Almost-sure path properties of (2, d, β)-superprocesses

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Abstract

We obtain exact almost-sure estimates for the short-time propagation of the closed support of (2, d, β)-superprocesses. Upper estimates are derived by solving a certain singular non-linear evolution equation, whereas lower estimates are obtained by the use of the branching-particle-system approximation.

Key words: Superprocess; Branching particle system; Modulus of continuity; Rate of convergence; Non-linear evolution equation

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1. Introduction

In this work, we study path properties of spatially and time-homogeneous superprocesses \( X_t \) with state space \( M_F = M_F(\mathbb{R}^d) \) of finite measures on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) furnished with the weak topology. We refer to Dawson (1992) for an introduction to superprocesses and their properties.

It is known (cf., e.g., Fitzsimmons, 1988; Dawson, 1992) that for a wide class of transition functions and a given initial measure \( \mu \in M_F(\mathbb{R}^d) \) we can construct a canonical measure-valued process \((\mathbb{D}, \mathcal{D}, (\mathcal{D}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_\mu)_{\mu \in M_F(\mathbb{R}^d)})\). Here \( \mathbb{D} := D([0,\infty), M_F(\mathbb{R}^d)), X_t : \mathbb{D} \to M_F(\mathbb{R}^d), X_t(\omega) = \omega(t), \mathcal{D}_t = \sigma\{X_s: 0 \leq s \leq t\}, \mathcal{D} = \bigvee \mathcal{D}_t \), where \( \mathbb{P}_\mu \) is a probability measure on \( \mathbb{D} \) with the Laplace transition functional \( \mathbb{P}_\mu(\exp\{-\langle X_t, \phi \rangle\}) \) of the following form:

\[
\mathbb{P}_\mu(\exp\{-\langle X_t, \phi \rangle\}) = \exp\left\{ -\int (V_\phi(x) \mu(dx)) \right\}, \tag{1.1}
\]

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where \( \phi \) belongs to the space \( b_{p} \mathcal{D} \) of bounded measurable non-negative functions on \( \mathbb{R}^{d} \). The family \( \{V_t\} \) is called the cumulant-, log-Laplace- or \( \psi \)-semigroup and \( v(t, x) := (V_t \phi)(x) \) is characterized as the unique mild solution of

\[
\frac{\partial v}{\partial t} = Av(t, x) + \Phi(v(t, x)); \quad v(0, x) := \phi(x) \tag{1.2}
\]

for some operator \( A \) and function \( \Phi \). Eq. (1.2) is called the log-Laplace equation. In this work, we adopt the terminology used in Dawson (1992). Choose

\[
A = -\frac{1}{2}(-A)^{\alpha/2} \tag{1.3}
\]

with \( A = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}, \alpha \in (0, 2] \), and

\[
\Phi(u) = -\gamma u^{1+\beta}, \tag{1.4}
\]

where \( \gamma \) is an arbitrary positive constant, and \( \beta \in (0, 1] \).

The superprocess corresponding to the log-Laplace equation (1.2) with the above choice of \( A \) and \( \Phi \) is called \((\alpha, d, \beta)-superprocess\). Operator \( A \) and function \( \Phi \) are interpreted respectively as the motion (spherically symmetric stable process in \( \mathbb{R}^d \) with exponent \( \alpha \)) and the local branching, whose distribution belongs to the domain of attraction of an asymmetric stable distribution with exponent \( 1 + \beta \). See also the description of the construction of superprocesses as limits of branching particle systems below, specifically, Formula (1.11). The class of \((2, d, 1)\)-superprocesses was treated in Dawson et al. (1989). Here we generalize some results of that work for \((2, d, \beta)\)-superprocesses. They satisfy Condition (1.3) with \( \alpha = 2 \) and Condition (1.4) (recall that \( \beta \in (0, 1] \)). We also assume without loss of generality that \( \gamma = 1/(1 + \beta) \); the case of an arbitrary positive \( \gamma \) is reduced to this special case by scaling space and time. As will be explained in more details below, the cases \( \beta = 1 \) and \( \beta < 1 \) differ in an important way, namely, the \((2, d, \beta)\)-superprocess has continuous trajectories if and only if \( \beta = 1 \).

Replacing \( A \) and \( \Phi \) in (1.2) by the expressions on the right-hand sides of (1.3) and (1.4) respectively, we get that the \((2, d, \beta)\)-superprocess \( X_t \) is described by the following log-Laplace equation:

\[
\frac{\partial v}{\partial t} = \frac{1}{2} Av - \frac{1}{1 + \beta} v^{1+\beta}; \quad v(0, x) := \phi(x). \tag{1.5}
\]

This approach to study superprocesses \( X_t \) by means of their log-Laplace (non-linear evolution) equations will be used in the Appendix to get estimates useful for describing the growth of the closed support \( S(X_t) \) of the superprocess \( X_t \): properties of some functionals of \( X_t \) will be derived from properties of some non-linear evolution equations that are variants of (1.5).

We first state the results on the growth of the closed support \( S(X_t) \). To this end, we introduce some auxiliary notation. If \( K_1 \) and \( K_2 \) are non-empty compact subsets of
In $\mathbb{R}^d$, let
\[ \rho_1(K_1, K_2) := \min \left[ \sup_{x \in K_1} \text{dist}(x, K_2), 1 \right] \]
and
\[ \rho(K_1, K_2) := \max(\rho_1(K_1, K_2), \rho_1(K_2, K_1)) ; \]
\[ \rho(K_1, \emptyset) := 1. \]
Here $\text{dist}(\cdot, \cdot)$ between a point and a set in $\mathbb{R}^d$ is that induced by the Euclidean metric in $\mathbb{R}^d$ and $\emptyset$ denotes an empty set.

It is not difficult to show that $\rho$ is the Hausdorff metric on the space $\mathcal{K}(\mathbb{R}^d)$ of all compact subsets of $\mathbb{R}^d$ (cf., e.g., Dugundji (1966, p. 205) and Cutler (1984)).

The following result generalizes Theorem 1.2 of Dawson et al. (1989) for the case of an arbitrary $\beta \in (0, 1]$.

**Theorem 1.1.** Assume that $X_0$ has compact support. Then $\{S(X_t), t \geq 0\}$ is a right continuous process taking values in $(\mathcal{K}(\mathbb{R}^d), \rho)$.

The proof of Theorem 1.1 is carried out in Section 2.

**Remark.** Note that for the case $\beta = 1$, Perkins (1990, Theorem 1.4) proved stronger results on the almost-sure path properties of $S(X_\cdot)$ including the existence of left-hand limits.

It turns out that the paths of the support process $S(X_\cdot)$ possess finer properties analogous to P. Lévy's global modulus of continuity for the Wiener process $w(\cdot)$. The proofs of the following Theorem 1.2 (global modulus of continuity for $(2, d, \beta)$-superprocesses) and Theorem 1.3 (local modulus of continuity for $(2, d, \beta)$-superprocesses) are also carried out in Section 2.

Now, we introduce the following real-valued process, which is often called the total mass process and plays an important role in the theory of measure-valued processes:
\[ M_t := X_t(\mathbb{R}^d) = \langle X_t, 1 \rangle, \]
where $I$ is the indicator function and $\langle \cdot, \cdot \rangle$ denotes the inner product.

It appears that the total mass process $M_t$ inherits some interesting properties of the $(2, d, \beta)$-superprocess $X_t$. In addition, it is not surprising that some path properties of the measure-valued process $X_t$ can be easily clarified by the consideration of analogous properties of sample paths of the real-valued process $M_t$. We study the total mass process $M_t$ in Propositions 1.10 and 1.11.

Now, we proceed with the formulation of the following theorem.

**Theorem 1.2.** Let $\mu \in M_F(\mathbb{R}^d)$ and $\kappa > \kappa(\beta, d) := (\beta d + 8\beta + 2)/(2(1 + \beta))$. Let $T > 0$ be fixed. Then for $\mathbb{P}_\mu$-a.e. $\omega$ there exists a $\delta(\omega, \kappa) > 0$ such that if $0 \leq s, t \leq T$ satisfy...
0 < t - s < \delta, then
\[ S(X_t) \subseteq S(X_s)^{g_{\kappa}(t-s)}, \]  
where
\[ g_{\kappa}(u) := \sqrt{(2/\beta)(1 + \beta)\min(1, u(\log(1/u) + \kappa\log\log(1/u)))} \text{ for } u \in (0, e^{-1}) \]
\[ g_{\kappa}(0) := 0. \]  
(1.6')

Here
\[ A^\varepsilon := \{ x \in \mathbb{R}^d : \text{dist}(x, A) \leq \varepsilon \} \]
denotes the \varepsilon-neighborhood of the set A.

Remarks. (i) Note that the result of Theorem 1.2 yields the compact support property, i.e., if \( X_0 \) has compact support, so does \( X_t \) for \( t > 0 \). It also implies that
\[ \mathbb{P}_{\delta_0} \left\{ \lim_{t \downarrow 0} \text{diam}(S(X_t)) = 0 \right\} = 1. \]

(ii) Note that for the special case \( \beta = 1 \) (so-called “continuous Brownian motion”) an analogous result was obtained in Dawson et al. (1989, Theorem 1.1). That theorem contains an analogue of (1.6) as well as some additional information on the path properties of (2, d, 1)-superprocesses, which are formulated in the non-standard setting. The result of Theorem 1.1 of Dawson et al. (1989) was reformulated in the historical setting and proved to be sharp (see Dawson and Perkins (1991, Theorem 8.7)). The result of our Theorem 1.2 for the case \( \beta = 1 \) is sharper in the other sense, namely, the upper estimate (1.6) is more accurate. A better bound for the growth of the closed support \( S(X_t) \) is attained by some refinement of the usual Borel–Cantelli arguments. For the case \( 0 < \beta < 1 \), Theorem 1.2 seems to be new. However, its proof is based on ideas going back to Paul Lévy (1937, pp. 168–172) that had been first used for the derivation of the global modulus of continuity for one-dimensional Wiener process \( w(\cdot) \):
\[ \mathbb{P} \left\{ \lim_{t \downarrow 0} \sup_{0 \leq s \leq 1-t} \sup_{0 \leq u \leq t} \frac{|w(s + u) - w(s)|}{\sqrt{2t \log(1/t)}} = 1 \right\} = 1. \]  
(1.7)

(iii) Note the difference of the constant in (1.6') and \( 2^{1/2} \) in (1.7), even in the case \( \beta = 1 \).

For the case when the left-hand end of the interval is fixed the following result is valid:

**Theorem 1.3.** Let \( \mu \in M_F(\mathbb{R}^d) \) and \( \kappa > \overline{\kappa}(\beta, d) := (d/(2/\beta)) + 1 \). Then for each fixed \( t \geq 0 \) and for \( \mathbb{P}_\mu \text{-a.e. } \omega \) there exists a \( \delta_*(\omega, \kappa) \) such that if \( 0 < s < \delta_*(\omega, \kappa) \), then
\[ S(X_{t+s}) \subseteq S(X_t)^{g_{\kappa}(s)}, \]  
(1.8)
where
\[ h_r(u) := \sqrt{2/\beta} \min(1, u(\log(1/u) + \kappa \log \log(1/u))) \quad \text{for } u \in (0, e^{-1}] \tag{1.8'} \]
\[ h_r(0) := 0. \]

**Remarks.**

(i) Let us point out that the versions of Theorems 1.2 and 1.3 are also true in the historical setting; these versions generalize Theorem 8.7 of Dawson and Perkins (1991). The historical versions of Theorems 1.2 and 1.3 can be found in Dawson et al. (1994). The technique of historical processes is also used in the proof of Proposition 3.2, which plays an auxiliary role in this work.

(ii) Note that for the special case \( \beta = 1 \) (continuous super-Brownian motion) a similar result in the non-standard setting was obtained in Dawson et al. (1989, Theorem 4.5) (cf. also Tribe (1989, Theorem 1.5.b)). Our result for this case is sharper in some sense (see comments to Theorem 1.2). For the case \( 0 < \beta < 1 \) our Theorem 1.3 seems to be new. The result of Theorem 1.3 is actually also true for \( \kappa \in [(d/(2/\beta)), (d/(2/\beta)) + 1] \). However, the proof of this fact involves different arguments using historical processes and is not included here (see Dawson et al. (1994) for more details).

(iii) Note that (1.8) can be viewed as the local modulus of continuity for the super-Brownian motion. However, the local modulus of continuity for \( w(\cdot) \) contains the iterated logarithm and hence has a different form than (1.7) and (1.8) (cf., e.g., Csörgő and Révész (1981, p. 41)) for a heuristic explanation of this phenomenon. Namely, for any fixed \( t_0 \geq 0 \),
\[ \mathbb{P} \left\{ \lim_{0 \leq h \leq t \to 0} \frac{\sup_{t \leq u \leq t} |w(t_0 + h) - w(t_0)|}{\sqrt{2t \log \log(1/t)}} = 1 \right\} = 1. \tag{1.7'} \]

The fact that the form of the local modulus of continuity for the super-Brownian motion resembles the global (not local) modulus of continuity for the Wiener process heuristically reflects the fact that even though we consider here propagation of the closed support of the super-Brownian motion on short time intervals, because of branching we must take into account the possibility of large increments of many individual Wiener processes. Thus, our situation is more similar in character to that of the global modulus of continuity for the Wiener process, where the possibility of large increments on a large number of short time intervals is considered, rather than to that of the local modulus of continuity for the Wiener process, where the consideration centers about the possibility of large increments on a single short time interval. See also Remark (ii) to Proposition 1.8.

Now, let us highlight the main methods used for the derivation of our results. First, in order to prove Theorems 1.2 and 1.3 (analogous to the results of Dawson et al. (1989)) which can be viewed as almost-sure upper estimates for the growth of the closed support \( S(X_t) \), some a priori upper estimates for the probability of the event that the superprocess \( X \) will visit the exterior of the closed ball \( \overline{B}(0, R) \) centered at the
origin with radius $R$ during the time period $[0, t]$ are needed. The following result generalizes Theorem 3.3.b of Dawson et al. (1989) for the case of an arbitrary $\beta \in (0, 1]$:

**Proposition 1.4.**

\[ \mathbb{P}_{a, \delta_0} \{ X_s(\bar{B}(0, R)^c) > 0 \text{ for some } s \leq t \} \]

\[ \leq C(\beta, d) \cdot a \cdot R^{-2/\beta} \cdot (R/\sqrt{t})^{d+(4/\beta)-2} \cdot \exp \{ -R^2/(2t) \} \]

(1.9)

if $R > 2\sqrt{t}$, where $A^c$ is the complement to set $A$, $\bar{A}$ is the closure of set $A$, $a > 0$ is any real, and $C(\beta, d)$ is a certain positive constant depending only on $\beta$ and $d$.

The proof of Proposition 1.4 is based on the investigation of properties of a certain non-linear evolution equation (cf. (4.5)) and follows along the same lines as that of Theorem 3.3.b of Dawson et al. (1989). A detailed proof is given in the Appendix.

Let us emphasize that the range of application of Proposition 1.4 is not confined to our Theorems 1.2 and 1.3. In particular, it was used to show that the Hausdorff dimension of the closed support $S(X_t)$ (support dimension) of the $(2, d, \beta)$-superprocesses is less than or equal to $2/\beta$ (cf. Theorem 9.3.3.5.b of Dawson (1993)). In addition, for the case $d > 2/\beta$ the just quoted result implies that the support dimension of the $(2, d, \beta)$-superprocesses is in fact equal to $2/\beta$, since the support dimension is greater than or equal to the carrying dimension of $X_t$ which in turn is greater than or equal to $2/\beta$ by Theorem 7.3.1 of Dawson (1992).

In order to prove Theorems 1.2 and 1.3 we apply the following corollary to Proposition 1.4.

**Corollary 1.5.** Let $\mu \in M_P(\mathbb{R}^d)$, and $R > (2/\beta)s^{1/2}$. Then

\[ \mathbb{P}_{x, \mu} \{ \rho_t(S(X_{t+u}), X_t) > R \} \cap \{ M_{t+u} \neq 0 \} \text{ for some } u \leq s \mid \mathcal{P}_t \]

\[ = \mathbb{P}_{x, \mu} \{ \rho_t(S(X_u), \mu) > R \} \cap \{ M_u \neq 0 \} \text{ for some } u \leq s \}

with $\mu = X_t$,

\[ \leq C(\beta, d)R^{-2/\beta} \cdot (R/\sqrt{s})^{d+(4/\beta)-2} \exp \{ -R^2/(2s) \} \cdot M_t, \quad \mathbb{P}_{x, \mu}\text{-a.s.} \]

(1.9')

The proof of Corollary 1.5 is carried out at the beginning of Section 2.

The second basic approach used in this work for the derivation of the exact almost-sure lower estimates for the growth of the closed support $S(X_t)$ is based on the fact that superprocesses can be constructed as limits of certain branching particle systems (BPS). Hereinafter, we consider branching particle systems as measure-valued processes such that measures at time $t$ are generated by all the living particles at time $t$. This method goes back to the works by Watanabe (1968) and Dawson (1978). Recent developments are reflected in Perkins (1988) and Dynkin (1991).

For the sake of simplicity we assume that $X_t$ starts with an atom at the origin at time $t = 0$:

\[ \mathbb{P} \{ X_0 = m \cdot \delta_0 \} = 1, \quad (1.10) \]
where \( m > 0 \). Note that the \((2, d, \beta)\)-superprocess \( X_t \) satisfying (1.10) can be approached by the following BPS \( Y_t^{(\eta)} \). Suppose that we start with \( \eta \) particles at time 0; each particle has constant mass \( m/\eta \). Hence, the initial distribution of BPS \( Y_0^{(\eta)} \) can be written down as follows:

\[
\mathbb{P}\{ Y_0^{(\eta)} = m \cdot \delta_0 \} = 1. \tag{1.10'}
\]

We also assume that each particle immediately starts to perform \( d \)-dimensional Brownian motion. At an exponentially distributed instant of time with mean \( \eta^{-\beta} \) the particle splits into a random number of offspring. Each newly-born particle is a copy of its parent and immediately starts to perform \( d \)-dimensional Brownian motion. The motions, lifetimes and branchings of all particles are independent of each other. The branching mechanism is assumed to be governed by the particle production generating function

\[
\psi_\beta(s) := s + \frac{1}{1 + \beta} (1 - s)^{1 + \beta}. \tag{1.11}
\]

Under these conditions, the branching particle system \( Y_t^{(\eta)} \) converges weakly to the \((2, d, \beta)\)-superprocess \( X \), as \( \eta \to \infty \) (cf., e.g., Dawson (1993, Theorem 4.6.2)).

**Remark.** Note that the function \( \psi_\beta(\cdot) \) describes the *local branching law*. This means that if a particle splits, then a random number \( v \) of particles are produced, and

\[
\mathbb{P}\{ v = n \} = \begin{cases} 1 & \text{if } n \neq 1, \\ \frac{1}{1 + \beta} \cdot \left(1 + \beta \right)^{-1} (-1)^n & \text{if } n = 1. \end{cases}
\]

Recall that the branching law is critical, i.e. \( \mathbb{E}v \equiv 1 \), and in the case \( \beta < 1 \) it belongs to the domain of the normal attraction of a real-valued asymmetric stable law with exponent \( 1 + \beta \).

Let \( K_t^{(\eta)} \) denote the number of initial particles of the BPS \( Y_t^{(\eta)} \), having living descendants at instant \( t \). Obviously, \( K_t^{(\eta)} \) can be viewed as the sum of \( \eta \) 0/1-valued independent identically distributed Bernoulli random variables with the probability of success \( Q_t^{(\eta)} \) in a single trial (i.e. survival of descendants of a single particle from their initial set at instant \( t \)) given by

\[
Q_t^{(\eta)} = \left(1 + \frac{\beta}{\beta + 1} t \eta^\rho \right)^{-1/\beta}. \tag{1.12}
\]

(cf. Zolotarev (1957, Section 5)). In addition,

\[
\mathbb{P}\{ K_t^{(\eta)} > 0 \} = 1 - (1 - Q_t^{(\eta)})^\eta, \tag{1.13}
\]

\[
\mathbb{P}\{ K_t^{(\eta)} = l \} = \binom{\eta}{l} (Q_t^{(\eta)})^l (1 - Q_t^{(\eta)})^{\eta - l}, \tag{1.13'}
\]

where \( 0 \leq l \leq \eta \).
Remark. Note that in fact the strong convergence of a slightly modified branching particle system to the superprocess \( X \) is valid (cf., e.g., Dawson and Perkins (1991, Theorem 3.9)). We will use some relevant arguments from Dawson and Perkins (1991) and Barlow and Perkins (1993) in Section 3 of this work. Namely, the above mentioned result on strong convergence will be used for the derivation of the lower bound of Theorem 3.1.

Now, consider the total variation distance between integer-valued probability measures \( \pi_1 \) and \( \pi_2 \) on \( \mathbb{Z} \):

\[
\sigma(\pi_1, \pi_2) := \sup_{\mathcal{A} \subset \mathbb{Z}} |\pi_1(\mathcal{A}) - \pi_2(\mathcal{A})|.
\]

The following result provides the exact rate of convergence in the classical Poisson theorem for the triangular array of 0/1-valued Bernoulli variables:

**Proposition 1.6.** For any fixed real \( t > 0 \), the following assertions are true as \( \eta \to \infty \):

(i) \( K_t^{(m)} \overset{d}{\to} \Pi\left( m \left( \frac{\beta}{\beta + 1} t \right)^{-1/\beta} \right) \),

where \( K_t^{(m)} \) and \( \Pi(\kappa) \) denote the number of initial particles of the BPS \( X_t^{(m)} \) having living descendants at instant \( t \) and a random variable having the Poisson distribution with parameter \( \kappa \), respectively.

(ii) \[
\frac{1}{32} \min\left( 1, \left( \frac{\beta}{\beta + 1} t \right)^{1/\beta} \right)^{1/2} \eta(Q_t^{(m)})^2 \leq \sigma\left( K_t^{(m)}, \Pi\left( \left( \frac{\beta}{\beta + 1} t \right)^{-1/\beta} \right) \right) \leq \eta(Q_t^{(m)})^2.
\]

**Remarks.** (i) The proof of the lower bound was obtained in Barbour and Hall (1984, Theorem 2), whereas the proof of the upper bound was obtained in Le Cam (1960).

(ii) Obviously, the above inequalities along with (1.12) imply that for any fixed positive \( t \), there exist positive constants \( C_1(t, \beta) < C_2(t, \beta) \) such that for all sufficiently large integer \( \eta \),

\[
C_1(t, \beta) \eta^{-1} \leq \sigma\left( K_t^{(m)}, \Pi\left( \left( \frac{\beta}{\beta + 1} t \right)^{-1/\beta} \right) \right) \leq C_2(t, \beta) \eta^{-1}.
\] (1.14)

Note that the results of Proposition 1.6 will be used in the proof of Proposition 1.10.

Let us point out that the branching-particle-system approach will be used in Section 3 for the derivation of almost-sure lower estimates for the asymptotic behavior of \( \sup_{0 \leq u \leq t} r(u) \) as \( t \to 0 \), where \( r(u) := \inf \{ R : S(X_u) \subseteq B(0, R) \} \) (cf. Theorem 3.1 of Section 3). In turn, a combination of upper and lower estimates (Theorems 1.3 and 3.1) enables us to establish the following result on the "almost-sure rate of convergence" in the local modulus of continuity for \( (2, d, \beta) \)-superprocesses starting from a pure atom (or a point source).
Theorem 1.7. Let Conditions (1.3), (1.4) and (1.10) be valid. Let $m > 0$ be fixed. Then for any $\varepsilon > 0$,
\[
P_m \left\{ 1 + \frac{1}{2} \left( \frac{d - 2}{2/\beta} - \varepsilon \right) \frac{\log \log 1/t}{\log 1/t} \leq \frac{\sup_{0 \leq u \leq t} r(u)}{\sqrt{(2/\beta) t \log 1/t}} \right. \\
\leq 1 + \frac{1}{2} \left( \frac{d}{2/\beta} + 1 + \varepsilon \right) \frac{\log \log 1/t}{\log 1/t} \quad \text{for all sufficiently small positive } t \right\} = 1.
\] (1.15)

As far as we know the result of Theorem 1.7 is new for any $\beta \in (0, 1]$. The proof of Theorem 1.7 is carried out at the end of Section 3.

Note that the result of Theorem 1.7 on the local propagation of the closed support of $(2, d, \beta)$-superprocesses is consistent with the following Proposition 1.8 which provides the exact almost-sure rate of convergence in Lévy's global modulus of continuity for $d$-dimensional Wiener process:

Proposition 1.8. For any integer $d \geq 1$, there exist positive constants $C_1(d) \leq C_2(d)$ such that for any positive $\varepsilon$,
\[
P \left\{ 1 + (C_1(d) - \varepsilon) \frac{\log \log 1/t}{\log 1/t} \leq \frac{\sup_{0 \leq s \leq 1 - t} \sup_{0 < u \leq t} |w(s + u) - w(s)|}{\sqrt{2t \log 1/t}} \right. \\
\leq 1 + (C_2(d) + \varepsilon) \frac{\log \log 1/t}{\log 1/t} \quad \text{for all sufficiently small positive } t \right\} = 1.
\]

Remarks. (i) Note that for $d = 1$ this result can be derived from (Chung et al., 1959, Theorem 2) (see also (Itô and McKeane, 1974, Problem on p. 38)) with $C_1(1) = C_2(1) = 5/4$. On the other hand, for $d \geq 2$ this result is true with $C_1(d) = d/4$ and $C_2(d) = 1 + d/4$ (the proof will be published in (Dawson et al., 1993)). Analogous almost-sure upper estimates for various processes taking values in separable Banach spaces can be derived from (Csáki and Csörgő, 1990, Theorem 2.2), while corresponding lower estimates follow from that related to the one-dimensional case by applying arguments similar to those used for establishing Relationship (6.5) of (Csáki and Csörgő, 1992). (ii) Note that there are some gaps between the coefficients of $\log \log 1/t$ in our result on the rate of convergence in the local modulus of continuity for $(2, d, \beta)$-superprocesses (cf. (1.15)) as well as in the result on the rate of convergence in the global modulus of continuity for $d$-dimensional Wiener process (Proposition 1.3).

Let us observe that Relationship (1.15) yields the following important corollary.
Corollary 1.9. Let Conditions (1.3), (1.4) and (1.10) be valid. Then

\[
\mathbb{P}_{m,d_0} \left\{ \sup_{0 \leq u \leq t} \frac{r(u)}{\sqrt{2/\beta \; t \log (1/t)}} - 1 = O \left( \frac{\log \log 1/t}{\log 1/t} \right) \right\} = 1; \quad t \downarrow 0 \; \; \Leftrightarrow \; \; (1.15')
\]

\[
\mathbb{P}_{m,d_0} \left\{ \lim_{t \downarrow 0} \frac{\sup_{0 \leq u \leq t} r(u)}{\sqrt{2/\beta \; t \log (1/t)}} = 1 \right\} = 1. \; \; (1.15'')
\]

Remarks. (i) It is clear that the estimate \(O(\log \log (1/t) / \log (1/t))\) is exact at least for \(d \geq 3\), since both coefficients under \(\log \log (1/t) / \log (1/t)\) in inequalities under the probability sign in (1.15) can be chosen to be positive. The analogy with Proposition 1.8 concerning the \(d\)-dimensional Wiener process suggests that the coefficient of \(\log (1/t) / \log (1/t)\) on the left-hand side of the inequality defining the event in (1.15) can probably be chosen to be positive for \(d = 1\) or 2 as well, and that the estimate \(O(\log \log (1/t) / \log (1/t))\) in (1.15') is also exact in these cases.

(ii) Note that for the case \(\beta = 1\) (1.15'') was obtained in Tribe (1989, Theorem 2.1). Our method of proof is similar to that of Tribe (1989), but we do not work in the non-standard setting that he used for proving the weak convergence.

(iii) Note that (1.15'') demonstrates that the constant \(\sqrt{2/\beta}\) which appears in the formulation of the local modulus of continuity for \((2, d, \beta) - \text{superprocesses}\) (cf. (1.8)-(1.8')) is in fact sharp. See also remarks to Theorem 1.2.

It should be pointed out that Formula (1.15'') demonstrates that for any \(\beta \in (0, 1]\) the closed support \(S(X_t)\) propagates continuously with the speed \(\sqrt{(2/\beta) \; t \log (1/t)}\). However, it is interesting to note that mechanisms of mass fluctuations in the cases \(\beta = 1\) and \(0 < \beta < 1\) are qualitatively different. We explain this by the consideration of the total mass process \(M_t\). It turns out that for \(\beta = 1\) sample paths of \(M_t\) are continuous with probability 1, whereas for \(0 < \beta < 1\) sample paths of \(M_t\) are elements of the càdlàg space but in fact discontinuous with probability 1. For this reason, we refer to the cases \(\beta = 1\) and \(0 < \beta < 1\) as the continuous super-Brownian motion and the discontinuous super-Brownian motion, respectively. The proof can be found in El Karoui and Roelly (1991, Theorem 7) and in Dawson (1993, Theorem 6.1.3). The difference between these two cases is analogous to the difference between the path properties of the Wiener process and stable processes with discontinuous paths. Indeed, one can easily obtain (cf., e.g., Section 6.2.2 of Dawson (1993)) that in the case of the continuous super-Brownian motion (i.e. \(\beta = 1\)) the generator \(\mathfrak{A}_1\) of the total mass process is as follows:

\[
\mathfrak{A}_1 f(x) = \frac{1}{2} x f''(x).
\]

On the other hand, in the case of the discontinuous super-Brownian motion (i.e. \(\beta \in (0, 1)\)) the generator \(\mathfrak{A}_\beta\) of the total mass process equals

\[
\mathfrak{A}_\beta f(x) = \int_0^\infty (f(x + u) - f(x) - u \cdot f'(x)) \cdot \nu_\beta(du),
\]
where the Lévy jump measure \( \nu_\beta^\nu(d\mu) \) on \( \mathbb{R}_+ \setminus \{0\} \) is defined as
\[
\nu_\beta^\nu(d\mu) := d\mu(- x \cdot C_\beta \cdot \mu^{-\beta - 1})
\]
with
\[
C_\beta := \frac{\sin(\pi \beta) \cdot \Gamma(\beta + 1)}{\pi \cdot (\beta + 1)}.
\]

Now, we proceed with the derivation of the Laplace transform for the total mass process \( M_t \).

**Proposition 1.10.**

\[
\psi_t(\lambda, \beta) := \mathbb{E} \exp \{- \lambda M_t\} = \exp \left\{- \frac{m \cdot \lambda}{(1 + (\beta/(1 + \beta))t \lambda^{-1} \beta)} \right\}.
\]

**Remark.** It should be mentioned that (1.16) was obtained in Dawson (1992, Formula (5.4.2)) (see also Dawson (1993, Formula (4.5.2))) by the use of analytical methods, namely, by solving Eq. (1.5) for the special case \( \phi \equiv \lambda \). Below we give an alternative, purely probabilistic proof based on the branching-particle-system approximation. This proof may be interesting in its own right, and on the other hand it involves many ideas which are often used in the branching-particle-system approach. Therefore, it is useful to develop this approach in detail for a simple example, at least from the methodological point of view.

**Proof of Proposition 1.10.** Without loss of generality we can assume that \( m = 1 \), i.e., that \( X_t \) starts with the \( \delta \)-function at the origin at time \( t = 0 \). Hence \( X_t \) can be approached by BPS \( Y_t^{(n)} \) which starts with \( \eta \) particles of the constant mass \( 1/\eta \) at time \( t = 0 \) (see (1.10')). Now, denote the number of living descendants of an individual particle from the initial set at time \( t \) by \( Z_t^{(n)}(i) \) (1 \( \leq \) \( i \) \( \leq \) \( \eta \)). Note that random numbers \( Z_t^{(n)}(i) \) represent the number of particles in clusters of age \( t \), where by a cluster we mean the set of descendants of a single initial particle. In turn, the random sum \( K_t^{(n)} \) (introduced above Formula (1.12)) represents the number of non-empty clusters of age \( t \).

Set \( \tilde{Z}_t^{(n)}(i) := (Z_t^{(n)}(i)|Z_t^{(n)}(i) > 0) \) and recall that the mass of an individual particle is equal to \( 1/\eta \). Hence, the total mass \( M_t^{(n)} \) of the branching particle system \( Y_t^{(n)} \) at time \( t \) can be written down as
\[
M_t^{(n)} = \frac{1}{\eta} \sum_{i=1}^{K_t^{(n)}} \tilde{Z}_t^{(n)}(i).
\]

Let \( \psi_t^{(n)}(\lambda) := \mathbb{E} \exp \{- \lambda M_t^{(n)}\} \). It suffices to establish the pointwise convergence of \( \psi_t^{(n)}(\lambda) \) to \( \psi_t(\lambda) \). The Laplace transform of \( Z_t^{(n)}(i) \) and the probability of non-extinction \( Q_t^{(n)} \) are given in Section 5 of Zolotarev (1957). In particular,
\[
\theta_t^{(n)}(\lambda) := \mathbb{E} \exp \{- \lambda Z_t^{(n)}\} = 1 - \left(1 - e^{-\lambda}\right)^{-\beta} + \frac{\beta}{\beta + 1} \left(t \eta^\beta \right)^{-1/\beta}.
\]
Note that for any $t \geq 0$ and $\eta \geq 1$,

$$\psi_t^{(n)}(0) = \lim_{\lambda \to 0} \psi_t^{(n)}(\lambda) = 1.$$ 

Now, let us proceed with the derivation of the representation for $\psi_t^{(n)}(\lambda) := E \exp \left\{ -\lambda \cdot M_t^{(n)} \right\}$. By (1.18), the Laplace transform $\tilde{\chi}_t^{(n)}(\cdot)$ of the total mass of an individual cluster of BPS $Y_t^{(n)}$, conditioned on non-extinction at time $t$, is given by

$$\tilde{\chi}_t^{(n)}(\lambda) := E \exp \left\{ -\lambda \cdot \frac{\hat{Y}_t^{(n)}}{\eta} \right\} = \left[ 1 - ((1 - e^{-\lambda/\eta})^{-\beta} + (\beta/(\beta + 1)) t \eta^\beta)^{-1/\beta} - 1 \right.$$ 

$$\left. \left[ Q_t^{(n)} \right]/Q_t^{(n)} \right].$$

(1.19)

Obviously, $\tilde{\chi}_t^{(n)}(\lambda)$ represents the Laplace transform of the total mass of a cluster of age $t$ (of the BPS $Y_t^{(n)}$), conditioned to be non-empty at time $t$. It is easily seen that for any fixed positive $t$ and for any $\lambda \in \mathbb{R}_+^+$,

$$\tilde{\chi}_t^{(n)}(\lambda) \to \tilde{\chi}(\lambda) := 1 - \lambda \left[ \frac{\beta t/(\beta + 1)}{(\beta t \lambda^\beta/(\beta + 1)) + 1} \right]^{1/\beta}, \quad \text{as } \eta \to \infty.$$ 

(1.19')

Note that (1.17) and the independence of clusters generated by different initial particles imply that

$$\psi_t^{(n)}(\lambda) = \sum_{l=0}^{\infty} \mathbb{P} \left\{ K_t^{(n)} = l \right\} \cdot (\tilde{\chi}_t^{(n)}(\lambda))^l,$$

(1.20)

since $\mathbb{P} \left\{ K_t^{(n)} = l \right\} = 0$ for any integer $l \geq \eta + 1$, by (1.13'). Combining (1.20) with the upper bound in (1.14) we get that

$$\left| \psi_t^{(n)}(\lambda) - \sum_{l=0}^{\infty} \mathbb{P} \left\{ K_t^{(n)} = l \right\} \cdot \left( \prod_{i=0}^{l-1} \left( \frac{\beta}{\beta + 1} t \right)^{-1/\beta} \right) \right| \leq \sigma \left( K_t^{(n)}, \prod_{i=0}^{l-1} \left( \frac{\beta}{\beta + 1} t \right)^{-1/\beta} \right) \sum_{l=0}^{\infty} (\tilde{\chi}_t^{(n)}(\lambda))^l \leq C_2(t, \beta) \eta^{-1} (1 - \tilde{\chi}_t^{(n)}(\lambda))^{-1}$$

$$\leq C_2(t, \beta) \eta^{-1} \left( 1 + \frac{(1 - e^{-\lambda/\eta})^{-\beta} - 1}{(\beta/(\beta + 1)) t \eta^\beta + 1} \right).$$

(1.21)
It follows from (1.19') that the rightmost expression in (1.21) is $O(\eta^{-1})$ as $\eta \to \infty$. Therefore, we reduce our problem to the derivation of the asymptotics of

$$
\sum_{i=0}^{\infty} \mathbb{P} \left\{ \prod \left( \left( \frac{\beta}{\beta + 1} \cdot t \right)^{-1/\beta} \right) = l \right\} \left( \hat{\chi}_l^{(0)}(\lambda) \right)^l
$$

$$
= \sum_{i=0}^{\infty} \frac{1}{i!} \left( \left( \frac{\beta}{\beta + 1} \cdot t \right)^{-1/\beta} \right)^i \cdot \left( \hat{\chi}_l^{(0)}(\lambda) \right)^i \cdot \exp \left\{ - \left( \frac{\beta}{\beta + 1} \cdot t \right)^{-1/\beta} \right\}
$$

$$
= \exp \left\{ - \left( \frac{\beta}{\beta + 1} \cdot t \right)^{-1/\beta} \cdot \left( \hat{\chi}_l^{(0)}(\lambda) - 1 \right) \right\}
$$

$$
= \exp \left\{ - \left( \frac{\beta}{\beta + 1} \cdot t \cdot \left( 1 + \frac{1 - e^{-\lambda/\eta}}{\beta(\beta + 1)t\eta^{\beta + 1}} \right) \right)^{-1/\beta} \right\}.
$$

Taking the limit of the rightmost expression in (1.22) as $\eta \to \infty$ we obtain (1.16).

**Remark.** For the special case $\beta = 1$ (continuous super-Brownian motion) the formula for the Laplace transform of the total mass takes on a simpler form:

$$
\nu_t(\lambda, 1) = \exp \left\{ - \frac{m \cdot \lambda}{1 + t\lambda/2} \right\}.
$$

(1.16')

To complete the introduction, let us note that the total mass process possesses an elegant scaling property which makes it easy to study distributions of clusters conditioned on non-extinction, which start with infinitely small mass. In this respect, set $M_t := M_t$ and let $A_{\lambda, \rho} := (M_{t, m} : M_{t, m} > 0)$. Let $A_{\lambda, \rho} := \lim_{m \to \infty} A_{\lambda, \rho}.

**Proposition 1.11.** The distributions of $\frac{\hat{M}_{t, 0}}{t^{1/\beta}}$ and $\hat{A}_{\lambda, 0}$ coincide.

**Remark.** Note that the scaling property for the total mass process can be derived from the scaling property for $(\alpha, d, \beta)$-superprocesses which was obtained in Dawson (1992, Section 6.5). However, the corresponding scaling property for the total mass process can be derived independently from the scaling property for $(\alpha, d, \beta)$-superprocesses.

**Proof.** Let us denote the Laplace transform $\mathbb{E} \exp \left\{ - \lambda \cdot \hat{M}_{t, 0} \right\}$ of $\hat{M}_{t, 0}$ by $\tau_t(\lambda)$. Taking into account the fact that

$$
\mathbb{P} \left\{ M_{t, m} = 0 \right\} = \lim_{\lambda \to \infty} \mathbb{E} \exp \left\{ - \lambda \cdot M_{t, m} \right\} = \exp \left\{ - \frac{m}{\left( \frac{\beta}{\beta + 1} \cdot t \right)^{1/\beta}} \right\}
$$
(see (1.16)) we easily obtain the result that
\[
\tau_t(\lambda) = \lim_{m \to 0} \frac{\exp\left\{- m \lambda \left(1 + \frac{\beta}{\beta + 1} t \lambda^\beta\right)^{-1/\beta}\right\} - \exp\left\{- m \left(\frac{\beta}{\beta + 1} t\right)^{-1/\beta}\right\}}{1 - \exp\left\{- m \left(\frac{\beta}{\beta + 1} t\right)^{-1/\beta}\right\}}
\]
\[
= \lim_{m \to 0} \frac{m \left(\frac{\beta}{\beta + 1} t\right)^{-1/\beta} - m \lambda \left(1 + \frac{\beta}{\beta + 1} t \lambda^\beta\right)^{-1/\beta}}{m \left(\frac{\beta}{\beta + 1} t\right)^{-1/\beta}} + O(m^2)
\]
\[
= 1 - \left(1 + \frac{\beta + 1}{\beta t \lambda^\beta}\right)^{-1/\beta}.
\]
(1.23)

It only remains to note that (1.23) yields that the Laplace transform of $\mathcal{M}_{t,0}/t^{1/\beta}$ is equal to
\[
1 - \left(1 + \frac{\beta + 1}{\beta t \lambda^\beta}\right)^{-1/\beta} = 1 - \left(1 + \frac{\beta + 1}{\beta \lambda^\beta}\right)^{-1/\beta},
\]
but the right-hand side of this equality coincides with the Laplace transform of $\mathcal{M}_{1,0}$. \qed

**Remark.** It is likely that the results of the subsequent Sections 2 and 3 remain valid under the following weaker restrictions on the function $\Phi$ (compare to (1.4)):
\[
\Phi(u) = -cu^2 + \int_0^\infty (1 - e^{-us} - us)n(ds).
\]
(1.4')

where $c \geq 0$ is any real, and $n(ds) \sim c(\beta)s^{-\beta-2}ds$ as $s \to 0$.

2. **Global and local moduli of continuity for paths of $(2, d, \beta)$-superprocesses**

In this section we prove Theorems 1.1–1.3. Recall that the key point for the proof of all these theorems is Corollary 1.5 to Proposition 1.4, which provides an upper bound for the probability of the event that the $(2, d, \beta)$-superprocess $X$, will visit the exterior of the ball $B(0, R)$ centered at the origin with radius $R$ during the time period $[0, t]$ (cf. (1.9) and (1.9')). Recall that the proof of Proposition 1.4 is deferred to the Appendix.

We start with the proof of Corollary 1.5.

**Proof of Corollary 1.5.** First, we approximate $X_t(dx)$ by atomic measures in $S(X_t)$,
\[
\mu_n = \sum_{i=1}^{N(n)} a^*_i \cdot \delta_{x^*_i} \quad \text{with} \quad x^*_i \in S(X_t) \quad \text{and} \quad \sum a^*_i = M_t.
\]
It is clear that
\[ \mu_n \frac{d}{dx} X_t \text{ as } n \to \infty. \]

Then \( \mathbb{P}_{\mu_n} \frac{d}{dx} \mathbb{P}_{X_t} \text{ as } n \to \infty \) by the Feller property of \((2, d, \beta)\)-superprocesses (cf. Dawson (1992, Proposition 5.5.5)). Since \( S(\mu) \) is closed, then for any \( \kappa > 0 \), the set
\[
\bigcup_{0 \leq u < s + \kappa} \left\{ \{ \rho_1(S(X_u), S(\mu)) > R \} \cap \{ M_u \neq 0 \} \right\}
\]
\[ = \{ v(\cdot) \in \mathbb{D}([0, \infty), M_K(\mathbb{R}^d)): v(u, (S(\mu)^R)^c) > 0 \text{ for some } 0 \leq u < s + \kappa \}
\]
is an open set in the Skorohod topology. Therefore, by the Markov property
\[
\mathbb{P}_{\mathbb{X}_0} \left\{ \{ \rho_1(S(X_t+u), S(X_t)) > R \} \cap \{ M_{t+u} \neq 0 \} \text{ for some } 0 \leq u \leq s \right\} = \mathbb{P}_{\mu_1} \left\{ \{ \rho_1(S(X_u), S(\mu)) > R \} \cap \{ M_u \neq 0 \} \text{ for some } 0 \leq u \leq s \right\}
w \leq \liminf_{n \to \infty} \mathbb{P}_{\mu_n} \left\{ \{ \rho_1(S(X_u), S(\mu)) > R \text{ for some } 0 \leq u < s + \kappa \right\}.
\]

Since \( \mathbb{P}_{\mu_n} = \star \mathbb{P}_{\varphi_1} \mathcal{K}_{\kappa_n} \), then \( X_t \frac{d}{dt} X^1 + \cdots + X_t^N \), where \( \{ X_t^i : i = 1, \ldots, N(n) \} \) are independent, and \( X_t^i \) is a version of the \((2, d, \beta)\)-superprocess with the initial measure \( \alpha_0^i \cdot \delta_{x_t^i} \) (here \( \star \) denotes the operation of convolution). Moreover,
\[
\{ \rho_1(S(X(u), S(\mu)) > R \text{ for some } 0 \leq u < s + \kappa \} \subset \bigcup_{i=1}^{N(n)} \{ \rho_1(S(X^i(u), \{ x_i \}) > R \text{ for some } 0 \leq u < s + \kappa \}.
\]

Therefore, by Proposition 1.4,
\[
\liminf_{n \to \infty} \mathbb{P}_{\mu_n} \left\{ \rho_1(S(X_u), S(\mu)) > R \text{ for some } 0 \leq u < s + \kappa \right\}
\]
\[ \leq \liminf_{n \to \infty} \sum_{i=1}^{N(n)} \mathbb{P}_{\varphi_1} \mathcal{K}_{\kappa_n} \left\{ \rho_1(S(X_u^i), \{ x_i \}) > R \text{ for some } 0 \leq u < s + \kappa \right\}
\]
\[ \leq c(\beta, d) R^{-2/\beta} \left( \frac{R}{(s + \kappa)^{1/2}} \right)^{d+(4/\beta)-2} \exp \left\{ -\frac{R^2}{2(s + \kappa)} \right\} \cdot M_t.
\]
The result follows since \( \kappa > 0 \) is arbitrary. \( \square \)

We now proceed with the following analytical lemma which describes properties of the functions \( g_\kappa(\cdot) \) and \( h_\kappa(\cdot) \) and is used in the proofs of Theorems 1.2 and 1.3. First we introduce the function \( \{ n_0(u): 0 < u < (\log 2)^2/2 \} \), where \( n_0(u) \) is the unique integer \( n \geq 1 \) such that
\[
(\log 2)^2(n + 1)^2/2^{n+1} \leq u < (\log 2)^2 n^2/2^n.
\]
Note that \( n_0(u) \to \infty \) as \( u \to 0 \).
Lemma 2.1. Let $\kappa(\beta, d) := (\beta d + 4\beta + 2)/(1 + \beta)$ and $\tilde{\kappa}(\beta, d) := (d/(2/\beta)) + 1$. Let the function $g_{\kappa}(\cdot)$ be defined by (1.6'), the function $h_{\kappa}(\cdot)$ be defined by (1.8'), $\kappa_1 := (\kappa + \kappa(\beta, d))/2$, and $\kappa_2 := (\kappa + \tilde{\kappa}(\beta, d))/2$.

(i) If $\kappa > \kappa(\beta, d)$ then for any fixed positive constant $C$, there exists a positive integer $n(g_{\kappa}, C)$ such that

$$g_{\kappa}(u) \geq g_{\kappa_1}(u) + Cg_{\kappa_1}(1/2^{n_0(u)})$$

for all $u$ with $n_0(u) \geq n(g_{\kappa}, C)$.

(ii) If $\kappa > \kappa(\beta, d)$ then for any fixed positive constant $C$, there exists a positive integer $n(h_{\kappa}, C)$ such that

$$h_{\kappa}(u) \geq h_{\kappa_2}(u) + Ch_{\kappa_2}(1/2^{n_0(u)})$$

for all $u$ with $n_0(u) \geq n(h_{\kappa}, C)$.

(iii) For any $\kappa > \kappa(\beta, d)$ there exists a constant $C(\beta, d, \kappa)$ such that for all $n \geq 2$,

$$\sum_{l=n+1}^{\infty} g_{\kappa}(1/2^{l}) \leq C(\beta, d, \kappa)g_{\kappa}(1/2^{n}).$$

Proof. Proof of (i). It is easy to show that for all sufficiently small positive $u$,

$$g_{\kappa}(u) - g_{\kappa_1}(u) \geq \frac{\kappa - \kappa(\beta, d)}{6} \log \log 1/u \cdot g_{\kappa_1}(u).$$

Now, note that the functions $g_{\kappa_1}(u)$ and $\log \log (1/u)/\log (1/u)$ are monotonically increasing and if $n = n_0(u)$, then $u \geq ((n + 1)^2(\log 2)^2/2)/2^n$. Therefore, we ascertain that the right-hand side of the above inequality is greater than or equal to

$$\text{Const}(\beta, d, \kappa)(n + 1)g_{\kappa_1}(1/2^n) \cdot \frac{\log(n \log 2 - \log((n + 1)^2(\log 2)^2/2))}{n \log 2}$$

$$\geq \text{Const}(\beta, d, \kappa)g_{\kappa_1}(1/2^n) \log n,$$

which implies (2.1).

Proof of (ii). This proof is exactly the same as that of (i) and is therefore omitted.

Proof of (iii). From the definition of $g_{\kappa}$ we have that

$$\sum_{l=n+1}^{\infty} g_{\kappa}(1/2^{l}) \leq C_1(\beta, d, \kappa) \cdot \sum_{l=n+1}^{\infty} \frac{\sqrt{l}}{2^{l/2}} \leq C_2(\beta, d, \kappa) \cdot \int_{n}^{\infty} x^{1/2} \cdot 2^{-x/2} \, dx.$$

The latter integral is relatively easily estimated by the use of Laplace's method, since it is equivalent as $n \to \infty$ (up to a certain positive constant) to $n^{1/2} \cdot 2^{-n/2}$. The latter expression in turn is equivalent as $n \to \infty$ (up to a certain positive constant) to $g_{\kappa}(1/2^n)$ which yields (2.1''). □

Now, consider some properties of the function $\rho_1$ (see the introduction for its definition). Note that $\rho_1$ is not symmetric and hence not a metric. However, it satisfies
the triangle inequality: \( \rho_1(A_1, A_3) \leq \rho_1(A_1, A_2) + \rho_1(A_2, A_3) \). It is easily seen that Formulas (1.6) and (1.8) can be respectively rewritten as follows:

\[
\rho_1(S(X_t), S(X_s)) \leq g_\kappa(t - s) \quad \text{if } 0 < t - s < \delta(\omega, \kappa) \text{ for } \mathbb{P}_\mu\text{-a.e. } \omega, \tag{2.2}
\]

and if \( t \geq 0 \) is fixed, then

\[
\rho_1(S(X_{t+s}), S(X_s)) \leq h_\kappa(s) \quad \text{if } 0 < s < \delta_\kappa(\omega, \kappa) \text{ for } \mathbb{P}_\nu\text{-a.e. } \omega. \tag{2.2'}
\]

**Proof of Theorem 1.2.** This proof will be carried out by analogy with that of Lévy’s global modulus of continuity for the Wiener process. Fix \( K > 0 \) and set

\[
\mathcal{B}_{K, r} := \{X_T \neq 0\} \cap \left\{ \sup_{0 \leq t \leq r} M_t \leq K \right\}.
\]

Note that the lifetime of \( X_t \) is finite, \( \mathbb{P}_\mu\text{-a.s.} \) (cf., e.g., Dawson (1993, Section 8.1) for the case \( \beta = 1 \)). On the other hand, for the case \( \beta < 1 \) the finiteness of the lifetime of \( X_t \) can be established by considering the following stopping times:

\[
\tau := \inf\{t: M_t = 0\},
\]

and

\[
\tau_\eta := \inf\{t: M_t^{(\eta)} = 0\},
\]

where \( M_t \) and \( M_t^{(\eta)} \) denote the total mass processes of the super-process \( X_t \) and of the BPS \( Y_t^{(\eta)} \) defined by (1.16) and (1.17), respectively. Indeed, it is clear that for any positive \( K \),

\[
\mathbb{P}_\mu\{\tau > K\} \leq \lim_{\eta \to \infty} \inf \mathbb{P}_\mu\{\tau_\eta > K\}.
\]

In addition,

\[
\mathbb{P}_\mu\{\tau_\eta > K\} \leq \text{Const} \eta Q_K^{(\eta)} = \text{Const} \left( \frac{1}{\eta^\beta} + \frac{\beta}{\beta + 1} \cdot K \right)^{-1/\beta},
\]

by (1.12). Hence, the probability \( \mathbb{P}_\mu\{\tau > K\} \) is the general term of a convergent series, which easily implies the finiteness of the lifetime of \( X_t \) in the case \( \beta < 1 \). Also, \( \{X_t: t \geq 0\} \in \mathcal{D}(\{0, \infty\}, M_\beta(\mathbb{R}^d)), \mathbb{P}_\mu\text{-a.s.} \) (cf., e.g., Dawson (1993, Theorem 4.6.2.c)). In particular, these facts imply that \( \sup_{t \geq 0} M_t < \infty, \mathbb{P}_\mu\text{-a.s.} \), and that it suffices to establish (2.2) only for \( \mathbb{P}_\mu\text{-a.e. } \omega \in \mathcal{B}_{K, r} \).

Now, let us show that for any \( K > 0 \), and \( 0 < T < \infty \),

\[
\mathbb{P}_\mu\{S(X_t) \leq S(X_s)^{g_\kappa(t-s)} \forall t \leq T \text{ and } 0 \leq t - s \text{ sufficiently small} \} \cap \mathcal{B}_{K, r} = 0.
\]

For notational simplicity we will write the proof in the case \( T = 1 \).
Given $\kappa > \kappa(\beta, d) = (\beta d + 8\beta + 2)/(2(1 + \beta))$, we first derive the following result for the grid,

$$\mathbb{P}_\mu\left\{ \mathcal{B}_{K,1} \cap \max_{0 < k = j - i \leq \lfloor \log N \rfloor^2} \max_{0 \leq i < j \leq N} \frac{\rho_1(S(X_{ij/N}), S(X_{ij/N}))}{g_\kappa(k/N)} \leq 1 \text{ for all } N = 2^n \text{ large enough} \right\}

= \mathbb{P}_\mu(\mathcal{B}_{K,1}). \quad (2.3)$$

Now, in order to establish (2.3), we first prove that the following probabilities are general terms of a convergent series (for $N = 2^n$) and then apply Borel–Cantelli arguments:

$$p_N := \mathbb{P}_\mu\left\{ \mathcal{B}_{K,1} \cap \max_{0 < k = j - i \leq \lfloor \log N \rfloor^2} \max_{0 \leq i < j \leq N} \frac{\rho_1(S(X_{ij/N}), S(X_{ij/N}))}{g_\kappa(k/N)} > 1 \right\}. \quad (2.4)$$

Note that we estimate the distance $\rho_1$ between closed supports $S(X_j)$ taken at points of the grid that lie much closer to each other than in the classical proof of Lévy's global modulus of continuity (compare $(\log N)^2$ in (2.4) with $N^\epsilon$ in Lévy's proof). It can be shown by the use of Corollary 1.5 that

$$p_N \leq C(\beta, d, K) \sum_{0 < k \leq \lfloor \log N \rfloor^2} N g_\kappa(k/N)^{-2/\beta} \left(g_\kappa(k/N)/(\sqrt{k/N})^{d+(4/\beta)} - 2\right) \exp\left\{-\frac{1 + \beta}{\beta} \left(\log N \right)^2 \left(\log \log N \right)^2\right\}.$$  

Keeping in mind that

$$g_\kappa(u) = \frac{2}{\sqrt{\beta}} (1 + \beta) \min\left(1, u \left(\log \frac{1}{u} + \kappa \log \log \frac{1}{u}\right)\right)$$

(cf. (1.6')) we easily obtain the result that the probability $p_N$ does not exceed

$$C_1(\beta, d, K, \kappa) \sum_{0 < k \leq \lfloor \log N \rfloor^2} N \left(\frac{k}{N} \log \frac{N}{k}\right)^{-1/\beta} \left(\log \frac{N}{k}\right)^{d/2 - 1 + 2/\beta} \exp\left\{-\frac{1 + \beta}{\beta} \left(\log \frac{N}{k} + \kappa \log \log \frac{N}{k}\right)\right\} \leq C_2(\beta, d, K, \kappa) \sum_{0 < k \leq \lfloor \log N \rfloor^2} k(\log N)^{-1 - \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta}} \leq C_3(\beta, d, K, \kappa)(\log N)^{4 - \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta}}.$$

Recall that $\kappa > (\beta d + 4\beta + 2)/(2(1 + \beta))$, which implies that the power of logarithm in the rightmost expression is less than $-1$. This implies that (taking $N = 2^n$)

$$p_{2^n} \leq C_4(\beta, d, K, \kappa) \cdot n^{1 - \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta}},$$

i.e., it is the general term of a convergent series. A subsequent application of the first Borel–Cantelli Lemma implies that for the subsequence $N = 2^n$ at most a finite
The number of events

\[ \left\{ \mathcal{B}_{k,1} \cap \max_{0 < k = j - i \leq (\log N)^2} \frac{\rho_1(S(X_{j/N}), S(X_{i/N}))}{g_k(k/N)} > 1 \right\} \]

occurs, which in turn implies (2.3).

The remainder of the proof is carried out by a slight modification of the standard technique (cf., e.g., McKean (1969, p. 16)) and is based on the just proved Equality (2.3) for the maximum over the grid, and the triangle inequality \( \rho_1(A_1, A_3) \leq \rho_1(A_1, A_2) + \rho_1(A_2, A_3) \).

Fix an arbitrary \( \kappa > \kappa(\beta, d) = (\beta d + 8\beta + 2)/(2(1 + \beta)) \) and recall that \( \kappa_1 = (\kappa + \kappa(\beta, d))/2 \). Then for \( \mathbb{P}_\mu \text{-a.e. } \omega \in \mathcal{B}_{K,1} \) there exists a positive integer \( n_1(\omega) \) such that

\[ \max_{0 < k = j - i \leq (\log N)^2} \frac{\rho_1(S(X_{j/N}), S(X_{i/N}))}{g_k(k/N)} \leq 1 \tag{2.5} \]

for any integer \( N = 2^n \) with \( n \geq n_1(\omega) \). In the remainder of the proof we fix a pair \( (\omega, n_1(\omega)) \), which satisfies (2.5).

Now, let \( \delta'(\omega, \kappa) \) be such that \( n_0(\delta'(\omega, \kappa)) \geq n_1(\omega) \). Let us consider a pair \( (s, t); 0 \leq s < t \leq 1 \) such that \( u := t - s < \delta'(\omega, \kappa) \), and choose \( n = n_0(t - s) \). Note that

\[ n_0(t - s) \geq n_0(\delta'(\omega, \kappa)) \geq n_1(\omega) \]

Therefore, for \( 0 < j - i \leq u \cdot 2^n < (n \log 2)^2 = (\log 2)^2 \), and hence we can apply (2.5) for the estimation of \( \rho_1(S(X_{j/2^n}(\omega)), S(X_{i/2^n}(\omega))) \).

We can then choose sequences of integers \( n < p_1 < p_2 < \cdots \) and \( n < q_1 < q_2 < \cdots \) such that the successive terms

\[ s_k := i \cdot 2^{n - k} - 2^{-p_1} - 2^{-p_2} - 2^{-p_k} \tag{2.6} \]

and

\[ t_k := j \cdot 2^{n - k} + 2^{-q_1} + 2^{-q_2} + 2^{-q_k} \tag{2.6'} \]

satisfy \( |s_k - s| \leq 2^{-p_k}, |t_k - t| \leq 2^{-q_k}, s \leq s_k \leq i \cdot 2^{n - k} < j \cdot 2^{n - k} \leq t_k \leq t. \) A subsequent application of the triangle inequality implies that

\[ \rho_1(S(X_i), S(X_j)) \leq \rho_1(S(X_{i/2^n}), S(X_j)) + \rho_1(S(X_{j/2^n}), S(X_{i/2^n})) + \rho_1(S(X_i), S(X_{j/i^n})), \tag{2.7} \]

where here and below \( X_i \) refers to \( X_i(\omega) \). To estimate the middle term on the right-hand side of (2.7) we apply the monotonicity of the function \( g_{k_1}(\cdot) \) and (2.3):

\[ \rho_1(S(X_{j/2^n}), S(X_{i/2^n})) \leq g_{k_1}((j - i)/2^n) \leq g_{k_1}(t - s). \tag{2.8} \]

Now, note that by the triangle inequality,

\[ \rho_1(S(X_{i/2^n}), S(X_j)) \leq \sum_{k=1}^{\infty} \rho_1(S(X_{s_k}), S(X_{s_{k+1}})), \tag{2.9} \]

\[ \rho_1(S(X_i), S(X_{j/2^n})) \leq \sum_{k=1}^{\infty} \rho_1(S(X_{t_{k+1}}), S(X_{i_k})). \tag{2.9'} \]
Applying (2.3) along with the fact that all \( \{ s_t \} \) 's and \( \{ t_k \} \) 's belong to the grid and the monotonicity of the function \( g_{\kappa_1}(\cdot) \), we ascertain that each of two series on the right-hand sides of (2.9) and (2.9') does not exceed \( \sum_{t=n+1}^{\infty} \theta_{\kappa_1}(1/2^t) \). Hence, by Lemma 2.1 (iii) we get the following bound for the first term on the right-hand side of (2.7):

\[
\rho_1(S(X_{i/2^n}), S(X_s)) \leq \sum_{t=n+1}^{\infty} g_{\kappa_1}(1/2^t) \leq C(\beta, d, \kappa_1) \cdot g_{\kappa_1}(1/2^n). \tag{2.10}
\]

The following estimate for the third term on the right-hand side of (2.7) is analogous to Estimate (2.10) (related to the first term) and is derived by the use of the same technique. Namely,

\[
\rho_1(S(X_t), S(X_{j/2^n})) \leq C(\beta, d, \kappa_1) \cdot g_{\kappa_1}(1/2^n). \tag{2.11}
\]

Combining (2.7), (2.8), (2.10) and (2.11) we obtain the result that

\[
\rho_1(S(X_t), S(X_s)) \leq g_{\kappa_1}(t - s) + 2 \cdot C(\beta, d, \kappa_1) \cdot g_{\kappa_1}(1/2^n) \tag{2.12}
\]

for all sufficiently large \( n \). Provided that \( n_0(t - s) \geq n(g_{\kappa_1}, 2C(\beta, d, \kappa_1)) \) (this function is defined in Lemma 2.1 (i), we can combine (2.12) with the inequality (2.1) of Lemma 2.1(i) to obtain that

\[
\rho_1(S(X_t(\omega)), S(X_s(\omega))) \leq g_{\kappa}(t - s) \tag{2.13}
\]

for any \( 0 \leq t - s \leq \delta(\omega, \kappa) \) such that \( n(\delta(\omega, \kappa)) \geq \max(n_1(\omega), n(g_{\kappa}, 2 \cdot C(\beta, d, \kappa_1))) \), which completes the proof of Theorem 1.2. \( \square \)

**Proof of Theorem 1.1.** Obviously, it suffices to demonstrate that \( \rho_1(S(X_t), S(X_s)):= \max(\rho_1(S(X_t), S(X_s)), \rho_1(S(X_s), S(X_t))) \to 0 \) as \( t \to s \). Note that Theorem 1.2 shows that if \( t \to s \) then \( \rho_1(S(X_t), S(X_s)) \to 0 \). On the other hand, the compactness of \( S(X_s) \) (cf. Remark (i) to Theorem 1.2) and the right continuity of \( X_t \) imply that \( \rho_1(S(X_s), S(X_t)) \to 0 \). \( \square \)

**Proof of Theorem 1.3.** This proof follows along the same lines as that of Theorem 1.2, but is much simpler. Set

\[
\mathcal{G}_{K,t} := \{ X_t \neq 0 \} \cap \left\{ \sup_{0 \leq u \leq t} M_u \leq K \right\}.
\]

Recall that the lifetime of \( X_t \) is finite, \( \mathbb{P}_\mu \)-a.s. and that \( \sup_{t \geq 0} M_t < \infty, \mathbb{P}_\mu \)-a.s. (cf. the beginning of the proof of Theorem 1.2). Therefore, it suffices to establish (2.2') only for \( \mathbb{P}_\mu \)-a.s. \( \omega \in \mathcal{G}_{K,t} \). Note that the right continuity of \( X_t \) and the facts that \( X_t \) is bounded away from zero and bounded from above on the set \( \mathcal{G}_{K,t} \) imply that for any fixed \( t \geq 0 \) the \( (2, d, \beta) \)-superprocess \( X_t \) is bounded away from zero and bounded from above in a certain neighborhood of \( t \) as well.
Now, given \( t \geq 0 \) and \( \kappa > \tilde{\kappa}(\beta, d) = (d/(2/\beta)) + 1 \) let us show that for any \( K > 0 \),

\[
\mathbb{P}_\mu \left\{ \mathcal{B}_{K,t} \cap \max_{0 < i \leq (\log N)^2} \frac{\rho_1(S(X_{t+i/N}), S(X_i))}{h_\kappa(i/N)} \leq 1 \text{ for all } N = 2^n \text{ large enough} \right\} = \mathbb{P}_\mu(\mathcal{B}_{K,t}).
\]

(2.14)

By analogy with (2.4), consider the following probabilities:

\[
q_N := \mathbb{P}_\mu \left\{ \mathcal{B}_{K,t} \cap \max_{0 < i \leq (\log N)^2} \frac{\rho_1(S(X_{t+i/N}), S(X_i))}{h_\kappa(i/N)} > 1 \right\}.
\]

It is easy to show that the probability \( q_N \) does not exceed

\[
\mathbb{P}_\mu \left\{ \max_{0 < i \leq (\log N)^2} \frac{\rho_1(S(X_{t+i/N}), S(X_0))}{h_\kappa((\log N)^2/N)} \right\}
\]

\[
\leq \mathbb{P}_\mu \left\{ X_s(\mathbb{B}(0,h_\kappa((\log N)^2/N))_{\varepsilon}) > 0 \text{ for some } s \leq (\log N)^2/N \right\}.
\]

The latter probability is bounded from above by

\[
C(\beta, d) K h_\kappa((\log N)^2/N)^{-2/\beta} (h_\kappa((\log N)^2/N)/\sqrt{(\log N)^2/N})^{d+(4/\beta)-2} \cdot \exp\{-h_\kappa^2((\log N)^2/N)/(2 \cdot ((\log N)^2/N))\}
\]

by Corollary 1.5.

Now, keeping in mind that

\[
h_\kappa(u) = \frac{2}{\sqrt{\beta}} \min\left(1, u \left(\log \frac{1}{u} + \kappa \log \log \frac{1}{u}\right)\right)
\]

(cf. (1.8')) we obtain the result that the above expression and hence \( q_N \) do not exceed

\[
C_1(\beta, d, \kappa) K (\log(N/(\log N)^2))^{-1-(\kappa/\beta - d/2 - 1/\beta)}
\]

\[
\leq C_2(\beta, d, \kappa) \cdot (\log N)^{-1-((\kappa/\beta - d/2 - 1/\beta)}.
\]

(2.15)

Note that the power of \( \log N \) in the expression on the right-hand side of (2.15) is less than \(-1\) which yields that for the subsequence \( N = 2^n \) it is the general term of a convergent series. Applying the first Borel-Cantelli Lemma we obtain the result that for the subsequence \( N = 2^n \) at most a finite number of events

\[
\mathbb{P}_\mu \left\{ \mathcal{B}_{K,t} \cap \max_{0 < i \leq (\log N)^2} \frac{\rho_1(S(X_{t+i/N}), S(X_i))}{h_\kappa(i/N)} > 1 \right\}
\]

occurs, which in turn implies (2.14).

The remainder of the proof involves an application of the just proved equality (2.14) for the maximum over the grid, the triangle inequality, Lemma 2.1(ii), Theorem 1.2 and arguments similar to those used for the derivation of (2.12) and (2.13).
Fix an arbitrary $\kappa > \bar{\kappa}(\beta, d) = (d/(2/\beta)) + 1$ and recall that $\kappa_2 = (\kappa + \bar{\kappa}(\beta, d))/2$. Subsequently fix any $\kappa_3 > \max(\kappa_2, \kappa(\beta, d))$.

Then by (2.14) and Theorem 1.2, for $\mathbb{P}_\mu$-a.s. $\omega \in \mathscr{H}_{K,1}$, there exists a positive integer $n_2(\omega)$ such that

$$\max_{0 < t \leq (\log N)^2} \frac{\rho_1(S(X_{t+i/N}), S(X_t))}{h_{n_2}(i/N)} \leq 1$$

for any integer $N = 2^n$ with $n \geq n_2(\omega)$, and

$$\rho_1(S(X_u), S(X_s)) \leq g_{\kappa_3}(u - s) \quad \text{if} \quad 0 < u - s < 2^{-n_2(\omega)}.$$  \hspace{1cm} (2.16')

In the remainder of the proof we fix a pair $(\omega, n_2(\omega))$, which satisfies (2.16'), (2.16'').

Now, let $\delta'_*(\beta, \kappa)$ be such that $n_0(\delta'_*(\beta, \kappa)) \geq n_2(\omega)$. Let us consider positive $u < \delta'_*(\beta, \kappa)$ and choose $n = n_0(u)$. Note that $n_0(u) \geq n_0(\delta'_*(\beta, \kappa)) \geq n_2(\omega)$. Therefore, for $0 < i \leq u \cdot 2^n < (\log 2)^2 = (\log 2^n)^2$, and hence we can apply (2.16') for the estimation of $\rho_1(S(X_{t+i/2^n}), S(X_t))$.

We can then choose a sequence of integers $n < q_1 < q_2 < \cdots$ such that the successive terms

$$u_k := i \cdot 2^{-n} + 2^{-q_1} + 2^{-q_2} + \cdots + 2^{-q_k}$$

satisfy $i \cdot 2^n \leq u_k \leq u$ and $|u_k - u| \leq 2^{-q_k}$ (compare to (2.6)–(2.6')). A subsequent application of the triangle inequality implies that

$$\rho_1(S(X_{t+i/2^n}), S(X_t)) \leq \rho_1(S(X_{t+i/2^n}), S(X_{t+i/2^n})) + \rho_1(S(X_{t+i/2^n}), S(X_{t+i/2^n})).$$  \hspace{1cm} (2.17)

To estimate the first term on the right-hand side of (2.17) we apply (2.14) and the monotonicity of the function $h_{\kappa_3}(\cdot)$:

$$\rho_1(S(X_{t+i/2^n}), S(X_t)) \leq h_{\kappa_2}(u).$$  \hspace{1cm} (2.18)

Now, note that by the triangle inequality

$$\rho_1(S(X_{t+i/2^n}), S(X_{t+i/2^n})) \leq \sum_{k=1}^{\infty} \rho_1(S(X_{t+u_{k+1}}, S(X_{t+u_k})).$$  \hspace{1cm} (2.19)

Applying (2.16'') and the monotonicity of the function $g_{\kappa_3}(\cdot)$, we ascertain that the series on the right-hand side of (2.19) does not exceed $\sum_{l=n+1}^{\infty} g_{\kappa_3}(1/2^l)$. Hence, by Lemma 2.1(iii) we get the following bound for the rightmost term on the right-hand side of (2.17):

$$\rho_1(S(X_{t+i/2^n}), S(X_{t+i/2^n})) \leq \sum_{l=n+1}^{\infty} g_{\kappa_3}(1/2^l) \leq C(\beta, d, \kappa_3) \cdot g_{\kappa_3}(1/2^n).$$  \hspace{1cm} (2.20)

Note that the rightmost expression in (2.20) is equivalent as $n \to \infty$ (up to a certain positive constant) to $h_{\kappa_2}(1/2^n)$. Combining these arguments with (2.17), (2.18) and
(2.20) we obtain the result that

$$\rho_1(S(X_{1+u}), S(X_t)) \leq h_{\kappa_2}(u) + C(\beta, d, \kappa_2, \kappa_3) \cdot h_{\kappa_2}(1/2^n)$$

(2.21)

for all sufficiently large n. Provided that $$n_0(u) \geq n(h_{\kappa_2}, C(\beta, d, \kappa_2, \kappa_3))$$ (this function is defined in Lemma 2.1(ii), we can combine (2.21) with Inequality (2.1') of Lemma 2.1(ii) to obtain that

$$\rho_1(S(X_{1+u}(\omega)), S(X_t(\omega))) \leq h_{\kappa_2}(u)$$

for any $$0 < u < \delta_{\kappa_2}(\beta, \kappa)$$ such that $$n_0(\delta_{\kappa_2}(\beta, \kappa)) \geq \max\{n_2(\omega), n(h_{\kappa_2}, C(\beta, d, \kappa_2, \kappa_3))\}$$, which completes the proof of Theorem 1.3. □

3. The exact almost-sure rate of convergence for local propagation of mass from a point source

In this section we first formulate and prove Theorem 3.1 and then derive the assertion of Theorem 1.7 that provides the exact almost-sure rate of convergence for the short-time propagation of the closed support $$S(X_t)$$ of $$X_t$$. Here we base the derivation of almost-sure lower estimates for the closed support of the $$(2, d, \beta)$$-superprocess $$X_t$$ from those related to the closed support of a certain branching particle system (BPS) $$\hat{\gamma}_t^{(n)}$$ that converges to $$X_t$$ as $$n \to \infty$$ in a strong sense. The derivation of the lower estimates for the closed support of $$\hat{\gamma}_t^{(n)}$$ involves an application of Lemma 3.1 below, some purely probabilistic arguments, and the well-known estimates for the tail behavior of d-dimensional Wiener process. Note that in order to obtain lower estimates of this section we need to impose the supplementary condition (1.10), which is the condition of non-degeneracy of the mass of the $$(2, d, \beta)$$-superprocess $$X_t$$ at the initial moment (in contrast to Section 2 devoted to the derivation of upper estimates).

Now, let us introduce the following family of continuous increasing functions on the interval $$[0, e^{-1}]$$:

$$v_\varepsilon(s) = \frac{2}{\beta} \left( \log \frac{1}{s} + \left( \frac{d-2}{2/\beta} \right) \log \log \frac{1}{s} \right) \text{ for } s \in (0, e^{-1}];$$

$$v_\varepsilon(0) = 0.$$  

**Theorem 3.1.** Let Conditions (1.3), (1.4) and (1.10) be valid. Then for any positive $$\varepsilon$$ less than one if $$d \leq 2$$ and less than $$(d-2)/(2/\beta)$$ if $$d > 3$$,

$$\mathbb{P}_{\varepsilon, \delta_0} \left\{ \sup_{0 < u \leq t} r(u) \geq v_\varepsilon(t) \text{ for all sufficiently small positive } t \right\} = 1$$  \hspace{1cm} (3.1)

(Recall that $$r(u) = \inf\{R: S(X_u) \subseteq B(0, R)\}$$.)
Proof. The cornerstone of the proof is the following proposition related to the behavior of $r(\cdot)$ at specific time instants.

**Proposition 3.2.** (compare to Ugbebor (1980, pp. 42-43)). Let $\gamma_n := \exp\{-n^\rho\}$, where $0 < \rho < 1/2$ is fixed. Then for any $\varepsilon > 0$,

$$\mathbb{P}_{\gamma_{n+1}}\{r(\gamma_{n+1}) \geq v_\varepsilon(\gamma_n) \text{ for all sufficiently large } n\} = 1.$$  

(3.1')

Indeed, (3.1') easily implies (3.1), since for $\gamma_{n+1} \leq t \leq \gamma_n$

$$\sup_{0 \leq u \leq t} r(u) \geq \sup_{0 \leq u \leq \gamma_{n+1}} r(u) \geq r(\gamma_{n+1}) \geq v_\varepsilon(\gamma_n) \geq v_\varepsilon(t), \quad \mathbb{P}_{\gamma_{n+1}}\text{-a.s.}$$

**Remark.** Note that Tribe (1989, Theorem 2.1) used a slightly different technique for the derivation of a lower bound. Namely, his lower bound followed from the lower bound for specific time instants with the subsequent application of almost-sure upper estimates. However, that approach is not sufficient to isolate the principal error term $\log \log(1/t)/\log(1/t)$ in (1.15).

**Proof of Proposition 3.2.** Set

$$\tau_n := C(d, \beta, \rho, \varepsilon) \cdot \gamma_n^{1/\beta} \cdot \left(\frac{1}{\gamma_n}\right)^{\varepsilon/\beta},$$

where $C(d, \beta, \rho, \varepsilon)$ is some positive constant which depends on the specified parameters.

Our first step is to show that the expression

$$\exp\{-v_\varepsilon(\gamma_n)^2/2\gamma_{n+1}\} (v_\varepsilon(\gamma_n)^2/2\gamma_{n+1})^{(d-2)/2} \geq \tau_{n+1}.$$  

(3.2)

Indeed, the expression on the left-hand side of (3.2) is greater than or equal to

$$C(d, \beta, \varepsilon) \cdot \exp\left\{-\frac{1}{\beta} \frac{\gamma_n}{\gamma_{n+1}} \cdot \log \frac{1}{\gamma_n}\right\} \cdot \exp\left\{-\frac{1}{\beta} \frac{\gamma_n}{\gamma_{n+1}} \cdot \left(\frac{d-2}{2/\beta - \varepsilon}\right) \cdot \log \frac{1}{\gamma_n}\right\}$$

$$\cdot \left(\frac{\gamma_n}{\gamma_{n+1}} \cdot \log \frac{1}{\gamma_n}\right)^{(d-2)/2}$$

$$\geq C(d, \beta, \rho, \varepsilon) \cdot (\gamma_n/\gamma_{n+1})^{1/\beta} \cdot \left(\log \frac{1}{\gamma_n}\right)^{(d-2)/2} \cdot \left(\log \frac{1}{\gamma_n}\right)^{-((d-2)/2 - \varepsilon/\beta) \gamma_n/\gamma_{n+1}}$$

(Note that

$$\frac{\gamma_n}{\gamma_{n+1}} = \exp\{n^\rho \cdot ((1+1/n)^\rho - 1)\} = \exp\left\{\frac{\rho}{n^{1-\rho}} + o\left(\frac{1}{n^{1-\rho}}\right)\right\} = 1 + \frac{\rho}{n^{1-\rho}} + o\left(\frac{1}{n^{1-\rho}}\right)$$

)
as \( n \to \infty \) and hence is greater than or equal to some positive constant \( C(\rho) \) for all sufficiently large \( n \), by our choice of \( \{\gamma_n\} \). In addition,

\[
\frac{\gamma_n^{1/\beta} \gamma_{n+1}^b}{\gamma_{n+1}^{1/\beta}} = \left( \frac{\gamma_n^{1/\beta} \gamma_{n+1}^b}{\gamma_{n+1}^{1/\beta}} \right)^{1/\beta} = \left( \frac{\gamma_n \gamma_{n+1}^b}{\gamma_{n+1}^{1/\beta} \gamma_{n+1}^b} \right)^{1/\beta}
\]

\[
= \left( 1 + \frac{\rho}{n^{1-\rho}} + o\left( \frac{1}{n^{1-\rho}} \right) \right)
\]

\[
\cdot \left( 1 - \frac{\rho}{n^{1-2\rho}} + o\left( \frac{1}{n^{1-2\rho}} \right) \right)^{1/\beta}
\]

as \( n \to \infty \) and hence is greater than or equal to some positive constant \( C(\beta, \rho) \) for all sufficiently large \( n \) (recall that \( 0 < \rho < \frac{1}{2} \)). Therefore, the expression on the left-hand side of (3.2) is greater than or equal to

\[
C(d, \beta, \rho, \varepsilon) \gamma_{n+1}^{1/\beta} \cdot \left( \log \frac{1}{\gamma_n} \right)^{-((d-2)/2)(\gamma_n/\gamma_{n+1} - 1)} \cdot \left( \log \frac{1}{\gamma_n} \right)^{(\varepsilon/\beta)(\gamma_n/\gamma_{n+1})}
\]

Now, it is easily seen that

\[
\left( \log \frac{1}{\gamma_n} \right)^{-((d-2)/2)(\gamma_n/\gamma_{n+1} - 1)} = (n^\rho)^{-((d-2)/2)(\rho/n^{1-\varepsilon} + o(1/n^{1-\varepsilon}))} \quad (\text{as } n \to \infty)
\]

\[
= \exp \left\{ -\rho^2 \cdot \frac{d-2}{2} \cdot \log n + o\left( \log n \right) \right\}
\]

and hence is greater than or equal to some positive constant \( C(\rho) \) for all sufficiently large \( n \) (recall that \( 0 < \rho < \frac{1}{2} \)). On the other hand,

\[
\frac{\left( \log \gamma_n \right)^{(\varepsilon/\beta)(\gamma_n/\gamma_{n+1})}}{\left( \log 1/\gamma_{n+1} \right)^{(\varepsilon/\beta)}} = \left( \log \frac{1}{\gamma_n} \right)^{((\varepsilon/\beta)(\gamma_n/\gamma_{n+1}) - 1)} \cdot \left( n^\rho \right)^{(\varepsilon/\beta)}
\]

\[
= n^{(\varepsilon/\beta)\rho(\rho/n^{1-\varepsilon} + o(1/n^{1-\varepsilon}))} \cdot \left( 1 - \frac{1}{n+1} \right)^{\rho^{1/\beta}} \quad (\text{as } n \to \infty)
\]

and hence is greater than or equal to some positive constant \( C(\beta, \rho, \varepsilon) \) for all sufficiently large \( n \). Therefore, the expression on the left-hand side of (3.2) is greater than or equal to

\[
C(d, \beta, \rho, \varepsilon) \gamma_{n+1}^{1/\beta} \cdot \left( \log \frac{1}{\gamma_{n+1}} \right)^{(\varepsilon/\beta)} := \tau_{n+1}
\]

for all sufficiently large \( n \), which completes the proof of the first step of Proposition 3.1.

Note that the first Borel–Cantelli Lemma implies that in order to establish (3.1') it suffices to prove that

\[
\sum_{n=1}^{\infty} P_{\cdot, \cdot, b} \{ r(\gamma_{n+1}) \leq v_\varepsilon(\gamma_n) \} < \infty.
\]

(3.3)
In order to get (3.3), we introduce a new, slightly modified branching particle system \( \tilde{Y}_t^{\eta} \) (compare to the branching particle system \( Y_t^{(\eta)} \) considered in the Introduction). Let \( \Pi(m \cdot \eta) \) denote the Poisson random variable with parameter \( m \cdot \eta \). Suppose that we start with a Poisson random number \( \Pi(m \cdot \eta) \) of particles located at the origin at time \( t = 0 \); each particle has constant mass \( 1/\eta \). Hence, the initial distribution of \( \tilde{Y}_0^{\eta} \) can be written down as follows:

\[
\tilde{Y}_0^{\eta} \overset{d}{=} \frac{\Pi(m \cdot \eta)}{\eta} \cdot \delta_0.
\]

In addition, each particle immediately starts to perform \( d \)-dimensional Brownian motion. At an exponentially distributed instant of time with mean \( \eta^{-\beta} \) the particle splits into a random number of offspring, and each newly-born particle starts to perform \( d \)-dimensional Brownian motion. The motions, lifetimes and branchings of all particles are independent of each other. The branching mechanism is governed by the particle production generating function \( \psi_\beta(\cdot) \) given by (1.11). It is clear that 

\[
\Pi(m \cdot \eta/\eta) \overset{p}{\rightarrow} m \text{ as } \eta \rightarrow \infty.
\]

Let \( H_t \) denote the historical process corresponding to the superprocess \( X_t \), starting with the initial measure \( X_0 = m \cdot \delta_0 \) (cf., e.g., Dawson and Perkins (1991) for the construction of historical processes). In particular, this means that for fixed \( t \), \( H_t \) is a random measure on \( C([0, \infty), \mathbb{R}^d) \) such that the induced random measure on \( \mathbb{R}^d, H_t(\{f: f(t) \in \cdot\}) \), has the same distribution as \( X_t \). Moreover, it is proved in Dawson and Perkins (1991, Theorem 3.9) that for \( \eta > 0 \) the induced random measure on \( \mathbb{R}^d, H_{\eta t}^{\eta + \epsilon}(\{f: f(t) \in \cdot\}) \), is almost surely a pure atomic random measure consisting of a finite number of atoms and its support has the same distribution as the support of the BPS \( \tilde{Y}_t^{\eta} \) defined in the previous paragraph. In addition, Lemma 2.2 of Barlow and Perkins (1993) yields that the closed support of \( X_t \) is stochastically larger than the closed support of \( H_{\eta t}^{\eta + \epsilon}(\{f: f(t) \in \cdot\}) \). Hence, we can conclude that for any fixed integer \( n \geq 1 \) and for any \( \eta > 0 \),

\[
\mathbb{P}_{m \cdot \delta_0} \left( \tau(\gamma_{n+1}) \leq v_t(\gamma_n) \right) \leq \mathbb{P}_{(110m, n) \cdot \delta_0} \left( S(\tilde{Y}_{\gamma_{n+1}}^{(\eta)}), \mathbb{B}(0, v_\eta(\gamma_n)) \right),
\]

(3.4)

where \( \mathbb{B}(0, v_\eta(\gamma_n)) \) is the closed ball centered at the origin with radius \( v_\eta(\gamma_n) \). Hence, in view of (3.4), in order to prove (3.3) it suffices to derive appropriate upper estimates for the closed support of the corresponding BPS \( \tilde{Y}_t^{(\eta)} \).

Let \( \{w_t(\cdot)\} \) denote independent copies of the standard \( d \)-dimensional Wiener process \( w(\cdot) \). Note that the following arguments are similar to the non-standard analysis arguments used in the proof of Theorem 2.1 of Tribe (1989). We describe these arguments on a heuristic level. First, the measure \( \tilde{Y}_{\gamma_{n+1}}^{(\eta)} \) of BPS at time \( t = \gamma_{n+1} \) can be represented as the union of the random number \( K_{\gamma_{n+1}}^{(\eta)} \) of non-empty clusters of age \( \gamma_{n+1} \) (see the initial paragraph of the proof of Proposition 1.10 for more details regarding the cluster picture). It is clear that the distribution of \( K_{\gamma_{n+1}}^{(\eta)} \) is Poisson with parameter \( m \cdot \eta \). In addition, for any integer \( n \geq 1 \) and for any integer \( l \geq 0 \),

\[
\mathbb{P}_{\tilde{Y}_{\gamma_{n+1}}^{(\eta)}} \{ \tilde{K}_{\gamma_{n+1}}^{(\eta)} = l \} = \sum_{k=1}^{\infty} e^{-m \eta} \left( \frac{m \eta}{k} \right)^k \frac{1}{k!} (Q_{\gamma_{n+1}}^{(\eta)})^l (1 - Q_{\gamma_{n+1}}^{(\eta)})^{k-l}.
\]

(3.5)
We proceed with the derivation of the Laplace transform $f_{\mathcal{K}^{(n)}_{\gamma_{n+1}}}(s)$ of the number of initial particles of BPS $\overline{Y}^{(n)}$ having living descendants at instant $\gamma_{n+1}$. Applying (3.5) we get that for any real $s \geq 0$,

$$f_{\mathcal{K}^{(n)}_{\gamma_{n+1}}}(s) = \sum_{k=0}^{\infty} e^{-s} \left( \sum_{k=0}^{\infty} \frac{(mn)^{k}}{k!} \binom{k}{l} (Q_{\gamma_{n+1}}^{(n)})^{l} (1 - Q_{\gamma_{n+1}}^{(n)})^{k-l} \right)$$

Note that the sum within the braces on the right-hand side of (3.6) is in fact the Laplace transform of the sum of Bernoulli variables. Therefore,

$$f_{\mathcal{K}^{(n)}_{\gamma_{n+1}}}(s) = e^{-s} \sum_{k=0}^{\infty} \frac{(mn)^{k}}{k!} \left( \sum_{l=0}^{k} \binom{k}{l} (Q_{\gamma_{n+1}}^{(n)})^{l} (1 - Q_{\gamma_{n+1}}^{(n)})^{k-l} \right).$$

Combining the above representation with (3.6) we obtain the result that for any real $s \geq 0$,

$$f_{\mathcal{K}^{(n)}_{\gamma_{n+1}}}(s) = e^{-s} \sum_{k=0}^{\infty} \frac{1}{k!} \{ mn(1 - Q_{\gamma_{n+1}}^{(n)}(1 - e^{-s})) \}^{k}$$

It is obvious, in view of (3.7), that the distribution of $\mathcal{K}^{(n)}_{\gamma_{n+1}}$ is Poisson with parameter $m \cdot \eta \cdot Q_{\gamma_{n+1}}^{(n)}$.

Now, we choose one and only one representative (hereinafter referred to as the tagged particle) from each surviving (i.e. non-empty) cluster at time $t$. It is clear that the exclusion from consideration of the non-tagged particles will give us a conservative lower bound. On the other hand, this simplification will enable us to exploit the independence of tagged particles (belonging to different clusters of age $t$). In this respect, the probability of the event that all the particles of BPS $\overline{Y}^{(n)}_{\gamma_{n+1}}$ remain inside the closed ball $B(0, v_{\epsilon}(y_{\epsilon}))$,

$$\mathbb{P}_{(\Pi(\gamma_{n+1}))} \cdot \delta_{\epsilon} \{ S(\overline{Y}^{(n)}_{\gamma_{n+1}}) \in B(0, v_{\epsilon}(y_{\epsilon})) \} \leq \mathbb{P} \left\{ \sup_{1 \leq i \leq \mathcal{K}^{(n)}_{\gamma_{n+1}}} \left| w_{i}(y_{n+1}) \right| \leq v_{\epsilon}(y_{\epsilon}) \right\}$$

$$\leq \sum_{l=0}^{\infty} \mathbb{P} \{ \mathcal{K}^{(n)}_{\gamma_{n+1}} = l \} \cdot \left( 1 - \mathbb{P} \{ \left| w(y_{n+1}) \right| > v_{\epsilon}(y_{\epsilon}) \} \right)^{l}$$

To estimate $\mathbb{P} \{ \left| w(y_{n+1}) \right| > v_{\epsilon}(y_{\epsilon}) \}$, we apply the self-similarity property for the Wiener process, $w(t) \overset{d}{=} w(kt)/\sqrt{k}$ with $k = 1/t$, the following well-known bounds for
the tail probabilities of the $d$-dimensional Wiener process

$$ C_d e^{-x^2/2} x^{d-2} \geq \Pr \left\{ \sup_{0 \leq s \leq 1} |w(s)| > x \right\} \geq \Pr \{ |w(1)| > x \} \geq C_2(d) e^{-x^2/2} x^{d-2}, \tag{3.9} $$

and the lower estimate (3.2). We easily get that

$$ \Pr \{ |w(\gamma_{n+1})| > v_c(\gamma_n) \} \geq \tau_{n+1} $$

for all sufficiently large $n$. Therefore, the probability on the left-hand side of (3.8) does not exceed

$$ \sum_{l=0}^{\infty} \Pr(K_{\gamma_{n+1}}^l = l) \exp \{ - l \cdot \tau_{n+1} \} = f_{K_{\gamma_{n+1}}}^l(\tau_{n+1}) \tag{3.10} $$

by definition, where $f_{K_{\gamma_{n+1}}}^l(\cdot)$ denotes the Laplace transform of the number of initial particles of BPS $\tilde{Y}^{(n)}$ having living descendants at instant $\gamma_{n+1}$ given by (3.7). Hence, the expression on the left-hand side of (3.10) is equal to

$$ f_{K_{\gamma_{n+1}}}^l(\tau_{n+1}) = \exp \{ m \eta Q_{\gamma_{n+1}}^{(n)} (exp \{ - \tau_{n+1} \} - 1) \}. \tag{3.11} $$

Note that by our choice of $\tau_{n+1}$,

$$ \exp \{ m \eta Q_{\gamma_{n+1}}^{(n)} (exp \{ - \tau_{n+1} \} - 1) \} \leq \exp \{ - m \eta Q_{\gamma_{n+1}}^{(n)} \tau_{n+1} + m \eta Q_{\gamma_{n+1}}^{(n)} \tau_{n+1}^2/2 \} $$

$$ \leq \exp \{ - m \eta Q_{\gamma_{n+1}}^{(n)} \tau_{n+1}/2 \} \tag{3.12} $$

(for all sufficiently large $n$).

Replacing $\tau_{n+1}$ by $C(d, \beta, \rho, \epsilon) \gamma_{n+1}^{1/\beta} \left( \log \frac{1}{\gamma_{n+1}} \right)^{\epsilon/\beta}$ and $Q_{\gamma_{n+1}}^{(n)}$ by the expression on the right-hand side of (3.12) in the rightmost expression in (3.12) and then making relatively simple transformations we obtain the result that

$$ \exp \{ - m \cdot \eta \cdot Q_{\gamma_{n+1}}^{(n)} \cdot \tau_{n+1}/2 \} $$

$$ \leq \exp \left\{ - \frac{C(d, \beta, \rho, \epsilon) m}{2} \gamma_{n+1}^{1/\beta} \left( \log \frac{1}{\gamma_{n+1}} \right)^{\epsilon/\beta} \left( \frac{1}{1/(\gamma_{n+1} \cdot \eta^\beta) + \beta(\beta + 1)} \right)^{1/\beta} \left( \log \frac{1}{\gamma_{n+1}} \right)^{\epsilon/\beta} \right\} $$

$$ \leq \exp \left\{ - \frac{C(d, \beta, \rho, \epsilon) m}{3 \beta(\beta + 1)} \left( \log \frac{1}{\gamma_{n+1}} \right)^{\epsilon/\beta} \right\} $$

(for $\eta \geq \eta(\gamma_n)$).

A combination of the above inequalities with (3.8) and (3.10)–(3.12) implies that for any $\eta \geq \eta(\gamma_n)$,

$$ \Pr_{(\Pi(\eta_0, \eta))} \cdot_0 \{ S(\tilde{Y}_{\gamma_{n+1}}^{(n)}) \in \overline{B}(0, v_c(\gamma_n)) \} \leq \exp \left\{ - \frac{C(d, \beta, \rho, \epsilon) m}{3 \beta(\beta + 1)} \left( \log \frac{1}{\gamma_{n+1}} \right)^{\epsilon/\beta} \right\}. \tag{3.13} $$

Note that the expression on the right-hand side of (3.13) does not depend on $\eta$ and that (3.4) implies that for any integer $n \geq 1$,

$$ \Pr_{m \cdot_0} \{ r(\gamma_{n+1}) \leq v_c(\gamma_n) \} \leq \liminf_{\eta \to \infty} \Pr_{(\Pi(\eta_0, \eta))} \cdot_0 \{ S(\tilde{Y}_{\gamma_{n+1}}^{(n)}) \in \overline{B}(0, v_c(\gamma_n)) \}. $$
Therefore, for all sufficiently large $n$,

$$
\mathbb{P}_{m, \delta_0} \{ r(\gamma_{n+1}) \leq v_\epsilon(\gamma_{-}) \} \leq \exp \left\{ - \frac{C(d, \beta, \rho, \epsilon) \cdot m}{3(\beta/(\beta + 1))^{1/\beta}} \left( \log \frac{1}{\gamma_{n+1}} \right)^{1/\beta} \right\}.
$$

(3.14)

Taking into account the fact that $\log(1/\gamma_{n+1}) = (n + 1)^{\rho}$ we easily obtain that the expression on the right-hand side of (3.14) is the general term of a convergent series. Therefore, the series (3.3) is convergent.

Proof of Theorem 1.7. In order to get an upper estimate we apply Relationship (1.8) of Theorem 1.3 with $t = 0$. Note that (1.10) implies that $S(X_0) = \{0\}$ and hence $S(X_0)^{h_\kappa(\omega)} = \mathbb{B}(0, h_\kappa(s))$. Therefore, for each $\kappa > (d/(2/\beta)) + 1$ and for each $0 < s < \delta_\kappa(\omega, \kappa)$,

$$
\mathbb{P}_{m, \delta_0} \{ S(X_u) \subseteq \mathbb{B}(0, h_\kappa(\omega)) \text{ for } 0 < u \leq s \} = 1.
$$

(3.15)

Obviously, (3.15) yields that for any positive $\epsilon$

$$
\mathbb{P}_{m, \delta_0} \left\{ \sup_{0 \leq u \leq t} r(u) \leq \sqrt{\frac{2}{\beta}} \cdot t \cdot \left( \log \frac{1}{t} + \left( \frac{d}{2/\beta} + 1 + \epsilon \right) \log \log \frac{1}{t} \right) \right\} = 1.
$$

(3.16)

Combining (3.1) and (3.16) we obtain the result that for any positive $\epsilon$,

$$
\mathbb{P}_{m, \delta_0} \left\{ \sqrt{\frac{2}{\beta}} \cdot t \cdot \left( \log \frac{1}{t} + \left( \frac{d}{2/\beta} - \epsilon \right) \log \log \frac{1}{t} \right) \leq \sup_{0 \leq u \leq t} r(u) \right\} \leq \sqrt{\frac{2}{\beta}} \cdot t \cdot \left( \log \frac{1}{t} + \left( \frac{d}{2/\beta} + 1 + \epsilon \right) \log \log \frac{1}{t} \right) \text{ for all sufficiently small positive } t.
$$

Dividing all three terms in inequality under the probability sign by $\sqrt{(2/\beta) \cdot t \cdot \log(1/t)}$ and keeping in mind that $\sqrt{1 + \theta} = 1 + \theta/2 + O(\theta^2)$ as $\theta \rightarrow 0$, we immediately obtain (1.15). □

4. Appendix

Proof of Proposition 1.4. Recall that our $(2, d, \beta)$-superprocess is characterized by its log-Laplace equation

$$
\frac{\partial v(t, x)}{\partial t} = \frac{1}{2} \cdot \Delta v - \frac{1}{1 + \beta} \cdot (v(t, x))^{1+\beta}
$$

(cf. (1.5)). Now, we split the proof of Proposition 1.4 into four steps.
Step 1: Passage from measure-valued processes to partial differential equations. Set

\[ P_t := \mathbb{P}_{\alpha_0} \{ X_s(\mathbb{B}(0, R)^c) > 0 \text{ for some } s \leq t \}. \tag{4.1} \]

Obviously, \( P_t = 1 - \mathbb{P}_{\alpha_0} \{ X_s(\mathbb{B}(0, R)^c) = 0 \text{ for all } s \leq t \}. \) Since \( X_t \) is an element of the càdlàg space then the latter probability can be represented in the following form:

\[ \mathbb{P}_{\alpha_0} \{ X_s(\mathbb{B}(0, R)^c) = 0 \text{ for any } s \leq t \} = \lim_{\theta \to -\infty} \mathbb{E}_{\alpha_0} \exp \left\{ -\theta \cdot \int_0^t \langle \psi, X_s \rangle \cdot ds \right\}, \tag{4.2} \]

where

\[ \psi(x) = \begin{cases} 
0, & \text{if } |x| \leq R; \\
|x|/R - 1, & \text{if } R < |x| \leq 2R; \\
1, & \text{if } |x| \geq 2R.
\end{cases} \]

Relationship (4.2) then follows from the fact that the expectation on its right-hand side can be represented as

\[ \mathbb{E}_{\alpha_0} \left( \exp \left\{ -\theta \cdot \int_0^t \langle \psi, X_s \rangle \cdot ds \right\} : A \right) + \mathbb{E}_{\alpha_0} \left( \exp \left\{ -\theta \cdot \int_0^t \langle \psi, X_s \rangle \cdot ds \right\} : A^c \right), \]

where \( A := \{ X_s(\mathbb{B}(0, R)^c) = 0 \text{ for all } s \leq t \}. \) We get that the first term is equal to \( \mathbb{P}_{\alpha_0} \{ X_s(\mathbb{B}(0, R)^c) = 0 \text{ for any } s \leq t \}, \) by the choice of the function \( \psi, \) and the integral \( \int_0^t \langle \psi, X_s \rangle \cdot ds \) presented in the second term is bounded away from 0 by the right continuity of \( X_s. \) Therefore, the second term decays exponentially to 0 as \( \theta \to -\infty. \)

Combining (4.1) and (4.2) we get that

\[ P_t = 1 - \lim_{\theta \to -\infty} \mathbb{E}_{\alpha_0} \exp \left\{ -\theta \cdot \int_0^t \langle \psi, X_s \rangle \cdot ds \right\}. \tag{4.3} \]

We estimate the latter expectation by the use of Theorem 3.1 of Iscoe (1986):

\[ \mathbb{E}_{\alpha_0} \exp \left\{ -\theta \cdot \int_0^t \langle \psi, X_s \rangle \cdot ds \right\} = \exp \{- \langle u_\theta(t, x), \alpha_0 \cdot \delta_0 \rangle \} = \exp \{- a \cdot u_\theta(t, 0) \}, \tag{4.4} \]

where \( \langle \cdot, \cdot \rangle \) stands for the inner product of continuous \( \mathbb{R}^d \)-valued functions on \([0, t]\) and finite measures on \( \mathbb{R}^d, \) and \( u_\theta(\cdot, \cdot) \) is the solution to the following Cauchy problem:

\[ \begin{aligned}
\frac{\partial u_\theta}{\partial t} &= \frac{1}{2} \cdot \Delta u_\theta - \frac{1}{1 + \beta} \cdot u_\theta^{1+\beta} + \theta \cdot \psi, \\
u_\theta(0, x) &= 0.
\end{aligned} \tag{4.5} \]
(Note that Theorem 3.1 of Iscoe (1986) requires the function $\psi$ to belong to a class of functions vanishing at infinity, which includes functions of compact support. In order to get estimates for the expectation on the right-hand side of (4.3) as stated, one takes a sequence of functions of compact support which monotonically increase to $\psi$ as, for example, in Lemma 3.4 of Iscoe (1988). This step is routine and is omitted here.)

Applying (4.3) and (4.4) we easily obtain the following upper estimate:

$$P_t = 1 - \lim_{\theta \to \infty} \exp \left\{- a \cdot u_\theta(t, 0) \right\} \leq a \lim_{\theta \to \infty} u_\theta(t, 0),$$  \hspace{1cm} (4.6)

where $u_\theta(\cdot, \cdot)$ is defined as the solution to (4.5).

**Step 2:** Here we will prove that

(i) $u_\theta(t, x) \uparrow \hat{u}_\theta(x)$ as $t \uparrow \infty$, where $\hat{u}_\theta(x)$ denotes the solution to the following equation.

$$\frac{1}{2} \Delta \hat{u}_\theta(x) - \frac{1}{1 + \beta} \cdot \hat{u}_\theta(x)^{1 + \beta} + \theta \cdot \psi(x) = 0$$  \hspace{1cm} (4.7)

(see Lemma 4.1 below). This means heuristically that $u_\theta(t, \cdot)$ approaches a stationary state at infinity.

(ii) $\hat{u}(x) \uparrow \hat{u}(x)$ as $\theta \uparrow \infty$, where $\hat{u}(x)$ is the solution to the following problem.

$$\frac{1}{2} \Delta \hat{u}_R(x) = \frac{1}{1 + \beta} \cdot \hat{u}_R(x)^{1 + \beta},$$

$$\hat{u}_R(x) \to \infty \text{ as } |x| \uparrow R$$  \hspace{1cm} (4.8)

(see Lemmas 4.2-4.3 below). It is then clear that $a \lim_{\theta \to \infty} u_\theta(t, 0)$ (and hence $P_t$) are bounded from above by $a \cdot \hat{u}_R(0)$. However, an application of the Feynman-Kac formula enables us to get a sharper estimate for $P_t$. Note that our arguments are similar to those of Dawson et al. (1989), where the special case $\beta = 1$ was considered, and of Iscoe (1986, 1988).

Now, we proceed with a series of lemmas.

**Lemma 4.1** (cf. Iscoe (1986, Theorem 3.3)). Function $u_\theta(t, x)$ increases to function $\hat{u}_\theta(x)$ as $t \uparrow \infty$. The convergence is uniform in $x$, and $\hat{u}_\theta(x)$ is the solution to (4.7).

**Remark.** Note the absence of the coefficient $(1 + \beta)/\beta$ under $u^{1 + \beta}$ in Formulas (3.4) and (3.7) of Iscoe (1986) that appears in our Formulas (4.5) and (4.7). However, its presence or absence does not affect the validity of the proof, since the case of an arbitrary positive coefficient $\gamma$ is reduced to the special case $\gamma = (1 + \beta)/\beta$ by scaling space and time.

**Lemma 4.2** (Compare to Iscoe (1988, Lemma 3.2)). Let $u_1(x)$ and $u_2(x)$ be non-negative functions in $\mathbb{R}^d$ which approach zero as $|x| \to \infty$ and satisfy the following equations:

$$\frac{1}{2} \Delta u_1 - \frac{u_1^{1 + \beta}}{1 + \beta} + \psi_1 = 0; \quad \frac{1}{2} \Delta u_2 - \frac{u_2^{1 + \beta}}{1 + \beta} + \psi_2 = 0$$
with \( \psi_1(x) \leq \psi_2(x) \) being non-negative functions in \( \mathbb{R}^d \) having compact supports, and approaching zero as \( |x| \to \infty \). Then \( u_1 \leq u_2 \). In particular, both equations have unique solutions.

**Proof.** Note that \( |u_1|^{1+\beta}/(1 + \beta) \leq \sup \psi_2 \), for otherwise at a point where \( |u_1| \) attains its maximum (i.e. \( \Delta u_1 \leq 0 \))

\[
0 = \frac{1}{2} \Delta u_1 - \frac{|u_1|^{1+\beta}}{1 + \beta} + \psi_1 \leq \frac{|u_1|^{1+\beta}}{1 + \beta} + \psi_2 < 0.
\]

Now suppose that \( u_1 \geq u_2 \) at some point \( x \in \mathbb{R}^d \). Then at a point where \( u_1 - u_2 \) attains a positive maximum

\[
0 \geq \frac{1}{2} \Delta (u_1 - u_2) = \frac{|u_1|^{1+\beta} - |u_2|^{1+\beta}}{1 + \beta} + (\psi_2 - \psi_1) > 0,
\]

which is absurd. The uniqueness follows by taking \( \psi_1 = \psi_2 \); this implies that \( u_1 \leq u_2 \) and by symmetry, \( u_2 \leq u_1 \). \( \square \)

Lemma 4.2 demonstrates that the solution to (4.7), \( \hat{u}_\theta(\cdot) \), increases to a certain function \( \hat{u}_R(x) \) as \( \theta \uparrow \infty \). This limit is specified by the following lemma.

**Lemma 4.3.** \( \hat{u}_\theta(x) \uparrow \hat{u}_R(x) \) as \( \theta \uparrow \infty \), where \( \hat{u}_R(x) \) is the solution to (4.8).

**Proof.** This proof is analogous to those of Proposition 3.5 in the case \( d = 1 \), and of Lemma 3.14 and Proposition 3.15 in the case \( d \geq 2 \) of Iscoe (1988) (where the special case \( \beta = 1 \) was treated) and therefore is omitted. However, note that in order to derive an upper estimate for \( P_t \) (essential for the proof of Proposition 1.4) we need a much weaker result, namely, that

\[
\hat{u}_\theta(x) \leq \hat{u}_R(x)
\]

for any \( x \) such that \( |x| < R \) (compare to Dawson et al. (1989, Formula (3.3.16))), which is easily obtained by the use of maximum-principle argument. \( \square \)

Now, we investigate some properties of \( \hat{u}_R(x) \).

**Lemma 4.4.** (i) The boundary problem (4.8) has the unique positive solution \( \hat{u}_R(x) \) which satisfies the following properties.

(ii) Self-similarity.

\[
\hat{u}_R(x) = R^{-2/\beta} \cdot \hat{u}_1(x/R);
\]

(iii) \[
\lim_{|x| \uparrow R} \frac{\hat{u}_1(x)}{((2 + \beta) \cdot (1 + \beta)/(\beta^2)^{1/\beta} \cdot (1 - |x|)^{-2/\beta} = 1.}
\]
Proof. Note that our arguments are similar to those of Lemma 3.6 of Dawson et al. (1989).

We start with proving (iii). For simplicity of notation, let us denote \( \tilde{u}_1(x) \) by \( u(x) \), and \( (2 + \beta) \cdot (1 + \beta) / \beta^2 )^{1/\beta} \cdot (1 - |x|)^{-2/\beta} \) by \( w_{\beta}(x) \). Obviously, (iii) is equivalent to:

\[
u(x) \sim w_{\beta}(x) \quad \text{as } x \uparrow 1.
\]

Set \( v_{\beta} := u/w_{\beta} \) and note that the solution to (4.8) is invariant under rotation, which enables us to get an equivalent representation of (4.8) in radial coordinates:

\[
u''(r) + \frac{d-1}{r} \cdot u'(r) = \frac{2}{1 + \beta} \cdot u^{1+\beta};
\]

\[
u(r) \rightarrow \infty \quad \text{as } r \uparrow 1.
\]

Now, taking square roots of both sides in (4.12) and integrating from \( r \) to 1 leads to the following inequality:

\[
u(1) = \lim_{r \uparrow 1} \nu(r) \quad \text{Obviously, } \nu(1) \text{ equals infinity by (4.11')}. Then the latter inequality obviously yields that

\[
u(1) = \lim_{r \uparrow 1} \nu(r) \quad \text{Obviously, } \nu(1) \text{ equals infinity by (4.11')}.
\]

Therefore, making the change of variables \( y = (1 - r)^{2/\beta} \cdot s \) turns (4.14) into

\[
u(1) = \lim_{r \uparrow 1} \nu(r) \quad \text{Obviously, } \nu(1) \text{ equals infinity by (4.11')}. Then the latter inequality obviously yields that

\[
u(1) = \lim_{r \uparrow 1} \nu(r) \quad \text{Obviously, } \nu(1) \text{ equals infinity by (4.11')}.
\]

Therefore, making the change of variables \( y = (1 - r)^{2/\beta} \cdot s \) turns (4.14) into
where \( \ell(r) = (1 - r)^{2\beta} \cdot \frac{u(r)}{u(0)} \). If \( \lim_{r \uparrow 1} \ell(r) = 0 \), then by Fatou's Lemma,

\[
\frac{2 \cdot u^{\beta/2}(0)}{\sqrt{(1 + \beta) \cdot (2 + \beta)}} \geq \lim \inf_{r \uparrow 1} \int_{\ell(r)}^{\infty} \frac{dy}{\sqrt{y^2 + \beta - (1 - r)^{2(2 + \beta)/\beta}}}
\]

\[
= \int_{0}^{\infty} y^{-(2 + \beta)(2/\beta)} \cdot dy = \infty,
\]

which is absurd. Hence,

\[
\lim \sup_{r \uparrow 1} v_{\beta}(r) > 0. \tag{4.15}
\]

Now, let us prove that \( \lim_{r \uparrow 1} \inf v_{\beta}(r) < + \infty \). To this end, we rewrite (4.11) in the integral form (compare with Dawson et al. (1989, pp. 156–157)):

\[
u(r) = u(\rho) + \rho^{d-1} \cdot u'(\rho) \cdot \int_{\rho}^{r} s^{1-d} \cdot ds + \frac{2}{\beta + 1} \cdot \int_{\rho}^{r} s^{1-d} \cdot \left( \int_{\rho}^{s} t^{d-1} \cdot u^{\beta+1}(t) \cdot dt \right) \cdot ds,
\]

where \( 0 < \rho \leq r < 1 \). Keeping in mind that \( u = v_{\beta} \cdot w_{\beta} \) we obtain the following equivalent representation of the latter formula:

\[
v_{\beta}(r) = \kappa_{\beta}(r) + \frac{2}{(\beta + 1)w_{\beta}(r)} \cdot \int_{\rho}^{r} s^{1-d} \cdot \left( \int_{\rho}^{s} v_{\beta}(t)w_{\beta}(t)t^{d-1} \cdot dt \right) \cdot ds, \tag{4.16}
\]

where

\[
\kappa_{\beta}(r) = \frac{u(\rho) + \rho^{d-1} \cdot u'(\rho) \cdot \int_{\rho}^{r} s^{1-d} \cdot ds}{w_{\beta}(r)}.
\]

It is easily seen that \( \kappa_{\beta}(r) \to 0 \) as \( r \uparrow 1 \). Now, we introduce

\[
g_{\beta}(r) := \frac{2}{(\beta + 1) \cdot w_{\beta}(r)} \cdot \int_{\rho}^{r} s^{1-d} \cdot \left\{ \int_{\rho}^{s} (w_{\beta}(t))^{1+\beta} \cdot t^{d-1} \cdot dt \right\} \cdot ds.
\]

To obtain its asymptotics as \( r \uparrow 1 \) we twice apply L'Hôpital's rule. After simple algebra we get that

\[
\lim_{r \uparrow 1} g_{\beta}(r) = 1. \tag{4.17}
\]

Now assume that \( \lim_{r \uparrow 1} v_{\beta}(r) = + \infty \). Then with \( r_{0} \) chosen such that \( v_{\beta}(r_{0}) = N > 2^{1/\beta} \), and \( v_{\beta}(r) \geq v_{\beta}(r_{0}) \), \( g_{\beta}(r) > \frac{1}{2} \) for \( r \geq r_{0} \) (4.16) yields that

\[
N = v_{\beta}(r_{0}) > N^{1+\beta/2} > N,
\]

which is absurd. Hence,

\[
\lim \inf_{r \uparrow 1} v_{\beta}(r) < + \infty. \tag{4.18}
\]

Now, if for all \( r \) sufficiently close to 1 function \( v_{\beta}(r) \leq C < 1 \) then (4.16) yields that \( v_{\beta}(r) \leq o(1) + C^{1+\beta} \cdot (1 + o(1)) \). Therefore, if we assume that \( C_{0} = \lim_{r \uparrow 1} \sup v_{\beta}(r) < 1 \), then with \( C = C_{0} \cdot (1 + \varepsilon) \), where \( 0 < \varepsilon < C_{0}^{\beta/(1+\beta)} - 1 \) we obtain that
\[ C_0 \leq C_1^{1+\beta} = C_1^{1+\beta}(1+\varepsilon)^{1+\beta} < C_0, \] which is absurd. Similarly, the assumption \( 1 < \lim_{r \to 1} \inf u_\beta(r) \geq 1 \) leads to a contradiction. Thus, we have proved that \( \lim_{r \to 1} \sup u(r)/w_\beta(r) \geq 1 \), and \( \lim_{r \to 1} \inf u(r)/w_\beta(r) \leq 1 \) (compare with (4.15) and (4.18)). Hence, it only remains to establish the coincidence of \( \lim_{r \to 1} \sup (u(r))/w_\beta(r) \) and \( \lim_{r \to 1} \inf (u(r))/w_\beta(r) \). To this end, the difference \( h_{\beta,c}(\cdot):= u_\beta(\cdot) - C \cdot w_\beta(\cdot) \) should be considered. Keeping in mind that the Laplacian in radial coordinates is represented by \( \Delta w_\beta(r) = w''_\beta(r) + ((d-1)/r) w'_r(r) \), applying (4.11) we derive the following equation for the function \( h_{\beta,c}(\cdot) \):

\[ \Delta h_{\beta,c} = \frac{2}{1+\beta} \cdot w'^{1+\beta}_\beta \cdot \left( \left( \frac{u(r)}{w_\beta(r)} \right)^{1+\beta} - C \cdot \left( 1 + \frac{(d-1) \cdot (1-r)}{r \cdot (2/\beta + 1)} \right) \right). \] (4.19)

The rest of the proof follows along the same lines as that of Dawson et al. (1989, Lemma 3.6) (cf. the paragraph below (3.2.30) therein). In particular, our formula (4.19) is analogous to (3.2.30) of that work.

(i) To establish uniqueness of the solution to (4.8), we follow the arguments of Proposition 3.15 of Iscoe (1988). Let us assume the existence of two different solutions \( \hat{u}_R^{(1)}(x) \) and \( \hat{u}_R^{(2)}(x) \). Set \( \hat{u}_R^{(3)}(x) := C^{2/\beta} \cdot \hat{u}_R^{(2)}(Cx) \). Then it is easily checked that

\[ \Delta \hat{u}_R^{(3)}(x) = \frac{2}{\beta + 1} \cdot C^{2/\beta + 2} \cdot \hat{u}_R^{(2)}(Cx)^{1+\beta} = \frac{2}{\beta + 1} \cdot (C^{2/\beta} \cdot \hat{u}_R^{(2)}(Cx))^{1+\beta} = \frac{2}{\beta + 1} \cdot \hat{u}_R^{(3)}(x)^{1+\beta}. \]

For \( 0 < C < 1 \) we obtain that \( \hat{u}_R^{(3)}(x) \) is finite on \( \partial B(0, R) \). If \( \hat{u}_R^{(1)} \) were ever less than \( \hat{u}_R^{(3)} \), then at a point where \( \hat{u}_R^{(1)} - \hat{u}_R^{(3)} \) attains its negative minimum,

\[ 0 \leq \Delta (\hat{u}_R^{(1)}(x) - \hat{u}_R^{(3)}(x)) = \frac{2}{\beta + 1} \cdot (\hat{u}_R^{(1)}(x)^{1+\beta} - \hat{u}_R^{(3)}(x)^{1+\beta}) < 0, \]

which is absurd. Therefore, \( \hat{u}_R^{(1)}(x) \geq \hat{u}_R^{(3)}(x) = C^{2/\beta} \cdot \hat{u}_R^{(2)}(Cx) \). Letting \( C \uparrow 1 \) we obtain \( \hat{u}_R^{(1)}(x) \geq \hat{u}_R^{(2)}(x) \). Since \( \hat{u}_R^{(1)}(x) \) and \( \hat{u}_R^{(2)}(x) \) are interchangeable, we get uniqueness.

(ii) The self-similarity follows from uniqueness, since \( R^{-2/\beta} \cdot u_1(x/R) \) is obviously a solution to (4.8).

**Step 3:** At this stage, we get some auxiliary estimates for \( \lim_{\theta \to \infty} u_\theta(t, 0) \), where \( u_\theta(\cdot, \cdot) \) is the solution to (4.5). Note that here we follow the arguments of the proof of Theorem 3.3 of Dawson et al. (1989) (cf. (3.3.17) therein).

Now, we apply the Feynman–Kac formula for the derivation of the following representation for \( u_\theta(\cdot, \cdot) \):

\[ u_\theta(t, 0) = \theta \cdot \mathbb{E}_{0, 0} \left\{ \int_0^t \psi(w_s) \cdot \exp \left\{ - \int_0^s \frac{u_\theta(t - \tau, w_\tau)^\beta}{\beta + 1} \cdot d\tau \right\} \cdot ds \right\}, \]

where \( w_s \) is the standard \( d \)-dimensional Wiener process. For \( R_1 < R \) we introduce

\[ \tau_{R_1} := \begin{cases} \inf \{s \leq t : |w_s| > R_1\} & \text{if such instant } s \text{ exists,} \\ t & \text{otherwise.} \end{cases} \]
Note that $\tau_{R_1}$ is a stopping time and $|w(\tau_{R_1})| = R_1$ almost surely. Since $u_\theta(s, x)$ is an even function of $x$, then we can replace $u(s, w_{1x})$ by $u(s, R_1)$. Now, applying the strong Markov property for the Wiener process we obtain that

$$u_\theta(t, 0) = \mathbb{E}_{0, 0}\left[ 1_{\{\tau_{R_1} \leq t\}} \cdot \exp\left\{ - \int_0^{\tau_{R_1}} \frac{u_\theta(t - \tau, w_x)}{\beta + 1} \, d\tau \right\} \cdot \mathbb{E}_{0, R_1}\left( \int_0^{\tau_{R_1}} \theta \cdot \psi(w_s) \cdot \exp\left\{ - \int_0^{s} \frac{u_\theta(t - \tau_{R_1} - w_v, w_x)}{\beta + 1} \, dv \right\} \, ds \right\} \right].$$

It is easily seen that the second factor is equal to $u_\theta(t - \tau_{R_1}, R_1)$ almost surely. Therefore,

$$u_\theta(t, 0) \leq \mathbb{E}_{0, 0}\left\{ 1_{\{\tau_{R_1} \leq t\}} \cdot u_\theta(t - \tau_{R_1}, R_1) \right\}. \quad (4.20)$$

Note that

$$u_\theta(t - \tau_{R_1}, R_1) \leq \hat{u}_R(R_1) \quad (4.21)$$

by Lemmas 4.1–4.3. Furthermore, by Lemma 4.4 we obtain that

$$\hat{u}_R(R_1) \sim ((2 + \beta)(1 + \beta)/\beta^2)^{1/\beta} \cdot R^{-2/\beta}(1 - R_1/R)^{-2/\beta} \quad (4.22)$$

as $R_1 \uparrow R$. Moreover, by (3.9)

$$\mathbb{P}\left\{ \sup_{0 \leq s \leq t} |w_s| > R_1 \right\} \leq C(d) \cdot (R_1/\sqrt{t})^{d - 2} \cdot \exp\left\{ - R_1^2/(2t) \right\}. \quad (4.23)$$

Combining (4.20)–(4.23) we obtain the result that

$$u_\theta(t, 0) \leq C(d, \beta) \cdot R^{-2/\beta} \cdot (1 - R_1/R)^{-2/\beta} \cdot (R_1/\sqrt{t})^{d - 2} \cdot \exp\left\{ - R_1^2/(2t) \right\}. \quad \text{(4.24)}$$

Taking into account the fact that the right-hand side of the latter inequality does not depend on $\theta$ we immediately obtain that

$$\lim_{\theta \to \infty} u_\theta(t, 0) \leq C(d, \beta) \cdot R^{-2/\beta} \cdot (1 - R_1/R)^{-2/\beta} \cdot (R_1/\sqrt{t})^{d - 2} \cdot \exp\left\{ - R_1^2/(2t) \right\}. \quad \text{(4.24)}$$

Step 4: At this stage, we need to choose $R_1$ in order to get an appropriate estimate for the left-hand side of (4.24). Set $\alpha := 1 - R_1/R$, and $K := R_1^2/(2t)$. Then

$$\exp\left\{ - R_1^2/(2t) \right\} = \exp\left\{ - R_1^2/(2t) \right\} \cdot \exp\left\{ \frac{R_1^2}{2t} \cdot (1 - R_1^2/R^2) \right\} \leq \exp\left\{ - R_1^2/(2t) \right\} \cdot \exp\{2 \cdot \alpha \cdot K\}.\quad \text{(4.24)}$$

We choose $\alpha$ to minimize the following expression.

$$\alpha^{-2/\beta} \cdot \exp\{2 \cdot \alpha \cdot K\} = (\alpha^{-1/\beta} \cdot e^{x \cdot K})^2.$$
After simple algebra we obtain the result that the minimum is attained at \( z = 1/(\beta \cdot K) \), and

\[
\left(1 - \frac{R_1}{R}\right)^{-2/\beta} \cdot \exp\left\{ - \frac{R_1^2}{2t} \right\} \leq \exp\left\{ - \frac{R^2}{2t} \right\} \left(\frac{\beta R^2}{2t}\right)^{2/\beta} \exp\left\{ 2/\beta \right\}.
\]

Combining the latter estimate with (4.24) we obtain the result that

\[
\lim_{\theta \to \infty} u_{\theta}(t, 0) \leq C_4(d, \beta) \cdot R^{-2/\beta} \cdot (R/\sqrt{t})^{d+(4/\beta)-2} \cdot \exp\left\{ - \frac{R^2}{2t} \right\}.
\]

Together with (4.6), this yields the assertion of Proposition 1.4. \(\square\)

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