Cartesian products of graphs as subgraphs of de Bruijn graphs of dimension at least three

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Abstract

Given a Cartesian product $G = G_1 \times \cdots \times G_m$ ($m \geq 2$) of nontrivial connected graphs $G_i$, and the base $d$, dimension $D$ de Bruijn graph $B(d, D)$, it is investigated under which conditions $G$ is (or is not) a subgraph of $B(d, D)$. We present a complete solution of this problem for the case $D \geq 4$. For $D = 3$, we give partial results including a complete solution for the case that $G$ is a torus, i.e., $G$ is the Cartesian product of cycles.

Keywords: de Bruijn graphs; Cartesian products of graphs; Interconnection networks; Subgraph embeddings; Hypercubes; Tori; Parallel and distributed computing

1. Introduction

In the context of parallel and distributed computation, the problem of embedding one interconnection network into another one is of fundamental importance and has gained considerable attention during the recent years. Among the various graphs that have been proposed as interconnection networks for parallel computers, Cartesian product graphs (as hypercubes, grids, and tori) and shuffle-oriented graphs (as shuffle-exchange graphs and de Bruijn graphs) are among the most popular ones. In the present paper, we deal with the problem of subgraph containment of Cartesian product graphs in de Bruijn graphs and present results completely settling several of the relevant subcases, thereby improving previous results of Andreae et al. [1] and Heydemann et al. [6, 7].

For general information on interconnection networks and, in particular, on containment and embedding results, we refer to [2, 3, 9, 10] and the literature mentioned there; for applications, e.g. in the field of parallel image processing and pattern recognition, see [5, 11–13].

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All graphs considered in this paper are simple, i.e. have no loops or multiple edges. If $G$ is a graph, then $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. Our terminology is standard; for graph-theoretic terminology not explained here, we refer to [4]. For graphs $G_i$ ($i = 1, \ldots, m$), the Cartesian product $G_1 \times \cdots \times G_m$ is the graph with vertex set $V(G_1) \times \cdots \times V(G_m)$ where two vertices $(a_1, \ldots, a_m), (b_1, \ldots, b_m)$ are joined by an edge if and only if there exists an $i \in \{1, \ldots, m\}$ such that $a_j = b_j$ for all $j \neq i$ and such that there is an edge of $G_i$ joining $a_i$ with $b_i$. For integers $d, D \geq 2$, the directed de Bruijn graph $\mathcal{B}(d, D)$ has as its vertex set the set of $D$-tuples $(a_1, \ldots, a_D)$ where the entries $a_i$ are integers with $0 \leq a_i \leq d - 1$, and there exists an arc from $(a_1, \ldots, a_D)$ to $(b_1, \ldots, b_D)$ if and only if $b_i = a_{i+1}$ (if $i = 1, \ldots, D - 1$). The corresponding undirected de Bruijn graph $B(d, D)$ results from $\mathcal{B}(d, D)$ by ignoring the orientations of the arcs, suppressing loops, and identifying each pair of multiple edges; $d$ is called the base of $B(d, D)$ and $D$ is its dimension. We remark that $D$ is equal to the diameter of $\mathcal{B}(d, D)$ (which is the same as the diameter of $B(d, D)$) and $d$ equals the indegree (and outdegree) of each vertex of $\mathcal{B}(d, D)$. A graph is nontrivial if it has at least two vertices. The symbol $K_n$ denotes the complete graph with $n$ vertices; $K_{n,m}$ denotes the complete bipartite graph with color classes of cardinality $n$ and $m$, respectively; $P_n(C_n)$ denotes the path (cycle) with $n$ vertices, where, in the case of a cycle, $n \geq 3$ is assumed; a cycle with $n$ vertices is called an $n$-cycle. We write $H \subseteq G$ to indicate that $H$ is a subgraph of $G$.

Tori, grids, and hypercubes can, in terms of the Cartesian product, be defined as follows. For $m \geq 2$, a graph $G$ is an $m$-dimensional grid (torus) if $G$ is the Cartesian product of $m$ nontrivial paths (cycles). The $m$-dimensional hypercube $H(m)$ is the graph $G_1 \times \cdots \times G_m$ where all $G_i$ are complete graphs $K_2$; for $m = 3$, $H(m)$ is called a cube.

For graphs $G, H$, a subgraph embedding $\varphi : G \to H$ is an injective mapping $\varphi : V(G) \to V(H)$ such that $\varphi(x)\varphi(y)$ is an edge of $H$ whenever $xy$ is an edge of $G$. In the present paper, we do not consider any kind of embeddings other than subgraph embeddings, and thus we several times just say “embedding” instead of “subgraph embedding”.

Given a Cartesian product $G = G_1 \times \cdots \times G_m$ ($m \geq 2$) of nontrivial connected graphs $G_i$ and a de Bruijn graph $B(d, D)$ such that $|G| = |B(d, D)| (= d^D)$, it was investigated in [1] under which conditions $G$ is a spanning subgraph of $B(d, D)$. A closely related question was examined by Heydemann et al. [6, 7] who, for grids, hypercubes, and (occasionally) tori, considered embeddings of $G = G_1 \times \cdots \times G_m$ into an optimal de Bruijn graph (with respect to $G$), i.e., into a de Bruijn graph $\mathcal{B}(d, D)$ with $\min\{d^{D-1}, (d - 1)^D\} < |G| < d^D$. In the present paper, we consider the more general question whether a given Cartesian product $G = G_1 \times \cdots \times G_m$ ($m \geq 2$) of nontrivial connected $G_i$ is a subgraph of a given de Bruijn graph $B(d, D)$, i.e., we do not restrict our investigations to spanning subgraphs or to optimal de Bruijn graphs. We do not attack the case $D = 2$ here since the methods to be employed for $D = 2$ (and also the expected results) appear to be quite different from the case $D \geq 3$. We now present
our main results (Theorems 1–4), always assuming that $G = G_1 \times \cdots \times G_m$ ($m \geq 2$) is a Cartesian product of nontrivial connected graphs $G_i$ and that $d, D$ are integers with $d \geq 2$, $D \geq 3$.

(1) Theorem 1 contains a series of conditions each of which implies the nonexistence of subgraph embeddings $\varphi : G \to B(d, D)$ other than the “trivial ones” (which are, roughly speaking, the embeddings obtained by observing that certain subgraphs of $B(d, D)$ isomorphic to $K_{d,d}$ or $K_{i,d-1}$ can easily be found). Among other results, Theorem 1 completely settles the case $m \geq 3$ and also the case $m = 2$, $G_1$ nonbipartite, $|G_2| \geq 3$; further, the case $G = G_1 \times K_2$ with $G_1$ being bipartite and not 2-connected is the only case left open by Theorem 1 for $D \geq 5$.

(2) Theorem 2 settles the before-mentioned case which was left open by Theorem 1 for $D = 5$.

(3) For $D = 4$, the only case left open by Theorem 1 is the case $G = G_1 \times K_2$ (where, in contrast to the case $D = 5$, $G_1$ may be arbitrary). Theorem 3 characterizes the graphs $G_1$ for which $G_1 \times K_2 \subseteq B(d, 4)$, thus providing a complete solution of our problem for $D = 4$. We also present some corollaries of Theorem 3.

(4) For $D = 3$, we do not have a complete solution of the problem but (in addition to the partial results for $D = 3$ mentioned in (1)) we have obtained a complete answer for a particularly interesting class of Cartesian product graphs, namely the class of tori: Theorem 4 states that, for a torus $G = G_1 \times \cdots \times G_m$, there exists a nontrivial subgraph embedding $\varphi : G \to B(d, 3)$ if and only if $G = C_n \times C_n$ for $n \equiv 2 \pmod{4}$ and $d \geq \max\{\frac{n}{2}, 5\}$.

Our results extend and improve several of the results previously obtained in [1, 6, 7]. For example, in the appendix of [6] a (nontrivial) embedding $\varphi : C_{10} \times C_{10} \to B(d, 3)$ for $d = 5$ is given which by means of our Theorem 4 has found its natural generalization.

The paper is organized as follows. In the remainder of this section, we collect some basic graph-theoretic definitions and notational conventions. Thereafter, in Section 2, we prepare the proofs of the main results by collecting a series of lemmas. The remaining sections contain the above-mentioned theorems and their proofs.

If $x, y$ are adjacent vertices of a graph, then the edge joining $x$ and $y$ is denoted by $xy$ or $x-y$; similarly, if in a digraph there exists an arc from a vertex $x$ to a vertex $y$, then this arc is denoted by $(x, y)$ or $x \rightarrow y$. For a graph $G$ and $X \subseteq V(G)$, we use the notation $G[X]$ for the subgraph of $G$ induced by $X$. For a graph $G$ with vertex set $V$, we define $G - X := G[V \setminus X]$ for $X \subseteq V$; we usually write $G - x$ instead of $G - \{x\}$. The degree of a vertex $v$ of a graph $G$ is denoted by $deg_G(v)$, and $\Delta(G)$ denotes the maximum degree of $G$. The distance of $x$ and $y$ in the graph $G$ is denoted by $d_G(x, y)$ (for $x, y \in V(G)$). As usual, a (directed) walk of length $k$ in a graph (digraph) $G$ is a sequence $(v_0, v_1, \ldots, v_k)$ of vertices of $G$ such that $v_{i-1} - v_i$ ($v_{i-1} \rightarrow v_i$) is an edge (arc) of $G$ ($i = 1, \ldots, k$); a (directed) walk is a (directed) trail if its edges (arcs) are pairwise distinct. For a graph $G$, the set of its components is denoted by $comp G$. The symbol $\mathbb{Z}_r$ denotes the integers modulo $r$. 
2. Preliminaries

In this section, we present a series of lemmas preparing the proofs of the subsequent theorems. Let \( C \) be a 4-cycle and assume that the digraph \( \tilde{C} \) results from \( C \) by assigning to each edge of \( C \) one of its two possible orientations. We call \( \tilde{C} \) a cycle of type \( t \) if \( t \) is the maximum length of a directed trail contained in \( \tilde{C} \). Then, of course, \( 1 \leq t \leq 4 \), and for each such \( t \) there is (up to isomorphism) just one cycle of type \( t \), namely, every cycle of type 1 is of the form \( x \rightarrow y \leftarrow z \rightarrow w \leftarrow x \) and cycles of type 2, 3, and 4 are of the form \( x \rightarrow y \leftarrow z \rightarrow w \leftarrow x \), \( x \rightarrow y \rightarrow z \rightarrow w \rightarrow x \), and \( x \rightarrow y \rightarrow z \rightarrow w \rightarrow x \), respectively. If \( x \rightarrow y \rightarrow z \rightarrow w \leftarrow x \) is a cycle of type 3, then \( x \rightarrow y \leftarrow z \rightarrow w \leftarrow x \) is called the corresponding cycle of type 1.

Lemmas 1-3 contain useful observations on 4-cycles in \( \tilde{B}(d, D) \); proofs can be found in [1], but alternatively the reader can readily prove these lemmas without consulting [1].

Lemma 1. \( \tilde{B}(d, D) \) contains no cycle of type 2.

Lemma 2. If \( \tilde{B}(d, D) \) contains a cycle of type 3, then it also contains the corresponding cycle of type 1.

Lemma 3. Let \( x \rightarrow y \leftarrow z \rightarrow w \leftarrow x \) be a cycle of type 1 contained in \( \tilde{B}(d, D) \) with \( x = (x_1, \ldots, x_D) \), \( y = (y_1, \ldots, y_D) \), \( z = (z_1, \ldots, z_D) \), \( w = (w_1, \ldots, w_D) \). Then \( x_i = z_i = y_{i-1} = w_{i-1} \) (\( i = 2, \ldots, D \)).

A 4-cycle \( C : x \rightarrow y \rightarrow z \rightarrow w \rightarrow x \) of \( B(d, D) \) is of type 4 if \( x \rightarrow y \rightarrow z \rightarrow w \rightarrow x \) or \( x \leftarrow y \leftarrow z \leftarrow w \leftarrow x \) is contained in \( \tilde{B}(d, D) \). An edge \( x \rightarrow y \) of \( B(d, D) \) is called a double edge if both \( x \rightarrow y \) and \( x \leftarrow y \) are arcs of \( \tilde{B}(d, D) \). A vertex \( a = (a_1, \ldots, a_D) \) of \( B(d, D) \) is \( k \)-periodic if \( a_i = a_{i+k} \) (\( 1 \leq i \leq D - k \)). Observe that each vertex of a cycle of type 4 is 4-periodic and each vertex of a double edge is 2-periodic (and thus also 4-periodic); these simple observations will be used several times without explicit mention.

Lemma 4. If \( D \geq 4 \), then every vertex of \( B(d, D) \) is contained in at most one cycle of type 4, and if \( D = 3 \), then every edge of \( B(d, D) \) is contained in at most one cycle of type 4. Similarly, if \( D \geq 4 \), then a cycle of type 4 and a double edge of \( B(d, D) \) cannot have a vertex in common, and if \( D = 3 \), then no edge of a cycle of type 4 is a double edge.

Proof. Let \( C : x \rightarrow y \rightarrow z \rightarrow w \rightarrow x \) be a cycle of type 4 in \( B(d, D) \) such that \( x \rightarrow y \rightarrow z \rightarrow w \rightarrow x \) is in \( \tilde{B}(d, D) \) and let \( x = (x_1, \ldots, x_D) \). If \( D \geq 4 \), then \( y = (x_2, \ldots, x_{D-1}, x_D, x_{D-3}, x_{D-2}) \), \( z = (x_3, \ldots, x_D, x_{D-3}, x_{D-2}, x_{D-1}) \), \( w = (x_4, \ldots, x_{D-2}, x_D, x_{D-3}, x_{D-1}) \), and thus \( y, z, w \) are uniquely determined by \( x \); moreover, \( x \) cannot be 2-periodic since this would imply \( x = z \). This proves
Lemma 4 for the case $D \geq 4$. Now let $D = 3$. Then $x = (x_1, x_2, x_3)$ and $y = (x_2, x_3, x_4)$ for some $x_4 \in \{0, \ldots , d - 1\}$. It follows that $z = (x_3, x_4, x_1)$ and $w = (x_4, x_1, x_2)$, which means that $z$ and $w$ are uniquely determined by $x$ and $y$. Moreover, $x - y$ cannot be a double edge since this would imply $x_1 = x_3, x_2 = x_4$, and thus we would have $x = z$. This settles the case $D = 3$.

Lemma 5. For $D \geq 4$, let $a = (a_1, \ldots , a_D), c = (c_1, \ldots , c_D)$ be a pair of opposite vertices of a cycle of type 4 contained in $B(d, D)$. Then $a_i \neq c_i$ for at least one $i \in \{2, \ldots , D - 1\}$.

Proof. Let $a \rightarrow x \rightarrow c \rightarrow y \rightarrow a$ be a cycle of type 4 contained in $\bar{B}(d, D)$. Then $x = (a_2, \ldots , a_D, a_{D-3})$, $c = (c_1, \ldots , c_D) = (a_3, \ldots , a_D, a_{D-3}, a_{D-2})$, and $y = (a_4, \ldots , a_D, a_{D-3}, a_{D-2}, a_{D-1})$. Suppose that $a_i = c_i$ for $i = 2, \ldots , D - 1$. Then it follows that $x = y$, which is impossible.

A graph $G$ is a ladder (closed ladder) if it is isomorphic to the Cartesian product $G_1 \times K_2$ for a path (cycle) $G_1$. For a ladder $L$ with $2n$ vertices choose the notations of the vertices such that $V(L) = \{\xi_1, \ldots , \xi_n, \eta_1, \ldots , \eta_n\}$, $E(L) = \{\xi_i \xi_{i+1}, \eta_i \eta_{i+1}: i = 1, \ldots , n - 1\} \cup \{\xi_i \eta_i: i = 1, \ldots , n\}$. Then the edge $\xi_i \eta_i$ is called the $i$th rung of $L$; $\xi_i \eta_i$ is an outer rung if $i = 1$ or $n$, and an inner rung otherwise; $\xi_n \eta_n$ is the last rung of $L$. Note that these definitions are dependent on the choice of the notations for the vertices of $L$ and thus, whenever we talk about the rungs of a ladder, we implicitly assume that, as above, some fixed choice of notations is given. A similar remark holds for closed ladders. Note also that ladders $L$ are bipartite graphs, so that it makes sense to talk about the two color classes of $L$.

A ladder $L \subseteq B(d, D)$ is simple if (i) $L$ has at least two rungs, (ii) no 4-cycle of $L$ is of type 4, and (iii) no inner rung is a double edge.

By iterated application of the Lemmas 1–3, one readily obtains the following lemma (which is a generalized, undirected version of Lemma 3).

Lemma 3'. For a simple ladder $L \subseteq B(d, D)$, let $xy$ be the first, $z'w'$ the second, and $zw$ the last rung of $L$, where the notations are chosen such that $x, z', z$ are in the same color class of $L$ and such that the 4-cycle $x \rightarrow y \rightarrow z' \rightarrow w' \rightarrow x$ is contained in $\bar{B}(d, D)$. (Note that this choice is always possible.) Let $x = (x_1, \ldots , x_D), y = (y_1, \ldots , y_D), z = (z_1, \ldots , z_D), w = (w_1, \ldots , w_D)$. Then $x_i = z_i = y_{i-1} = w_{i-1} (i = 2, \ldots , D)$.

The proof of the next lemma is based on the following observation. Let $L \subseteq B(d, D)$ be a simple ladder and $k$ be a positive integer with $k < D$. Then at most two of the four vertices on the outer rungs of $L$ are $k$-periodic because, otherwise, by Lemma 3' there would be a $k$-periodic vertex $a = (a_1, \ldots , a_D)$ on the first rung and a $k$-periodic vertex $b = (b_1, \ldots , b_D)$ on the last rung such that $a_i = b_i (i = 2, \ldots , D)$ or $a_{i-1} = b_{i-1} (i = 2, \ldots , D)$, which, in either case, would imply the contradiction $a = b$. 
Lemma 6. If \( D \geq 5 \), then
(i) every ladder \( L \subseteq \mathcal{B}(d,D) \) contains at most one cycle of type 4 and at most one rung which is a double edge, and \( L \) never contains both a cycle of type 4 and a rung which is a double edge,
(ii) for a closed ladder \( L \subseteq \mathcal{B}(d,D) \), none of its 4-cycles is of type 4.

**Proof.** Assuming that the lemma is false, one obtains from Lemma 4 that there exists a simple ladder \( L' \subset L \) such that the first rung as well as the last rung of \( L' \) is a double edge or an edge of a cycle of type 4, contradicting the observation in the paragraph before Lemma 6. \( \Box \)

A walk \( (x_0,x_1,\ldots,x_t) \) of \( \mathcal{B}(d,D) \) is an alternating walk if \( x_i + x_{i+1} \in \mathcal{B}(d,D) \) for all even \( i \), \( x_i - x_{i+1} \in \mathcal{B}(d,D) \) for all odd \( i \) or, conversely, \( x_i - x_{i+1} \in \mathcal{B}(d,D) \) for all even \( i \), \( x_i + x_{i+1} \in \mathcal{B}(d,D) \) for all odd \( i \).

Lemma 7. For \( D \geq 3 \), let \( e=aa' \), \( f=bb' \) be double edges of \( \mathcal{B}(d,D) \) and let \( (x_0=a, x_1,\ldots,x_t=b) \) be an alternating walk of \( \mathcal{B}(d,D) \). Then \( e=f \); further, \( a=b, a'=b' \) if \( t \) is even, and \( a=b', a'=b \) otherwise.

**Proof.** We consider the case that \( x_i \rightarrow x_{i+1} \in \mathcal{B}(d,D) \) for all even \( i \) and \( x_i \leftarrow x_{i+1} \in \mathcal{B}(d,D) \) for all odd \( i \); the remaining case can be treated analogously. Let \( a=(a_1,\ldots,a_D) \), \( a'=(a'_1,\ldots,a'_D) \), \( b=(b_1,\ldots,b_D) \), \( b'=(b'_1,\ldots,b'_D) \), and \( x_i=(x_{i,1},\ldots,x_{i,D}) \) for \( i=0,\ldots,t \).

Since \( e, f \) are double edges, the vertices \( a, b, a', b' \) are 2-periodic with \( a'_1=a_2, a'_2=a_1, b'_1=b_2, b'_2=b_1 \). By induction on \( i=0,\ldots,t \), one obtains \( x_{i,j}=a_j \) (\( j=2,\ldots,D \)) for all even \( i \) and \( x_{i,j}=a_{j+1} \) (\( j=1,\ldots,D-1 \)) for all odd \( i \). Hence \( b=(b_1,a_2,\ldots,a_D) \) if \( t \) is even and \( b=(a_2,\ldots,a_D,b_D) \) if \( t \) is odd. Now the 2-periodicity of \( b \), together with the assumption \( D \geq 3 \), yields the assertion. \( \Box \)

For a Cartesian product \( G_1 \times \cdots \times G_m \), a subgraph \( H \) of \( G_1 \times \cdots \times G_m \) is called 1-dimensional if there exists an \( i \) such that, for any pair of vertices \( a=(a_1,\ldots,a_m), b=(b_1,\ldots,b_m) \) of \( H \), \( a_j=b_j \) for all \( j \neq i \). A 4-cycle \( C \subseteq G_1 \times \cdots \times G_m \) which is not 1-dimensional is called 2-dimensional.

A proof of the next lemma was given in [1], however, in order to make the paper self-contained, we also present the proof here.

Lemma 8. Let \( G=G_1 \times \cdots \times G_m \) be the Cartesian product of \( m \geq 2 \) nontrivial connected graphs and let \( \sim \) be an equivalence relation on the vertex set of \( G \). Assume that \( a \sim c \) for all vertices \( a, c \) which form a pair of opposite vertices on some 2-dimensional 4-cycle of \( G \). Then the partition of \( V(G) \) corresponding to \( \sim \) consists of at most two classes. Moreover, if \( G \) is bipartite and \( A \) is a color class of \( G \), then \( a \sim a' \) for all \( a, a' \in A \).

**Proof.** Since every connected graph contains a spanning tree, it clearly suffices to prove the lemma for the case that the graphs \( G_i \) are trees. Then the graph \( G \) is bipartite since
it is the Cartesian product of bipartite graphs. Let $A$ be a color class of $G$ and pick $a, b \in A$. We claim that $a \sim b$ (which implies the lemma). Note that, in order to prove our claim, it suffices to consider the case when $d_G(a, b) = 2$ since the general case can be settled by iterated application of the distance-two case. Let $P = (a, x, b)$ be a path of $G$. If $a, b$ is a pair of opposite vertices on a 2-dimensional 4-cycle of $G$ then we are done; otherwise $P$ is a 1-dimensional subpath of $G$ and (since the graphs $G_i$ are nontrivial) there exists a 1-dimensional subpath $P' = (a', x', b')$ of $G$ such that $P \cap P' = \emptyset$ and $aa', xx', bb' \in E(G)$. Hence $a \sim x' \sim b$.  

**Lemma 9.** Let $G = G_1 \times G_2$ be the Cartesian product of connected graphs $G_i$ with $|G_i| \geq 3$ $(i = 1, 2)$. Assume that $G$ is bipartite and let $a, b$ be vertices of distinct color classes. Then $G - \{a, b\}$ is connected.

**Proof.** Suppose that $G - \{a, b\}$ is disconnected and let $(x_1, x_2), (y_1, y_2)$ be vertices of $G - \{a, b\}$ which are in distinct components. Since $|G_i| \geq 3$ $(i = 1, 2)$, we can choose paths $P_i \subseteq G_i$, $P_i \subseteq G_2$ such that $|P_i| \geq 3$ and $x_i, y_i \in P_i$ $(i = 1, 2)$. Let $G' := P_1 \times P_2$. Then, clearly, $G' - \{a, b\}$ is disconnected. However, since $|P_i| \geq 3$ $(i = 1, 2)$, this can only happen if $a, b$ are the neighbors of a vertex of degree 2 in $G'$. (We leave the easy proof of the latter statement to the reader.) But then $a, b$ are in the same color class of $G$, which is a contradiction. 

Let $G$ be a bipartite connected graph and assume that $\varphi : G \to B(d, D)$ is a subgraph embedding with the following property: there exist $x_1, \ldots, x_{D-1} \in \{0, \ldots, d-1\}$ such that one color class of $G$ is mapped into the set \{$(\xi, x_1, \ldots, x_{D-1}) : \xi \in \{0, \ldots, d-1\}$\} while the other is mapped into the set \{$(x_1, \ldots, x_{D-1}, \eta) : \eta \in \{0, \ldots, d-1\}$\}. Then $\varphi$ is called a trivial subgraph embedding.

**Proposition 1.** Given a bipartite connected graph $G$ and integers $d, D$ with $d \geq 2$ and $D \geq 3$, there exists a trivial subgraph embedding $\varphi : G \to B(d, D)$ if and only if $d \geq \max\{|A|, |B|\}$, where $A, B$ denote the color classes of $G$.

**Proof.** If $\varphi : G \to B(d, D)$ is a trivial subgraph embedding, then $d \geq \max\{|A|, |B|\}$ immediately follows from the fact that $\varphi$ is injective. For the proof of the converse, assume $d \geq \max\{|A|, |B|\}$. Then there exist injective mappings $f : A \to \{0, \ldots, d-1\}$ and $g : B \to \{0, \ldots, d-1\}$. Putting $x_1 := 1$ and $x_i := 0$ $(i = 2, \ldots, D - 1)$, we define $\varphi : G \to B(d, D)$ by

$$\varphi(v) = \begin{cases} (f(v), x_1, \ldots, x_{D-1}) & \text{for } v \in A, \\ (x_1, \ldots, x_{D-1}, g(v)) & \text{for } v \in B. \end{cases}$$

Taking into account that $D \geq 3$ and $x_1 \neq x_2$, one now easily verifies that $\varphi$ is a trivial subgraph embedding.  

3. Conditions excluding the existence of nontrivial embeddings

Our main result in this section is Theorem 1, which contains a series of conditions each of which implies the nonexistence of subgraph embeddings $G_1 \times \cdots \times G_m \rightarrow B(d,D)$ other than the trivial ones. We remark that one common feature of the conditions (i)–(v) listed in Theorem 1 is that they are independent of the value of $d$. For $a=(a_1,\ldots,a_D)$, $b=(b_1,\ldots,b_D)$, we write $a \sim b$ if $a_i=b_i$ $(i=2,\ldots,D-1)$.

**Theorem 1.** For $m \geq 2$ let $G=G_1 \times \cdots \times G_m$ be a Cartesian product of nontrivial connected graphs $G_i$ and let $d,D$ be integers with $d \geq 2$, $D \geq 3$. Assume that one of the following conditions holds:

(i) $m \geq 3$,

(ii) there exist $i,j \in \{1,\ldots,m\}$, $i \neq j$, such that $G_i$ is nonbipartite and $|G_j| \geq 3$,

(iii) $D \geq 4$ and there exist $i,j \in \{1,\ldots,m\}$, $i \neq j$, such that $|G_i|, |G_j| \geq 3$,

(iv) $D \geq 5$ and at least one $G_i$ is 2-connected,

(v) $D \geq 5$ and at least one $G_i$ is nonbipartite.

Then there do not exist any subgraph embeddings $\varphi : G \rightarrow B(d,D)$ if $G$ is nonbipartite. For bipartite $G$, there do not exist any subgraph embeddings $\varphi : G \rightarrow B(d,D)$ other than the trivial ones.

**Proof.** Let $G \subseteq B(d,D)$. We first treat the case that $G$ is bipartite and then settle the nonbipartite case.

(1) $G$ is bipartite: By (i)–(v) we may assume that one of the following conditions holds.

$m \geq 3$,

$m=2$, $|G_1|, |G_2| \geq 3$, $D \geq 4$,

$m=2$, $G_1$ is 2-connected, $D \geq 5$.

We first show that each of these conditions implies the following.

If $a, c$ is a pair of opposite vertices of a 2-dimensional 4-cycle $C$ of $G$, then $a \sim c$.

If (1) or (2) holds, then (by arguments based on Lemmas 1–5) statement (4) can be obtained as in the proof of [1], Theorem 7; in order to make the paper self-contained, we give a short sketch of the argument. If $C$ is not of type 4, then $C$ is a simple ladder with exactly two rungs and $a \sim c$ immediately follows from Lemma 3’. Thus we may assume that $C$ is of type 4. If (1) holds, then $C$ is contained in a cube $H \subseteq G$. Let $y$ be a vertex of $H$ such that $d_H(a,y)=d_H(c,y)=2$. Then $a \sim y \sim c$ since, by Lemma 4, $a, y$ (as well as $c, y$) is a pair of opposite vertices of a 4-cycle which is not of type 4. If (2) holds, then $C$ is contained in a grid $H=Q_1 \times Q_2 \subseteq G$ where $Q_1, Q_2$ are paths with three vertices. Then it follows from Lemma 4 that, with the exception of $C$, all
other 4-cycles of $H$ are not of type 4, from which one easily deduces a contradiction (by means of the Lemmas 3' and 5 together with the transitivity of the relation $\sim$).

Assume now that (3) holds. Then (as a consequence of the fact that $G_1$ is bridgeless) one obtains that $C$ is a 2-dimensional 4-cycle of a closed ladder $L \subseteq G$. Hence $a \sim c$ by Lemma 6(ii) and Lemma 3'.

Let $A, B$ be the color classes of the bipartite graph $G$. Then, by (4) in conjunction with Lemma 8, one obtains

$$a \sim a', \quad b \sim b' \quad \text{for all} \quad a, a' \in A \quad \text{and} \quad b, b' \in B.$$  

(5)

Let $a_2, \ldots, a_{D-1}$ and $b_2, \ldots, b_{D-1}$ denote the inner entries of the vertices of $A$ and $B$, respectively, and put $a^* := (b_2, a_2, \ldots, a_{D-1}, b_{D-1})$, $b^* := (a_2, b_2, \ldots, b_{D-1}, a_{D-1})$. Note that, if $a \in A$, $b \in B$ and if $a \rightarrow b$ is an arc of $B(d, D)$, then $a = (x, a_2, \ldots, a_{D-1}, b_{D-1})$, $b = (a_2, b_2, \ldots, b_{D-1}, y)$ for some $x, y \in \{0, \ldots, d - 1\}$; similarly, if $b \rightarrow a$ is an arc of $B(d, D)$ for $a \in A$, $b \in B$, then $a = (b_2, a_2, \ldots, a_{D-1}, x)$, $b = (y, b_2, \ldots, b_{D-1}, a_{D-1})$ for some $x, y \in \{0, \ldots, d - 1\}$. Consequently, if $a \in A$, $b, b' \in B$ and if $b \rightarrow a$, $a \rightarrow b'$ are arcs of $B(d, D)$, then $a \sim a^*$ (and a similar statement holds for $b^*$). Let $G' := G - \{a^*, b^*\}$. Note that $G'$ is connected; indeed, if (2) holds, this follows from Lemma 9 and, otherwise, this follows from the fact that each of the conditions (1) and (3) implies the 3-connectedness of $G$.

Summarizing one concludes from the connectedness of $G'$ that either all edges of $G'$ are oriented (as arcs of $B(d, D)$) from $A$ to $B$ or all edges of $G'$ are oriented from $B$ to $A$, and we may assume that the former holds. Hence all members of $A$ are of the form $(x, a_2, \ldots, a_{D-1}, b_{D-1})$ and all members of $B$ are of the form $(a_2, \ldots, a_{D-1}, b_{D-1}, y)$, which proves that $G$ is trivially embedded in $B(d, D)$.

(II) $G$ is nonbipartite: Then at least one $G_i$ is nonbipartite and thus contains an odd cycle $C_n$. Hence in order to prove the theorem, it is sufficient to establish the correctness of the following statements.

$$P_3 \times C_n \not\subseteq B(d, D) \quad \text{for odd } n,$$  

(6)

$$P_3 \times C_n \not\subseteq B(d, D) \quad \text{for odd } n \quad \text{and} \quad D \geq 5.$$  

(7)

For the proof of (6) suppose that, for some odd $n$, there is a subgraph embedding $\varphi : P_3 \times C_n \rightarrow B(d, D)$. We use the notations for the vertices of $\varphi(P_3 \times C_n)$ as indicated in Fig. 1, where the left and right margins have to be identified.

In the sequel, whenever we use terms like “vertical edge”, “horizontal edge”, “up”, “down”, etc., we use these expressions as suggested by Fig. 1. In particular, there are two rows of vertical edges, the upper row and the lower row. We say that two vertical edges of the same row are neighbors if they are incident with a common horizontal edge. The following is an immediate consequence of the Lemmas 1 and 2.

Let $e, f$ be vertical edges of the same row such that $e$ and $f$ are neighbors and such that neither $e$ nor $f$ is a double edge. Then (in $B(d, D)$) one of the edges $e, f$ is oriented upwards and the other downwards.  

(8)
Since $n$ is odd, statement (8) implies that

each row of vertical edges contains at least one double edge. \hfill (9)

From Lemma 7 one obtains

if $e, f$ are distinct double edges of $B(d, D)$, then the distance between $e$ and $f$ must be at least 2. \hfill (10)

Consequently, for each vertical double edge, its two neighbors of the same row are not double edges, and thus one obtains, by making use of the fact that $n$ is odd, that the lower row of vertical edges must contain a double edge $e$ such that the (uniquely determined) orientations of the two neighbors of $e$ are distinct, i.e., one neighbor is oriented upwards and the other downwards. We may assume that $e$ is the edge $\varphi(1,1)\varphi(2,1)$. By (10), the edge $f := \varphi(0, 1)\varphi(1, 1)$ is not a double edge; we assume that $f$ is oriented downwards. (The case of $f$ being oriented upwards can be reduced to the case of $f$ being oriented downwards by exchanging each vertex $(x_1, \ldots, x_d)$ of $B(d, D)$ with the vertex $(x_d, x_{d-1}, \ldots, x_1)$.)

By (10), none of the vertical edges $f$ shown in Fig. 2 is a double edge, and by (8) the neighbors of $f$ are oriented upwards.

Moreover, we (clearly) may assume that the neighbors of $e$ are oriented as displayed in Fig. 2.

By (9) there exists a minimal $k$ with $3 \leq k \leq n - 1$ such that $\varphi(1, k)\varphi(2, k)$ or $\varphi(1, k)\varphi(2, k)$ is a double edge. We claim that the following holds.

For $j \in \{1, \ldots, k - 1\}$, if $j$ is odd, then the arc $\varphi(1, j) \rightarrow \varphi(1, j + 1)$ is in $\overline{B}(d, D)$ and, if $j$ is even, then $\varphi(1, j) \rightarrow \varphi(1, j + 1)$ is in $\overline{B}(d, D)$. \hfill (11)

For the proof of (11), pick $j \in \{1, \ldots, k - 1\}$ and let $A_0$ be the 4-cycle of $\varphi(P_3 \times C_n)$ formed by $\varphi(0, j), \varphi(1, j), \varphi(1, j + 1), \varphi(0, j + 1)$; similarly, let $A_2$ be the 4-cycle formed by $\varphi(0, j), \varphi(1, j), \varphi(1, j + 1), \varphi(2, j + 1)$. Then by Lemma 4 there exists an $r \in \{0, 2\}$ such that $A_r$ is not of type 4. Note that (by (8) $\overline{B}(d, D)$ contains the arcs $\varphi(r, j) \rightarrow \varphi(1, j), \varphi(1, j + 1) \rightarrow \varphi(r, j + 1)$ if $j$ is odd, and $\overline{B}(d, D)$ contains the arcs $\varphi(r, j) \leftarrow \varphi(1, j), \varphi(1, j + 1) \leftarrow \varphi(r, j + 1)$, otherwise. Further, one of the edges $\varphi(r, j)\varphi(1, j)$,
φ(1,j + 1)φ(r,j + 1) is not a double edge, and thus assertion (11) follows from the Lemmas 1 and 2. However, (11) contradicts Lemma 7, and thus we have proved (6).

For the proof of (7) suppose that $L := P_2 \times C_n \subseteq B(d,D)$ with $n$ odd and $D \geq 5$. By the same arguments as used above for the proof of (9) one finds that at least one rung of $L$ is a double edge. Consequently, by Lemma 6, there is exactly one such rung, say $e = xy$, and no 4-cycle of $L$ is of type 4. Let $e'$ be any rung of $L$ distinct from $e$ and let $L_1, L_2$ denote the two (distinct) subladders of $L$ which both have $e$ and $e'$ as outer rungs. Then application of Lemma 3' to $L_1$ and $L_2$ yields $x_i = y_i$ (i = 2,...,D - 1) where $x = (x_1,\ldots,x_D)$, $y = (y_1,\ldots,y_D)$. But then $x = y$ since $e = xy$ is a double edge. This contradiction establishes (7).

4. Solution for $D \geq 5$

For $D \geq 5$, the next theorem supplements Theorem 1, thus yielding a complete result for $D \geq 5$. We need some preparation. For a graph $H$ let $x$ be a vertex or an edge of $H$. For $\mathcal{C} \subseteq \text{comp}(H - x)$, we use $\sum \mathcal{C}$ as a short hand for the sum $\sum |C|$ taken over all $C \in \mathcal{C}$. (Note that, if $x$ is a vertex, then $H - x$ was defined in Section 1, and if $x$ is an edge, then $H - x$ denotes the graph with vertex set $V(H)$ and edge set $E(H) \setminus \{x\}$.) Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \text{comp}(H - x)$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, $\mathcal{C}_1 \cup \mathcal{C}_2 = \text{comp}(H - x)$ and such that $|\sum \mathcal{C}_1 - \sum \mathcal{C}_2|$ is minimum; assume further that $\sum \mathcal{C}_1 \leq \sum \mathcal{C}_2$. Then we put $\lambda (x) = \sum \mathcal{C}_2$ if $x$ is an edge and $\lambda (x) = 1 + \sum \mathcal{C}_2$, otherwise, and define $\lambda (H) = \min \{\lambda (x)\}$ where the minimum is taken over all vertices and edges of $H$. For a bipartite Cartesian product $G = G_1 \times \cdots \times G_m$ of $m \geq 2$ nontrivial connected graphs, let $A, B$ denote the color classes of $G$ and define $\mu (G)$ by

$$
\mu(G) = \begin{cases} 
\lambda (G_1) & \text{if } m = 2, G_1 \text{ is not 2-connected, and } G_2 = K_2, \\
\lambda (G_2) & \text{if } m = 2, G_2 \text{ is not 2-connected, and } G_1 = K_2, \\
\max \{|A|,|B|\} & \text{otherwise.}
\end{cases}
$$
Theorem 2. For $m \geq 2$ let $G = G_1 \times \cdots \times G_m$ be a Cartesian product of nontrivial connected graphs $G_i$ and let $d, D$ be integers with $d \geq 2$, $D \geq 5$. Then $G \subseteq B(d, D)$ if and only if $G$ is bipartite and $d \geq \mu(G)$.

Before we prove Theorem 2, we provide some basic definitions which are useful not only for the proof of Theorem 2 but also for more general situations. We always assume $D \geq 3$ although parts of the forthcoming discussion are valid for $D = 2$, too.

For an edge $e = ab$ of $B(d, D)$ assume that $a = (x, a_1, \ldots, a_{D-1})$, $b = (a_1, \ldots, a_{D-1}, y)$. Then $(a_1, \ldots, a_{D-1})$ is called a support of $e$. Note that the support of an edge $e$ is uniquely determined if $e$ is not a double edge; if, on the other hand, $e$ is a double edge, say, $e = ab$ with $a = (x, y, x, \ldots)$, $b = (y, x, y, \ldots)$, then there are exactly two $(D - 1)$-tuples which are a support of $e$, namely, the 2-periodic $(D - 1)$-tuples $(x, y, \ldots)$ and $(y, x, \ldots)$.

Now let $G = G_1 \times K_2$ for an arbitrary (i.e., not necessarily connected and nontrivial) graph $G_1$ where $V(G) = \{(v, i) : v \in V(G_1), i = 0, 1\}$, $E(G) = \{(v, 0)(w, 0), (v, 1)(w, 1) : vw \in E(G_1)\} \cup \{(v, 0), (v, 1) : v \in V(G_1)\}$. Let $\varphi : G \rightarrow B(d, D)$ be a fixed subgraph embedding. For $v \in V(G_1)$, let $e_v$ denote the edge $(v, 0)(v, 1)$. Then a support of $\varphi(e_v)$ is called a support of $v$. For $s = (a_1, \ldots, a_{D-1})$ with $a_i \in \{0, \ldots, d - 1\}$, let $V_s$ denote the set of vertices of $G_1$ having support $s$. Then the components of $G_1[V_s]$ are called the $s$-components of $G_1$ (provided that $V_s \neq \emptyset$). Let $\Sigma$ be the set of all $L \subseteq V(G_1)$ such that, for some $s$, $L$ is the vertex set of some $s$-component. If $L$ is an $s$-component, $s$ is called a support of $L$. Let $L_1, L_2 \in \Sigma$ with $L_1 \neq L_2$. Then $L_1, L_2$ are called neighbors of the first kind if $L_1 \cap L_2 \neq \emptyset$; if $L_1 \cap L_2 = \emptyset$ and if there exists an edge of $G_1$ connecting a vertex of $L_1$ with a vertex of $L_2$, then $L_1, L_2$ are called neighbors of the second kind; $L_1, L_2$ are neighbors if they are neighbors of the first or second kind. We use the following notational convention: 2-periodic $k$-tuples are denoted by $(x, y, \ldots)$ whenever the value of $k$ is clear from the context. A similar convention is used for 4-periodic $k$-tuples where, for $k = 2$ and 3, $(x, y, z, u, \ldots)$ denotes the $k$-tuples $(x, y)$ and $(x, y, z)$, respectively. The following statements are direct consequences of the definitions.

Every vertex of $G_1$ is contained in at most two sets $L \in \Sigma$ (and in at least one such set). (12)

Let $L_1, L_2 \in \Sigma$ are neighbors, then $L_1$ and $L_2$ do not possess a common support. (13)

If $L_1, L_2 \in \Sigma$ are neighbors of the first kind, then there exists a $v \in V(G_1)$, together with distinct $x, y \in \{0, \ldots, d - 1\}$, such that $L_1 \cap L_2 = \{v\}$, $\varphi(e_v) = (x, y, \ldots)(y, x, \ldots)$ and such that the $(D - 1)$-tuples $(x, y, \ldots)$ and $(y, x, \ldots)$ are the uniquely determined supports of $L_1$ and $L_2$, respectively. (14)
The next statement immediately follows from the above definitions together with Lemma 3' (and the fact that \( \phi \) is injective).

Let \( L_1, L_2 \in \Sigma \) be neighbors of the second kind with \( v \in L_1, w \in L_2, \) and \( uvw \in E(G_1) \). Then the 4-cycle \( \phi(v,0) - \phi(v,1) - \phi(w,1) - \phi(w,0) - \phi(v,0) \) of \( B(d,D) \) is of type 4 and there exist \( a, a_2, a_3, a_4 \in \{0, \ldots, d-1\} \) such that

\[
\phi(v) = (a_1, a_2, a_3, a_4, \ldots) \quad \text{and} \quad \phi(w) = (a_3, a_4, a_1, a_2, \ldots)
\]

are the uniquely determined supports of \( L_1 \) and \( L_2 \), respectively. Moreover, \( a_1 \neq a_3 \) or \( a_2 \neq a_4 \). \hfill (15)

We next show the following.

If \( D \geq 5 \), then each \( L \in \Sigma \) has at most one neighbor. \hfill (16)

For the proof of (16) let \( L_1, L_2, L'_1 \in \Sigma \) be such that \( L_2, L'_2 \) are neighbors of \( L_1 \). If \( L_1, L_2 \) as well as \( L_1, L'_2 \) are neighbors of the first kind, then it follows from (14) that \( L_1 \cap L_2 = L_1 \cap L'_2 = \{v\} \), and thus \( L_2 = L'_2 \) by (12). Hence we may assume that \( L_1, L_2 \) are neighbors of the second kind. Let \( v, w, a_i \ (i = 1, \ldots, 4) \) be as in (15). It follows that \( L_1, L'_2 \) are neighbors of the second kind, too, since (by (15) and because \( D \geq 5 \)) the uniquely determined support \((a_2, a_3, a_4, a_1, \ldots)\) of \( L_1 \) is not 2-periodic. Let \( v' \in L_1, w' \in L'_2 \) such that \( v'w' \in E(G_1) \). Since \((a_2, a_3, a_4, a_1, \ldots)\) is the uniquely determined support of \( L_1 \), one concludes from (15), together with the fact that \( D \geq 5 \), that \( \phi(e_{v'}) = \phi(e_w) \) holds and that \((a_4, a_1, a_2, a_3, \ldots)\) is the support of \( L'_2 \). Thus we have \( L_2 \cap L'_2 \neq \emptyset \) (since \( \phi(e_{v'}) = \phi(e_w) \) implies \( w' = w \)); moreover, \( L_2, L'_2 \) have the same support and, therefore, \( L_2 = L'_2 \) by (13).

Proof of Theorem 2. Since all other cases are already covered by Theorem 1 and Proposition 1, we may assume that \( m = 2 \) and \( G = G_1 \times K_2 \) hold and that \( G_1 \) is bipartite and not 2-connected. For the proof of Theorem 2, we have to show that

\[
G \subseteq B(d,D) \quad \text{if and only if} \quad d \geq \lambda(G_1).
\]

For the proof of the "only if" part assume that \( \phi: G \to B(d,D) \) is a subgraph embedding. We choose the notations as in the paragraph before (12). By (16) and by the connectedness of \( G_1 \) we have \( |\Sigma| \leq 2 \) (so that we can distinguish three cases as follows).

Case 1: \( |\Sigma| = 1 \). In this case \( |G_1| \leq d \) since in \( B(d,D) \) there are at most \( d \) pairwise disjoint edges with a given support, and thus (since \( \lambda(G_1) \leq |G_1| \)) we have \( \lambda(G_1) \leq d \). This settles case 1 and thus (in the next two cases) we may assume \( |\Sigma| = 2 \), say \( \Sigma = \{L_1, L_2\} \).

Case 2: \( L_1 \cap L_2 \neq \emptyset \). Then there exist \( v, x, y \) as in (14). Application of (15) to the graph \( G_1 \setminus v \) shows that there cannot be an edge of \( G_1 \) joining a vertex of \( L_1 \setminus \{v\} \) with a vertex of \( L_2 \setminus \{v\} \). Put \( \mathcal{C}_i = \{C \in \text{comp}(G_1 - v): C \cap L_i \neq \emptyset\} \quad (i = 1, 2) \).
Then $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$, $\mathcal{G}_1 \cup \mathcal{G}_2 = \text{comp}(G_1 - v)$, and $\sum \mathcal{G}_i = |L_i| - 1 \leq d - 1$ since $|L_i| \leq d$ ($i = 1, 2$). Hence $\lambda(v) = 1 + (d - 1) = d$ and thus $\lambda(G_1) \leq d$.

**Case 3:** $L_1 \cap L_2 = \emptyset$. It follows from (15), together with $D \geq 5$, that $L_1, L_2$ are joined by exactly one edge $f$. Hence $G_1 - f$ has exactly two components, namely, $G_1[L_1]$ and $G_1[L_2]$. Let $\mathcal{G}_i = \{G_i[L_i]\}$ ($i = 1, 2$). Then $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$, $\mathcal{G}_1 \cup \mathcal{G}_2 = \text{comp}(G_1 - f)$, $\sum \mathcal{G}_i = |L_i| \leq d$ ($i = 1, 2$) and, therefore, $\lambda(G_1) \leq \lambda(f) \leq d$.

Thus we have settled the "only if" part of (17). For a proof of the converse, assume $d \geq \lambda(G_1)$. Choose the notations such that $V(G) = \{(v, i): v \in V(G_i), i = 0, 1\}$, $E(G) = \{(v, 0)(w, 0), (v, 1)(w, 1): vw \in E(G_1)\} \cup \{(v, 0)(v, 1): v \in V(G_i)\}$. We separately discuss two cases.

**Case 1:** There is a vertex $v$ of $G_1$ with $\lambda(G_1) = \lambda(v)$. Let $\mathcal{G}_1, \mathcal{G}_2$ be as in the definition of $\lambda(v)$. Then $\sum \mathcal{G}_1 \leq \sum \mathcal{G}_2 = \lambda(v) - 1 = \lambda(G_1) - 1 \leq d - 1$. For $i = 1, 2$ let $H_i$ be the graph induced in $G_1$ by the set $\{x \in V(G_1): x = v$ or $x \in C$ for some $C \in \mathcal{G}_i\}$. Then $H_1 \cap H_2 = \emptyset$, $|H_i| \leq d$ ($i = 1, 2$); further, there is no edge of $G_1$ joining a vertex of $H_1 - v$ with $H_2 - v$.

Recall that $G_1$ (and thus also each $H_i$) is bipartite and let $A_i \cup B_i = V(H_i)$ be a corresponding partition into color classes, where we assume $v \in A_i$ ($i = 1, 2$). Let $\varphi: V(G) \rightarrow B(d, D)$ be an injective mapping such that the following hold (for $u \in V(G)$ and $\varphi(u) = (a_1, \ldots, a_D)$).

\[
(a_1, \ldots, a_D) = \begin{cases}
(0, 1, \ldots) & \text{if } u = (v, 0), \\
(1, 0, \ldots) & \text{if } u = (v, 1),
\end{cases}
\]

\[
(a_1, \ldots, a_{D-1}) = \begin{cases}
(0, 1, \ldots) & \text{if } u \in (A_1 \times \{0\}) \cup (B_1 \times \{1\}), \\
(1, 0, \ldots) & \text{if } u \in (A_2 \times \{1\}) \cup (B_2 \times \{0\}),
\end{cases}
\]

\[
(a_2, \ldots, a_D) = \begin{cases}
(0, 1, \ldots) & \text{if } u \in (A_1 \times \{1\}) \cup (B_1 \times \{0\}), \\
(1, 0, \ldots) & \text{if } u \in (A_2 \times \{0\}) \cup (B_2 \times \{1\}).
\end{cases}
\]

Such a mapping $\varphi$ exists since $|A_i| + |B_i| - |H_i| \leq d$ ($i = 1, 2$). Clearly, each such mapping defines a subgraph embedding $G \rightarrow B(d, D)$.

**Case 2:** There is an edge $f$ of $G_1$ with $\lambda(G_1) = \lambda(f)$. If $G_1 - f$ is connected, then $|G_1| = \lambda(f) = \lambda(G_1) \leq d$ and thus $G_1 \times K_2 \subseteq K_{d,d} \subseteq B(d, D)$. Otherwise, $G_1 - f$ has exactly two components $H_1, H_2$ and $|H_1|, |H_2| \leq \lambda(f) \leq d$. Let $f = vw$ with $v \in H_1$, $w \in H_2$ and let $A_1 \cup B_1 = V(H_i)$ ($i = 1, 2$) be a partition into color classes with $v \in A_1$, $w \in A_2$. Let $\varphi: V(G) \rightarrow B(d, D)$ be an injective mapping such that, for $\varphi(u) = (a_1, \ldots, a_D)$, the following equations hold:

\[
(a_1, \ldots, a_D) = \begin{cases}
(0, 0, 1, 1, \ldots) & \text{if } u = (v, 0), \\
(0, 1, 1, 0, \ldots) & \text{if } u = (v, 1), \\
(1, 1, 0, 0, \ldots) & \text{if } u = (w, 1), \\
(1, 0, 0, 1, \ldots) & \text{if } u = (w, 0),
\end{cases}
\]
Such a mapping \( \varphi \) exists since \(|A_i| + |B_i| = |H| \leq d\), and each such \( \varphi \) defines a subgraph embedding \( G \to B(d, D) \).

5. Solution for \( D = 4 \)

In the preceding sections, we have presented results completely solving the case \( D \geq 5 \). Moreover, by Theorem 1 large parts of the case \( D = 4 \) are settled: it is precisely the case \( G = G_1 \times G_2 \) with \( G_1 = K_2 \) or \( G_2 = K_2 \) which remains unsettled. It turns out that, for \( D = 4 \), this case is much more difficult to handle than the corresponding case for \( D \geq 5 \) since, in contrast to the case \( D \geq 5 \), there exist trickier ways to construct Cartesian product subgraphs \( G_1 \times K_2 \) of \( B(d, 4) \). For an illustration of this, we refer to Fig. 3, which shows that, in contrast to \( B(d, D) \) for \( D \geq 5 \), \( B(d, 4) \) may contain nonbipartite Cartesian product graphs. Thus simple answers cannot be expected for \( D = 4 \). Nevertheless, our next theorem provides a characterization of the graphs \( G_1 \) for which \( G_1 \times K_2 \subseteq B(d, 4) \) holds. We need some preparation.

As usual, a partition of a set \( S \) is a set of nonempty, pairwise disjoint subsets of \( S \) covering all elements of \( S \). For a partition \( \mathcal{P} \) of the vertex set of a graph \( H \), the graph \( H/\mathcal{P} \) is defined as follows: \( \mathcal{P} \) is the vertex set of \( H/\mathcal{P} \), and distinct \( T, U \in \mathcal{P} \) are joined by an edge of \( H/\mathcal{P} \) if and only if there exist vertices \( v \in T, w \in U \) such that \( vw \in E(H) \). In particular, if \( H = (V, E) \) is a graph and \( M \subseteq E \) is a matching, then \( \mathcal{P} := M \cup \{ \{x\}: x \text{ is not incident with an edge of } M \} \) is a partition of \( V \) (where each edge of \( M \) is regarded as a 2-element subset of \( V \)).

In this case we say that the graph \( H/\mathcal{P} \) results from \( H \) by contraction of \( M \) (and we identify the one-element set \( \{x\} \in \mathcal{P} \) with their corresponding vertices \( x \)). The next definitions are crucial.

**Definition 1** \((d\text{-bundle graph of the first kind})\). For an integer \( d \geq 2 \) and a graph \( H = (V, E) \), let \( \mathcal{K} \) be a partition of \( V \); the elements of \( \mathcal{K} \) are called classes. For each \( K \in \mathcal{K} \), let \( \mathcal{B}(K) \) be a partition of \( K \); the elements of \( \mathcal{B}(K) \) are called bundles. Assume that the following conditions hold.

1. There are at most \( d \) classes, each class consists of at most \( d \) bundles, and each bundle contains at most \( d \) vertices,
2. for each pair of distinct classes \( K, L \) there is at most one edge of \( H \) connecting a vertex of \( K \) with a vertex of \( L \).
(1.3) for each class $K$ and each bundle $B \in \mathcal{B}(K)$ there exists at most one edge of $H$ joining a vertex of $B$ with a vertex contained in $V \setminus K$, and if $|B| = d$, then there exists exactly one such edge,

$(1.4)$ for each class $K$ and distinct bundles $B, C \in \mathcal{B}(K)$ there is at most one edge connecting a vertex of $B$ with a vertex of $C$,

$(1.5)$ for each bundle $B$ and each $v \in B$ there is at most one $w \in V \setminus B$ such that $vw \in E(H)$,

$(1.6)$ $H$ is bipartite.

Let $M$ be the set of edges of $H$ connecting different classes and note that (by (1.3)) $M$ is a matching of $H$. Then the graph $G$ resulting from $H$ by contraction of $M$ is called a $d$-bundle graph of the first kind (and each graph isomorphic to $G$ is called a $d$-bundle graph of the first kind, too). Further, $H$ is called the underlying graph of $G$, and the pair $(\mathcal{X}, (\mathcal{B}(K))_{K \in \mathcal{X}})$ is called a $d$-bundle decomposition of $H$ of the first kind.

For example, the cycle $C_9$ is a 3-bundle graph of the first kind. This can be seen as follows. Let $H$ be a 12-cycle and choose the notations such that $V(H) = \{v_0, \ldots, v_{11}\}$ and $E(H) = \{v_0v_1, v_{10}v_{11}, v_{11}v_0\}$. Putting $L_i = \{v_{4i}, v_{4i+1}, v_{4i+2}, v_{4i+3}\}$ for $i = 0, 1, 2$, $B_{i,j} = \{v_{4i+2j}, v_{4i+2j+1}\}$ for $i = 0, 1, 2$ and $j = 0, 1$, $\mathcal{X} = \{L_0, L_1, L_2\}$ and $\mathcal{B}(L_i) = \{B_{i,0}, B_{i,1}\}$ for $i = 0, 1, 2$, one finds that the conditions (1.1)–(1.6) hold. The corresponding matching $M$ consists of the three edges $v_3v_4, v_7v_8, v_{11}v_0$, and the graph resulting from $H$ by contraction of $M$ is a 9-cycle.

Fig. 3. An example showing that $C_9 \times K_2$ is a subgraph of $\mathcal{B}(d,4)$ for $d = 3$. 
Definition 2 (d-bundle graph of the second kind). Let \( d \geq 2 \) be an integer and \( G = (V, E) \) a graph. Let \( K_0, K_1 \) be disjoint subsets of \( V \) such that \( V = K_0 \cup K_1 \), we stress the fact that \( K_0 = \emptyset \) or \( K_1 = \emptyset \) is not excluded. \( K_0 \) and \( K_1 \) are called classes. Let \( B_i \) be a partition of \( K_i \) (i = 0, 1). The elements of \( B_i \) are called bundles (i = 0, 1). Assume that the following conditions hold.

(II.1) Each \( B_i \) consists of at most \( d \) bundles, and each bundle contains at most \( d \) vertices; further, if \( B_i \) consists of exactly \( d \) bundles, then \( |B| \leq d - 1 \) for at least one bundle \( B \in B_i (i = 0, 1) \),

(II.2) there is no edge of \( G \) joining distinct bundles of the same \( K_i \),

(II.3) for each \( B \in B_0 \) and \( C \in B_1 \) there is at most one edge joining \( B \) with \( C \),

(II.4) every vertex has at most one neighbor outside its own bundle,

(II.5) if \( |B_0| = |B_1| = d \), then there exists at least one pair of bundles \( B \in B_0 \), \( C \in B_1 \) such that \( |B| \leq d - 1 \), \( |C| \leq d - 1 \) and such that there is no edge joining a vertex of \( B \) with a vertex of \( C \),

(II.6) \( G \) is bipartite.

Then \( G \) is called a \( d \)-bundle graph of the second kind and the quadruple \((K_0, K_1, B_0, B_1)\) is called a \( d \)-bundle decomposition of \( G \) of the second kind.

Now our result reads as follows.

Theorem 3. Let \( d \geq 2 \) be an integer and let \( G_1 \) be a connected graph. Then \( G_1 \times K_2 \subseteq B(d, 4) \) if and only if \( G_1 \) is a \( d \)-bundle graph of the first or second kind.

The somewhat technical proof of Theorem 3 is given in Section 7 of this paper, where we also give examples showing that \( d \)-bundle graphs of the first kind exist which are not \( d \)-bundle graphs of the second kind, and vice versa. Here we restrict ourselves to the presentation of three corollaries of Theorem 3.

Corollary 1. Let \( G_1 \) be a connected graph and assume that \( G_1 \times K_2 \subseteq B(d, 4) \). Then \( G_1 \) has at most \( d^3 - \frac{1}{2}(d^2 + d) \) vertices if \( d \geq 3 \) and at most 6 vertices if \( d = 2 \).

Proof. By Theorem 3, \( G_1 \) is a \( d \)-bundle graph of the first or second kind. We first consider the case that \( G_1 \) is a \( d \)-bundle graph of the first kind and choose the notations as in the corresponding definition. (In particular, \( H \) denotes the underlying graph of \( G_1 \).) Then \( |V(G_1)| = |V(H)| - |M| \). From (1.1), (1.3) one obtains \( |V(H)| \leq d^2(d - 1) + 2|M| \), and thus \( |V(G_1)| \leq d^2(d - 1) + |M| \). Moreover, we have \( |M| \leq \frac{1}{2}d(d - 1) \) by (1.1) and (1.2). Hence \( |V(G_1)| \leq d^2(d - 1) + \frac{1}{2}d(d - 1) = d^3 - \frac{1}{2}(d^2 + d) \). If \( G_1 \) is a \( d \)-bundle graph of the second kind, then \( |V(G_1)| \leq 2d^2 - 2 \) immediately follows from the corresponding definition. Summarizing we have \( |V(G_1)| \leq f(d) \) for \( f(d) = \max\{d^3 - \frac{1}{2}(d^2 + d), 2d^2 - 2\} \), which implies the corollary. (For this, note that \( f(2) = 6 \) and \( f(d) = d^3 - \frac{1}{2}(d^2 + d) \) for \( d \geq 3 \) as can easily be verified.) \( \square \)

Corollary 2. For \( d \geq 2 \), let \( G_1 \) be a connected graph with \( G_1 \times K_2 \subseteq B(d, 4) \). Then \( \Delta(G_1) \leq 2(d - 1) \), and \( G_1 \) has at most \( \frac{1}{2}(d^2 - d) \) vertices of degree greater than \( d \).
Proof. Let $G_i$ be a $d$-bundle graph of the first kind and choose the notations as in the corresponding definition. Pick $v \in V(G_i)$. Then it follows from the definitions that $\deg_{G_i} v \leq d$ if $v \in V(H)$ and $\deg_{G_i} v \leq 2(d-1)$ if $v \in M$ and thus (because $|M| \leq \frac{1}{2} (d^2 - d)$) the assertion follows. In the case that $G_i$ is a $d$-bundle graph of the second kind, one even obtains the stronger result $\Delta(G_i) \leq d$ (as an immediate consequence of the definition). $\square$

Corollary 3. For $d \geq 3$, let $n = d^3 - d^2 - d + 1$. Then $P_n \times K_2 \subseteq B(d,4)$.

Proof. Let $H$ be a path with $d^2(d-1)$ vertices and choose the notations such that $V(H) = \{v_1, v_2, \ldots, v_{d^2(d-1)}\}$ and $E(H) = \{v_i v_{i+1}: 1 \leq i \leq d^2(d-1) - 1\}$. For each $j \in \{1, 2, \ldots, d\}$, we call the set $\{v_i: (j-1)d < i \leq jd(d-1)\}$ a class; and for each $k \in \{1, 2, \ldots, d^2\}$, we call the set $\{v_i: (k-1)(d-1) < i \leq k(d-1)\}$ a bundle. Then it is obvious that conditions (I.1) (I.6) hold and that the resulting $d$-bundle graph of the first kind is a path with $n$ vertices. $\square$

By a refinement of the preceding proof it can be shown that, for $d \geq 3$, $B(d,4)$ contains even $P_f(\delta) \times K_2$ as a subgraph where $\delta = d^3 - \frac{1}{2}(d^2 + \delta)$. Moreover, we have $P_6 \times K_2 \subseteq B(2,4)$ since it can easily be seen that $P_6$ is a 2-bundle graph of the second kind. These results in particular show that the bound of Corollary 1 is sharp. We omit the details of the proofs, but we remark that the mentioned results are proved (though in a different, more direct way) in a forthcoming paper of Hintz [8] on ladders in de Bruijn graphs.

6. Tori

As our final result we present a complete solution for the case that $G_i \times \cdots \times G_m$ is a torus (for $D \geq 3$). By Theorem 1 we may restrict ourselves to 2-dimensional bipartite tori and to the case $D = 3$. Let $T(r,s)$ denote the Cartesian product of two cycles of length $r$ and $s$, respectively, assuming that the vertex set of $T(r,s)$ is the set of all pairs $(i,j)$ with $i \in \mathbb{Z}_r$, $j \in \mathbb{Z}_s$, where distinct vertices $(i_1,j_1)$, $(i_2,j_2)$ are adjacent if either $i_1 = i_2$ and $j_1, j_2$ differ by one (mod $s$) or $j_1 = j_2$ and $i_1, i_2$ differ by one (mod $r$). When discussing subgraph embeddings $\varphi: T(r,s) \to B(d,3)$ we use terms like “vertical edge”, “horizontal edge”, “left”, “right”, etc., the meaning of which can always be obtained by consulting Fig. 4; note that in Fig. 4 the upper and lower margin, as well as the left and right margin, have to be identified.

Assume that $\varphi: T(r,s) \to B(d,3)$ is a subgraph embedding and let $C \subseteq B(d,3)$ be a 4-cycle of the form

$$C: \varphi(i,j) - \varphi(i,j+1) - \varphi(i+1,j+1) - \varphi(i+1,j) - \varphi(i,j). \tag{18}$$

Assume that $C$ is of type 4. Then $C$ is called an $R$-cycle ($L$-cycle) if “its top edge is right (left) pointing”, i.e. $\bar{B}(d,3)$ contains the arc $\varphi(i,j) \to \varphi(i,j+1)(\varphi(i,j) \leftarrow$
Fig. 4. Notations for the vertices of a torus \( \varphi(T(r,s)) \) contained in \( B(d,3) \).

Fig. 5. Types of squares of \( \varphi(T(r,s)) \), where the labelling of the vertices of the \( L \)-, \( U \)- and \( D \)-cycle is understood to be analogous to the labelling of the \( R \)-cycle.

\[ \varphi(i,j) \quad \varphi(i,j+1) \]
\[ \varphi(i+1,j) \quad \varphi(i+1,j+1) \]

\( \varphi(i,j+1) \); see Fig. 5. Similarly, if \( C \) is not of type 4, then it is a \( U \)-cycle (\( D \)-cycle) if its edges are oriented in \( B(d,3) \) as shown in Fig. 5, where one of the edges may be a double edge. (Recall that, by Lemmas 4 and 7, \( C \) contains no double edge if it is of type 4, and at most one double edge, otherwise.) For short, a 4-cycle \( C \) as in (18) is called a square of \( \varphi(T(r,s)) \).

If squares \( C_1, C_2 \) of \( \varphi(T(r,s)) \) have just one edge (vertex) in common, then \( C_1, C_2 \) are neighbors (diagonal neighbors). The next lemma contains a crucial observation.

**Lemma 10.** If \( C_1, C_2 \) are neighbors, then not both \( C_1 \) and \( C_2 \) are \( U \)-cycles and, similarly, not both are \( D \)-cycles.

**Proof.** Supposing the contrary, it suffices (by symmetry) to consider the case that \( C_1, C_2 \) are \( U \)-cycles having a horizontal edge \( e \) in common. Then \( e \) is a double edge and thus (by Lemma 7)

\[ \text{any other double edge of } \varphi(T(r,s)) \text{ has distance from } e \text{ at least 2.} \quad (19) \]

Let \( C'_1 \) and \( C'_2 \) be the left neighbors of \( C_1 \) and \( C_2 \), respectively. Then one concludes from (19) that one of \( C'_1, C'_2 \) is an \( R \)-cycle and the other is a \( D \)-cycle, and we may assume that the cycles \( C_1, C'_1 \) form a configuration as shown in the shaded area of Fig. 6. (The case that the \( D \)-cycle is above the \( R \)-cycle can be treated analogously.) With the aid of (19) and Lemma 4, one obtains that the squares of the leftmost column of Fig. 6 must be marked \( U, L, U \) as shown in the figure: to obtain this, first conclude that the upper square must be marked \( U \), thereafter conclude that the middle square must be marked \( L \), and finally consider the lower square. By (19), the edge \( g \) is not a double edge and, by Lemma 7, the same holds for \( h \). Hence, the middle square of
As a consequence of Lemma 10 one obtains the following.

**Lemma 11.** The diagonal neighbors of L-cycles are R-cycles, and vice versa.

**Proof.** Let $C$ be an L-cycle. Then Lemma 4 implies that the left and right neighbors of $C$ are D-cycles and that the upper and lower neighbors of $C$ are U-cycles. Thus (by Lemma 10) a diagonal neighbor $C'$ of $C$ cannot be a U- or D-cycle, and thus must be an R-cycle. A similar argument settles the case of an R-cycle.

By Lemma 10, if no square of $\varphi(T(r,s))$ is of type 4, then $\varphi$ is of type DU, i.e., the toroidal grid of Fig. 4 consists of U-cycles and D-cycles arranged like the black and white squares of a toroidal chess board (cf. Fig. 7). Note that in this case the corresponding embedding $\varphi : T(r,s) \rightarrow B(d,3)$ is trivial (in the sense defined in Section 2). On the other hand, if there exists at least one square of $\varphi(T(r,s))$ which is of type 4, then it follows from Lemmas 4 and 11 that $\varphi$ is of type RUDL, i.e., the squares of the toroidal grid of Fig. 4 are arranged as shown in Fig. 7. (Note that each embedding of type RUDL is nontrivial.) Next we will be concerned with a detailed inspection of the type RUDL.
Let \( \varphi : T(r,s) \rightarrow B(d,3) \) be an embedding of type \( RUDL \). By symmetry, we may assume that the cycle \( \varphi(0,0) - \varphi(0,1) - \varphi(1,1) - \varphi(1,0) - \varphi(0,0) \) is an \( R \)-cycle; moreover, \( r \leq s \) may be assumed. Note that both \( r \) and \( s \) are even. Let

\[
\varphi(i,j) = (x_{i,j}, \beta_{i,j}, \gamma_{i,j}) \quad \text{for } i \in \mathbb{Z}_r, j \in \mathbb{Z}_s
\]

and recall that in expressions like \( \varphi(i,j) \), \( x_{i,j}, \ldots \) operations of the first (second) index are to be taken modulo \( r(s) \). We claim that

\[
\beta_{i,j} = \begin{cases} 
\beta_{i,j,0} & \text{if } i \equiv j \mod 2, \\
\beta_{i-j,0} & \text{otherwise}.
\end{cases}
\]

and

\[
x_{i,j} = \begin{cases} 
\beta_{i,j-1} & \text{if } i \equiv 0 \mod 2, \\
\beta_{i,j+1} & \text{otherwise}.
\end{cases}
\]

\[
\gamma_{i,j} = \begin{cases} 
\beta_{i,j+1} & \text{if } i \equiv 0 \mod 2, \\
\beta_{i,j-1} & \text{otherwise}.
\end{cases}
\]

Indeed, formula (21) immediately follows from Lemma 3' by considering the diagonals formed by the \( D \)- and \( U \)-cycles of an embedding of type \( RUDL \), and the other two formulas follow from the observation that, informally speaking, for an embedding of type \( RUDL \), the even rows of horizontal edges are oriented from left to right, while the odd rows are oriented from right to left. We next show that from (21), (22) one obtains

\[ r = s. \] (23)

Indeed, we have \( \beta_{i,j} = \beta_{i,j+r} \) by application of (21), together with the fact that \( r \) is even. Hence \( \varphi(i,j) = \varphi(i,j+r) \) by (22) and, therefore, the supposition \( r < s \) would contradict the injectivity of \( \varphi \). Hence (23). We next show that

\[ r = 4h + 2 \text{ for some integer } h \geq 1. \] (24)

For the proof of (24) suppose that \( r = 4h, h \geq 1 \). Then it follows from (21), (22) that \( \varphi(2h, 2h) = (\beta_{1,0}, \beta_{0,0}, \beta_{-1,0}) = \varphi(0,0) \), contradicting the injectivity of \( \varphi \). Hence (24).

We next show that

\[ \beta_{i,0} \neq \beta_{j,0} \text{ if } i \equiv j \mod 2, i \neq j. \] (25)

For the proof suppose the contrary. Assume that \( |\{ \beta_{i,0} : i \text{ even} \}| < r/2 \). (The case \( |\{ \beta_{i,0} : i \text{ odd} \}| < r/2 \) can be settled by similar arguments.) Note that, by (21), \( \beta_{i,i} = \beta_{2i,0} \) (\( i = 0, \ldots , r-1 \)) and thus there must exist \( i_1, i_2, i_3 \) (\( 0 \leq i_1 < i_2 < i_3 \leq r-1 \)) such that \( \beta_{i_1,i_1} = \beta_{i_2,i_2} = \beta_{i_3,i_3} \). Moreover (by (21)) we have \( \beta_{i_1,i_1} = \beta_{1,0}, \beta_{i_2,i_2} = \beta_{-1,0} \) (\( i = 0, \ldots , r-1 \)) and thus (by (22)) we have \( |\{ \varphi(i_1,i_1), \varphi(i_2,i_2), \varphi(i_3,i_3) \}| \leq 2 \), contradicting the injectivity of \( \varphi \). Hence (25). Our next result states that

\[ d \geq \max \left\{ \frac{r}{2}, 5 \right\}. \] (26)
For the proof note that, by (25), \(d > r/2\) and thus (because of (24)) it remains to settle the case \(r = 6\). Thus suppose \(r = 6, d \leq 4\), put \(\beta_i = \beta_{i,0}\). Note that it follows from (21), (22) that all vertices of \(\varphi(T(r,r))\) are of the form \((\beta_i, \beta_j, \beta_k)\) with \(i \neq j \neq k \pmod{2}, i \neq k\). Further by (25), together with the hypothesis \(d \leq 4\), one obtains \(|\{\beta_0, \beta_2, \beta_4\} \cap \{\beta_1, \beta_3, \beta_5\}| \geq 2\), say, \(\beta_\kappa = \beta_\mu, \beta_\nu = \beta_\lambda\) with \(\kappa, \nu\) even, \(\mu, \lambda\) odd, \(\kappa \neq \nu, \mu \neq \lambda\). But then \((\beta_\lambda, \beta_\nu, \beta_\mu) = (\beta_\nu, \beta_\mu, \beta_\lambda)\), which means that there exist less than 36 distinct triples \((\beta_i, \beta_j, \beta_k)\) with \(i \neq j \neq k \pmod{2}, i \neq k\), in contradiction to the injectivity of \(\varphi\). Hence (26).

So far we have found that every nontrivial embedding \(\varphi: T(r,s) \to B(d,3)\) must be of type \(RUDL\) and we have derived a series of properties that \(\varphi\) must have. Next we show the existence of such embeddings.

For \(r = 4h + 2, h \geq 1\), let \(T(r,r)\) be given and assume \(d \geq \max\{r/2, 5\}\). For each \(i \in \mathbb{Z}_r\), choose \(\beta_i \in \{0, \ldots, d-1\}\) arbitrarily. For \(i,j \in \mathbb{Z}_r\), let

\[
\beta_{i,j} = \begin{cases} 
\beta_{i+j} & \text{if } i \equiv j \pmod{2} \\
\beta_{i-j} & \text{otherwise}
\end{cases}
\]

(27)

and define \(x_{i,j}, \gamma_{i,j}\) such that (22) holds. (All operations of indices are taken modulo \(r\).) Define \(\varphi: V(T(r,r)) \to V(B(d,3))\) by

\[
\varphi(i,j) := (x_{i,j}, \beta_{i,j}, \gamma_{i,j}).
\]

(28)

Then one finds

\[
\varphi(i,j) = \begin{cases} 
(\beta_{i-j+1}, \beta_{i+j}, \beta_{i-j-1}) & \text{if } i \equiv 0, j \equiv 0 \pmod{2}, \\
(\beta_{i-j+1}, \beta_{i-j}, \beta_{i-j+1}) & \text{if } i \equiv 0, j \equiv 1 \pmod{2}, \\
(\beta_{i-j+1}, \beta_{i-j}, \beta_{i-j+1}) & \text{if } i \equiv 1, j \equiv 0 \pmod{2}, \\
(\beta_{i-j+1}, \beta_{i-j}, \beta_{i-j+1}) & \text{if } i \equiv 1, j \equiv 1 \pmod{2}.
\end{cases}
\]

(29)

As a consequence of (29) one obtains that \(\varphi\) is a subgraph embedding of type \(RUDL\) if and only if \(\varphi\) is injective; moreover, for injective \(\varphi\), the cycle \(\varphi(0,0) - \varphi(0,1) - \varphi(1,1) - \varphi(1,0) - \varphi(0,0)\) is an \(R\)-cycle. (We leave the (easy) proofs of these facts to the reader.)

We claim that \(\varphi\) is injective if and only if the \(\beta_i (i \in \mathbb{Z}_r)\) are chosen such that the following conditions (30) and (31) hold.

If \(i \equiv j \pmod{2}, i \neq j\), then \(\beta_i \neq \beta_j\). (30)

There exist no \(i, k, i', k' \in \mathbb{Z}_r\) such that \(i \equiv 0, i' \equiv 1 \pmod{2}, k \equiv i - 2\) or \(i + 2, k' \equiv i' - 2\) or \(i' + 2 \pmod{r}\) and such that \(\beta_i = \beta_{i'}, \beta_k = \beta_{k'}\). (31)

For the proof of our claim assume first that (30) and (31) hold. Let \(i, j, k, l \in \mathbb{Z}_r\) with \(\varphi(i,j) = \varphi(k,l)\). In order to prove that \(\varphi\) is injective, we have to show \(i \equiv k, j \equiv l \pmod{r}\). We first consider the case \(i - j \equiv k - l \pmod{2}\). If \(i \equiv k \pmod{2}\), say, \(i \equiv 0\) and \(k \equiv 1 \pmod{2}\), then one concludes from (29), (30) (together with the assumptions \(\varphi(i,j) = \varphi(k,l)\), \(i - j \equiv k - l \pmod{2}\)) that \(i - j + 1 \equiv k - l - 1\), \(i - j - 1 \equiv k - l + 1\), \(i - j \equiv k - l \pmod{2}\), \(i \equiv k \pmod{2}\), \(i - 1 \equiv k - 1 \pmod{2}\), \(i + 1 \equiv k + 1 \pmod{2}\), \(i + 1 \equiv k - 1 \pmod{2}\), etc.
or \(i + j - 1 \equiv k + l + 1, \ i + j + 1 \equiv k + l - 1 (\mod r)\). In either case, this implies \(2 \equiv -2 (\mod r)\), which contradicts \(r - 4h + 2, \ h \geq 1\). Hence \(i \equiv k (\mod 2)\), and thus also \(j \equiv l (\mod 2)\). Hence (by (29), (30) and the assumption \(\varphi(i, j) = \varphi(k, l)\)) one obtains \(i - j \equiv k + l, \ i-j \equiv k - l (\mod r)\). Hence \(i - k \equiv 0\) or \(r/2 (\mod r)\), but (because of \(r/2 = 2h + 1\) and \(i \equiv k (\mod 2)\)) the latter is impossible. Hence \(i \equiv k, \ j \equiv l (\mod r)\) and thus we have settled the case \(i - j \equiv k - l (\mod 2)\). Now assume \(i - j \neq k - l (\mod 2)\), say, \(i - j, \ k \neq 1 (\mod 2)\). But then it follows from (29), together with the assumption \(\varphi(i, j) = \varphi(k, l)\), that \(i-j, \ k \neq 1 (\mod 2)\), and \(k + l + 1\) form a quadruple of elements of \(\mathbb{Z}_r\) having the property described in (31), which is a contradiction.

Now, conversely, assume that \(\varphi\) is injective. Then, by the remarks in the paragraph after (29), \(\varphi\) is a subgraph embedding of type RUDL with the additional property that \(\varphi(0, 0) - \varphi(0, 1) - \varphi(1, 1) - (1, 0) - \varphi(0, 0)\) is an \(R\)-cycle. From this one finds that (25) must hold. Hence (30), and thus it remains to show (31). To this end, let \(\mathcal{F} = \{(i, j, k): i, j, k \in \mathbb{Z}_r, \ j \neq i (\mod 2), \ k \equiv i - 2 or i + 2 (\mod r)\}\) and observe that \(\varphi(x, y) \in \{(\beta_1, \beta_2, \beta_3): (i, i, k) \in \mathcal{F}\}\) for all \(x, y \in \mathbb{Z}_r\). Further, \(|\mathcal{F}| = r^2 = |\mathbb{Z}_r \times \mathbb{Z}_r|\), and thus we conclude from the injectivity of \(\varphi\) that \((\beta_1, \beta_2, \beta_3) \neq (\beta_i, \beta_j, \beta_k)\) whenever \((i, j, k), (\lambda, \mu, \nu)\) are distinct members of \(\mathcal{F}\). Now, in order to show (31), let \(i, i', k, k' \in \mathbb{Z}_r\) such that \(i \equiv 0, \ i' \equiv 1 (\mod 2), \ k \equiv i - 2 or i + 2 (\mod r)\). Then \((i, i', k, k')\) are distinct members of \(\mathcal{F}\) and thus \((\beta_1, \beta_2, \beta_3) \neq (\beta_i, \beta_j, \beta_k)\). Hence \(\beta_i \neq \beta_i' or \beta_k \neq \beta_k'\), which shows that (31) holds.

If the choice of the \(\beta_i\) \((i \in \mathbb{Z}_r)\) is such that (30) and (31) hold, then we call this choice an admissible choice of the \(\beta_i\), and we say that \(\varphi\) results from an admissible choice of the \(\beta_i\) if \(\varphi\) is defined by the formulas (22), (27), and (28) for some admissible choice of the \(\beta_i\).

Thus, summarizing, we have found that the function \(\varphi\) defined by (22), (27), and (28) is a subgraph embedding of type RUDL if and only if \(\varphi\) results from an admissible choice of the \(\beta_i\) \((i \in \mathbb{Z}_r)\). We now show that an admissible choice of the \(\beta_i\) is always possible. If \(r = 6\), then \(d \geq 5\), and thus we can choose the \(\beta_i\) such that \(\{\beta_0, \beta_2, \beta_4\}\) and \(\{\beta_1, \beta_3, \beta_5\}\) both are 3-element sets and such that these sets have just one element in common. But then (30) and (31) hold, and we are done. If \(r = 4h + 2 for h \geq 2, \ d \geq r/2,\) and thus we can choose the \(\beta_i\) \((i = 0, ..., r - 1)\) such that \(\{\beta_i: \ i \equiv 0 (\mod 2)\} = \{\beta_i: \ i \equiv 1 (\mod 2)\} = \{0, ..., r/2 - 1\}\) and such that \(\beta_i = \beta_i'\) for \(i \equiv 0, i' \equiv 1 (\mod 2)\) if and only if \(i' \equiv 2i + 1 (\mod r)\). Then (30) and (31) readily follow. (We leave the proof to the reader.)

Thus the results of this section can be summarized as follows.

**Theorem 4.** For a torus \(G\) and integers \(d, D\) with \(d \geq 2, D \geq 3\) there exists a nontrivial subgraph embedding \(\varphi: G \rightarrow B(d, D)\) if and only if \(D = 3, G = C_n \times C_n \) for \(n \equiv 2 (\mod 4)\), and \(d \geq \max\{n/2, 5\}\).

As our final remark, we mention that it follows from the considerations which have led to Theorem 4 that every nontrivial subgraph embedding \(\varphi: C_n \times C_n \rightarrow B(d, 3)\) is of
7. The proof of Theorem 3 and a remark on $d$-bundle graphs

Proof of Theorem 3. The proof consists of three parts.

Part 1. We show that, if $G_1$ is a connected graph with $G_1 \times K_2 \subseteq B(d, 4)$, then $G_1$ must be a $d$-bundle graph of the first or second kind.

For this purpose, let $\varphi : G_1 \times K_2 \rightarrow B(d, 4)$ be a subgraph embedding. We use all notations that were introduced in the paragraphs between Theorem 2 and its proof. In particular, for all $x, y, z \in \{0, \ldots, d - 1\}$, the set of vertices of $G_1$ having support $(x, y, z)$ is denoted by $V_{(x, y, z)}$. In the present situation, statements (14) and (15) read as follows.

If $L_1, L_2 \in \Sigma$ are neighbors of the first kind, then there exists a $v \in V(G_1)$, together with distinct $x, y \in \{0, \ldots, d - 1\}$, such that $L_1 \cap L_2 = \{v\}$, $\varphi(e_v) = (x, y, y, y)(y, x, y, x)$ and such that $(x, y, x)$ and $(y, x, y)$ are the uniquely determined supports of $L_1$ and $L_2$, respectively. \hfill (14')

Let $L_1, L_2 \in \Sigma$ be neighbors of the second kind with $v \in L_1, w \in L_2$, and $vw \in E(G_1)$. Then the 4-cycle $\varphi(v, 0) - \varphi(v, 1) - \varphi(w, 1) - \varphi(w, 0) - \varphi(v, 0)$ of $B(d, 4)$ is of type 4 and there exist $a_1, a_2, a_3, a_4 \in \{0, \ldots, d - 1\}$ such that $\varphi(e_v) = (a_1, a_2, a_3, a_4)(a_2, a_3, a_4, a_1)$, $\varphi(e_w) = (a_3, a_4, a_1, a_2)$ $(a_4, a_1, a_2, a_3)$ and such that $(a_2, a_3, a_4)$ and $(a_4, a_1, a_2)$ are the uniquely determined supports of $L_1$ and $L_2$, respectively. Moreover, $a_1 \neq a_3$ or $a_2 \neq a_4$. \hfill (15')

We also need the following statement, which can easily be verified.

Let $L_1, L_2 \in \Sigma$ be neighbors of the first kind, $L_1 \cap L_2 = \{v\}$, and let $u \in L_1$, $w \in L_2$ with $uw \in E(G_1)$. Then $u = v$ or $w = v$. \hfill (32)

In the sequel, statements (14'), (15'), and (32) will sometimes be used without being mentioned explicitly. We claim that there exists a subgraph embedding $\varphi' : G_1 \times K_2 \rightarrow B(d, 4)$ with the following property.

For all $x, y \in \{0, \ldots, d - 1\}$, if there are $d$ distinct vertices of $G_1$ having $(x, y, x)$ as a support (with respect to $\varphi'$), then there exists a vertex $v$ of $G_1$ with $\varphi'(e_v) = (x, y, y, y)(y, x, y, x)$. \hfill (33)

For the proof of our claim, let $A$ be the set of all pairs $(x, y)$ with $x, y \in \{0, \ldots, d - 1\}$ and $|V_{(x,y,x)}| \geq d$. For $(x, y) \in A$ put $V'_{(x,y,x)} := \{\varphi(v, i) : v \in V_{(x,y,x)}, i \in \{0, 1\}\}$ and
\[ W(x, y) := \{(x, y, x, \xi), (\xi, x, y, x) : \xi \in \{0, \ldots, d - 1\}\}. \] Then \( |V'(x, v, x) - d| \) and \( |V'(x, y, x) - d| \) are comparable sets of these sets we obtain \( x \neq y \) and \( W'(x, y, x) = W(x, y) \). In particular, we have \((x, y, x, y, y, x, y, x) \in \mathcal{V}'_{(x, y, x)} \). Hence there are \( v, w \in V_{(x, y, x)} \) and \( t, u \in \{0, 1\} \) such that \((x, y, x, y, y, x, y, x) = \varphi(t, t) \) and \((y, y, x, y, x) = \varphi(w, u) \). We define \( \tilde{t} \in \{0, 1\} \) and \( \tilde{z} \in \{0, \ldots, d - 1\} \) by \( \{t, \tilde{t}\} = \{0, 1\} \) and \( \varphi(t, \tilde{t}) = (z, x, y, x) \), and we put \( \varphi'(v, \tilde{t}) := (y, x, y, x) \) and \( \varphi'(w, u) := (z, x, y, x) \). This procedure is carried out for all pairs \((x, y) \in A \). For all remaining vertices of \( G_1 \times K_2 \) let \( \varphi' \) have the same values as \( \varphi \). Then it can easily be verified that \( \varphi' : G_1 \times K_2 \to B(d, 4) \) is a subgraph embedding satisfying (33).

We assume in the sequel that the given subgraph embedding \( \varphi \) itself has the property (33). We now distinguish two cases.

Case 1: At least one vertex of \( G_1 \) has a 2-periodic support. We show that, in this case, \( G_1 \) is a \( d \)-bundle graph of the first kind. Since \( G_1 \) is connected, it follows from (14') and (15') that the supports of all vertices of \( G_1 \) are 2-periodic. Let \( W \) be the set of all triples \((v, x, y) \) satisfying \( v \in V(G_1) \), \( x, y \in \{0, \ldots, d - 1\} \), and \( v \in V_{(x, y, x)} \). We define a graph \( H \) with vertex set \( W \) as follows. Any pair of distinct vertices \((v, x, y), (v', x', y') \in W \) with \( v = v' \) is joined by an edge, and vertices \((v, x, y), (v', x', y') \in W \) with \( v \neq v' \) are joined by an edge if and only if \( x = x' \) and \( \nu v' \) is an edge of \( G_1 \).

For each \((v, x, y) \in W \), there exists exactly one element \( t \in \{0, 1\} \) such that \( \varphi(v, t) = (x, y, x, \xi) \) and \( \varphi(v, 1 - t) = (\eta, x, y, x) \) for some \( \xi, \eta \in \{0, \ldots, d - 1\} \). Define \( f(v, x, y) := t \). We claim that

\[ f(v, x, y) \neq f(w, a, b) \quad \text{for all edges } (v, x, y)(w, a, b) \in E(H). \] (34)

This is obvious if \( v = w \), and it is a consequence of Lemmas 1 and 2 if neither \( \varphi(e_v) \) nor \( \varphi(e_w) \) is a double edge. By symmetry, it remains to consider the case that \( v \neq w \), \( x \neq y \), and \( \varphi(e_v) = (x, y, x, y)(y, x, y, x) \). Suppose \( f(v, x, y) = f(w, a, b) =: t \). From \( v \neq w \) we obtain, by definition of \( H \), that \( a = x \) and \( \nu v \in E(G_1) \). Hence \( \varphi(w, t) = (x, b, x, \xi) \) for some \( \xi \in \{0, \ldots, d - 1\} \), and \( \varphi(v, t) \varphi(w, t) \) is an edge of \( B(d, 4) \). Since \( \varphi(v, t) = (x, y, x, y) \), we conclude \( x = y \). This contradiction proves (34). From (34) it follows that \( H \) is bipartite.

For \( x, y \in \{0, \ldots, d - 1\} \), we put \( K_x := \{(v, x, y) \in W : \xi = x \} \) and \( B_{x, y} := \{(v, x, y) \in \mathcal{W} : \xi = x, \eta = y \} \). Let \( \mathcal{K} := \{K_x : K_x \neq \emptyset\} \), and for each \( K_x \in \mathcal{K} \) let \( \mathcal{B}(K_x) := \{B_{x, y} : B_{x, y} \neq \emptyset\} \). Then, obviously, \( \mathcal{K} \) is a partition of \( W \) and \( \mathcal{B}(K_x) \) is a partition of \( K_x \) for each \( K_x \in \mathcal{K} \). We claim that \( (\mathcal{K}, (\mathcal{B}(K_x))_{K_x \in \mathcal{K}}) \) is a \( d \)-bundle decomposition of \( H \) of the first kind. In order to show this, it remains to verify the conditions (1.1)–(1.5). The injectivity of \( \varphi \) immediately establishes (1.1). For the proof of (1.2), let \( K_x, K_y \) be distinct classes and let \((v, x, y) \in K_x \) and \((w, a, b) \in K_y \) with \((v, x, y)(w, a, b) \in E(H) \). Then \( v = w \) because \( x \neq a \). Hence \( v \in V_{(x, y, x)} \cap V_{(a, b, a)} \) and, therefore, \((x, y, x, y)(y, x, y, x) = \varphi(e_v) = (a, b, a, b)(b, a, b, a) \). Since \( x \neq a \), we obtain \( b = x \) and \( y = a \). Hence \( y, b \) are uniquely determined by \( x, a \). The injectivity of \( \varphi \), together with \( \varphi(e_v) = (x, a, x, a)(x, x, x, a) \) and \( v = w \), shows that also \( v, w \) are uniquely determined by \( x, a \). Thus (1.2) holds. For the proof of the first part of (1.3), let \( B_{x, y} \) be a bundle, let \( K_a \) be a class distinct from...
Let \( K_x \), and let \((v, x, y) \in B_{x,y}, (w, a, b) \in K_a \) with \((v, x, y)(w, a, b) \in E(H)\). As in the proof of (1.2) we obtain \( y = a \), which means that the class \( K_a \) is uniquely determined by \( B_{x,y} \). The first part of (1.3) is now an immediate consequence of (1.2). For the second part of (1.3), let \( B_{x,y} \) be a bundle with \(|B_{x,y}| = d\). From the hypothesis that \( \varphi \) has the property (33) we obtain that there exists a vertex \( v \) of \( G_1 \) with \( \varphi(e_v) = (x, y, x, y)(y, x, y, x) \); in particular, we have \( x \neq y \). Hence \((v, x, y) \in B_{x,y}, (v, y, x) \in K_y, \) and \((v, x, y)(v, y, x) \) is an edge of \( H \). Thus (1.3) holds. For the proof of (1.4), let \( B_{x,y}, B_{a,b} \) be two distinct bundles of a class \( K_x, \) and let \((v, x, y) \in B_{x,y}, (w, x, b) \in B_{a,b} \) with \((v, x, y)(w, x, b) \in E(H)\). Then \( y \neq b \), and since not both \((x, y, x)\) and \((x, b, x)\) can be supports of \( v \), we have \( v \neq w \). Hence \( v w \in E(G_1) \). From (14'), (15') we conclude that \( \varphi(e_v) = (b, x, y, x)(x, y, x, b) \) and \( \varphi(e_w) = (y, x, b, x)(x, b, x, y) \). Therefore, \( v, w \) are uniquely determined by \( x, y, b \). Thus, (1.4) holds. For the proof of (1.5), let \((u, x, y), (w, a, b) \in \mathcal{W} \) with \((u, x, y)(w, a, b) \in E(H)\) and \((a, b) \neq (x, y)\). We distinguish two cases. Assume first that \( \varphi(e_v) \) is a double edge of \( B(d, 4) \), i.e., \( x \neq y \) and \( \varphi(e_v) = (x, y, x, y)(y, x, y, x) \). From this, together with (14'), (15'), and the definition of \( H \), one easily derives \( w = v, a = y, \) and \( b = x \), which means that \( (w, a, b) \) is uniquely determined by \( (v, x, y) \). Now assume that \( \varphi(e_v) \) is not a double edge, say \( \varphi(e_v) = (\xi, x, y, x)(x, y, x, \eta) \) with \( \xi \neq y \) or \( \eta \neq y \). Then one easily concludes that \( w = v, \ v w \in E(G_1), a = x, b \neq y, b = \xi = \eta, \ \varphi(e_v) = (b, x, y, x)(x, y, b, x) \), and \( \varphi(e_w) = (y, x, b, x)(x, b, x, y) \). Therefore, \( (w, a, b) \) is uniquely determined by \( (v, x, y) \), and (1.5) has been established.

Let \( G' \) be the \( d \)-bundle graph of the first kind defined by \( H \) and \((\mathcal{X}, (B(K))_{K \in \mathbb{K}})\), and let \( M \) be the matching of \( H \) which is contracted in the corresponding definition. Obviously, \( M \) consists exactly of those edges \((v, x, y)(v', x', y') \) of \( H \) which satisfy \( v = v' \).

We define a mapping \( \varphi: V(G') \to V(G_1) \) as follows. For \( x \in V(G') \), let \( \varphi(x) := v \), where \( x = (v, x, y)(v', x', y') \) if \( x \in M \), and \( x = (v, x, y) \) otherwise. Then, by taking into account (14'), (15'), and (32), it is not difficult to verify that \( \varphi \) is an isomorphism of the graphs \( G' \) and \( G_1 \). Thus, we have shown that \( G_1 \) is a \( d \)-bundle graph of the first kind.

Case 2: No vertex of \( G_1 \) has a 2-periodic support. We show that, in this case, \( G_1 \) is a \( d \)-bundle graph of the second kind. Most arguments are similar to those used in Case 1, but easier. Note that it follows from the hypothesis of Case 2 that the support of each vertex of \( G_1 \) is uniquely determined. By (15'), the connectedness of \( G_1 \) implies that there are fixed distinct elements \( a, b \in \{0, \ldots, d - 1\} \) such that the support of any vertex of \( G_1 \) is of the form \((a, \xi, b) \) or \((b, \xi, a)\) with \( \xi \in \{0, \ldots, d - 1\} \).

For each \( x \in V(G_1) \), there exists exactly one \( f(x) \in \{0, 1\} \) such that \( \varphi(v, f(v)) \) is of the form \((x, y, z, \xi) \) with \( \xi \in \{0, \ldots, d - 1\} \), where \((x, y, z) \) denotes the support of \( v \). It follows from Lemmas 1 and 2 that \( f(v) \neq f(w) \) for all edges \( v w \) of \( G_1 \), which shows that \( G_1 \) is bipartite.

Let \( K_0 \) be the set of all vertices of \( G_1 \) having a support of the form \((a, \xi, b) \) and let \( K_1 := V(G_1) \setminus K_0 \). For \( i \in \{0, 1\} \) and \( x \in \{0, \ldots, d - 1\} \), let \( B_{i,x} \) be the set of all vertices of \( K_i \) having a support of the form \((\xi, x, \eta) \). Put \( \mathcal{B}_i := \{B_{i,x}: B_{i,x} \neq \emptyset\} \) for \( i \in \{0, 1\} \). Then, obviously, \( K_0 \cup K_1 = V(G_1), K_0 \cap K_1 = \emptyset, \) and \( \mathcal{B}(K_i) \) is a partition of \( K_i \) for \( i \in \{0, 1\} \). We claim that \((K_0, K_1, \mathcal{B}_0, \mathcal{B}_1)\) is a \( d \)-bundle decomposition of \( G_1 \) of the second kind; we have to verify the conditions (II.1)–(II.5). For the proof of (II.1), we
first remark that clearly each $\mathcal{B}_i$ consists of at most $d$ bundles and each bundle contains at most $d$ vertices. Moreover, we have $|B_{0,a}| \leq d - 1$ or $|B_{0,b}| \leq d - 1$ since otherwise there would exist $v \in B_{0,a}$, $w \in B_{0,b}$, $t, u \in \{0, 1\}$ with $\varphi(v, t) = (a, a, b, b) = \varphi(w, u)$. Analogously, one obtains $|B_{1,a}| \leq d - 1$ or $|B_{1,b}| \leq d - 1$. Thus (II.1) holds. Condition (II.2) is an immediate consequence of (15'). For the proof of (II.3), let $B_{0,x} \in \mathcal{B}_0$, $B_{1,y} \in \mathcal{B}_1$, and let $v \in B_{0,x}$, $w \in B_{1,y}$ with $vw \in E(G_1)$. Then (15') implies $\varphi(e_v) = (y, a, x, b)(a, x, b, y)$ and $\varphi(e_w) = (y, b, a)(b, y, a, x)$, which shows that $v, w$ are uniquely determined by $x, y$. Thus (II.3) holds. For the proof of (II.4), let $B_{i,x}$ be a bundle and let $v \in B_{i,x}$, $w \in V(G_1) \backslash B_{i,x}$ with $vw \in E(G_1)$. By symmetry, we may assume $i = 0$. Then $\varphi(e_v) = (\xi, a, x, b)(a, x, b, y)$ with uniquely determined $\xi, \eta \in \{0, \ldots, d - 1\}$, and (15') implies $\xi = \eta$ and $\varphi(e_w) = (a, x, b, a)(b, \xi, a, x)$, showing that $w$ is uniquely determined by $v$. Thus (II.4) holds.

For the proof of (II.5), assume $|\mathcal{B}_0| = |\mathcal{B}_1| = d$. If $|B_{0,a}| \leq d - 1$, $|B_{1,b}| \leq d - 1$ and if there is no edge joining a vertex of $B_{0,a}$ with a vertex of $B_{1,b}$, then we are done. Now suppose that $|B_{0,a}| = d$ or $|B_{1,b}| = d$ or that there is an edge between $B_{0,a}$ and $B_{1,b}$. We claim the following.

The set $\{\varphi(v, 0), \varphi(v, 1): v \in B_{0,a} \cup B_{1,b}\}$ contains at least one of the two vertices $(a, b, b, a), (a, a, b, b)$ and also at least one of the two vertices $(b, b, a, a), (b, a, a, b)$.

Indeed, if $|B_{0,a}| = d$, then (35) follows from the fact that the set $\{\varphi(v, 0), \varphi(v, 1): v \in B_{0,a}\}$ must contain all vertices of $B(d, 4)$ that are of the form $(a, a, b, \xi)$ or $(\xi, a, a, b)$; the case $|B_{1,b}| = d$ is settled analogously. If, finally, there are $v \in B_{0,a}$, $w \in B_{1,b}$ with $vw \in E(G_1)$, then (15') yields $\varphi(e_v) = (b, a, a, b)(a, a, b, b)$ and $\varphi(e_w) = (a, b, a, b)$.

From (35) we conclude $||\{\varphi(v, 0), \varphi(v, 1): v \in B_{0,b}\}|| \leq 2d - 1$ since at least one of the vertices $(a, b, b, a), (a, a, b, b)$ is not available. This implies $|B_{0,b}| \leq d - 1$. Analogously, we obtain $|B_{1,a}| \leq d - 1$. Moreover, there do not exist $v \in B_{0,a}$, $w \in B_{1,a}$ with $vw \in E(G_1)$ since this would imply $\varphi(e_v) = (a, a, b, b)(a, b, a, a)$ and $\varphi(e_w) = (b, b, a, a)$.

Thus (II.5) has been established, and we have shown that $G_1$ is a $d$-bundle graph of the second kind.

Part 2: We show that, if $G_1$ is a $d$-bundle graph of the first kind, then $G_1 \times K_2 \subseteq B(d, 4)$. We remark that in this part, as well as in the subsequent part 3, the assumption that $G_1$ is connected is not required.

Assume that $G_1$ is a $d$-bundle graph of the first kind and let $H, (\mathcal{X}, (\mathcal{B}(K))_{K \in \mathcal{X}})$, and $M$ be given as in the corresponding definition. Because of $|\mathcal{X}| \leq d$ there exists an injective mapping $\sigma: \mathcal{X} \rightarrow \{0, \ldots, d - 1\}$. From (1.1), (1.2), and (1.3) it follows that, for each $K \in \mathcal{X}$, there exists an injective mapping $\tau_K: \mathcal{B}(K) \rightarrow \{0, \ldots, d - 1\}$ with the following property.

For each bundle $B \in \mathcal{B}(K)$ and each class $L \neq K$, if there is an edge of $H$ joining a vertex of $B$ with a vertex of $L$, then $\tau_K(B) = \sigma(L)$.
From (1.1), (1.3)–(1.5) we obtain that, for each \( K \in \mathcal{K} \) and each \( B \in \mathcal{B}(K) \), there exists an injective mapping \( \rho_B : B \to \{0, \ldots, d - 1\} \), with the following two properties.

For each vertex \( v \in B \), the existence of a vertex \( w \in V(H) \setminus K \) with \( vw \in E(H) \) is equivalent to \( \rho_B(v) = \tau_K(B) \). \hspace{1cm} (37)

For each vertex \( v \in B \) and each bundle \( C \in \mathcal{B}(K) \setminus \{B\} \), if \( v \) has a neighbor in \( C \), then \( \rho_B(v) = \tau_K(C) \). \hspace{1cm} (38)

Since \( H \) is bipartite, there exists a function \( f : V(H) \to \{0, 1\} \) such that \( f(v) \neq f(w) \) for all edges \( vw \in E(H) \).

We define a mapping \( \psi : V(H) \times \{0, 1\} \to B(d, 4) \) as follows. For given \( v \in V(H) \) and \( t \in \{0, 1\} \), let \( K \) and \( B \) denote the class and the bundle of \( v \), respectively; then we put

\[
\psi(v, t) := \begin{cases} 
(\sigma(K), \tau_K(B), \sigma(K), \rho_B(v)) & \text{if } t = f(v), \\
(\rho_B(v), \sigma(K), \tau_K(B), \sigma(K)) & \text{if } t \neq f(v).
\end{cases}
\]

We next show the following statements.

For distinct \((v, t), (w, u) \in V(H) \times \{0, 1\}\), the equality \( \psi(v, t) = \psi(w, u) \) holds if and only if \( vw \in M \) and \( t = u \). \hspace{1cm} (39)

For any \( v \in V(H) \), the vertices \( \psi(v, 0), \psi(v, 1) \) are neighbors in \( B(d, 4) \). \hspace{1cm} (40)

For any \( vw \in E(H) \setminus M \) and any \( t \in \{0, 1\} \), the vertices \( \psi(v, t), \psi(w, t) \) are neighbors in \( B(d, 4) \). \hspace{1cm} (41)

For the proof of (39), let \( K, B, L, C \) denote the class of \( v \), the bundle of \( v \), the class of \( w \), and the bundle of \( w \), respectively. Suppose first \( \psi(v, t) = \psi(w, u) \). Assume \( t = f(v) \) and \( u = f(w) \). Then we have \( (\sigma(K), \tau_K(B), \sigma(K), \rho_B(v)) = (\sigma(L), \tau_L(C), \sigma(L), \rho_C(w)) \). Since \( \sigma \) is injective, we obtain \( K = L \), which, together with the injectivity of \( \tau_K \), leads to \( B = C \); finally, the injectivity of \( \rho_B \) implies \( v = w \). But then \( t = f(v) = f(w) = u \), which contradicts the hypothesis that \( (v, t), (w, u) \) are distinct. Thus, we have settled the case \( t = f(v), u = f(w) \). Analogously, the assumption \( t \neq f(v), u \neq f(w) \) leads to a contradiction, and thus, by symmetry, it remains to consider the case \( t = f(v), u \neq f(w) \). Then we have \( (\sigma(K), \tau_K(B), \sigma(K), \rho_B(v)) = (\sigma(L), \tau_L(C), \sigma(L), \rho_C(w)) \), i.e., \( \rho_B(v) = \sigma(L) = \tau_K(B) \) and \( \rho_C(w) = \sigma(K) = \tau_L(C) \). Since \( \rho_B(v) = \tau_K(B) \), we conclude from (37) that \( v \) has a neighbor \( w' \in V(H) \setminus K \); further, \( \tau_K(B) = \sigma(L) \) and (36), together with the injectivity of \( \sigma \), lead to \( w' \in L \), which in particular means \( L \neq K \). Analogously, we conclude from \( \rho_C(w) = \tau_L(C) = \sigma(K) \) that \( w \) has a neighbor \( v' \in K \). Now (1.2) implies \( v' = v \) and \( w' = w \) and therefore \( vw \in E(H) \). Hence we have \( vw \in M \). Further, \( f(v) \neq f(w) \), which implies \( u = f(v) = t \).
Thus the "only if" statement of (39) has been proved. If, conversely, $vw \in M$ and $t = u$, then we have $f(v) \neq f(w)$ since $v, w$ are neighbors. By symmetry, we may assume $t = f(v) \neq f(w)$. From (36) and (37) we obtain $\rho_B(v) = \tau_K(B) = \sigma(L)$ and $\rho_C(w) = \tau_L(C) = \sigma(K)$, and the definition of $\psi$ yields $\psi(v, t) = (\sigma(K), \tau_K(B), \sigma(K), \rho_B(v)) = (\rho_C(w), \sigma(L), \tau_L(C), \sigma(L)) = \psi(w, t)$. This settles (39). Statement (40) is obvious. For the proof of (41) we may assume, by symmetry, that $t = f(v) \neq f(w)$. Since $vw \notin M$, the vertices $v, w$ belong to the same class $K$. Let $B, C$ denote the bundle of $v, w$, respectively. Then we have $\psi(v, t) = (\sigma(K), \tau_K(B), \sigma(K), \rho_B(v))$ and $\psi(w, t) = (\tau_K(B), \sigma(K), \rho_B(v), \sigma(K))$. For $B = C$, these two vertices are clearly neighbors in $B(d, 4)$. For $B \neq C$, (38) leads to $\rho_B(v) = \tau_K(C)$ and $\rho_C(w) = \tau_K(B)$. Then $\psi(w, t) = (\tau_K(B), \sigma(K), \rho_B(v), \sigma(K))$, and this vertex is a neighbor of $\psi(v, t)$. Thus (41) has been established.

From (39)–(41) it follows that we obtain a subgraph embedding $\varphi : G_1 \times K_2 \to B(d, 4)$ by the following prescription: for $z \in V(G_1)$ and $t \in \{0, 1\}$, define $\varphi(z, t) := \psi(z, t)$ if $z$ is a vertex of $H$ not incident with an edge of $M$, and $\varphi(z, t) := \psi(v, t)$ if $z = vw \in M$.

Part 3: We show that, if $G_1$ is a $d$-bundle graph of the second kind, then $G_1 \times K_2 \subseteq B(d, 4)$. Let $(K_0, K_1, B_0, B_1)$ be a $d$-bundle decomposition of the second kind of $G_1$. For $i \in \{0, 1\}$, put $B'_i := B_i \cup \{0\}$ if $|B_i| \leq d - 1$, and $B'_i := \emptyset$ if $|B_i| = d$. Note that $|B'_i| \leq d$. Because of (II.1) and (II.5), we can choose fixed $B' \in B'_0, C' \in B'_1$ such that $|B'| \leq d - 1, |C'| \leq d - 1$ and such that there is no edge of $G_1$ joining a vertex of $B'$ with a vertex of $C'$. We choose injective mappings $\sigma_i : B'_i \to \{0, \ldots, d - 1\} (i \in \{0, 1\})$ with

$$\sigma_0(B') = 0 \quad \text{and} \quad \sigma_1(C') = 1. \quad (42)$$

From (II.1), (II.3), (II.4), and the choice of $B', C'$ it follows that, for each $i \in \{0, 1\}$ and each $B \in B_i$, there exists an injective mapping $\rho_B : B \to \{0, \ldots, d - 1\}$ with the following properties (43) and (44).

For each $v \in B$ and each $C \in B_{1-i}$, if $v$ has a neighbor in $C$, then $\rho_B(v) = \sigma_{1-i}(C). \quad (43)$

If $B = B'$, then $\rho_B(v) \neq 1$ for all $v \in B$; and if $B = C'$, then $\rho_B(v) \neq 0$ for all $v \in B. \quad (44)$

Since $G_1$ is bipartite, there exists a function $f : V(G_1) \to \{0, 1\}$ such that $f(v) \neq f(w)$ for all edges $vw$ of $G_1$.

We define a mapping $\varphi : V(G_1) \times \{0, 1\} \to B(d, 4)$ as follows. For given $v \in V(G_1)$ and $t \in \{0, 1\}$, let $K_i$ and $B$ denote the class and the bundle of $v$, respectively; then we put

$$\varphi(v, t) := \begin{cases} (i, \sigma_i(B), 1 - i, \rho_B(v)) & \text{if } t = f(v), \\ (\rho_B(v), i, \sigma_i(B), 1 - i) & \text{if } t \neq f(v). \end{cases}$$
We next show the following statements.

- $\varphi$ is injective. (45)

- For any $v \in V(G_1)$, the vertices $\varphi(v, 0), \varphi(v, 1)$ are neighbors in $B(d, 4)$. (46)

- For any $vw \in E(G_1)$ and any $t \in \{0, 1\}$, the vertices $\varphi(v, t), \varphi(w, t)$ are neighbors in $B(d, 4)$. (47)

For the proof of (45), let $(v, t), (w, u) \in V(G_1) \times \{0, 1\}$ with $\varphi(v, t) = \varphi(w, u)$. Let $K_i, B, K_j, C$ denote the class of $v$, the bundle of $v$, the class of $w$, and the bundle of $w$, respectively. We first consider the case $t = f(v), u = f(w)$. Then $(i, \sigma_i(B), 1 - i, \rho_B(v)) = \varphi(v, t) = \varphi(w, u) = (j, \sigma_j(C), 1 - j, \rho_C(w))$. From this we obtain $i = j$; the injectivity of $\sigma_i$ yields $B = C$; and the injectivity of $\rho_B$ leads to $v = w$. Moreover, we have $t = f(v) = f(w) = u$ and therefore $(v, t) = (w, u)$. The case $t \neq f(v), u \neq f(w)$ can be treated analogously. By symmetry, it remains to consider the case $t = f(v), u \neq f(w)$. Then we have $(i, \sigma_i(B), 1 - i, \rho_B(v)) = \varphi(v, t) = \varphi(w, u) = (\rho_C(w), j, \sigma_j(C), 1 - j)$. From this we obtain $\sigma_i(B) = j$ and $\rho_B(v) = 1 - j$ and, therefore, $\{\sigma_i(B), \rho_B(v)\} = \{0, 1\}$. Similarly, we find $\{\sigma_j(C), \rho_C(w)\} = \{0, 1\}$. (By (42) and (44), we conclude $\{B, C\} \cap \{B', C'\} = \emptyset$. Since $\sigma_0, \sigma_1$ are injective, we obtain $\sigma_i(B) = 1 - i$ and $\sigma_j(C) = 1 - j$. But now we arrive at the contradiction $j = \sigma_i(B) = 1 - i = \sigma_j(C) = 1 - j$.

Thus (45) has been proved. Statement (46) is obvious. For the proof of (47), let again $K_i, B, K_j, C$ denote the class of $v$, the bundle of $v$, the class of $w$, and the bundle of $w$, respectively. Since $vw$ is an edge, we may assume $t = f(v) \neq f(w)$. Then we have $\varphi(v, t) = (i, \sigma_i(B), 1 - i, \rho_B(v))$ and $\varphi(w, t) = (\sigma_j(C), j, \rho_C(w), 1 - j)$. If $i = j$, then (II.2) implies $B = C$, and then clearly $\varphi(v, t), \varphi(w, t)$ are neighbors. If $i \neq j$, then $j = 1 - i$, and (43) yields $\rho_B(v) = \sigma_j(C), \rho_C(w) = \sigma_i(B)$, which shows that $\varphi(v, t), \varphi(w, t)$ are neighbors. Thus (47) is settled.

From (45)–(47) it follows that $\varphi$ is a subgraph embedding of $G_1 \times K_2$ into $B(d, 4)$. This completes the proof of Theorem 3.

We close with the following proposition concerning $d$-bundle graphs.

**Proposition 2.** For any integer $d \geq 4$, there exist connected $d$-bundle graphs of the first kind that are not $d$-bundle graphs of the second kind and, vice versa, there exist connected $d$-bundle graphs of the second kind that are not $d$-bundle graphs of the first kind.

**Proof.** In the proofs of Corollaries 1 and 3 of Theorem 3, it was shown that the path with $d^3 - d^2 - d + 1$ vertices is a connected $d$-bundle graph of the first kind and that every $d$-bundle graph of the second kind has at most $2d^2 - 2$ vertices. Since $d^3 - d^2 - d + 1 > 2d^2 - 2$ for $d \geq 4$, this proves the first statement.

In order to prove the second statement, we define a graph $G = (V, E)$ as follows. For each $i \in \{0, 1\}$ and each $j \in \{1, \ldots, d - 1\}$, we put $B_{i,j} := \{(i, j, k) : k \in \{1, \ldots, d - 1\}\}$; for each $i \in \{0, 1\}$, we put $K_i := \bigcup_{j=1}^{d-1} B_{i,j}$; and we put $V := K_0 \cup K_1$. Moreover, we
define $E := E_1 \cup E_2 \cup E_3$, where

$$E_1 := \{(0,j,j)(1,j,j) : j \in \{1,\ldots,d-1\}\},$$
$$E_2 := \{(i,j,k)(i,j,k) : i \in \{0,1\}, j,k \in \{1,\ldots,d-1\}, j \neq k\},$$
$$E_3 := \{(0,j,k)(1,k,j) : j,k \in \{1,\ldots,d-1\}, j \neq k\}.$$ 

We first show that $G$ is connected. Clearly, the subgraph $G[B_{i,j}]$ is connected for each $i \in \{0,1\}$ and each $j \in \{1,\ldots,d-1\}$. Moreover, for any $j,k \in \{1,\ldots,d-1\}$, there is an edge $e \in E_1 \cup E_3$ joining a vertex of $B_{0,j}$ with a vertex of $B_{1,k}$. Hence $G$ is connected.

We next show that $G$ is bipartite. For this purpose, consider

$$X := \{(0,j,j): j \in \{1,\ldots,d-1\}\} \cup \{(1,j,k): j,k \in \{1,\ldots,d-1\}, j \neq k\},$$
$$Y := \{(1,j,j): j \in \{1,\ldots,d-1\}\} \cup \{(0,j,k): j,k \in \{1,\ldots,d-1\}, j \neq k\}.$$ 

Obviously, $X,Y$ define a decomposition of $G$ into color classes. Putting $B_i := \{B_{i,j} : j \in \{1,\ldots,d-1\}\}$ for $i \in \{0,1\}$, it is now easy to verify that $(K_0,K_1,B_0,B_1)$ is a $d$-bundle decomposition of $G$ of the second kind. Hence $G$ is a connected $d$-bundle graph of the second kind, and it remains to show that $G$ is not a $d$-bundle graph of the first kind. For this purpose, assume the contrary. Then there exists a graph $H$, together with a $d$-bundle decomposition of the first kind $(\mathcal{X},(\mathcal{K})_{K \in \mathcal{X}})$ of $H$, such that $G$ is isomorphic to the graph $G'$ which results from $H$ by contraction of the corresponding matching $M$. Note that (1.3), (1.5), and the definition of $M$ imply the following.

The distance in $G'$ between any two distinct $e,f \in M$ is at least three. \hfill (48)

Let $\psi : G \rightarrow G'$ be an isomorphism. For $v \in V(G)$ with $\psi(v) = xy \in M$, we call each of the two vertices $x,y \in V(H)$ a representative of $v$, and for $v \in V(G)$ with $\psi(v) \in V(H)$ we call $\psi(v)$ the (only) representative of $v$. If $vw$ is an edge of $G$, then we conclude from (48), together with the definition of $M$, that there exists exactly one edge $e$ of $H$ which joins a representative of $v$ with a representative of $w$ and that there exists exactly one class $K$ of $H$ which contains representatives of both $v$ and $w$. We call $e$ and $K$ the representative and the $H$-class of $vw$, respectively. The following observation is crucial.

If $C$ is a cycle in $G$ of length 8, then there exists a class of $H$ containing a representative of each vertex of $C$. \hfill (49)

For the proof of (49), we first remark that (obviously) there exists exactly one cycle $C'$ in $H$ such that each edge of $C'$ is either the representative of an edge of $C$ or an element of $M$ which is a vertex of the 8-cycle $\psi(C) \subseteq G'$. Further, it is clear that $C'$ contains the representatives of all eight edges of $C$. Thus, (49) is settled if some class of $H$ contains all vertices of $C'$. Assume that $C'$ meets more than one class of $H$; then, by (1.2), $C'$ must meet at least three classes of $H$. Therefore, the 8-cycle $\psi(C) \subseteq G'$ contains at least three elements of $M$. But this contradicts (48). Hence (49) holds.
Next we remark that, for all $j, k$ with $j \neq k$, the following 8-cycle is contained in $G$:

$$(0,j,j)(1,j,j)(1,j,k)(0,k,k)(1,k,k)(1,k,j)(0,j,k)(0,j,j). \quad (50)$$

Putting $j = 1$ in (50), we see that, for any $k \in \{2, \ldots, d-1\}$, there exists an 8-cycle in $G$ containing the edges $(0,1,1)(1,1,1)$ and $(0,k,k)(1,k,k)$. Therefore, by (49), the $H$-classes of all edges of the form $(0,j,j)(1,j,j)$ are equal. Considering again (50), we see that every edge of $G$ is contained in some 8-cycle which contains an edge of the form $(0,j,j)(1,j,j)$. Hence, by (49), there exists a class $K$ of $H$ which is the $H$-class of all edges of $G$. Since every vertex of $G$ is incident with some edge, it follows that $K$ contains representatives of all vertices of $G$. But this implies $2(d-1)^2 = |V(G)| \leq |K| \leq d^2$, and therefore $d^2 - 4d + 2 < 0$, contradicting the hypothesis $d \geq 4$. Hence $G$ is not a $d$-bundle graph of the first kind. □

References


