant T_2$.

Let us discuss now the possible intervals according to the type of their bounds. First, it is not possible to have a relation $G(F_1', J_1', F_2', J_2') \leqslant (F_1'' \sqcup F_2'')$, because the lowest inner vertex is present in the first element and not in the second one, which would contradict injectivity.

Let us study each of the three remaining cases.

Case $[\sqcup, \sqcup]$: One can apply Proposition 2.1. The interval $[F_1' \sqcup F_2', F_1'' \sqcup F_2'']$ is non-empty if and only if $F_1' \leqslant F_1''$ and $F_2' \leqslant F_2''$. If these conditions are fulfilled, this interval is isomorphic to the claimed product.

Case $[G, G]$: *The interval* $[G(F_1', J_1', F_2', J_2'), G(F_1'', J_1'', F_2'', J_2'')]$. Let $F_1'' = f_1'' \sqcup T_1''$ where $T_1'' = F_1''[J_1'']$ is a tree and similarly let $F_2'' = f_2'' \sqcup T_2''$ where $T_2'' = F_2''[J_2'']$ is a tree. Then $G(F_1'', J_1'', F_2'', J_2'')$ is equal to $f_1'' \sqcup f_2'' \sqcup (T_1'' \vee T_2'')$. One can then decompose the interval as a product by Proposition 2.1. By applying Proposition 2.2 to the interval under $T_1'' \vee T_2''$, the product interval is non-empty if and only if one has $F_1' \leqslant F_1''$ and $F_2' \leqslant F_2''$ and the parts J_1'' and J_2'' contain respectively the parts J_1' and J_2'. When these conditions are satisfied, Proposition 2.3 shows that this interval is isomorphic to the claimed product.

Case $[\sqcup, G]$: *The interval* $[F_1' \sqcup F_2', G(F_1'', J_1'', F_2'', J_2'')]$. Let $F_1'' = f_1'' \sqcup T_1''$ where $T_1'' = F_1''[J_1'']$ is a tree and similarly let $F_2'' = f_2'' \sqcup T_2''$ where $T_2'' = F_2''[J_2'']$ is a tree. Then $G(F_1'', J_1'', F_2'', J_2'')$ is equal to $f_1'' \sqcup f_2'' \sqcup (T_1'' \vee T_2'')$. One can then decompose the interval as a product by Proposition 2.1. By applying Proposition 2.2 to the interval under $T_1'' \vee T_2''$, the product interval is non-empty if and only if one has $F_1' \leqslant F_1''$ and $F_2' \leqslant F_2''$. □

3. Rank property

Say that a finite poset is *ranked* if it has a unique minimal element $\hat{0}$ and all maximal chains have the same length. Note that this definition differs slightly from the usual definition which requires the uniqueness of the maximal element.

Proposition 3.1. *The poset* For(I) *is ranked by the number of inner vertices.*

Proof. The proof is by induction on the cardinal of I. The proposition is true by inspection for small I.

Fix a maximal interval $[E, T]$ where T is a tree on I and E is the forest without inner vertices. Consider a maximal chain $E = E_0 \leqslant \cdots \leqslant F = E_{k-1} \leqslant E_k = T$ in $[E, T]$. It is clear from Lemma 1.1 that the length k is at most the number of inner vertices of T.

Let us proceed according to the properties of F.

Assume first that F contains the lowest inner vertex of T. By maximality, there should be no element between F and T, and one can conclude by induction hypothesis and Proposition 2.3 that either $F_1 = T_1$ and F_2 has just one vertex less than T_2 or the similar situation obtained by exchanging 1 and 2 holds.

Assume on the contrary that F does not contain the lowest inner vertex of T. By maximality, there should be no element between F and T, and one can conclude by induction hypothesis and Proposition 2.4 that $F_1 = T_1$ and $F_2 = T_2$.

Therefore, in both cases, the number of inner vertices of F is the number of inner vertices of T minus one. By induction and Proposition 2.1, the length of all maximal chains of $[E, F]$ is the number of inner vertices of F.

This implies that the length of all maximal chains of $[E, T]$ is the number of inner vertices of T. All trees on I have the same number of inner vertices. The proposition is proved. $\square$

Note that the corank function in For(I) is given by the number of trees minus one.

4. Invariants of intervals

For a standard reference on posets, see [3].

4.1. M-polynomials and Z-polynomials

Let P be a ranked poset with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$. Let crk be the corank function on P, which is defined by $\operatorname{crk}(a) = \operatorname{rk}(\hat{1}) - \operatorname{rk}(a)$. The degree of the poset is $\deg(P) = \operatorname{crk}(\hat{0})$. One defines the M-polynomial of the poset P, which is a generating function for the Möbius function, as follows:

$$M(P) = \sum_{a \leqslant b} \mu(a, b) x^{\operatorname{crk}(a)} y^{\operatorname{crk}(b)}. \tag{1}$$

In the same way, one defines the Z-polynomial, which is a generating function for the zeta function, as follows:

$$Z(P) = \sum_{a \leqslant b} x^{\operatorname{crk}(a)} y^{\operatorname{crk}(b)}. \tag{2}$$

The characteristic polynomial is defined to be

$$\chi(P) = \sum_b \mu(\hat{0}, b) y^{\mathrm{crk}(b)}. \tag{3}$$

The cardinal polynomial is the generating function for the corank:

$$\mathrm{Card}(P) = \sum_a x^{\mathrm{crk}(a)}. \tag{4}$$

The Möbius number is $\mu(P) = \mu(\hat{0}, \hat{1})$. The characteristic polynomial is recovered as the coefficient of $x^{\mathrm{crk}(\hat{0})}$ in the M-polynomial. In turn, the characteristic polynomial contains the Möbius number as constant coefficient. The M-polynomial also contains the information of the cardinal polynomial. The cardinal polynomial is also determined by the Z-polynomial. The following proposition is classical.

Proposition 4.1. *Let P_1 and P_2 be two such ranked posets and $P_1 \times P_2$ their product. Then*

$$Z_{P_1 \times P_2} = Z_{P_1} Z_{P_2} \quad \text{and} \quad M_{P_1 \times P_2} = M_{P_1} M_{P_2}. \tag{5}$$

Lemma 4.2. *The value at $y = 1$ of the M-polynomial is* 1.

Proof. This is an immediate consequence of the definition of the Möbius function and the existence of $\hat{1}$. □

4.2. Z-polynomials of special intervals

Let F be a forest and T be a tree on the set I. Assume that $F \leqslant T$ and the image of $V(F)$ in $V(T)$ does not contain the lowest inner vertex of T. We keep the notations of Section 2.4.

Theorem 4.3. *The Z-polynomial of the special interval $[F, T]$ (denoted by Z) is determined by the Z-polynomials of the intervals $[F_1, T_1]$ and $[F_2, T_2]$ (denoted by Z_1 and Z_2). One has*

$$Z = xyZ_1Z_2 + \partial_x(xZ_1)\partial_x(xZ_2) + x\partial_y(yZ_1)\partial_y(yZ_2). \tag{6}$$

Proof. The sum defining Z is split in three parts, according to the three different kinds of subintervals in $[F, T]$ listed in Proposition 2.4.

The first part is given by

$$\sum_{\substack{F_1' \leqslant F_1'' \\ F_2' \leqslant F_2''}} x^{1+\mathrm{crk}(F_1')+\mathrm{crk}(F_2')} y^{1+\mathrm{crk}(F_1'')+\mathrm{crk}(F_2'')},$$

which is xyZ_1Z_2. The second part is given by

$$\sum_{\substack{F_1'\leqslant F_1''\\ F_2'\leqslant F_2''}} \sum_{J_1',J_2'} x^{\mathrm{crk}(F_1')+\mathrm{crk}(F_2')} y^{\mathrm{crk}(F_1'')+\mathrm{crk}(F_2'')}$$
$$= \sum_{\substack{F_1'\leqslant F_1''\\ F_2'\leqslant F_2''}} (1+\mathrm{crk}(F_1'))(1+\mathrm{crk}(F_2'))x^{\mathrm{crk}(F_1')+\mathrm{crk}(F_2')} y^{\mathrm{crk}(F_1'')+\mathrm{crk}(F_2'')},$$

which is $\partial_x(xZ_1)\partial_x(xZ_2)$. The last part is given by

$$\sum_{\substack{F_1'\leqslant F_1''\\ F_2'\leqslant F_2''}} \sum_{J_1'',J_2''} x^{1+\mathrm{crk}(F_1')+\mathrm{crk}(F_2')} y^{\mathrm{crk}(F_1'')+\mathrm{crk}(F_2'')}$$
$$= \sum_{\substack{F_1'\leqslant F_1''\\ F_2'\leqslant F_2''}} (1+\mathrm{crk}(F_1''))(1+\mathrm{crk}(F_2''))x^{1+\mathrm{crk}(F_1')+\mathrm{crk}(F_2')} y^{\mathrm{crk}(F_1'')+\mathrm{crk}(F_2'')},$$

which is $x\partial_y(yZ_1)\partial_y(yZ_2)$. This concludes the proof of the theorem. □

4.3. *M-polynomials of special intervals*

Let F be a forest and T be a tree on the set I. Assume that $F\leqslant T$ and the image of $V(F)$ in $V(T)$ does not contain the lowest inner vertex of T. We keep the notations of Section 2.4.

Theorem 4.4. *The M-polynomial of the special interval* $[F,T]$ (*denoted by* M) *depends only on the M-polynomials of the intervals* $[F_1,T_1]$ *and* $[F_2,T_2]$ (*denoted by* M_1 *and* M_2). *One has*

$$M = xyM_1M_2 + (1-x)\partial_x(xM_1)\partial_x(xM_2). \tag{7}$$

Proof. By induction on the degree of $[F,T]$.

Formula (7) is correct if $F=T_1\sqcup T_2$ and $T=T_1\vee T_2$, which is the only possible case of degree 1.

The sum which defines M is split in three parts, according to the three different kinds of subintervals in $[F,T]$ listed in Proposition 2.4. The first part is given by

$$\sum_{\substack{F_1'\leqslant F_1''\\ F_2'\leqslant F_2''}} \mu(F_1',F_1'')\mu(F_2',F_2'')x^{1+\mathrm{crk}(F_1')+\mathrm{crk}(F_2')} y^{1+\mathrm{crk}(F_1'')+\mathrm{crk}(F_2'')},$$

which is xyM_1M_2. The second part is given by

$$\sum_{\substack{F_1'\leqslant F_1''\\ F_2'\leqslant F_2''}} \sum_{\substack{J_1'\\ J_2'}} \mu(F_1',F_1'')\mu(F_2',F_2'')x^{\mathrm{crk}(F_1')+\mathrm{crk}(F_2')}y^{\mathrm{crk}(F_1'')+\mathrm{crk}(F_2'')}$$
$$= \sum_{\substack{F_1'\leqslant F_1''\\ F_2'\leqslant F_2''}} (1+\mathrm{crk}(F_1'))(1+\mathrm{crk}(F_2'))\mu(F_1',F_1'')\mu(F_2',F_2'')$$
$$\times\ x^{\mathrm{crk}(F_1')+\mathrm{crk}(F_2')}y^{\mathrm{crk}(F_1'')+\mathrm{crk}(F_2'')},$$

which is $\partial_x(xM_1)\partial_x(xM_2)$.

The computation of the third part is more complicated. It is given by

$$\sum_{\substack{F_1'\leqslant F_1''\\ F_2'\leqslant F_2''}} \left(\sum_{\substack{J_1''\\ J_2''}} \mu([F_1'\sqcup F_2', G(F_1'',J_1'',F_2'',J_2'')]) \right) x^{1+\mathrm{crk}(F_1')+\mathrm{crk}(F_2')}y^{\mathrm{crk}(F_1'')+\mathrm{crk}(F_2'')}. \qquad (8)$$

Let us simplify the inner summation. Fix a part J_1'' and a part J_2'' and assume that $\mathrm{crk}(G(F_1'',J_1'',F_2'',J_2''))\neq 0$. This excludes only the case when $G(F_1'',J_1'',F_2'',J_2'') = T$.

Then $G(F_1'',J_1'',F_2'',J_2'')$ is not a tree. Let K_1'' be the complement of J_1'' in I_1 and K_2'' be the complement of J_2'' in I_2.

Denote the trees $F_1''[J_1'']$ and $F_2''[J_2'']$ by T_1'' and T_2''. Then G can be uniquely decomposed as

$$F_1''[K_1'']\sqcup F_2''[K_2'']\sqcup (T_1''\vee T_2'')$$

and $F_1'\sqcup F_2'$ can also be decomposed as

$$F_1'[K_1'']\sqcup F_2'[K_2'']\sqcup (F_1'[J_1'']\sqcup F_2'[J_2'']).$$

Hence the interval $[F_1'\sqcup F_2', G(F_1'',J_1'',F_2'',J_2'')]$ is isomorphic to the product

$$[F_1'[K_1''],F_1''[K_1'']]\times[F_2'[K_2''],F_2''[K_2'']]\times[F_1'[J_1'']\sqcup F_2'[J_2''], T_1''\vee T_2''].$$

The inner sum can be rewritten as

$$\sum_{\substack{J_1''\\ J_2''}} \mu([F_1'[K_1''],F_1''[K_1'']])\mu([F_2'[K_2''],F_2''[K_2'']])\mu([F_1'[J_1'']\sqcup F_2'[J_2''], T_1''\vee T_2'']),$$

where the sum runs over the set of pairs (J_1'',J_2'') of parts of F_1'' and F_2''. The parts K_1'', K_2'' and trees T_1'', T_2'' depend on the pair (J_1'',J_2'') as before.

As stated below in Corollary 4.5, it follows from the induction hypothesis that for all special intervals $[F^\natural, T^\natural]$ of smaller degree, one has

$$\mu([F^\natural,T^\natural]) = -(\deg([F_1^\natural,T_1^\natural])+1)(\deg([F_2^\natural,T_2^\natural])+1)\mu([F_1^\natural,T_1^\natural])\mu([F_2^\natural,T_2^\natural]). \qquad (9)$$

Hence using this consequence of the induction hypothesis, the inner sum is

$$-\sum_{\substack{J_1''\\J_2''}} \mu([F_1'[K_1''],F_1''[K_1'']])\mu([F_2'[K_2''],F_2''[K_2'']])((\deg([F_1'[J_1''],T_1''])+1)$$
$$\times(\deg([F_2'[J_2''],T_2''])+1)\mu([F_1'[J_1''],T_1''])\mu([F_2'[J_2''],T_2''])),$$

where the sum runs over the set of pairs (J_1'',J_2'') of parts of F_1'' and F_2''.

The opposite of this sum can be decomposed as the product of

$$\sum_{J_1''} \mu([F_1'[K_1''],F_1''[K_1'']])(\deg([F_1'[J_1''],T_1''])+1)\mu([F_1'[J_1''],T_1''])$$
$$= \mu([F_1',F_1'']) \sum_{J_1''} (\deg([F_1'[J_1''],T_1''])+1),$$

and the similar sum over J_2''. But the sum of $\deg([F_1'[J_1''],T_1''])+1$ over J_1'' is equal to $\operatorname{crk}(F_1')+1$. Indeed, it is $\deg([F_1',F_1''])+\operatorname{crk}(F_1'')+1$, as the number of trees of F_1'' is $\operatorname{crk}(F_1'')+1$.

So the inner sum is equal to

$$-(\operatorname{crk}(F_1')+1)(\operatorname{crk}(F_2')+1)\mu([F_1',F_1''])\mu([F_2',F_2'']). \tag{10}$$

Hence the third part (8) is equal, up to a polynomial in x corresponding to special intervals with maximal element T, to

$$-\sum_{\substack{F_1'\leqslant F_1''\\F_2'\leqslant F_2''}} (1+\operatorname{crk}(F_1'))(1+\operatorname{crk}(F_2'))\mu([F_1',F_1''])\mu([F_2',F_2''])$$
$$\times x^{1+\operatorname{crk}(F_1')+\operatorname{crk}(F_2')}y^{\operatorname{crk}(F_1'')+\operatorname{crk}(F_2'')}, \tag{11}$$

which is $-x\partial_x(xM_1)\partial_x(xM_2)$.

Therefore the full sum M is equal to the expected formula, up to a polynomial in x. By Lemma 4.2 the value of M at $y=1$ is 1, and the value of the right-hand side of formula (7) at $y=1$ is also 1. Hence formula (7) stands exactly. The induction step is done and the theorem is proved. $\square$

Corollary 4.5. *The characteristic polynomial of the interval* $[F,T]$ (*denoted by* χ) *depends only on the characteristic polynomials of the intervals* $[F_1,T_1]$ *and* $[F_2,T_2]$ (*denoted by* χ_1 *and* χ_2). *One has*

$$\chi(y)=(y-(\deg_1+1)(\deg_2+1))\chi_1(y)\chi_2(y), \tag{12}$$

where $\deg_1=\deg([F_1,T_1])$ *and* $\deg_2=\deg([F_2,T_2])$. *As a special case, one has*

$$\mu=-(\deg_1+1)(\deg_2+1)\mu_1\mu_2 \tag{13}$$

for Mőbius numbers.

Proof. The characteristic polynomial is the coefficient of the maximal power of x in the M-polynomial. Considering the coefficient of $x^{\deg_1+\deg_2+1}$ in Eq. (7) and using the fact that the x-degrees of M_1 and M_2 are $\deg_1$ and $\deg_2$, respectively, one gets the relation (12) between characteristic polynomials. Then, as the constant term of the characteristic polynomial is the Möbius number, the relation (12) between Möbius numbers follows. $\square$

4.4. Factorization of characteristic polynomials

Let F, F' be forests on the set I with $F \leqslant F'$. Let V be the image of $V(F)$ in $V(F')$. Let us call *marked vertices* the elements of V. To each non-marked vertex $v \in V(F') \backslash V$, there correspond two subtrees T_1 and T_2. Let d_1 (resp. d_2) be the number of leaves of T_1 (resp. T_2) minus the number of marked vertices of T_1 (resp. T_2). One associates to the non-marked vertex v its *exponent* which is the integer $d_1 d_2$. The exponents of the pair (F, F') are the exponents of the non-marked vertices of F'. We remark that the exponents of (F, F') only depend on F' and the set V of marked inner vertices, not on which F is mapped to F' using V.

Theorem 4.6. *The characteristic polynomial of the interval* $[F, F']$ *has only positive integer roots*, *which are the exponents of the pair* (F, F').

Proof. The proof is by induction on the cardinal of I and $\deg([F, F'])$.

The statement is true for $\deg([F, F']) = 0$ or I a singleton, in which cases the set of exponents is empty.

If F' is not a tree, then using Proposition 2.1, the statement is a consequence of the induction hypothesis.

If F' is a tree and V contains the bottom vertex of F', then the statement follows from the induction hypothesis by using Proposition 2.3.

If F' is a tree and V does not contain the bottom vertex of F', then the statement follows from the induction hypothesis by using Corollary 4.5.

The theorem is proved. $\square$

Some more examples are given in Fig. 5.

5. Partitive posets

5.1. Definition and product

A *partitive poset* is a ranked poset P with $\hat{0}$ and $\hat{1}$ together with a ranked-poset map f_P from P to a subposet of the partition lattice of a finite set I (possibly with shifted rank function). The only examples which will be used here are all intervals $[F, F']$ where F and F' are forests on I with the map defined by the partition of the label set according to trees. One can define the product of two partitive posets (P_1, I_1, f_1) and (P_2, I_2, f_2). As a ranked poset, it is the usual product $P_1 \times P_2$. The

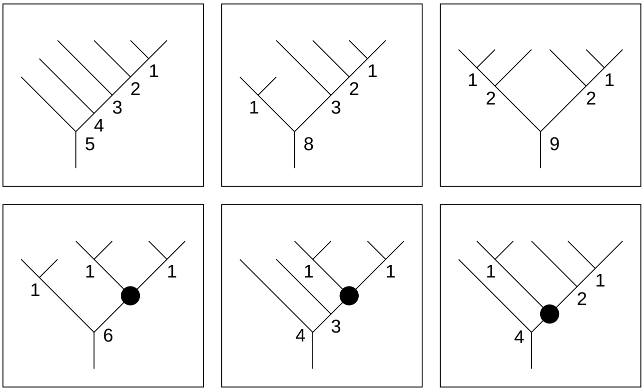

Fig. 5. Other examples of exponents.

composition of $f_1 \times f_2$ with the inclusion map of partition lattices (induced by disjoint union of partitions) defines a poset map from $P_1 \times P_2$ to the partition lattice of $I_1 \sqcup I_2$.

Proposition 5.1. *The interval* $[F_1 \sqcup F_2, F_1' \sqcup F_2']$ *is isomorphic as a partitive poset to the product of* $[F_1, F_1']$ *and* $[F_2, F_2']$.

Proof. This is an easy reformulation of Proposition 2.1. $\square$

5.2. Twisted product of partitive posets

Consider two partitive posets P_1, P_2 such that their respective $\hat{1}$ are mapped to a partition with only one part. Choose for each of these posets a part of the image of their $\hat{0}$ in the corresponding partition lattice. These chosen parts are denoted by K_1 and K_2. The twisted product of P_1 and P_2 is defined as follows. As a poset, it is simply $P_1 \times P_2$. The map to a partition lattice differs from the map for the usual product by gathering the parts containing K_1 and K_2 to a single part. It is easy to see that, up to isomorphism of partitive poset, this construction does not depend on the choices made. Let F be a forest and T be a tree on the set I with $F \leqslant T$. The binary tree T defines a partition $I = I_1 \sqcup I_2$ and two subtrees T_1 on I_1 and T_2 on I_2. Assume that the image of $V(F)$ contains the lowest inner vertex of T. Recall the notations of 2.3.

Proposition 5.2. *The interval* $[F, T]$ *is isomorphic as a partitive poset to the twisted product of the intervals* $[F_1, T_1]$ *and* $[F_2, T_2]$.

Proof. This is an easy reformulation of Proposition 2.3. $\square$

5.3. The $\bigvee$-product of partitive posets

Consider two partitive posets P_1, P_2 such that their respective $\hat{1}$ are mapped to a partition with only one part. The $\bigvee$ product of P_1 and P_2 is defined as follows. The underlying set is the disjoint union of $P_1 \times P_2$ with elements denoted by $\{a_1 \sqcup a_2\}$ and the set $\{G(a_1, J_1, a_2, J_2)\}$ where a_1 and a_2 are elements of P_1 and P_2, respectively and J_1 and J_2 are parts of the image of a_1 and a_2. The order relation is given by the following relations:

1. $(a_1 \sqcup a_2) \leqslant (a_1' \sqcup a_2')$ if $a_1 \leqslant a_1'$ and $a_2 \leqslant a_2'$.
2. $G(a_1, J_1, a_2, J_2) \leqslant G(a_1', J_1', a_2', J_2')$ if $a_1 \leqslant a_1'$, $a_2 \leqslant a_2'$ and the part J_1' (resp. J_2') contains the part J_1 (resp. J_2).
3. $(a_1 \sqcup a_2) \leqslant G(a_1', J_1', a_2', J_2')$ if $a_1 \leqslant a_1'$ and $a_2 \leqslant a_2'$.

The map to a partition lattice is defined as follows. An element $a_1 \sqcup a_2$ is mapped to the disjoint union of the partitions associated to a_1 and a_2. An element $G(a_1, J_1, a_2, J_2)$ is mapped to the partition obtained from the disjoint union of the partitions associated to a_1 and a_2 by gathering the two parts J_1 and J_2 to a single part. The result is a partitive poset, called the $\vee$-product of P_1 and P_2. Let F be a forest and T be a tree on the set I. Assume that $F \leqslant T$ and the image of $V(F)$ in $V(T)$ does not contain the lowest inner vertex of T. We keep the notations of Section 2.4.

Proposition 5.3. *The interval $[F, T]$ is isomorphic as a partitive poset to the $\vee$-product of the intervals $[F_1, T_1]$ and $[F_2, T_2]$.*

Proof. This is essentially a reformulation of Proposition 2.4. $\square$

5.4. Marked trees

Let F, F' be forests on the set I with $F \leqslant F'$.

Theorem 5.4. *Up to isomorphism of partitive posets, the interval $[F, F']$ depends only on the pair (F', V) where V is the subset of marked inner vertices of F' associated to F.*

Proof. By induction on the degree of $[F, F']$ and the cardinal of I. This is clear if the degree is zero or the cardinal of I is one.

If F' is not a tree, then the proposition follows from the induction hypothesis and Proposition 5.1.

If F' is a tree and V contains the lowest inner vertex of F', the statement follows from the induction hypothesis and Proposition 5.2.

If F' is a tree and V does not contain the lowest inner vertex of F', this follows from the induction hypothesis and Proposition 5.3. $\square$

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