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Asymptotic expansions of singular solutions for (3 + 1)-D Protter problems

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Abstract

Four-dimensional boundary value problems for the nonhomogeneous wave equation are studied, which are analogues of Darboux problems in the plane. The smoothness of the right-hand side function of the wave equation is decisive for the behavior of the solution of the boundary value problem. It is shown that for each $n \in \mathbb{N}$ there exists such a right-hand side function from C^n, for which the uniquely determined generalized solution has a strong power-type singularity at one boundary point. This singularity is isolated at the vertex of the characteristic cone and does not propagate along the cone. The present article describes asymptotic expansions of the generalized solutions in negative powers of the distance to this singularity point. Some necessary and sufficient conditions for existence of regular solutions, or solutions with fixed order of singularity, are derived and additionally some a priori estimates for the singular solutions are given.

Keywords: Wave equation; Boundary value problems; Generalized solution; Singular solutions; Propagation of singularities; Asymptotic expansion

1. Introduction

In the present paper some boundary value problems for the wave equation in $\mathbb{R}^4$ (having three space variables and one time variable) are studied, which are multidimensional analogues

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of Darboux problems in the plane. The usual Darboux problem is formulated in a characteristic triangle with two characteristic segments and a noncharacteristic one. The boundary values are prescribed on one characteristic and on the noncharacteristic segment. 50 years ago M.H. Protter [25] formulated and studied a three-dimensional analogue of the planar Darboux problems in a 3-D domain Ω, bounded by two characteristic cones Σ_1 and Σ_2 and a plane region Σ_0. Now, it is known that the multidimensional Protter problem P1 with the data on Σ_0 and Σ_1 is not well-posed, in contrast to the plane Darboux problem. The reason is that its adjoint homogeneous problem P1* has an infinite number of nontrivial classical solutions (Tong Kwang-Chang [27], Popivanov and Schneider [21], Khe Kan Cher [19]). What is now the situation around these problems? The uniqueness of a classical solution of problem P1 in $\mathbb{R}^4$ is proved (Garabedian [9]). On the other hand, a necessary condition for the existence of a classical solution for problem P1 is the orthogonality of the right-hand side function f to all solutions of the homogeneous problem P1*. To avoid this infinite number of necessary conditions in the frame of classical solvability, the concept of *generalized solutions* of problem P1 with a possible singularity on the cone Σ_2 was introduced [22]. Many authors studied these problems using different methods, e.g., Wiener–Hopf method, special Legendre functions, a priori estimates, nonlocal regularization, etc. (see [22] and references there, further [2,7,14,19,23,24]). On the other hand, in $\mathbb{R}^4$ another analogue of the classical Darboux problem in the same domain Ω is given (Bazarbekov [4]). Some different statements on Darboux type problems in $\mathbb{R}^3$ are known also [5,16,17]. Together with P1 its adjoined problem P1* has been studied for existence of bounded or unbounded solutions for the wave equation in $\mathbb{R}^3$ and $\mathbb{R}^4$ as well as for the Euler–Poisson–Darboux equation [2,11,14,15, 18,19,24]. Another aspect of the multidimensional analogue of some well-known 2-D transonic problems have been proposed also by Protter (cf. [25]). For the linear Gellerstedt equation of mixed type Protter gives a 3-D analogue to the planar Guderley–Morawetz problem, but even in the linear case a general understanding of the situation is not at hand. The uniqueness result in the linear case is given in [3] (see also [5,16]). In [20] is shown that the nonexistence Pohozaev type principle is valid for the semi-linear Protter problem in $\mathbb{R}^n$ for the case of supercritical Sobolev exponent.

According to the solvability problem, it is shown that for each $n \in \mathbb{N}$ there exists a right-hand side function $f \in C^n(\overline{\Omega})$ of the wave equation, for which the uniquely determined generalized solution has a strong power-type singularity like r^{-n} at the origin. This singularity is isolated at the vertex of the characteristic cone and does not propagate along the cone. It is interesting to find the reason why such a singularity occur for very smooth right-hand side functions f in contrast to the Cauchy problem, for example. Concerning Protter problems, the existence of singular solutions for both, wave and degenerate hyperbolic equations, were announced by Popivanov and Schneider in 1988 and the proofs are given in [21,22]. First a priori estimates for singular solutions of Protter problems, concerning the wave equation in $\mathbb{R}^3$, were obtained [22]. In the case of the wave equation in $\mathbb{R}^{m+1}$ the existence of solutions in the domain Ω_ε ($\Omega_\varepsilon \to \Omega$ and $\Sigma_{2,\varepsilon}$ approximates Σ_2 for $\varepsilon \to 0$), which grow up in the cone $\Sigma_{2,\varepsilon}$ like $\varepsilon^{-(n+m-2)}$ and $\varepsilon^{-(n+m-1)}$, respectively, are noted (Aldashev [1]). It is obvious that for $m = 3$ these results can be compared with the results from Corollary 3.4. Some other questions require to describe the exact order of singularity and to find some a priori estimates for the singular solutions. The answers for some Protter problems were given, after deriving necessary and sufficient conditions for the existence of solutions with fixed order of the singularity (see [23] in $\mathbb{R}^3$, [24] in $\mathbb{R}^4$ and Remarks 3.5 and 3.6).

According to the ill-posedness of Protter problems (see Remark 3.8), there have appeared some possible regularization methods in the case of the wave equation, involving either lower order terms [2,10] or some other kinds of perturbations like integro-differential terms, or nonlocal ones [7].

This paper is organized in Introduction and five more sections. In Section 2, the Protter problems P1, P1*, P2, P2* are formulated and the result on the existence of an infinite number of classical solutions to the homogeneous problems P1* and P2* (Lemma 2.1) is given. In Section 3, the main results of the paper are formulated and discussed, concerning the existence and the uniqueness (Theorem 3.1), asymptotic expansion of the *generalized solution* of problem P1 and the exact behavior of the singularity under the orthogonality conditions, given on the right-hand side function of the wave equation (Theorem 3.2). Necessary and sufficient conditions for the existence of only bounded solutions are given in Corollary 3.3. Some figures are also presented, which show the effects appearing near the singularity point. In Section 4, the 2-D boundary value problems P1.1 and P1.2 which correspond to the $(3+1)$-D problem P1, are discussed. Finally, these 2-D problems are transferred to an integral equation, which is invertible. Using special functions, some exact formulas for the solution of P1.2 are given (Lemma 4.3). Section 5 contains the most technical part of the paper. After detailed study of the relations in the exact formulas (Lemma 5.1), the results concerning the asymptotic expansions of the generalized solution of the 2-D problem P1.1 are proved (Theorem 5.2). Additionally, some corresponding a priori estimates are derived. In the final Section 6, the proofs of the main results (Theorem 3.2) are given using the results of Section 5.

2. Preliminaries

In the present paper, boundary value problems for the wave equation in $\mathbb{R}^4$

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = f(x,t) \tag{2.1}$$

with points $(x,t) = (x_1, x_2, x_3, t)$ are studied in the domain

$$\Omega = \left\{(x,t)\colon\ 0 < t < 1/2,\ t < \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 - t\right\},$$

bounded by the two characteristic cones

$$\Sigma_1 = \left\{(x,t)\colon\ 0 < t < 1/2,\ \sqrt{x_1^2 + x_2^2 + x_3^2} = 1 - t\right\},$$
$$\Sigma_2 = \left\{(x,t)\colon\ 0 < t < 1/2,\ \sqrt{x_1^2 + x_2^2 + x_3^2} = t\right\}$$

and the ball $\Sigma_0 = \{t = 0, \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\}$, centered at the origin $O\colon x = 0, t = 0$. The right-hand side function f of (2.1) satisfies some smoothness conditions in Ω, which will be fixed later. The following multidimensional analogues of Darboux problems were proposed by M.H. Protter [25]:

Problem P1. Find a solution of the wave equation (2.1) in Ω which satisfies the boundary conditions

$$\text{P1:}\quad u|_{\Sigma_0} = 0, \quad u|_{\Sigma_1} = 0.$$

Problem P1*. Find a solution of the wave equation (2.1) in Ω which satisfies the adjoint boundary conditions

$$\text{P1}^*\text{:}\quad u|_{\Sigma_0} = 0, \quad u|_{\Sigma_2} = 0.$$

Problems P2 and P2*. Find a solution of the wave equation (2.1) in Ω which satisfies the boundary conditions

$$\text{P2:}\quad u_t|_{\Sigma_0}=0,\quad u|_{\Sigma_1}=0$$

or the adjoint boundary conditions

$$\text{P2}^*:\quad u_t|_{\Sigma_0}=0,\quad u|_{\Sigma_2}=0,$$

respectively.

Protter [25] formulated and studied the analogues of P1 and P1* in $\mathbb{R}^3$ as multidimensional generalizations of the Darboux problem in the plane. In contrast to the plane Darboux problems, the multidimensional problems P1 and P2 are not well-posed, because their adjoint homogeneous problems P1* and P2* have an infinite number of classical solutions [19,21,27].

Let Y_n^m with $n=0,1,2,\dots$ and $m=1,\dots,2n+1$, be the orthonormal system of *spherical functions* in $\mathbb{R}^3$. They are defined usually on the unit sphere $S^2:=\{(x_1,x_2,x_3)\colon x_1^2+x_2^2+x_3^2=1\}$ in spherical polar coordinates (see [13]). Expressed in Cartesian coordinates here, one can define them by

$$Y_n^{2m}(x_1,x_2,x_3)=C_n^m\frac{d^m}{dx_3^m}P_n(x_3)\,\mathrm{Im}\bigl\{(x_2+ix_1)^m\bigr\},\quad \text{for } m=1,\dots,n \tag{2.2}$$

and

$$Y_n^{2m+1}(x_1,x_2,x_3)=C_n^m\frac{d^m}{dx_3^m}P_n(x_3)\,\mathrm{Re}\bigl\{(x_2+ix_1)^m\bigr\},\quad \text{for } m=0,\dots,n, \tag{2.3}$$

where C_n^m are constants and P_n are the *Legendre polynomials*. Recall that the functions Y_n^m form a complete orthonormal system in $L_2(S^2)$ (see [13]). For convenience in the discussions that follow, we extend the spherical functions out of S^2 radially, keeping the same notation Y_n^m for the extended function, i.e. $Y_n^m(x):=Y_n^m(x/|x|)$ for $x\in\mathbb{R}^3\setminus\{0\}$.

For $n,k\in\mathbb{N}\cup\{0\}$ define the functions

$$H_k^n(x,t)=\sum_{i=0}^{k}A_{k,i}^n\frac{t(|x|^2-t^2)^{n-1-k-i}}{|x|^{n-2i+1}},\qquad E_k^n(x,t)=\sum_{i=0}^{k}B_{k,i}^n\frac{(|x|^2-t^2)^{n-k-i}}{|x|^{n-2i+1}}, \tag{2.4}$$

where the coefficients are

$$A_{k,i}^n:=(-1)^i\frac{(k-i+1)_i(n-k-i)_i}{i!(n-i+\frac12)_i},\qquad A_{k,0}^n=1;$$
$$B_{k,i}^n:=(-1)^i\frac{(k-i+1)_i(n+1-k-i)_i}{i!(n-i+\frac12)_i},\qquad B_{k,0}^n=1,$$

with $(a)_i:=a(a+1)\cdots(a+i-1)$ and $(a)_0:=1$. Special representations read as (Khe Kan Cher [19])

$$H_k^n(x,t)=t|x|^{n-2k-3}\bigl(1-t^2/|x|^2\bigr)^{n-2k-1}F\bigl(n-k+1/2,-k,3/2;t^2/|x|^2\bigr),$$
$$E_k^n(x,t)=|x|^{n-2k-1}\bigl(1-t^2/|x|^2\bigr)^{n-2k}F\bigl(n-k+1/2,-k,1/2;t^2/|x|^2\bigr),$$

where $F=F(a,b,c;x)$ is the Gauss hypergeometric function. Further, define the functions

$$V_{k,m}^n(x,t)=H_k^n(x,t)Y_n^m(x),\qquad W_{k,m}^n(x,t)=E_k^n(x,t)Y_n^m(x). \tag{2.5}$$

These functions are classical solutions of the homogeneous adjoint Protter problems.

Lemma 2.1. [24] *The functions* $V^n_{k,m}(x,t)$ *are classical* $C^2(\overline{\Omega})$ *solutions of the homogeneous problem* P1* *for* $k = 0, \dots, [(n-1)/2] - 2$, *and* $W^n_{k,m}(x,t)$ *are classical solutions of the homogeneous problem* P2* *for* $k = 0, \dots, [n/2] - 2$.

The corresponding solutions to $V^n_{0,m}$ and $W^n_{0,m}$ for the three-dimensional case are known [21]. On the other hand, a necessary condition for the existence of classical solution for the problem P1 is the orthogonality of the right-hand side function f to all functions $V^n_{k,m}(x,t)$. To avoid an infinite number of necessary conditions in the frame of classical solvability, we introduce *generalized solutions* for the problem P1, eventually with a singularity at the origin (similarly to [22] and [24]).

Definition 2.2. A distribution $u = u(x,t)$ is called a generalized solution of the problem P1 in Ω, if the following conditions are satisfied:

(1) $u \in C^1(\overline{\Omega} \setminus O)$, $u|_{\Sigma_0 \setminus O} = 0$, $u|_{\Sigma_1} = 0$, and
(2) the identity

$$\int_{\Omega} (u_t w_t - u_{x_1} w_{x_1} - u_{x_2} w_{x_2} - u_{x_3} w_{x_3} - f w)\, dx\, dt = 0 \tag{2.6}$$

holds for all $w \in C^1(\overline{\Omega})$ such that $w|_{\Sigma_0} = 0$ and $w = 0$ in a neighborhood of Σ_2.

3. The asymptotic expansions of the generalized solution

In this section, the main results are discussed while the proofs will be given in the last section. Let the right-hand side function $f \in C^1(\overline{\Omega})$ of Eq. (2.1) be a harmonic polynomial of order l with $l \in \mathbb{N} \cup \{0\}$ and having the representation

$$f(x,t) = \sum_{n=0}^{l} \sum_{m=1}^{2n+1} f^m_n\big(|x|, t\big) Y^m_n(x). \tag{3.1}$$

First, the following result on the existence and the uniqueness of the generalized solution of problem P1 is valid.

Theorem 3.1. *The problem* P1 *has at most one generalized solution in the domain* Ω. *Suppose that the right-hand side* $f \in C^1(\overline{\Omega})$ *has the form* (3.1) *where* $l \in \mathbb{N} \cup \{0\}$. *Then, the unique generalized solution* $u(x_1, x_2, x_3, t)$ *of the problem* P1 *in* Ω *exists and has the form*

$$u(x,t) = \sum_{n=0}^{l} \sum_{m=1}^{2n+1} u^m_n\big(|x|, t\big) Y^m_n(x) \in C^2(\overline{\Omega} \setminus O). \tag{3.2}$$

Suppose that the right-hand side function f is fixed as a harmonic polynomial (3.1) of order l. According to the results in the 3-D case, we expect that the corresponding *generalized solution* $u(x,t)$ may have a power type singularity at the origin. In this paper the asymptotic expansion of the *generalized solution* of problem P1 at the origin O: $x=0, t=0$ will be found. Denote by $\beta^n_{k,m}$ the parameters

$$\beta^n_{k,m} := \int\limits_{\Omega} V^n_{k,m}(x,t) f(x,t)\,dx\,dt, \tag{3.3}$$

where $n=1,\dots,l$; $k=0,\dots,[\frac{n-1}{2}]$ and $m=1,\dots,2n+1$, then the main assertion is given.

Theorem 3.2. *Suppose that the right-hand side function $f\in C^1(\overline{\Omega})$ has the form* (3.1)*. Then the unique generalized solution $u(x,t)$ of problem* P1 *belongs to $C^2(\overline{\Omega}\setminus O)$ and has the following asymptotic expansion at the singular point O: $x=0, t=0$*

$$u(x,t) = \sum_{p=1}^{l} \left(|x|^2+t^2\right)^{-p/2} F_p(x,t) + F(x,t),$$

where:

(i) *the function $F\in C^2(\overline{\Omega}\setminus O)$ and satisfies the a priori estimate*

$$\left|F(x,t)\right| \leqslant C\|f\|_{C^1(\Omega)}, \quad (x,t)\in\Omega,$$

with constant C independent on f and $\|f\|_{C^k(\Omega)} = \sum_{|\alpha|\leqslant k}\max_{\overline{\Omega}} |D^\alpha f(x,t)|$;

(ii) *the functions F_p satisfy the equalities*

$$F_p(x,t) = \sum_{k=0}^{[(l-p)/2]} \sum_{m=1}^{2p+4k+1} \beta^{p+2k}_{k,m} F^{p+2k}_{k,m}(x,t), \quad p=1,\dots,l, \tag{3.4}$$

with functions $F^n_{k,m}\in C^2(\overline{\Omega}\setminus O)$ bounded and independent on f;

(iii) *if at least one of the constants $\beta^{p+2k}_{k,m}$ in* (3.4) *is different from zero, then for the corresponding function $F_p(x,t)$ there exists a direction $(\alpha,1) := (\alpha_1,\alpha_2,\alpha_3,1)$ with $(\alpha,1)t\in\Sigma_2$ for $0<t<1/2$, such that*

$$\lim_{t\to+0} F_p(\alpha t,t) = c_p = \text{const} \neq 0.$$

Corollary 3.3. *Suppose that the right-hand side function $f\in C^1(\overline{\Omega})$ has the form* (3.1) *and satisfies the orthogonality conditions*

$$\int\limits_{\Omega} V^n_{k,m}(x,t) f(x,t)\,dx\,dt = 0 \tag{3.5}$$

for all $n=1,\dots,l$; $k=0,\dots,[\frac{n-1}{2}]$ and $m=1,\dots,2n+1$. Then the unique generalized solution $u(x,t)$ of problem P1 *belongs to $C^2(\overline{\Omega}\setminus O)$, is bounded and fulfills the a priori estimate*

$$\sup_{\overline{\Omega}} |u| \leqslant C\|f\|_{C^1(\Omega)}. \tag{3.6}$$

Theorem 3.2 gives an asymptotic expansion of the *generalized solution*. In the case of orthogonality conditions (3.5) Corollary 3.3 shows that the solution is bounded and satisfies an a priori estimate. The case (iii) of Theorem 3.2 clarifies the significance of the above orthogonality conditions (3.5) and shows the exact behavior of the *generalized solution* if some of conditions (3.5) are not satisfied. In other words, for fixed indexes (n, k, m) the corresponding condition (3.5) "controls" one power-type singularity.

Finally, as a consequence of Theorem 3.2, without any orthogonality conditions on f, the following result is obtained.

Corollary 3.4. *The generalized solution u of problem* P1 *with a right-hand side function $f \in C^1(\overline{\Omega})$ in the form* (3.1) *satisfies the a priori estimate*

$$\left|u(x,t)\right| \leqslant C\Bigl(\max_{\overline{\Omega}} |f|\Bigr)\bigl(|x|^2 + t^2\bigr)^{-l/2}. \tag{3.7}$$

In Corollaries 3.3 and 3.4 both extreme cases are studied: all orthogonality conditions (3.5) are fulfilled and no one of them is satisfied. In each of them the exact behavior of the solution in the corresponding case is shown. The estimate (3.7), presented here is analogous to known estimates for Protter problems in $\mathbb{R}^3$ [22] and in $\mathbb{R}^m$ [1]. It is interesting that singularities of the *generalized solutions* are isolated at the origin O and do not propagate in the direction of the bicharacteristics on the characteristic cone Σ_2. Traditionally, it is assumed that the wave equation, with sufficiently smooth right-hand side cannot have a solution with an isolated singular point (Hörmander [12, Chapter 24.5]).

The main results for the existence of a singularity depending on orthogonality conditions at the point $(0, 0)$ in coordinates $r = |x|$ and t is illustrated by the following figures: only one of orthogonality conditions (3.5) is not fulfilled and the solution tends to $\pm\infty$ (Fig. 1), all the orthogonality conditions (3.5) are satisfied and the solution is bounded but not continuous at $(0, 0)$ (Fig. 2) and respectively the solution is continuous (Fig. 3).

Remark 3.5. We mention some differences between the results given here for the problem P1 and the results from [24] for problem P2, both in $\mathbb{R}^4$: First of all, the explicit asymptotic expansion

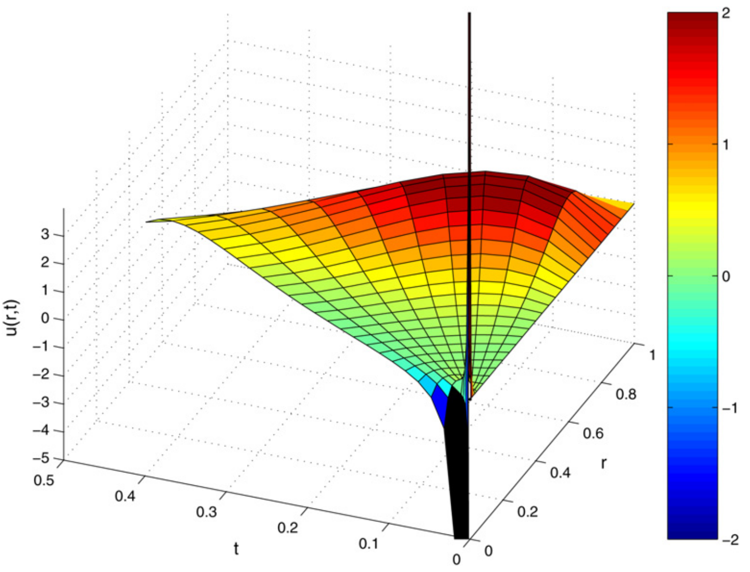

Fig. 1. One orthogonality condition is not fulfilled.

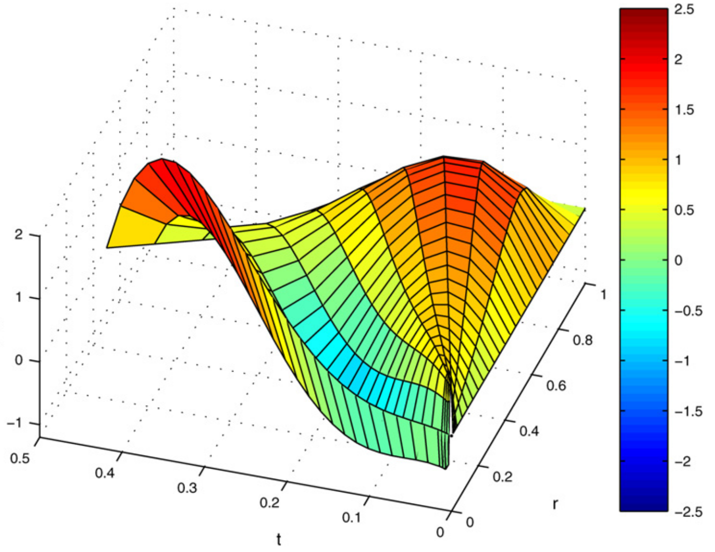

Fig. 2. All the orthogonality conditions are fulfilled and the solution is bounded but not continuous at (0, 0).

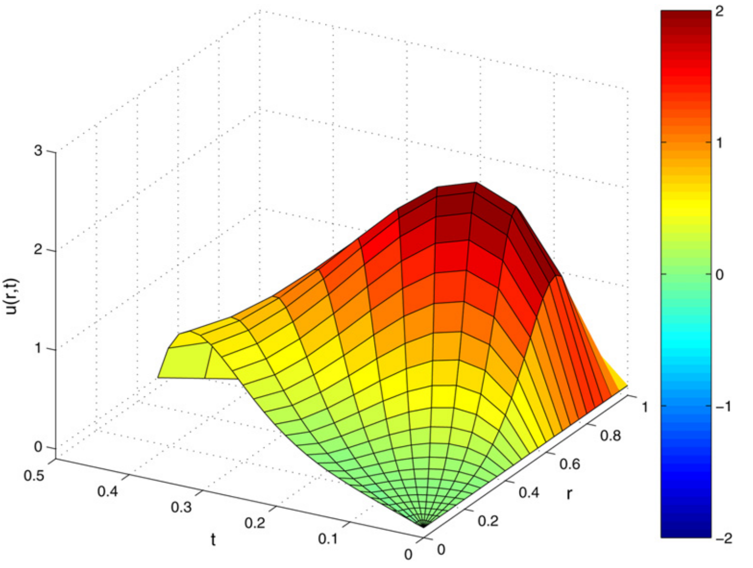

Fig. 3. All the orthogonality conditions are fulfilled and the solution is continuous.

here has no analogue in [24], where we have only the behavior of the singularities. Secondly, according to Corollary 3.3, if the orthogonality conditions are fulfilled, the *generalized* solution is bounded, while for the solution in [24, Theorem 1.1] some logarithmic singularities are possible. Finally, comparing the power of singularity of the *generalized solution* in Corollary 3.4 here and Corollary 1.1 from [24], one can see that in the worst case without any orthogonality conditions, the power in the estimate (3.7) is $(|x|^2 + t^2)^{-l/2}$, while in the analogous estimate in [24] it is $(|x|^2 + t^2)^{-(l+1)/2}$.

Remark 3.6. Let us point out the difference between the cases in $\mathbb{R}^3$ (see [23]) and $\mathbb{R}^4$ here. In both of them the study of Protter problem P1 is based on the properties of the special Legendre functions. Instead of Legendre polynomials P_n here, in the three-dimensional case [23] the Legendre functions P_ν with non-integer indexes $\nu = n - 1/2$ are used (for their properties see [8]). Both these techniques one can easily modify to get similar results for the $(m+1)$-dimensional problems: for even m (analogous to [23]) or for odd m (see [24] and the present results). Some different kind of results for Protter problems in $\mathbb{R}^{m+1}$ are presented in [1,2].

Remark 3.7. Let us mention one obvious consequence of Theorem 3.2 and the arguments above, concerning construction of functions orthogonal to the solutions $V_{k,m}^n$ of the adjoint homogeneous problem P1*. Take an arbitrary $C^2(\overline{\Omega})$ function $U(x,t)$ satisfying only the boundary conditions P1. Then applying the wave operator $\Box$, the result function $F := \Box U$ is orthogonal in $L_2(\Omega)$ to all the functions $V_{k,m}^n$, $n = 1, 2, \ldots.$

Finally, we formulate some still open questions, that naturally arise from the previous works on the Protter problem and the discussions above.

Open Problems. 1. To study the more general case when the right-hand side function $f \in C^1(\overline{\Omega})$, i.e. it can be expanded in an infinite Fourier series instead of the finite sum (3.1):

∘ Find some appropriate conditions for the function f under which there exists a generalized solution.

∘ What kind of singularity can have the generalized solution? The a priori estimates, obtained in [10,22], show that the *generalized solutions* of problem P1 (including the singular ones), can have at most an exponential growth as $|x| \to 0$. The natural question is: are there any singular solutions of these problems with exponential growth as $|x| \to 0$ or do all such solutions have only polynomial growth?

∘ To find some appropriate conditions for the function f under which the Protter problem has only regular, bounded or even classical solutions.

2. To study the Protter problems for degenerate hyperbolic equations. Up to now it is only known that some singular solutions exist [21]:

∘ We do not know what is the exact behavior of the singular solution even when the right-hand side function f is a finite sum like (3.1). Can one prove some a priori estimates for *generalized solutions*?

∘ Is it possible to find some orthogonality conditions for the function f, as here, under which only bounded solutions exist?

3. What happens with the ill-posedness of the Protter problems in a more general domain when the maximal symmetry is lost, if the cone Σ_2 is replaced by another characteristic one, but with the vertex away from the origin (as in [25])?

4. Why there appears a singularity for such smooth right-hand side even for the wave equation? Can we numerically model this phenomenon?

Remark 3.8. Define the operator $T : u_f \mapsto f \in C^1(\overline{\Omega})$, where u_f is the unique classical solution to Protter problem P1 for the right-hand side function f. According to Lemma 2.1 $\dim \operatorname{coker} T = \infty$. This means that T is not Fredholm operator in $C^1(\overline{\Omega})$ for example, but one could expect it to be semi-Fredholm there. Actually, in the present article we build some suitable infinite-dimensional subspaces $K_n \subset C^1(\overline{\Omega})$ $(n \in \mathbb{N})$ of functions, such that T is continuously invertible on K_n (see Theorem 3.2, with different notations) and K_n approximate $\{\operatorname{span}\{V^n_{k,m}\}\}^\perp$ (see Lemma 2.1) in some appropriate sense. However, it is not clear how to find $R(T)$. The last question is connected to the Open Problem 1.

4. Auxiliary results

Some results for solving integral equations and for finding the connection between the functions $H^n_i(r,t) := H^n_i(x,t)$, with $r = |x|$, defined in (2.4) and the Legendre polynomials P_n will be formulated.

Lemma 4.1. [24] *Suppose that the function $F \in C^1(0,1/2]$ satisfies $F(1/2) = 0$. Then all solutions $\lambda \in C^1(0,1/2]$ of the integral equation*

$$\int_\xi^{\frac{1}{2}} \lambda'(\xi_1) P_n\left(\frac{\xi}{\xi_1}\right) d\xi_1 = -F(\xi)$$

are given by

$$\lambda(\xi) = \lambda\left(\frac{1}{2}\right) + F(\xi) + \int_\xi^{\frac{1}{2}} P'_n\left(\frac{\xi_1}{\xi}\right) \frac{F(\xi_1)}{\xi_1} d\xi_1. \tag{4.1}$$

The proof is derived in [24], where some formulas from [26] have been used.

Lemma 4.2. [24] *Define the functions*

$$h_k(\xi,\eta) = \int_\eta^\xi s^k P_n\left(\frac{\xi\eta + s^2}{s(\xi+\eta)}\right) ds. \tag{4.2}$$

Then the equality

$$r^{-1} h_{n-2i-2}\left(\frac{r+t}{2}, \frac{r-t}{2}\right) = c^n_i H^n_i(r,t) \tag{4.3}$$

holds for $i = 0, \dots, [\frac{n-1}{2}]$ with some nonzero constants c^n_i.

Recall that

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n = \sum_{k=0}^{[\frac{n}{2}]} a_{2k} x^{n-2k}, \quad a_{2k} \neq 0, \tag{4.4}$$

is the Legendre polynomial of order n.

Using spherical coordinates, Protter problems can suitably be treated. Let (r,θ,φ) be the spherical coordinates in $\mathbb{R}^3$: $0 \leqslant \theta < \pi$, $0 \leqslant \varphi < 2\pi$ and $r > 0$ with

$$x_1 = r\sin\theta\cos\varphi, \qquad x_2 = r\sin\theta\sin\varphi, \qquad x_3 = r\cos\theta. \tag{4.5}$$

Then the (3 + 1)-D problem P1 can be transformed to a more easier one. Written in the new coordinates, the wave equation becomes

$$u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r^2 \sin^2\theta} u_{\varphi\varphi} - u_{tt} + \frac{2}{r} u_r + \frac{\cos\theta}{r^2 \sin\theta} u_\theta = f. \tag{4.6}$$

The two-dimensional spherical functions, expressed in terms of θ and φ in the traditional definition (see [13]), are $Y_n^m(\theta, \varphi) := Y_n^m(x_1, x_2, x_3)$, $x \in S^2$, $n = 0, 1, 2, \ldots$, $m = 1, \ldots, 2n + 1$ (see (2.2) and (2.3)), and satisfy the differential equation

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} Y_n^m \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} Y_n^m + n(n+1) Y_n^m = 0. \tag{4.7}$$

In the special case when the right-hand side function f of the wave equation (4.6) has the form

$$f(r, \theta, \varphi, t) = f_n^m(r, t) Y_n^m(\theta, \varphi),$$

according to Theorem 3.1 we may look for a solution of the same form,

$$u(r, \theta, \varphi, t) = u_n^m(r, t) Y_n^m(\theta, \varphi).$$

Then the problem P1 is reduced to the two-dimensional equation

$$u_{rr} + \frac{2}{r} u_r - u_{tt} - \frac{n(n+1)}{r^2} u = f \tag{4.8}$$

in the domain $D = \{(r, t)\colon 0 < t < 1/2,\ t < r < 1 - t\}$, bounded by

$$S_0 = \{(r, t)\colon t = 0,\ 0 < r < 1\},$$
$$S_1 = \{(r, t)\colon 0 < t < 1/2,\ r = 1 - t\}, \qquad S_2 = \{(r, t)\colon 0 < t < 1/2,\ r = t\}.$$

Problem P1.1. Find a solution of Eq. (4.8) in the domain D which satisfies the boundary conditions

$$\text{P1.1:} \quad u|_{S_0} = 0, \quad u|_{S_1} = 0. \tag{4.9}$$

Substituting $v = ru(r, t)$, $g = rf(r, t)$ and $\xi = (r + t)/2$; $\eta = (r - t)/2$, implies the equation

$$v_{\xi\eta} - \frac{n(n+1)}{(\xi + \eta)^2} v = g \tag{4.10}$$

in the domain $D_1 = \{(\xi, \eta) \in \mathbb{R}^2\colon 0 < \eta < \xi < 1/2\}$ with boundary conditions

$$\text{P1.2:} \quad v(\eta, \eta) = 0, \quad v(1/2, \eta) = 0, \quad \eta \in (0, 1/2].$$

Lemma 4.3. *The solution* $v(\xi, \eta)$ *of problem* P1.2 *is given by*

$$v(\xi, \eta) = \tau(\xi) + \int_\xi^{\frac{1}{2}} \tau(\xi_1) \frac{\partial}{\partial\xi_1} P_n\left(\frac{(\xi - \eta)\xi_1 + 2\xi\eta}{\xi_1(\xi + \eta)} \right) d\xi_1$$
$$- \int_\xi^{\frac{1}{2}} \left(\int_0^\eta P_n\left(\frac{(\xi - \eta)(\xi_1 - \eta_1) + 2\xi_1\eta_1 + 2\xi\eta}{(\xi_1 + \eta_1)(\xi + \eta)} \right) g(\xi_1, \eta_1)\, d\eta_1 \right) d\xi_1, \tag{4.11}$$

where

$$\tau(\xi) = G(\xi) + \int_{\xi}^{\frac{1}{2}} P_n'\left(\frac{\xi_1}{\xi}\right)\frac{G(\xi_1)}{\xi_1}\,d\xi_1 \tag{4.12}$$

and

$$G(\xi) = \int_{\xi}^{\frac{1}{2}} \left(\int_0^{\xi} P_n\left(\frac{\xi_1\eta_1 + \xi^2}{(\xi_1+\eta_1)\xi}\right) g(\xi_1,\eta_1)\,d\eta_1\right) d\xi_1. \tag{4.13}$$

Proof. Problem P1.2 can be solved explicitly. For Eq. (4.10) the function

$$R(\xi_1,\eta_1;\xi,\eta) = P_n\left(\frac{(\xi-\eta)(\xi_1-\eta_1) + 2\xi_1\eta_1 + 2\xi\eta}{(\xi_1+\eta_1)(\xi+\eta)}\right) \tag{4.14}$$

is the Riemann function (see Copson [6]). Following Aldashev [1] let the function $v(\xi,\eta)$ be constructed as a solution of Goursat's problem for Eq. (4.10) in the triangle D_1 with boundary values $v(1/2,\eta) = 0$ and $v(\xi,0) = \tau(\xi)$, i.e.,

$$v(\xi,\eta) = \tau(\xi) + \int_{\xi}^{\frac{1}{2}} \tau(\xi_1)\frac{\partial}{\partial\xi_1}R(\xi_1,0;\xi,\eta)\,d\xi_1 - \int_{\xi}^{\frac{1}{2}} \left(\int_0^{\eta} R(\xi_1,\eta_1;\xi,\eta)g(\xi_1,\eta_1)\,d\eta_1\right) d\xi_1.$$

The unknown function $\tau \in C^1(0,1/2]$ is found by using the boundary condition $v(\xi,\xi) = 0$, $\xi \in [0,1/2]$. For this reason, define the function $G(\xi)$ by (4.13). Then the unknown function $\tau(\xi)$ will satisfy the equation

$$\int_{\xi}^{\frac{1}{2}} \tau'(\xi_1)P_n\left(\frac{\xi}{\xi_1}\right) d\xi_1 = -G(\xi), \quad \xi \in (0,1/2]. \tag{4.15}$$

Using Lemma 4.1 and the fact that $\tau(1/2) = v(1/2,0) = 0$, this integral equation can be solved and one comes to expression (4.12) for the function $\tau(\xi)$. Thus, formulas (4.11), (4.13) and (4.12) give the generalized solution of P1.2. □

Remark 4.4. In the case $n = 0$ Eq. (4.10) is simply $v_{\xi\eta} = g$ and the generalized solution of P1.2 is obviously in conformity with the above formulas, since $P_0(x) \equiv 1$.

5. The asymptotic expansion in two-dimensional cases

We will discus in this section Problem P1 only for right-hand side functions f, which have the special form:

$$f(x,t) = f_n^m(r,t)Y_n^m(\theta,\varphi),$$

where $n, m \in \mathbb{N} \cup \{0\}$ and $m \leqslant 2n + 1$. Then in view of Theorem 3.1, the unique *generalized solution* u of problem P1 has the same form:

$$u(x,t) = u_n^m(r,t)Y_n^m(\theta,\varphi)$$

and the functions u_n^m and f_n^m satisfy Eq. (4.8) and the boundary conditions P1.1. After the change of the variables $\xi = (r+t)/2$; $\eta = (r-t)/2$ and the substitutions

$$v(\xi,\eta) = (\xi+\eta)u_n^m(\xi+\eta,\xi-\eta), \qquad g(\xi,\eta) = (\xi+\eta)f_n^m(\xi+\eta,\xi-\eta),$$

according to Lemma 4.3, the solution is determined by formulae (4.11), (4.13) and (4.12). Assume that the right-hand side function f in Eq. (4.8) is such that $rf(r,t) \in C^1(\overline{D})$ and thus $g(\xi,\eta) = rf(r,t)|_{r=\xi+\eta, t=\xi-\eta} \in C^1(\overline{D}_1)$.

In order to study the exact behavior of the function $v(\xi,\eta)$ and the solution $u = r^{-1}v$ of P1.1, let us now examine the behavior of $\tau(\xi)$ given by (4.12). Obviously, the integral in the right-hand side of (4.12) blows up when $\xi \to 0$, if no other conditions on $G(\xi)$ are imposed. One could keep it under control by imposing some orthogonality conditions on the function G. In the case when these orthogonality conditions are absent, an asymptotic expansion at the point of singularity $(0,0)$ will be found. To do this one needs a more detailed study of the function $G(\xi)$ and that of $\tau(\xi)$ around the point $\xi = 0$.

Lemma 5.1. *Let $g \in C^1(\overline{D}_1)$ and $n \in \mathbb{N}$. Then the function $G(\xi)$, given by formula* (4.13) *belongs to $C^2(0,1/2] \cap C^1[0,1/2]$ and $G(\xi) = d_n\xi + H(\xi)$, where $|d_n| \leqslant C\|g\|_{C^1(D_1)}$ and $|H(\xi)| \leqslant C\xi^{1+\varepsilon}\|g\|_{C^1(D_1)}$, $\varepsilon > 0$, with a constant C independent on f. In addition, the constant d_n is zero for all even numbers n.*

Proof. According to relation (4.13) between the functions $G(\xi)$ and $g = rf(r,t)$, the smoothness of the function $G(\xi)$ away from $\xi = 0$ is obvious. Changing in (4.13) the variables $\eta_1 = t\xi$ one has

$$G(\xi) = \xi \int_{\xi}^{\frac{1}{2}} \left(\int_0^1 P_n\left(\frac{t\xi_1 + \xi}{\xi_1 + t\xi}\right) g(\xi_1, t\xi)\, dt \right) d\xi_1. \tag{5.1}$$

Then, for the function

$$G_1(\xi) := \int_{\xi}^{\frac{1}{2}} \left(\int_0^1 P_n\left(\frac{t\xi_1 + \xi}{\xi_1 + t\xi}\right) dt \right) g(\xi_1, 0)\, d\xi_1 \tag{5.2}$$

the estimate

$$\left|G(\xi) - \xi G_1(\xi)\right| \leqslant C\xi^2\|g\|_{C^1(D_1)},$$

holds obviously and the constant C is independent on f. To study the behavior of the function $G_1(\xi)$ at $\xi = 0$ let compute for $0 < \xi < \xi_1 \leqslant 1/2$ the integral

$$I_n(\xi,\xi_1) := \int_0^1 P_n\left(\frac{t\xi_1 + \xi}{\xi_1 + t\xi}\right) dt$$

$$= \int_0^1 P_n(t)\,dt + \sum_{k=0}^{[n/2]} a_{2k} \int_0^1 \frac{\xi(1-t^2)}{\xi_1 + t\xi} \sum_{m=0}^{n-2k-1} \left(\frac{t\xi_1 + \xi}{\xi_1 + t\xi}\right)^m t^{n-2k-1-m}\,dt$$

$$= c_n + H^1(\xi, \xi_1),$$

where $c_n = \int_0^1 P_n(t)\,dt$ and $|H^1(\xi, \xi_1)| \leqslant C(\xi/\xi_1)$. Actually, $(t\xi_1 + \xi) \leqslant (\xi_1 + t\xi)$ and then all the terms in the second sum are bounded. Thus

$$G_1(\xi) = d_n + H^2(\xi), \qquad d_n = \int_0^1 P_n(t)\,dt \int_0^{\frac{1}{2}} g(\xi_1, 0)\,d\xi_1,$$

$$\left|H^2(\xi)\right| = \left| \int_\xi^{\frac{1}{2}} H^1(\xi, \xi_1) g(\xi_1, 0)\,d\xi_1 - c_n \int_0^\xi g(\xi_1, 0)\,d\xi_1 \right| \leqslant C\xi^\varepsilon \|g\|_{C^1(D_1)}, \quad \varepsilon > 0.$$

To complete the proof of Lemma 5.1, it is enough to see that for $n \geqslant 1$

$$c_n = \frac{1}{2^n n!} \int_0^1 \frac{d^n}{dt^n} \left\{ \left(t^2 - 1\right)^n \right\} dt = \frac{1}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k+1} \frac{d^{n-1}}{dt^{n-1}} t^{2k} \bigg|_{t=0}.$$

Each term in the last sum is zero, except the only one for which $2k = n - 1$, if n is an odd number, while $c_{2m} = 0$ for $n = 2m$. □

Denote by β_k^n the constants

$$\beta_k^n := \int_0^{\frac{1}{2}} \left(\int_t^{1-t} H_k^n(r, t) f(r, t) r^2\,dr \right) dt \quad \text{for } k = 0, \dots, \left[\frac{n-1}{2}\right]. \tag{5.3}$$

Theorem 5.2. *Let $rf(r,t) \in C^1(\overline{D})$. Then the generalized solution $u(r,t)$ of problem* P1.1 *belongs to $C^2(\overline{D} \setminus (0,0))$ and has the following asymptotic expansion at* $(0,0)$

$$u(r,t) = \sum_{k=0}^{[\frac{n-1}{2}]} r^{-1}(r+t)^{-(n-2k-1)} \beta_k^n F_k^n(r,t) + F^n(r,t), \tag{5.4}$$

where $F_k^n, F^n \in C^2(\overline{D} \setminus (0,0))$ and $|F_k^n(r,t)| \leqslant C$, $|F^n(r,t)| \leqslant C\|rf\|_{C^1(D)}$ with functions F_k^n and a constant C independent on f, and $F_k^n(t,t) \equiv \text{const} \neq 0$.

Proof. In this case, Theorem 3.1 gives the unique generalized solution $u(r,t)$. Let us begin the study of the behavior of the function $v(\xi, \eta)$, given by the representation (4.11) from Lemma 4.3. According to the smoothness of the function $G(\xi)$ in (4.11), the smoothness of $v(\xi, \eta)$ away from the point $(0,0)$ (and than the smoothness of $u(r,t)$ away from $r = 0$) is obvious. Let find now the expansion of $v(\xi, \eta)$ at $(0,0)$.

In the case $n=0$ the assertion of the theorem actually is that $u(r,t)$ is a bounded function, which follows directly in view of

$$\left|u(r,t)\right| = \left| r^{-1} \int\limits_{(r-t)/2}^{(r+t)/2} \left(\int\limits_{(r+t)/2}^{1/2} g(\xi_1,\eta_1)\,d\xi_1 \right) d\eta_1 \right| \leqslant \|g\|_{C(D_1)}.$$

Now, let $n \geqslant 1$. Using Lemma 5.1 it follows that $G(\xi) = d_n\xi + H(\xi)$ with the estimates $|d_n| \leqslant C\|g\|_{C^1(D_1)}$ and $|H(\xi)| \leqslant C\xi^{1+\varepsilon}\|g\|_{C^1(D_1)}$, $\varepsilon > 0$. In addition, for each even number n we have $d_n = 0$ and therefore $d_n P_n(0) = 0$ for all natural numbers n (obviously, $P_n(0) = 0$ when n is an odd number). On the other hand, Lemma 4.2 gives the equality

$$\begin{aligned}
&\int\limits_0^{\frac12} \xi^{n-2i-2} G(\xi)\,d\xi \\
&\quad = \int\limits_0^{\frac12} \left\{ \int\limits_0^{\xi_1} \left(\int\limits_{\eta_1}^{\xi_1} \xi^{n-2i-2} P_n\left(\frac{\xi_1\eta_1 + \xi^2}{\xi(\xi_1+\eta_1)} \right) d\xi \right) g(\xi_1,\eta_1)\,d\eta_1 \right\} d\xi_1 \\
&\quad = \int\limits_0^{\frac12} \left(\int\limits_0^{\xi} h^n_{n-2i-2}(\xi,\eta) g(\xi,\eta)\,d\eta \right) d\xi = c^n_i \int\limits_0^{\frac12} \left(\int\limits_t^{1-t} H^n_i(r,t) f(r,t) r^2\,dr \right) dt,
\end{aligned} \tag{5.5}$$

where $c^n_i \neq 0$. Therefore, the notations (5.3) lead to

$$\int\limits_0^{\frac12} \xi^{n-2i-2} G(\xi)\,d\xi = c^n_i \beta^n_i \quad \text{for } i = 0, \ldots, \left[\frac{n-1}{2}\right]. \tag{5.6}$$

Recalling the representation (4.12) of $\tau(\xi)$, first we find the derivative of the Legendre polynomial P_n from (4.4)

$$P'_n(x) = \sum_{k=0}^{[\frac{n-1}{2}]} (n-2k) a_{2k} x^{n-2k-1}, \quad a_{2k} \neq 0. \tag{5.7}$$

Therefore

$$\begin{aligned}
\int\limits_{\xi}^{\frac12} P'_n\left(\frac{\xi_1}{\xi}\right) \frac{G(\xi_1)}{\xi_1}\,d\xi_1 &= \sum_{k=0}^{[\frac{n-1}{2}]} a_{2k}(n-2k)\xi^{-n+2k+1} \int\limits_{\xi}^{\frac12} \xi_1^{n-2k-2} G(\xi_1)\,d\xi_1 \\
&= \sum_{k=0}^{[\frac{n-1}{2}]} a_{2k}(n-2k) c^n_k \beta^n_k \xi^{-n+2k+1} - \int\limits_0^{\xi} P'_n\left(\frac{\xi_1}{\xi}\right) \frac{G(\xi_1)}{\xi_1}\,d\xi_1.
\end{aligned} \tag{5.8}$$

Then it follows the estimate

$$\left| \int\limits_0^{\xi} P'_n\left(\frac{\xi_1}{\xi}\right) \frac{H(\xi_1)}{\xi_1}\,d\xi_1 \right| = \left| \sum_{k=0}^{[\frac{n-1}{2}]} a_{2k}(n-2k)\xi^{-n+2k+1} \int\limits_0^{\xi} \xi_1^{n-2k-2} H(\xi_1)\,d\xi_1 \right|$$

$$\leqslant C_1\|g\|_{C^1(D_1)}\sum_{k=0}^{[\frac{n-1}{2}]}\xi^{-n+2k+1}\int_0^{\xi}\xi_1^{n-2k-1+\varepsilon}\,d\xi_1$$
$$\leqslant C\xi^{1+\varepsilon}\big\|rf(r,t)\big\|_{C^1(D)}.$$

From (4.12) and (5.8) follows the asymptotic expansion of $\tau(\xi)$ at $\xi=0$:

$$\begin{aligned}\tau(\xi)&=G(\xi)+\int_{\xi}^{\frac12}P_n'\left(\frac{\xi_1}{\xi}\right)\frac{G(\xi_1)}{\xi_1}\,d\xi_1\\&=d_n\xi+H(\xi)+\sum_{k=0}^{[\frac{n-1}{2}]}a_{2k}(n-2k)c_k^n\beta_k^n\xi^{-n+2k+1}-\int_0^{\xi}P_n'\left(\frac{\xi_1}{\xi}\right)\frac{d_n\xi_1+H(\xi_1)}{\xi_1}\,d\xi_1\\&=\sum_{k=0}^{[\frac{n-1}{2}]}a_{2k}(n-2k)c_k^n\beta_k^n\xi^{-n+2k+1}+d_n\xi\big\{1-P_n(1)+P_n(0)\big\}+H_1(\xi)\\&=\sum_{k=0}^{[\frac{n-1}{2}]}a_{2k}(n-2k)c_k^n\beta_k^n\xi^{-n+2k+1}+H_1(\xi),\end{aligned}\tag{5.9}$$

where $|H_1(\xi)|\leqslant C\xi^{1+\varepsilon}\|g\|_{C^1(D_1)}$, using that $P_n(1)=1$ and $d_nP_n(0)=0$ for each number n. Substituting $\tau(\xi)$ in the formula (4.11) from Lemma 4.3, one gets the expansion

$$\begin{aligned}v(\xi,\eta)&=\sum_{k=0}^{[\frac{n-1}{2}]}a_{2k}(n-2k)c_k^n\beta_k^n\xi^{-n+2k+1}+H_1(\xi)\\&\quad+\int_{\xi}^{\frac12}\tau(\xi_1)\frac{\partial}{\partial\xi_1}P_n\left(\frac{(\xi-\eta)\xi_1+2\xi\eta}{\xi_1(\xi+\eta)}\right)d\xi_1+F_1(\xi,\eta),\end{aligned}\tag{5.10}$$

where the function F_1 is smooth (see (4.11)) and $|F_1(\xi,\eta)|\leqslant C\xi\|g\|_{C^1(D_1)}$. Then

$$\begin{aligned}&\int_{\xi}^{\frac12}\tau(\xi_1)\frac{\partial}{\partial\xi_1}P_n\left(\frac{(\xi-\eta)\xi_1+2\xi\eta}{\xi_1(\xi+\eta)}\right)d\xi_1\\&\quad=\sum_{k=0}^{[\frac{n-1}{2}]}a_{2k}(n-2k)c_k^n\beta_k^n\int_{\xi}^{\frac12}\xi_1^{-n+2k+1}\frac{\partial}{\partial\xi_1}P_n\left(\frac{(\xi-\eta)\xi_1+2\xi\eta}{\xi_1(\xi+\eta)}\right)d\xi_1\\&\qquad+\int_{\xi}^{\frac12}H_1(\xi_1)\frac{\partial}{\partial\xi_1}P_n\left(\frac{(\xi-\eta)\xi_1+2\xi\eta}{\xi_1(\xi+\eta)}\right)d\xi_1.\end{aligned}$$

Obviously,

$$\int_{\xi}^{\frac{1}{2}} \xi_1^{1+\varepsilon} \left| \frac{\partial}{\partial \xi_1} P_n\left(\frac{(\xi-\eta)\xi_1 + 2\xi\eta}{\xi_1(\xi+\eta)} \right) \right| d\xi_1 \leqslant \frac{C\xi\eta}{\xi+\eta} \int_{\xi}^{\frac{1}{2}} \xi_1^{\varepsilon-1}\, d\xi_1 \leqslant C'\xi. \tag{5.11}$$

Thus

$$\begin{aligned} v(\xi,\eta) = {} & \sum_{k=0}^{[\frac{n-1}{2}]} a_{2k}(n-2k) c_k^n \beta_k^n \xi^{-n+2k+1} \\ & \times \left\{ 1 + \xi^{n-2k-1} \int_{\xi}^{\frac{1}{2}} \xi_1^{-n+2k+1} \frac{\partial}{\partial \xi_1} P_n\left(\frac{(\xi-\eta)\xi_1 + 2\xi\eta}{\xi_1(\xi+\eta)} \right) d\xi_1 \right\} + H_2(\xi,\eta), \end{aligned}$$

where $|H_2(\xi,\eta)| \leqslant C\xi \|g\|_{C^1(D_1)}$. To complete the proof, it needs only to estimate the expression

$$\left| \int_{\xi}^{\frac{1}{2}} \xi_1^{-n+2k+1} \frac{\partial}{\partial \xi_1} P_n\left(\frac{(\xi-\eta)\xi_1 + 2\xi\eta}{\xi_1(\xi+\eta)} \right) d\xi_1 \right| \leqslant C\xi^{-n+2k+1}. \tag{5.12}$$

Finally, let us return to the generalized solution $u(r,t)$ of problem P1.1 and to the coordinates r and t:

$$u(r,t) = r^{-1} v\left(\frac{r+t}{2}, \frac{r-t}{2} \right) = r^{-1} \sum_{k=0}^{[\frac{n-1}{2}]} \beta_k^n (r+t)^{-n+2k+1} F_k^n(r,t) + F^n(r,t),$$

where the independent on f functions $F_k^n(r,t)$ are given by

$$F_k^n(r,t) := 2^{n-2k-1}(n-2k) a_{2k} c_k^n G_k^n\left(\frac{r+t}{2}, \frac{r-t}{2} \right), \tag{5.13}$$

with

$$G_k^n(\xi,\eta) := 1 + \xi^{n-2k-1} \int_{\xi}^{\frac{1}{2}} \xi_1^{-n+2k+1} \frac{\partial}{\partial \xi_1} P_n\left(\frac{(\xi-\eta)\xi_1 + 2\xi\eta}{\xi_1(\xi+\eta)} \right) d\xi_1. \tag{5.14}$$

Obviously, $F_k^n(r,t)$ are bounded (see (5.12)). Therefore

$$F_k^n(t,t) = 2^{n-2k-1}(n-2k) a_{2k} c_k^n \neq 0. \qquad \square$$

Remark 5.3. In the case, when all orthogonality conditions

$$\beta_i^n = \int_0^{\frac{1}{2}} \left(\int_t^{1-t} H_i^n(r,t) f(r,t) r^2\, dr \right) dt = 0 \quad \text{for } i = 0, \ldots, \left[\frac{n-1}{2} \right] \tag{5.15}$$

are fulfilled, expansion (5.4) directly gives $|u(r,t)| \leqslant C\|f\|_{C^1(D)}$.

6. Proofs of the main results

Let first mention that we omit the proof of Theorem 3.1, which is similar to the proof of Theorems 3.2 and 3.3 from [24] and Theorem 2.3 from [22].

Now, using the last two results from the previous section it is not hard to prove Theorem 3.2.

Proof of Theorem 3.2. When the right-hand side function $f(x,t)$ has the form (3.1), due to Theorem 3.1, the unique generalized solution $u(x_1,x_2,x_3,t)$ is given by (3.2). Moreover, the functions $u_n^m(r,t)$ are solutions of problem P1.1 with right-hand side $f_n^m \in C^1(\overline{D})$, where the Fourier coefficients f_n^m are given by

$$f_n^m(r,t) := \int_0^{\pi}\left(\int_0^{2\pi} f(r,\theta,\varphi,t) Y_n^m(\theta,\varphi)\,d\varphi\right)\sin\theta\,d\theta. \tag{6.1}$$

Obviously,

$$\max_{\overline{D}}\left|f_n^m(r,t)\right| \leqslant 4\max_{\overline{\Omega}}\left|f(x,t)\right|. \tag{6.2}$$

Using the definition of functions $V_{k,m}^n$ from Lemma 2.1 and (6.1), the identity

$$\int_0^{\frac{1}{2}}\left(\int_t^{1-t} H_k^n(r,t) f_n^m(r,t) r^2\,dr\right)dt = \int_{\Omega} V_{k,m}^n(x,t) f(x,t)\,dx\,dt,$$

implies the connection $\beta_k^n = \beta_{k,m}^n$ between the constants from (3.3) and (5.3). Therefore, one can apply Theorem 5.2 for the functions $u_n^m(r,t)$ and $f_n^m(r,t)$ with these values of β_k^n . Using all this and (6.2), one finds the expansion

$$u_n^m(r,t) = \sum_{k=0}^{[\frac{n-1}{2}]} r^{-1}(r+t)^{-(n-2k-1)}\beta_{k,m}^n F_k^{n,m}(r,t) + F^{n,m}(r,t),$$

where $|F^{n,m}(r,t)| \leqslant C\|f\|_{C^1(\Omega)}$, $|F_k^{n,m}(r,t)| \leqslant C$ and $F_k^{n,m}(t,t) = \text{const} \neq 0$. Summing up over n and m one gets the desired expansion.

To prove property (iii), notice that for a fixed direction $(\alpha, 1) := (\alpha_1,\alpha_2,\alpha_3,1)$ with $\alpha_1 = \sin\theta_0\cos\varphi_0$, $\alpha_2 = \sin\theta_0\sin\varphi_0$ and, $\alpha_3 = \cos\theta_0$, for the functions $F_{k,m}^n$ from (3.4) we have

$$F_{k,m}^n(\alpha t,t) := 2^{(2k-n+2)/2} F_k^{n,m}(t,t) Y_n^m(\varphi_0,\theta_0). \tag{6.3}$$

Therefore

$$\lim_{t\to+0} F_p(\alpha t,t) = \sum_{k=0}^{[(l-p)/2]}\ \sum_{m=1}^{2p+4k+1} C_{p,k,m}\beta_{k,m}^{p+2k} Y_{p+2k}^m(\varphi_0,\theta_0), \tag{6.4}$$

with constants $C_{p,k,m} \neq 0$. Thus the property (iii) follows from the fact that the spherical functions Y_n^m are linearly independent. □

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