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Chiral ring of strange metals: The multicolor limit

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Abstract

The low energy limit of a dense 2D adjoint QCD is described by a family of $\mathcal{N} = (2, 2)$ supersymmetric coset conformal field theories. In previous work we constructed chiral primaries for a small number $N < 6$ of colors. Our aim in the present note is to determine the chiral ring in the multicolor limit where N is sent to infinity. We shall find that chiral primaries are labeled by partitions and identify the ring they generate as the ring of Schur polynomials. Our findings impose strong constraints on the possible dual description through string theory in an AdS_3 compactification.

1. Introduction

Throughout the last few years, low dimensional examples of dualities between conformal field theories and gravitational models in Anti-de Sitter (AdS) space have received quite some attention. There are at least two motivations for such developments. On the one hand, many low dimensional critical theories can actually be realized in condensed matter systems. As these are often strongly coupled, the AdS/CFT correspondence might provide intriguing new analytic tools to compute relevant physical observables, much as it does for models of particle physics. On the other hand, low dimensional incarnations of the AdS/CFT correspondence might also offer new views on the very working of dualities between conformal field theories and gravita-

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tional models in AdS backgrounds. This applies in particular to the AdS_3/CFT_2 correspondence since there exist many techniques to solve 2-dimensional models directly, without the use of a dual gravitational theory. Recent examples in this direction include the correspondence between certain two-dimensional coset conformal field theories and higher spin gauge theories [2,3], see also [4–7] for examples involving supersymmetric conformal field theories and [8,9] for a more extensive list of the vast literature on the subject. It would clearly be of significant interest to construct new examples of the AdS_3/CFT_2 correspondence which involve full string theories in AdS_3.

In 2012, Gopakumar, Hashimoto, Klebanov, Sachdev and Schoutens [10] studied a two-dimensional adjoint QCD in which massive Dirac fermions Ψ are coupled to an $SU(N)$ gauge field. The fermions were assumed to transform in the adjoint rather than the fundamental representation of the gauge group. In the strongly coupled high density region of the phase space, the corresponding infrared fixed point is known to develop an $\mathcal{N} = (2, 2)$ superconformal symmetry. For a very small number $N \leq 3$ of colors, the fixed points belong to the series of $\mathcal{N} = (2, 2)$ superconformal minimal models and hence they are very well studied. But in order to compare with tree level string theory, one needs to explore the multicolor limit in which N goes to infinity. This regime is much less understood. Note that the central charge $c_N = (N^2 - 1)/3$ of fixed point theories grows quadratically with the rank $N - 1$ of the gauge group. While this is very suggestive of a string theory dual, there exist very little further clues on the appropriate choice of the 7-dimensional compactification manifold M^7 of the relevant AdS background.

A central clue for discriminating between potential gravitational duals of the infrared fixed point is expected to come from the chiral ring [11], i.e. the algebra of operators ϕ satisfying the *BPS* bound $h(\phi) = Q(\phi)$. Here h denote the scaling weight and Q the U(1) R-charge of ϕ, respectively. A certain subset of so-called regular chiral primaries is easy to construct and some of them had appeared in [10] already. But once we leave the territory of minimal models, these do not exhaust the set of chiral primary operators. In a previous paper we pushed the study of chiral primaries to $N > 3$ and constructed all such operators for $N = 4$ and $N = 5$. In both cases, we found new chiral primary operators that we dubbed *exceptional*. The total number of such exceptional chiral primaries can be shown to grow very rapidly with N.[1] On the other hand, the examples we reported on satisfy the BPS condition $h = Q$ only for one special value of N. Therefore it is not evident that exceptional chiral primaries contribute to the chiral ring of the multicolor limit. This is the question we are about to address with the present paper.

The main result of our analysis is that the large N limit of the chiral ring receives contributions only from regular chiral primaries. The latter can be counted quite easily. As described in [1], they are labeled by partitions or Young diagrams. Moreover, their operator product expansions may be argued to agree with the product of Schur polynomials. This provides a complete description of the chiral ring in the large N limit.

The plan of this short note is as follows. In the next section we shall introduce the model and review some of the key results from [1]. Section 3 contains the main new results of this paper. There we shall show that chiral primaries can only contribute in the limit $N \to \infty$ if they are regular. The operator products of regular chiral primaries are discussed in the concluding section along with a few open problems that should be addressed in future studies of the model.

[1] In particular, we observed that the number of exceptionals grows faster than the number of regular chiral primaries whose number grows as 2^N. For example, while there is a single exceptional at $N = 4$ along with 7 regulars, the $N = 8$ theory possesses 153 exceptional chiral primaries which outnumber the 125 regulars.

2. Review of background material

The role of this section is to review the definition of the model and the construction of its state space. We shall also recall a few central results on chiral primaries, including the construction of regular chiral primaries, that have been discussed in [10] and then extended in [1].

2.1. The coset model

The model we start with is a 2-dimensional version of QCD with fermions in the adjoint representations, i.e.

$$\mathcal{L}(\Psi, A) = \mathrm{Tr}\left[\overline{\Psi}(i\gamma^\mu D_\mu - m - \mu\gamma^0)\Psi\right] - \frac{1}{2g_{\mathrm{YM}}^2}\,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}\ . \tag{2.1}$$

Here, A denotes an $\mathrm{SU}(N)$ gauge field with field strength F and gauge coupling g_{YM}. The complex Dirac fermions Ψ transform in the adjoint of the gauge group and D_μ denote the associated covariant derivatives. The two real parameters m and μ describe the mass and chemical potential of the fermions, respectively.

We are interested in the strongly coupled high density regime of the theory, i.e. in the regime of very large chemical potential $\mu \gg m$ and g_{YM}. As is well known, we can approximate the excitations near the zero-dimensional Fermi surface by two sets of relativistic fermions, one from each component of the Fermi surface. These are described by the left- and right-moving components of massless Dirac fermions. At strong gauge theory coupling, the resulting (Euclidean) Lagrangian reads

$$\mathcal{L}_{\mathrm{eff}}(\psi, \overline{\psi}, A) = \mathrm{Tr}\left(\overline{\psi}^*\partial\,\overline{\psi} + \psi^*\bar{\partial}\,\psi + A_z[\psi^*, \psi] + A_{\bar{z}}[\overline{\psi}^*, \overline{\psi}]\right)\ . \tag{2.2}$$

Here we have dropped the term involving the field strength F, using that $g_{\mathrm{YM}} \to \infty$. Upon integrating out the two components A_z and $A_{\bar{z}}$ of the gauge field we obtain the constraints

$$J(z) := [\psi^*, \psi] \sim 0 \quad , \quad \bar{J}(\bar{z}) := [\overline{\psi}^*, \overline{\psi}] \sim 0\ . \tag{2.3}$$

These constraints are to be implemented on the state space of the $N^2 - 1$ components of the complex fermion ψ such that all the modes $J_n, n > 0$, of $J(z) = \sum J_n z^{-1-n}$ vanish on physical states, as is familiar from the standard Goddard–Kent–Olive coset construction [12].

In order to describe the chiral symmetry algebra of the resulting conformal field theory we shall start with the unconstrained model which we refer to as the *numerator* theory. It is based on $M = N^2 - 1$ complex fermions $\psi_\nu, \nu = 1, \dots, M$. These give rise to a Virasoro algebra with central charge $c_{\mathrm{N}} = N^2 - 1$, where the subscript N stands for numerator. We can decompose each complex fermion into two real components $\psi^n_\nu, n = 1, 2$, such that $\psi_\nu = \psi^1_\nu + i\psi^2_\nu$. From time to time we shall combine ν and n into a single index $\alpha = (\nu, n)$. Let us recall that the $2M$ real fermions ψ_α can be used to build $\mathrm{SO}(2M)$ currents $K_{\alpha\beta}$ at level $k = 1$. The central charge of the associated Virasoro field coincides with the central charge c_{N} of the original fermions. The $\mathrm{SO}(2M)_1$ current algebra generated by the modes of $K_{\alpha\beta}$ forms the numerator in the coset construction.

The algebra generated by the constraints (2.3) forms the denominator of the coset construction. According to the usual free fermion constructions of current algebras, the fields J that were introduced in eq. (2.3) form an $\mathrm{SU}(N)$ current algebra at level $k = 2N$. The components $J_\nu = j^1_\nu + j^2_\nu$ can be written as a sum of $\mathrm{SU}(N)$ currents j^n_ν with $\nu = 1, \dots, M$ and $n = 1, 2$. The latter are obtained as bilinears of the real fermions $\psi^n_\nu, n = 1, 2$, that we used in our description of the numerator theory. Through the Sugawara construction we obtain a Virasoro algebra with central charge $c_{\mathrm{D}} = 2(N^2-1)/3$, where the subscript D stands for denominator. Now we have assembled all the elements that are needed in defining the coset chiral algebra

$$\mathcal{W}_N := \mathrm{SO}(2N^2-2)_1/\mathrm{SU}(N)_{2N}\ . \tag{2.4}$$

The parameter N keeps track of the gauge group $\mathrm{SU}(N)$. The algebra $\mathcal{W}_N$ is a key element in our subsequent analysis. It is larger than the chiral symmetry considered in [10] which uses the subalgebra $\mathrm{SU}(N)_N \times \mathrm{SU}(N)_N \subset \mathrm{SO}(2N^2-2)_1$ to encode symmetries of the numerator theory.

Before we conclude this short review of the underlying model, let us recall that the chiral algebra $\mathcal{W}_N$ contains a $\mathrm{U}(1)$ current. It is constructed as

$$J(z) = \tfrac{1}{3}\sum_{\nu,\mu} \psi^1_\nu(z)\psi^2_\mu(z)\kappa^{\nu\mu} \tag{2.5}$$

where $\kappa^{\nu\mu}$ denotes the Killing form of $\mathrm{SU}(N)$. The zero mode of this current turns out to measure the R-charge of fields in $\mathcal{N}=(2,2)$ superconformal low energy limit of 2D adjoint QCD. It will therefore play a very important role in the subsequent analysis.

2.2. The state space

Our second aim is to discuss the state space of the coset model. We shall start by discussing the sectors of the chiral algebra $\mathcal{W}_N$ before concluding with a few comments on the modular invariant partition function of the model. Since the Ramond (R) and Neveu–Schwarz (NS) sector of an $\mathcal{N}=(2,2)$ superconformal field theory are related by spectral flow [11], our discussion will focus on the NS sector.

Let us denote the state space that is created with chiral fields of the numerator theory in the NS sector by $\mathcal{H}^{\mathrm{NS}}$. Under the action of the denominator chiral algebra $\mathrm{SU}(N)_{2N}$ the space $\mathcal{H}^{\mathrm{NS}}$ decomposes as

$$\mathcal{H}^{\mathrm{NS}} \cong \bigoplus_{a\in\mathcal{J}_N} \mathcal{H}^{\mathrm{C}}_{\{a\}} \otimes \mathcal{H}^{\mathrm{D}}_a\ . \tag{2.6}$$

Here, $\mathcal{H}^{\mathrm{D}}_a$ denotes the sectors of the denominator algebra $\mathrm{SU}(N)_{2N}$ and $a \in \mathcal{J}_N$ is the corresponding weight. We shall consider $\mathcal{J}_N$ as the set of $N-1$ tuples

$$a = [\lambda_1, \ldots, \lambda_{N-1}] \quad \text{with} \quad \sum_s^{N-1} \lambda_s \leq 2N\ . \tag{2.7}$$

Alternatively, the elements of $\mathcal{J}_N$ may be thought of as $\mathrm{SU}(N)$ Young diagrams $Y = Y_a$. Given $a = [\lambda_1, \ldots, \lambda_{N-1}]$ the length of the ith row is

$$Y_a = (l_1, \ldots, l_{N-1}) \quad \text{is} \quad l_i = \sum_{s=i}^{N-1} \lambda_s\ . \tag{2.8}$$

Of course it is just as easy to reconstruct $a = a(Y)$ from a Young diagram Y. The factor $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ has been introduced to denote sectors of the coset chiral algebra $\mathcal{W}_N$. It will become clear momentarily why we placed the index a in brackets $\{\cdot\}$.

As usual in the coset construction, for $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ not to be empty, the label a must satisfy certain selection rules. In addition, some of the spaces $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ carry equivalent representations of $\mathcal{W}_N$. In order to describe the relevant selection rules and field identifications, we need to introduce the following map γ

$$\gamma([\lambda_1, \ldots, \lambda_{N-1}]) = [2N - \sum_{s=1}^{N-1} \lambda_s, \lambda_1, \ldots, \lambda_{N-2}] . \tag{2.9}$$

Obviously, γ maps elements $a \in \mathcal{J}_N$ back into $\mathcal{J}_N$ and it obeys $\gamma^N = id$. One can show that two sectors $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ and $\mathcal{H}^{\mathrm{C}}_{\{b\}}$ of the coset chiral algebra are isomorphic provided that the weights a and b are related to each other by repeated application of γ or, equivalently,

$$\mathcal{H}^{\mathrm{C}}_{\{a\}} \cong \mathcal{H}^{\mathrm{C}}_{\{\gamma(a)\}} \quad \text{for} \quad a \in \mathcal{J}_N . \tag{2.10}$$

The isomorphism respects the action of the coset chiral algebra $\mathcal{W}_N$ on the sectors $\mathcal{H}^{\mathrm{C}}_{\{a\}}$. In order to state the selection rules we recall that the conformal weight $h^{\mathrm{D}} : \mathcal{J}_N \to \mathbb{R}$ is given by

$$h^{\mathrm{D}}(a) = \frac{C_2(a)}{3N} , \tag{2.11}$$

where the quadratic Casimir of an SU(N) representation $a = [\lambda_1, \ldots, \lambda_{N-1}]$ takes the form

$$C_2(a) = \frac{1}{2}\left[-\frac{n^2}{N} + nN + \sum_{i=1}^{r} (l_i^2 + l_i - 2il_i) \right] .$$

Here, we have used the row length parameters l_i introduced in eq. (2.8) and n denotes the total number of boxes $|Y| = n = \sum_i l_i$ in the Young diagram $Y = Y_a$. With these notations let us introduce the so-called monodromy charge

$$Q_\gamma(a) \equiv h^{\mathrm{D}}(\gamma(a)) - h^{\mathrm{D}}(a) \bmod 1 . \tag{2.12}$$

For the carrier space $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ of the coset algebra to be non-vanishing, the label a should be taken from the set $\mathcal{J}^0_N$ of SU$(N)_{2N}$ labels a with vanishing monodromy charge $Q_\gamma(a) = 0$,

$$\mathcal{H}^{\mathrm{C}}_{\{a\}} \cong \emptyset \quad \text{if} \quad Q_\gamma(a) \neq 0 . \tag{2.13}$$

Since representations of the coset chiral algebra $\mathcal{W}_N$ are invariant under the action (2.9) of the identification group $\mathbb{Z}_N$, isomorphism classes of representations of the coset chiral algebra are labeled by orbits $\{a\} \in \mathcal{O}_N = \mathcal{J}^0_N/\mathbb{Z}_N$.

A very useful way to parametrize elements of $\mathcal{J}^0_N$, i.e. SU$(N)_{2N}$ weights a with vanishing monodromy charge, through a pair of Young diagrams Y' and Y'' was described in [1]. Following that approach, we introduce two SU(N) Young diagrams Y' and Y'' with equal number $n' = |Y'| = |Y''|$ of boxes, subject to the additional conditions

$$r' + c'' \leq N \quad , \quad r'' + c' \leq 2N \tag{2.14}$$

where r', r'' and c', c'' denote the numbers of rows and columns of Y', Y'', respectively. Let us denote the row lengths of the Young diagrams Y' and Y'' by

$$Y' = (l'_1, \ldots, l'_{r'}) \quad , \quad Y'' = (l''_1, \ldots, l''_{r''}) .$$

As before, we arrange the l'_i and l''_i in decreasing order, i.e. $l'_i \geq l'_{i+1}$ etc. so that the largest entries are $l'_1 = c'$ and $l''_1 = c''$. From these two Young diagrams we can build a new diagram $Y = Y(Y', Y'') = (l_1, \ldots, l_{N-1})$ through

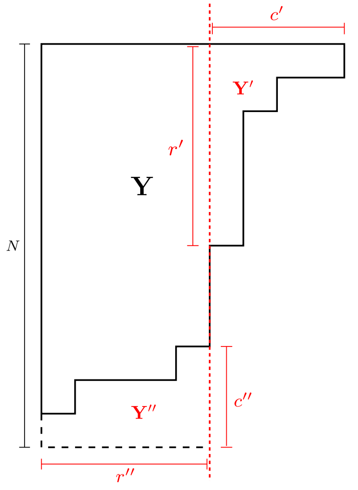

Fig. 1. Dissecting the Young diagram Y by the red dashed line identifies Young diagrams Y' and Y'' as soon as their numbers of boxes match. This can happen exactly once. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$l_i = \begin{cases} r'' + l'_i & \text{for} \quad i = 1, \ldots, r' \\ r'' & \text{for} \quad i = r' + 1, \ldots, N - l''_1 \\ r'' - k & \text{for} \quad i = N - l''_k + 1, \ldots, N - l''_{k+1}, \qquad k = 1, \ldots, r'' - 1 \\ 0 & \text{for} \quad i = N - l''_{r''} + 1, \ldots, N - 1 \,. \end{cases} \tag{2.15}$$

This prescription extends a construction in [13] and it gives a special family of so-called composite representations $Y = \bar{Y}''Y'$ in the sense of [14]. The latter have been defined without the additional condition $|Y'| = |Y''|$. It is not too difficult to show that all diagrams $Y = Y(Y', Y'')$ obtained in this way correspond to an $\mathrm{SU}(N)_{2N}$ weight $a = a(Y) = a(Y', Y'')$ with vanishing monodromy charge. Conversely, any such weight arises from a suitably chosen pair (Y', Y'').

The inverse procedure of obtaining diagrams Y' and Y'' from a given diagram $Y = Y_a = (l_1, \ldots, l_{N-1})$ satisfying the zero monodromy charge condition

$$\sum_i l_i \equiv 0 \bmod N, \tag{2.16}$$

goes as follows. One defines $r'' := \frac{1}{N} \sum_i l_i$, $c' := l_1 - r''$. Then the entries of the small Young diagrams Y' and Y'' can be written as

$$\begin{cases} Y' := (l_1 - r'', l_2 - r'', \ldots, l_{N-1} - r''), \\ Y''^{\mathrm{T}} := (r'', r'' - l_{N-1}, \ldots, r'' - l_1). \end{cases} \tag{2.17}$$

In both expressions the order of entries is non-decreasing and all non-positive entries are to be skipped from the end of these strings.

The prescriptions (2.15) and (2.17) might appear somewhat heavy at first, but they possess a very simple pictorial representation, see Fig. 1. Suppose we are given the two Young diagrams Y' and Y''. Then we need to flip Y'' and place it on the bottom line of the image which is N boxes below the top line. This Young diagram has to start in the first column and hence will extend over r'' columns. We now fill all the boxes above the flipped diagram before we attach

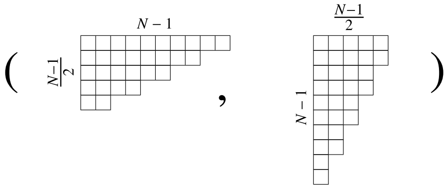

Fig. 2. The pair $(Y'(a_*), Y''(a_*))$ corresponding to $a_* = [2, 2, \dots, 2]$, N assumed odd.

the second Young diagram Y' on the right hand side. Conversely, if we are given Y, we must first construct the flipped Y''. It is made from all the boxes that are needed to fill the space below the Young diagram Y, including the N-th row. On the right hand side, we include as many columns r'' as are needed for the flipped Y'' to possess as many boxes as the Young diagram Y' that appears to the right of the r''th column. This can be done by increasing the number of columns one by one until the appropriate r'' is found. If no appropriate choice of r'' exists, the original Young diagram Y does not correspond to a sector with vanishing monodromy charge.

With the map $Y(Y', Y'')$ and its inverse well understood, we want to mention two properties of Y that become relevant later on. To begin with, it is evident from the geometric construction we described in the previous paragraph that the diagram $Y = Y(Y', Y'')$ possesses $|Y| = n = r''N$ boxes. Furthermore, one can show [1] that the value of the quadratic Casimir in the corresponding representation of $\mathrm{SU}(N)$ is given by

$$C_2(a(Y', Y'')) = C_2(Y(Y', Y'')) = n'N + C_2(Y') - C_2(Y''), \tag{2.18}$$

where $C_2(Y')$ and $C_2(Y'')$ are the quadratic Casimirs of the $\mathrm{SU}(N)$ representations associated with Y' and Y'', respectively.

At this point, we have explained everything there is to know about the formula (2.6). Even though we wrote that the summation index a is taken out of $\mathcal{J}_N$, we should keep in mind that the summands are trivial unless $a \in \mathcal{J}_N^0$ simply because the corresponding spaces $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ vanish. Hence, we can think of the summation as running over pairs (Y', Y'') of Young diagrams subject to the conditions (2.14). Finally, while the representation spaces $\mathcal{H}^{\mathrm{D}}_a$ of the denominator current algebra depend on the weight $a = a(Y', Y'')$, the sectors $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ are invariant under the action (2.9) of γ and hence only depend on the $\mathbb{Z}_N$ orbit $\{a\} = \{a(Y', Y'')\}$ of a.

The decomposition (2.6) is just used to build the representations $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ of the coset chiral algebra $\mathcal{W}_N$ but it does not tell us yet how these sectors are combined with those of the right moving chiral algebra in order to build a fully consistent conformal field theory. The relevant modular invariant partition function was described in [1]. For our purposes, it suffices to consider the case when N is prime. With this assumption, all but one of the sectors $\mathcal{H}^{\mathrm{C}}_{\{a\}}$ carry irreducible representations of the chiral algebra. Only the sector $\mathcal{H}^{\mathrm{C}}_{\{a_*\}}$ for $a_* = [2, 2, \dots, 2]$ can be decomposed into several irreducibles $\mathcal{H}^{\mathrm{C}}_{\{a_*\};\nu}$. The range of the index ν and other features of this decomposition are described in [1]. The full state space of the conformal field theory has been argued to take the form

$$\mathscr{H}^{\mathrm{C}} = \frac{1}{N} \bigoplus_{Y', Y''}{}' \mathcal{H}^{\mathrm{C}}_{\{a(Y', Y'')\}} \otimes \overline{\mathcal{H}}^{\mathrm{C}}_{\{a(Y', Y'')\}} \oplus \mathscr{H}^{\mathrm{C}}_{\mathrm{fix}} . \tag{2.19}$$

Here, we sum over all pairs (Y', Y'') of Young diagrams that obey the conditions (2.14) with the exception of the unique pair that gives the Young diagram $Y = Y_{a_*}$, see Fig. 2. Because of the identification rule (2.10), the sum over Y' and Y'' gives each term with multiplicity N. This degeneracy is removed when we divide by N. The term $\mathscr{H}^{\mathrm{C}}_{\mathrm{fix}}$ is built out of the sectors $\mathcal{H}^{\mathrm{C}}_{\{a_*\};\nu}$ and their right moving counterparts. The precise form, which can be found in [1], will not be relevant in the subsequent analysis. Indeed, as we shall argue, this sector of the state space cannot contribute any chiral primaries to the large N limit.

2.3. Regular chiral primaries

As we explained in the introduction, there exist two different classes of chiral primary fields which we referred to as regular and exceptional. While there is no general construction of the exceptional ones so far, the regular chiral primaries may be listed explicitly for any value of N. In fact, after we have introduced our parametrization of elements in $\mathcal{J}^0_N$ through pairs (Y', Y'') of Young diagrams, this task is really easy. It turns out that all sectors $\{a(Y', Y'' = Y')\}$ of the coset chiral algebra contain precisely one (regular) chiral primary field $\phi_{\mathrm{cp}}\{a\}$. This field is to be found among the ground states of the sector.

For ground states in the coset sector $\mathcal{H}^{\mathrm{C}}_{\{a(Y',Y')\}}$ there exists a simple formula to compute the exact conformal weight. By eq. (2.18) the quadratic Casimir of a representative $a = a(Y', Y')$ is simply $C_2(a) = n'N$ with $n' = |Y'| = |Y''|$. This implies that the conformal weight of the ground states takes the form

$$h(\phi\{a\}) = \frac{C_2(a)}{6N} = \frac{n'}{6}\,, \tag{2.20}$$

for $a = a(Y', Y')$. Since the diagonal sectors $a\,(Y', Y')$ contain a chiral primary ground state, its conformal weight and U(1) charge are given by $n'/6$. Let us stress once again that the diagonal or regular sectors do not contain any further chiral primaries among the $\mathcal{W}_N$ descendents [1].

Before we conclude this section we need to add a few comments that will later allow us to enumerate regular chiral primaries, i.e. the orbits of diagonal sectors $a(Y', Y')$. We should stress that most elements in such an orbit $\{a(Y', Y'' = Y')\}$ are not obtained from diagonal pairs $(Y', Y'') = (Y', Y')$. So, if we would like to decide whether the sector $\mathcal{H}^{\mathrm{C}}_{\{b\}}$ contains a chiral primary field, we need to construct the pair $(Y'(a), Y''(a))$ for each element a in the orbit $\{b\}$ of the element $b \in \mathcal{J}^0_N$ and check whether at least one of these pairs satisfies the condition $Y'(a) = Y''(a)$. Let us note that the orbit $\{a_*\}$ of the weight $a_* = [2, 2, \ldots, 2]$ consists of a single element a_* and $Y'(a_*) \neq Y''(a_*)$ for any prime $N > 2$. Hence, this special orbit does not contain a regular chiral primary unless $N = 2$.

More importantly, one can show that most orbits $\{b\}$ contain at most one representative $a \in \{b\}$ such that $Y'(a) = Y''(a)$. This follows from the following expression for the action of γ^k in the weights of (Y', Y''),

$$\begin{aligned}
\gamma^k(Y') \quad &:= (2(N-k) + l_{N-k+1} - r'', \ldots, 2(N-k) + l_{N-1} - r'',\\
&\qquad\qquad 2(N-k) - r'', l_1 - 2k - r'', \ldots, l_{N-k-1} - 2k - r'')\\
\gamma^k(Y''^{\mathrm{T}}) &:= (r'' + 2k - l_{N-k}, \ldots, r'' + 2k - l_1, r'' - 2(N-k),\\
&\qquad\qquad r'' - 2(N-k) - l_{N-1}, \ldots, r'' - 2(N-k) - l_{N-k+2}).
\end{aligned}$$

In both expressions the order of entries is non-decreasing and all non-positive entries are to be skipped from the end of these strings. The Y^{T} is used to denote the transpose of a Young diagram Y. If we now require $Y' = Y''$ and $\gamma^k(Y') = \gamma^k(Y'')$ it is easy to infer that the only solutions satisfying these two constraints are the Young diagrams of rectangular shape. These correspond to the orbits $\{a_\nu\}$ of the weight $a_\nu = [0, \dots, 0, N, 0, \dots, 0]$ for $\nu = 1, \dots, N-1$ where the only non-zero entry N can appear in any position ν, i.e. $\lambda_\nu = N$. As we have just demonstrated, field identifications can map a_ν to $a_{N-\nu}$. Both of these weights are associated with diagonal pairs $(Y', Y'' = Y')$ of Young diagrams.

In conclusion we have argued that regular chiral primary fields of the coset conformal field theory are associated with Young diagrams Y' such that $r' + c' \leq N$. The correspondence is one-to-one with the exception of the Young diagrams Y'_ν and $Y'_{N-\nu}$ which correspond to one and the same regular chiral primary.

3. Chiral primaries at large N

We now address the central goal of this work, namely to construct the chiral ring in the limit of large N. As we are about to vary N, many of the objects we encountered in the previous section will carry an additional label N. This applies in particular to the quadratic Casimir $C_2^{(N)}$, the sectors $\mathcal{H}^{\mathrm{C},(N)}$ of the coset chiral algebra as well as the maps $a_N = a_N(Y', Y'')$ and $Y_N = Y_N(Y', Y'')$ that associate a weight a or a Young diagram Y to a pair of Young diagrams Y' and Y''.

Since the coset model is built from representations of the coset chiral algebra, we should first explain how to take the large N limit of the sectors $\mathcal{H}^{\mathrm{C},(N)}_{\{a\}}$. In the previous section we learned how to parametrize the allowed values of a in terms of two Young diagrams Y' and Y''. In taking the limit, we keep these Young diagrams fixed, i.e. we define

$$\mathcal{H}_{\{Y',Y''\}} \equiv \lim_{N\to\infty} \mathcal{H}^{\mathrm{C},(N)}_{\{a_N(Y',Y'')\}} \,.$$

Let us stress that the Young diagram $Y = Y_a$ that we construct from Y' and Y'' depends on the value of N. This is why it was so important to place a subscript $_N$ on the corresponding SU(N) weight $a = a_N$. One can show that the sectors $\mathcal{H}_{\{Y',Y''\}}$ are well defined. In particular, the dimension of the subspaces with fixed conformal weight h stabilizes as we send N to infinity. We are now trying to find those pairs (Y', Y'') for which the space $\mathcal{H}_{\{Y',Y''\}}$ contains chiral primaries. Our claim is that this happens if and only if $Y' = Y''$. As we reviewed in the previous subsection, such diagonal pairs of Young diagrams are associated with regular chiral primaries.

In order to establish these claims let us consider any of the summands

$$\mathcal{H}^{\mathrm{NS}}_a = \mathcal{H}^{\mathrm{C},(N)}_{\{a\}} \otimes \mathcal{H}^{\mathrm{D}}_a$$

in the decomposition (2.6). The space $\mathcal{H}^{\mathrm{NS}}_a$ comes equipped with the action of several commuting operators. To begin with, we mention the zero modes of the coset Virasoro field and the U(1) currents, i.e. $L_0 = L_0^G - L_0^H$ and Q. In addition, we can also introduce the fermion number operator K_0 which is defined by

$$K_0 = \sum_{r\geq 1/2} \psi^1_{\mu,-r}\psi^1_{\nu,r}\kappa^{\mu\nu} + \psi^2_{\mu,-r}\psi^2_{\nu,r}\kappa^{\mu\nu} \,.$$

K_0 commutes with Q and L_0 and hence can be measured simultaneously on $\mathcal{H}^{\mathrm{NS}}_a$.

Proposition 3.1. *The conformal weight h_ϕ of states ϕ in the subspace $\mathcal{H}^{\mathrm{NS}}_a$ of the NS-sector is bounded from below by*

$$h_\phi \geq \frac{K_\phi}{2} - \frac{C_2^{(N)}(a)}{3N} \,. \tag{3.1}$$

Similarly, the U(1) charge Q_ϕ of the state ϕ is bounded from above by

$$|Q| \leq \frac{K_\phi}{6} \,. \tag{3.2}$$

In both inequalities, the number K_ϕ denotes the fermion number, i.e. the eigenvalue of the fermion number operator K_0 on the state ϕ.

The two inequalities follow straightforwardly from the fact that the complex fermion multiplets Ψ and Ψ^* have conformal weight $h_\Psi = 1/2$ and that their real and imaginary part ψ^1 and ψ^2 possess U(1) charge $|Q_{\psi^j}| = 1/6$. In the first relation, the two sides are equal in case the construction of ϕ does not involve any derivatives of the fermionic fields. The second relation becomes an equality for states ϕ that are built from ψ^1 or ψ^2 and its derivatives only.

There is another simple proposition we need to discuss. Before we state it, let us recall from [1] that a sector $\mathcal{H}^{\mathrm{C},(N)}_{\{a\}}$ of the coset model can only contain a chiral primary if

$$\min_{b\in\{a\}} \left(C_2^{(N)}(b)\right) \equiv 0 \bmod N \,,$$

i.e. the minimum $C_2(b)$ assumed in the orbit $\{a\}$ of a must be divisible by N, at least when N is odd. Under the action of the identification current, the value of the quadratic Casimir can only shift by an integer multiple[2] of N so that a sector $\mathcal{H}^{(N)}_{\{a\}}$ can only contain a chiral primary if

$$C_2^{(N)}(a) \equiv 0 \bmod N \,.$$

As we explained before, when we vary N we are instructed to keep Y' and Y'' fixed. Let us assume that N_0 is the minimal number for which the two inequalities

$$r' + c'' \leq N_0 \quad , \quad r'' + c' \leq 2N_0$$

are satisfied. Then Y' and Y'' define a sector of the coset theory for all $N \geq N_0$. We can use the rules stated above to construct a diagram $Y_N(Y', Y'')$ for all $N \geq N_0$. The associated representation is denoted by $a_N = a_N(Y', Y'')$, as before.

Proposition 3.2. *The family of sectors $\mathcal{H}^{\mathrm{C},(N)}_{\{a_N(Y',Y'')\}}$ can only contain a chiral primary if the two Young diagrams Y' and Y'' of $SU(N_0)$ belong to representations with the same value of the quadratic Casimir element,*

$$C_2^{(N_0)}(Y') = C_2^{(N_0)}(Y'') \,.$$

To prove this statement we recall from eq. (2.18) that the value of the quadratic Casimir element in the representation a_N of SU(N) is given by

$$C_2^{(N)}(a_N(Y', Y'')) = n'N + C_2^{(N)}(Y') - C_2^{(N)}(Y'') \,.$$

For the sector a_N to contain a chiral primary, the right hand side must be divisible by N. Since the first term is, we need to determine the conditions under which the $C_2^{(N)}(Y') - C_2^{(N)}(Y'')$ is a multiple of N. The difference of the Casimir reads

[2] The precise amount of this shift is $C_2(\gamma^i(a)) - C_2(a) = 3N\left(\sum_{j=1}^{i-1} l_{N-j} + i\,(N - r'' - i)\right)$.

$$C_2^{(N)}(Y') - C_2^{(N)}(Y'') = \frac{1}{2}\left(\sum_i (l_i'^2 + l_i' - 2il_i') - (l_i''^2 + l_i'' - 2il_i'')\right)$$
$$= C_2^{(N_0)}(Y') - C_2^{(N_0)}(Y'') \,, \tag{3.3}$$

and thus does not depend on N. Hence it clearly cannot be divisible by (a sufficiently large)[3] N unless the difference vanishes. This is what we had to prove.

Our third proposition is a little more difficult to prove, but it is absolutely crucial for what we are about to establish.

Proposition 3.3. *For the states ϕ in the sector $\mathcal{H}_a^{\mathrm{NS}}$, the fermion number satisfies the inequality*

$$K_\phi \geq n' \,.$$

The number n' is determined by the choice of a. It is computed from the associated Young diagram $Y = Y_a$ by, see eq. (2.17),

$$n' = \sum_{i=1}^{N-1} \theta\left(l_i - \frac{1}{N}\sum_{i=1}^{N-1} l_i\right),$$

where $\theta(x)$ denotes the Heaviside step-function. If the representation a is diagonal, i.e. $Y'(a) = Y''(a)$, the above formula simplifies to

$$n' = \sum_{i=1}^{n/N}\left(l_i - \frac{n}{N}\right) \,, \quad \textit{where} \quad n = \sum_{i=1}^{N-1} l_i \,.$$

We will first give a somewhat heuristic graphical argument using Young diagrams before we outline a formal proof of this proposition. Let us recall that all our fermions transform in the tensor product of the fundamental and the dual fundamental representations. These correspond to Young diagrams that consist of a single box and a single column of maximal length $N-1$, respectively.

We need to show that it takes at least n' fermionic fields in order to build a state in the representation a, i.e. the first time the representation a appears in the tensor power $\mathbf{adj}^{\otimes K}$ is for $K = n'$. For the SU(N) Lie algebra, the adjoint representation decomposes as $\mathbf{adj} = \square \otimes \bar{\square}$. Here $\bar{\square}$ denotes the (Young diagram of) the dual fundamental representation, i.e. a column of $N-1$ boxes.

The graphical proof goes as follows. In order to build the Young diagram Y we start with the K columns $\bar{\square}$ of size $N-1$. If we put them all side by side, we would obtain a rectangular Young diagram of size $(N-1) \times K$. As we increase the number K, the diagram Y'' starts to cut into the rectangle, see Fig. 3. This means that we have to remove a few boxes from those columns and move them to one of the previous columns. But since all these have maximal length, every time we take out one of the boxes and move it to a full column to the left, we lose an entire column with N boxes. It is easy to see that in total we need to move $n' - r''$ boxes which make us lose $N(n' - r'')$ boxes altogether. Hence we need $K = r'' + n' - r'' = n'$ fermions to begin with. Of course, this number is also sufficient since we can build Y' from the n' fundamentals $\square$. This concludes the graphical proof.

[3] A rough estimate for what 'sufficiently large' means is given by $N > N_0(N_0^2 - 1)/12$.

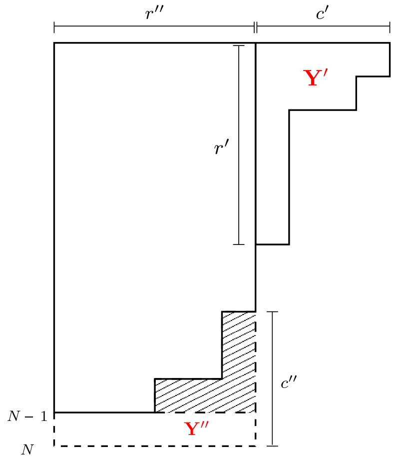

Fig. 3. The Young diagram Y is obtained from a rectangle of size $N \times r''$ by attaching the Young diagram Y' and removing (a reflected version of) the Young diagram Y''. Note that $r' + c'' \leq N$ is needed for the resulting diagram to be a Young diagram of $\mathrm{SU}(N)$. The shaded region shows those boxes from the tensor power of the dual fundamental that must be moved to the left.

Let us now go through a somewhat more formal argument. There is an explicit decomposition [15] found by studying so-called walled Brauer algebras [16,17] which provides us with the irreducible content of the K-th tensor power of the $\mathrm{SU}(N)$ adjoint representation, at least up to $2K \leq N$,

$$\mathbf{adj}^{\otimes K} = \sum_{n'=0}^{K} b_{n'}^{(K)} \sum_{Y', Y'' \vdash n'} \frac{n'!}{\prod_{(l',m') \in Y'} h\,(l', m')} \frac{n'!}{\prod_{(l'',m'') \in Y''} h\,(l'', m'')} \cdot a\,(Y', Y'') \,. \tag{3.4}$$

Here we use the notation $Y', Y'' \vdash n'$ to express that both Y' and Y'' are partitions of n', the products run over all boxes in Y' and Y'' and the integers $h\,(l, m)$ denote the length of a hook that is associated to the box (l, m), see Fig. 4. A formal definition can be found in Appendix A. The multiplicities $b_{n'}^{(K)}$ are of combinatorial nature and explicitly given by

$$b_{n'}^{(K)} := \sum_{i=0}^{K-n'} (-1)^{i+K+n'}\, i! \binom{K}{n'} \binom{K-n'}{i} \binom{i+n'}{i} \,. \tag{3.5}$$

The derivation of formula (3.4) is discussed in more detail in Appendix A. What is most important for us right now is that a representation a composed out of two small Young diagrams of n' boxes can appear on the right hand side of eq. (3.4) only when we start with the product of $K = n'$ adjoints on the left hand side of eq. (3.4). This concludes the proof of Proposition 3.3.

Now let us combine the previous three statements. According to Proposition 3.2, the sectors that can contribute a chiral primary which has finite weight in the large N limit have $C_2(a) \sim n'N$. Hence, the two inequalities in Proposition 3.1 become

$$h_\phi \geq \frac{K_\phi}{2} - \frac{n'}{3} = \frac{n'}{6} + \frac{K_\phi - n'}{2} \,, \tag{3.6}$$

$$|Q_\phi| \leq \frac{K_\phi}{6} = \frac{n'}{6} + \frac{K_\phi - n'}{6} \,. \tag{3.7}$$

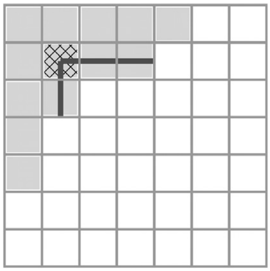

Fig. 4. For each box (l, m) in the Young diagram Y, one can draw a hook. In the figure we have shaded the box and indicated the hook by the thick line. The length of the hook, i.e. the length of the thick line, is denoted by $h(l, m)$.

In the second step we have sightly rewritten the bounds. From Proposition 3.3 we know that the quantity $K_\phi - n'$ is non-negative. Hence, the equality $h_{\phi_{\text{cp}}} = Q_{\phi_{\text{cp}}}$ between the conformal weight and U(1) charge of a chiral primary ϕ_{cp} can only be satisfied for $K_{\phi_{\text{cp}}} = n'$. This implies that both the weight and the U(1) charge of such chiral primaries,

$$h_{\phi_{\text{cp}}} = \frac{n'}{6} \quad \text{and} \quad Q_{\phi_{\text{cp}}} = \frac{n'}{6}\,, \tag{3.8}$$

saturate the bounds given in Proposition 3.1. As we explained in the text below Proposition 3.1, this implies that the state ϕ_{cp} is constructed from the fermionic fields ψ^1_ν only without any derivatives and components of ψ^2. States with these features must transform in the anti-symmetric tensor power of the adjoint representation. It is actually possible to work out the precise content of the anti-symmetrized part of the k-th power of adjoint representation for values of $k \leq N-1$,

$$\left\{\mathbf{adj}^{\otimes k}\right\}_{\text{antisymm}} = \sum_{n'=1}^{k} d^{(k)}_{n'} \sum_{Y' \vdash n'} a\,(Y', Y')\,. \tag{3.9}$$

A more detailed discussion and the precise values of the coefficients $d^{(n')}_k$ can be found in Appendix B. What is most important about formula (3.9), at least in our present context, is that all representations that appear in the decomposition are of the form $a(Y', Y'') = a(Y', Y')$. Now we only need to recall from Section 2.3 that such diagonal sectors are associated with regular chiral primaries to establish our central claim: The chiral primaries of the large N limit are regular. Let us stress once again that for any given finite value of N, chiral primaries can be constructed that do not satisfy eqs. (3.8) and hence are not regular.

4. Discussion, conclusion and open problems

In the preceding section we proved that chiral primary fields in the low energy limit of multi-color adjoint QCD are regular in the sense we defined in Section 2.3. We have seen before that such regular chiral primaries are in one-to-one correspondence with Young diagrams Y', at least if we approach the multi-color limit through a sequence of prime numbers N. In case N is prime, the only contribution to the state space (2.19) that is not simply a diagonal product of left- and right-movers is the term $\mathscr{H}_{\text{fix}}$ which does not contain any regular chiral primaries, see Section 2.3. Furthermore, as we approach the large N theory, the only orbits $\{a\}$ that are associated with two different diagonal pairs $(Y', Y'' = Y')$, namely the orbits $\{a_\nu\}$, see next to last paragraph in Section 2.3, give rise to regular chiral primaries of weight $h = \nu(N-\nu)/6$. Hence, they are not part of the spectrum of chiral primaries as N tends to infinity. For all remaining regular chiral primaries, the orbit is associated with a unique Young diagram Y'. Combining all these facts, we introduce the symbol $\phi_{\mathrm{cp}}(Y')$ to denote the unique chiral primary

$$\phi_{\mathrm{cp}}(Y') \in \mathcal{H}_{Y',Y''} \otimes \overline{\mathcal{H}}_{Y',Y''} \quad \text{with} \quad h(\phi_{\mathrm{cp}}(Y')) = n' = |Y'| \,.$$

As usual in $\mathcal{N} = (2, 2)$ supersymmetric theories, the chiral fields form a chiral ring which closes under operator product expansions. It is not difficult to argue that the chiral ring at large N must be isomorphic to a standard graded ring of symmetric functions $\Lambda_R = \oplus_{i\in\mathbb{N}} \Lambda_R^{(i)}$, which is a ring of formal infinite sums of monomials. Its Hilbert–Poincaré series

$$\sum_{i\in\mathbb{N}} \dim(\Lambda_R^{(i)})\, t^i := \prod_{i=1}^{\infty} \frac{1}{1-t^i} \,, \tag{4.10}$$

i.e. the function which generates dimensions of subspaces of grade i, is the generating function of integer partitions. The operator product of two chiral primaries at large N thus takes the form

$$\phi_{\mathrm{cp}}(Y_1') \cdot \phi_{\mathrm{cp}}(Y_2') = \sum_{Y_3'} \mathcal{C}_{Y_1',Y_2'}^{Y_3'}\, \phi_{\mathrm{cp}}(Y_3'), \tag{4.11}$$

where $\mathcal{C}_{Y_1',Y_2'}^{Y_3'}$ are the Littlewood–Richardson coefficients [18,19]. The ring is freely generated, e.g. by the elementary symmetric polynomials e_k, $k = 1, 2, \dots$ corresponding to those chiral primaries whose Young diagrams Y' consist of only one column. Obviously, there is exactly one such generator at each grade. The construction of generators representing chiral primaries $\phi_{\mathrm{cp}}(Y')$ corresponding to other Young diagrams $Y' \neq e_k$ is then performed with the help of the second Jacobi–Trudi identity.

In the special case of fusion with a chiral primary corresponding to the partition f_n of one row with n boxes, Pieri's formula implies

$$\phi_{\mathrm{cp}}(Y_1') \cdot \phi_{\mathrm{cp}}(f_n) = \sum_{Y_3'} \phi_{\mathrm{cp}}(Y_3'), \tag{4.12}$$

where the summation goes only over Young diagrams obtained from Y_1' by adding n boxes, no two in the same column. From this formula one can see that an iterative fusion of the vacuum with the lowest non-trivial chiral primary $\mathcal{C}_\square$ precisely generates the Young lattice (the lattice of Young diagrams ordered by inclusion). This is a nice way to picture a subring of the chiral ring generated by the grade 1 generator alone (see Fig. 5).

This concludes our discussion of the results in this paper. Let us recall that the main motivation of our work stems from the desire to constrain the dual higher spin or string theory. Let us recall that the family of "strange metal" coset models we analyzed in this work has a matrix-like structure in which the central charge behaves as $c \sim N^2$. While theories with a vector-like dependence $c \sim N$ have been argued to be dual to Vasiliev higher spin theories in AdS, the dual of strange metal coset theories with chiral algebra $\mathcal{W}_N$ is believed to possess a much larger symmetry than Vasiliev theory and could well be a string theory. The background geometry of such a potential dual string theory is severely constrained by the result we reported above. Since the spectrum of chiral primaries does not depend on the string length, we have argued that the background geometry should give rise to chiral primaries which are in one-to-one correspondence with partitions or Young diagrams. While we do not have any concrete proposal for now, we

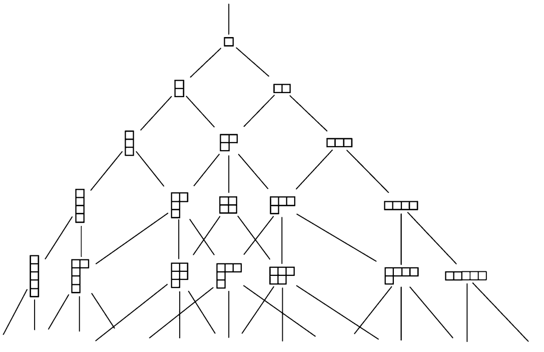

Fig. 5. Hasse diagram of the Young lattice.

want to point out that infinite families of AdS_3 backgrounds with at least $\mathcal{N} = (2, 0)$ supersymmetry have been constructed in [20]. It would be interesting to scan those solutions or apply the methods of Donos et al. in order to find a geometry that gives rise to the desired chiral ring. Let us note in passing that one chiral (left-moving) half of the strange metal coset theory was recently argued to arise in string theory on near horizon geometries of certain fast rotating black holes in an AdS space [21]. The constructions of Berkooz et al. provide the entire state space of the chiral strange metal coset, obviously including all the chiral primaries we described above. In case the aforementioned results or methods do not suffice to identify a dual string background, one might obtain valuable additional constraints on the dual theory by decomposing the spectrum of the strange metal coset theory into representations of higher spin symmetries, much along the lines of [22]. We plan to come back to these issues in future research.

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Appendix A. Tensor powers of the SU(N) adjoint representation

In this appendix we discuss the phylogeny of the formula (3.4) and we briefly outline its derivation.

Let us start with the following very well known formula for decomposing the k-th tensor power of the SU(N) fundamental representation into irreducible SU(N) representations

$$\square^{\otimes k} = \sum_{Y \vdash k} \frac{k!}{\prod_{(i,j)\in Y} h(i,j)}\, a(Y)\,. \tag{A.1}$$

Here $Y \vdash k$ denotes a Young diagram $Y = (l_1, \dots l_r)$ which has k boxes, i.e. is a partition of k, we use $a(Y)$ to label the corresponding $\mathrm{SU}(N)$ representation and

$$h(i,j) := l_j^{\mathrm{T}} - i + l_i - j + 1$$

is defined as the length of a hook (i, j) belonging to the given partition $Y = (l_1, \dots l_r)$. The product runs over the boxes of the Young diagram Y.

The raison d'être of formula (A.1) is the renowned Schur–Weyl duality [23]: The image of the action of the symmetric group S_k on the k-th tensor power of the $\mathrm{GL}_N(\mathbb{C})$ fundamental representation space can be identified with the centralizer algebra of $\mathrm{GL}_N(\mathbb{C})$ and vice versa. It means that under the joint action of S_k and $\mathrm{GL}_N(\mathbb{C})$, the tensor power decomposes into a direct sum of tensor products of irreducible modules for these two groups thus yielding formula (A.1). The coefficient

$$\frac{k!}{\prod_{(i,j)\in Y} h(i,j)}$$

is just the dimension of a corresponding representation of the symmetric group S_k.

It turns out that for a tensor power of the adjoint representation $\mathbf{adj}^k = \square^k \otimes \bar{\square}^k$ a similar correspondence holds, only that now the symmetric group algebra gets replaced by a more sophisticated structure known as the *walled Brauer algebra* [16,17]. The associated decomposition reads

$$\mathbf{adj}^{\otimes k} = \sum_{m=0}^{k} b_m^{(k)} \sum_{Y', Y'' \vdash m} \frac{m!}{\prod_{(i',j')\in Y'} h(i',j')} \frac{m!}{\prod_{(i'',j'')\in Y''} h(i'',j'')} \cdot a(Y', Y'')\,. \tag{A.2}$$

Here $a(Y', Y'')$ denotes an $\mathrm{SU}(N)$ representation generated from two Young diagrams Y' and Y'' according to (2.15), $Y', Y'' \vdash m$ means that Y' and Y'' are Young diagrams corresponding to partitions of m. The products in (A.2) run over boxes of the Young diagrams Y' and Y''. The range of validity here is $k \leq \lfloor \frac{N}{2} \rfloor$, otherwise not all of the listed representations $a(Y', Y'')$ are allowed to appear on the right hand side which results in a reshuffling of the remaining multiplicities. The multiplicities

$$b_m^{(k)} := \sum_{i=0}^{k-m} (-1)^{i+k+m}\, i! \binom{k}{m} \binom{k-m}{i} \binom{i+m}{i} \equiv \frac{k!}{m!} \sum_{i=0}^{k-m} \frac{(-1)^i}{i!} \binom{k-i}{m} \tag{A.3}$$

are actually the most interesting feature of formula (A.2). They reflect the fact that the new algebra replacing the symmetric group algebra in this case is not just a direct product of two copies of the latter. We refer the reader to [15] for background on walled Brauer algebras as well as the representation-theoretic discussion of decomposition formulas, such as the one displayed above.

There is a simple way to argue that the coefficients $b_m^{(k)}$ should have the form (A.3). Indeed, let us notice that they can be rewritten as

$$b_m^{(k)} = (-1)^{k+m} \binom{k}{m}\, {}_2F_0(m+1, -(k-m);\,|\,1) \tag{A.4}$$

where ${}_2F_0$ is the hypergeometric function of type $(2, 0)$. It is now straightforward to see that the coefficients $b_m^{(k)}$ actually satisfy the recursion relation

$$b_m^{(k)} = \frac{k(k-1)}{k-m}(b_m^{(k-2)} + b_m^{(k-1)}) \tag{A.5}$$

with the initial conditions $b_m^{(m-1)} = 0$, $b_m^{(m)} = 1$. This nice recursion readily suggests a way to proceed in proving the decomposition (A.2) with coefficients (A.3). Acting by induction, the inductive step is just to apply the Littlewood–Richardson rule [18,19] for multiplying all the Young diagrams present in the decomposition of the $(k-2)$-nd adjoint power by another two adjoint representations, carefully factoring out two hook multipliers which describe adding boxes to 'small' Young diagrams Y' and Y''. The relation (A.5) then allows to disentangle the obtained expression bringing it to the needed k-th step's outfit. For the original combinatorial proof involving a generalization of the Schensted insertion algorithm, see [24].

Appendix B. Antisymmetric part of the powers of adjoints

Before we begin with our discussion of eq. (3.9) let us give the precise statement and introduce a bit of additional notation. According to eq. (3.9), the part of the decomposition of the SU(N) adjoint power which transforms in the totally antisymmetric representation of the permutation group S_k is given by

$$\left\{\mathbf{adj}^{\otimes k}\right\}_{\text{antisymm}} = \sum_{m=1}^{k} d_m^{(k)} \sum_{Y' \vdash m} a\left(Y', Y'\right). \tag{B.1}$$

Here $a\left(Y', Y'\right)$ denotes an SU(N) representation generated from two Young diagrams $Y' = Y''$ according to (2.15), $Y' \vdash m$ means that Y' is a Young diagram satisfying $|Y'| = m$, i.e. is a partition of m.

The coefficients $d_m^{(k)}$ read

$$d_m^{(k)} := \sum_{1 \le m_1 < \cdots < m_{k-1} \le m-1} r_{m_1} r_{m_2 - m_1} \ldots r_{m_{k-1}-m_{k-2}} r_{m - m_{k-1}} \tag{B.2}$$

where r_m are expressed as

$$\begin{aligned} r_m &:= \frac{1}{m!}\left(\frac{d}{dq}\right)^{m-1}\left[\left(\frac{q}{\frac{\phi^3(q^2)}{\phi(q)\phi(q^4)} - 1}\right)^m\right]_{q=0} \\ &\equiv \frac{1}{m!}\left(\frac{d}{dq}\right)^{m-1}\left[\left(\frac{q}{\phi(-q)-1}\right)^m\right]_{q=0} \end{aligned} \tag{B.3}$$

and ϕ is the Euler function

$$\phi(q) := \prod_{i=1}^{\infty}(1 - q^i). \tag{B.4}$$

The range of validity of the formula (B.1) is restricted by $k \le N - 1$, otherwise not all of the listed representations $a\left(Y', Y'\right)$ are allowed to appear on the right hand side which results in a reshuffling of the remaining multiplicities.

We checked this formula by direct computation up to $k = 9$. Unfortunately, we were not able to find it in the literature. One immediate aspect to notice is that upon applying the Lagrange inversion formula, the coefficients $d_m^{(k)}$ turn out to be just coefficients of the series expansion of $q^k(\phi)$, where $q(\phi)$ denotes the function inverse to $\phi(-q) - 1$ around $q = 0$.

The parts of the $\mathrm{SU}(N)$ adjoint powers' decomposition transforming in other representations of the symmetric group will, of course, involve the 'non-diagonal' representations $a(Y', Y'')$, with $Y' \neq Y''$, to yield the full decomposition (A.1) when summed up over all representations of the symmetric group S_k. It is tempting to speculate that the multiplicities in those partial decompositions may be characterized by other modular forms replacing the Dedekind eta $\eta(q) = q^{\frac{1}{24}}\phi(q)$. The exact formulae of this type will be discussed elsewhere.

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