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Direct limits of infinite-dimensional Lie groups compared to direct limits in related categories

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Abstract

Let G be a Lie group which is the union of an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of Lie groups (all of which may be infinite-dimensional). We study the question when $G = \varinjlim G_n$ in the category of Lie groups, topological groups, smooth manifolds, respectively, topological spaces. Full answers are obtained for G the group $\mathrm{Diff}_c(M)$ of compactly supported C^∞-diffeomorphisms of a σ-compact smooth manifold M; and for test function groups $C^\infty_c(M, H)$ of compactly supported smooth maps with values in a finite-dimensional Lie group H. We also discuss the cases where G is a direct limit of unit groups of Banach algebras, a Lie group of germs of Lie group-valued analytic maps, or a weak direct product of Lie groups.

Keywords: Infinite-dimensional Lie group; Direct limit group; Direct limit; Inductive limit; Test function group; Diffeomorphism group; Current group; Compact support; Group of germs; Silva space; k_ω-Space; Differentiability; Smoothness; Continuity; Non-linear map

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Introduction

It frequently happens that an infinite-dimensional Lie group G of interest is a union $G = \bigcup_{n\in\mathbb{N}} G_n$ of an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of Lie groups which are easier to handle. Typically, each G_n is finite-dimensional, a Banach–Lie group, or at least a Fréchet–Lie group, while G is modelled on a more complicated locally convex space (e.g., an LB-space, LF-space, or a Silva space). Then good tools of infinite-dimensional calculus are available to establish differentiability properties of mappings on the groups G_n, while properties of mappings (or homomorphisms) on G are more elusive and can be difficult to access.

In this article, we consider mappings $f : G \to X$, where X is a topological space or smooth manifold; and also homomorphisms $f : G \to H$, where H is a topological group or Lie group. We analyze the question when continuity (or smoothness) of $f|_{G_n}$ for each n implies continuity (or smoothness) of f. To rephrase this problem in category-theoretical terms, let $\mathbb{TOP}$ be the category of topological spaces and continuous maps, $\mathbb{TG}$ the category of topological groups and continuous homomorphisms, $\mathbb{MFD}_\infty$ the category of smooth manifolds (modelled on real locally convex spaces) and smooth maps, and $\mathbb{LIE}$ the category of Lie groups (modelled on real locally convex spaces) and smooth homomorphisms. We consider a Lie group G which is a union $G = \bigcup_{n\in\mathbb{N}} G_n$ for an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of Lie groups, such that all inclusion maps $i_{n,m} : G_m \to G_n$ (for $m \leqslant n$) and $i_n : G_n \to G$ are smooth homomorphisms. Thus $\mathcal{S} := ((G_n)_{n\in\mathbb{N}}, (i_{n,m})_{n\geqslant m})$ is a direct system in $\mathbb{LIE}$, and we can consider $\mathcal{S}$ also as a direct system in $\mathbb{TOP}$, $\mathbb{TG}$ and $\mathbb{MFD}_\infty$, forgetting extraneous structure. We are asking whether $G = \varinjlim G_n$ (more precisely, $(G, (i_n)_{n\in\mathbb{N}}) = \varinjlim \mathcal{S}$) holds in $\mathbb{TOP}$, $\mathbb{MFD}_\infty$, $\mathbb{TG}$, respectively, $\mathbb{LIE}$.

If each G_n is a finite-dimensional Lie group and $G = \varinjlim G_n$ the direct limit Lie group constructed in [21] (see also [18,44,45] for special cases), then $G = \varinjlim G_n$ holds in each of the preceding categories [21, Theorem 4.3]. The goal of this article is to shed light on the case where the Lie groups G_n are infinite-dimensional. Then the situation changes drastically, and some of the direct limit properties can get lost.

We are interested both in general techniques for the investigation of direct limit properties, and a detailed analysis of the properties of concrete groups. In the following, we summarize some of the main results.

Tools for the identification of direct limits

Direct limit properties of $G = \bigcup_{n\in\mathbb{N}} G_n$ are accessible provided that G admits a "direct limit chart" composed of charts of the groups G_n (see Definition 2.1 for details). If this is the case, then the following information becomes available:

- if $G = \varinjlim G_n$ as a topological group, then $G = \varinjlim G_n$ as a Lie group (Theorem 2.6);
- $G = \varinjlim G$ as a topological space if and only if $L(G) = \varinjlim L(G_n)$ as a topological space (Theorem 3.3(a));
- if $L(G)$ is smoothly regular, then $G = \varinjlim G_n$ as a smooth manifold if and only if $L(G) = \varinjlim L(G_n)$ as a smooth manifold (Theorem 3.3(b)).

The existence of a direct limit chart is a very weak requirement, which is satisfied by all relevant examples known to the author. Another result (Proposition 11.8) provides a criterion (also satisfied by all relevant examples inspected so far) which ensures that $G = \varinjlim G_n$ as a topological group. This criterion is closely related to investigations in [53]. In this paper, a condition was formulated which facilitates a quite explicit ("bamboo-shoot") description of the group topology on a direct limit of topological groups.

Direct limit properties of the prime examples

We now summarize our results concerning concrete examples of direct limit groups.

1. Groups of compactly supported functions or diffeomorphisms. Let $\mathrm{Diff}_c(M)$ be the group of all C^∞-diffeomorphisms γ of a σ-compact, non-compact finite-dimensional smooth manifold M which are compactly supported (i.e., $\gamma(x) = x$ for all x outside some compact subset of M). Then $\mathrm{Diff}_c(M)$ is a Lie group modelled on the LF-space $C^\infty_c(M, TM)$ of compactly supported smooth vector fields (see [42] or [17]). It is a union $\mathrm{Diff}_c(M) = \bigcup_K \mathrm{Diff}_K(M)$ of the Lie subgroups $\mathrm{Diff}_K(M)$ of diffeomorphisms supported in K (which are Fréchet–Lie groups), for K ranging through the directed set of compact subsets of M. Given a finite-dimensional Lie group H, we are also interested in the "test function group" $C^\infty_c(M, H)$ of compactly supported smooth H-valued maps, which is a Lie group modelled on the LF-space $C^\infty_c(M, L(H))$ (see [14]). It is a union $C^\infty_c(M, H) = \bigcup_K C^\infty_K(M, H)$ of groups of mappings supported in a given compact set K. Now assume that M and H are non-discrete. Table 1 (compiled from Propositions 5.3, 5.4, 7.3 and 7.6) describes in which categories $C^\infty_c(M, H) = \varinjlim C^\infty_K(M, H)$, respectively, $\mathrm{Diff}_c(M) = \varinjlim \mathrm{Diff}_K(M)$ holds.

It was known before that the direct limit topology does not make $\mathrm{Diff}_c(M)$ a topological group (see [53, Theorem 6.1]), and Yamasaki's Theorem [56, Theorem 4] implies that the direct limit topology does not make $C^\infty_c(M, H)$ a topological group. Hence, the Lie group topologies must

Table 1

Category\Group	$C^\infty_c(M, H)$	$\mathrm{Diff}_c(M)$
Lie groups	yes	yes
Topological groups	yes	yes
Smooth manifolds	no	no
Topological spaces	no	no

be properly coarser than the direct limit topologies (as asserted in the last line of the table). The other results compiled in the table are new.

2. Weak direct products of Lie groups. The weak direct product $\prod^*_{n\in\mathbb{N}} G_n$ of a sequence $(G_n)_{n\in\mathbb{N}}$ of Lie groups is defined as the group of all $(g_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} G_n$ such that $g_n = 1$ for all but finitely many n; it carries a natural Lie group structure [19, Section 7]. We shall see that $\prod^*_{n\in\mathbb{N}} G_n = \varinjlim \prod_{k=1}^n G_k$ as a topological group and as a Lie group (Proposition 4.5). The direct limit properties in the categories of topological spaces or smooth manifolds depend on the sequence $(G_n)_{n\in\mathbb{N}}$ (see Remark 4.6).

3. Direct limits of unit groups of Banach algebras. Consider a sequence $A_1 \subseteq A_2 \subseteq \cdots$ of unital Banach algebras (such that all inclusion maps are continuous homomorphisms of unital algebras) and equip $A := \bigcup_{n\in\mathbb{N}} A_n$ with the locally convex direct limit topology. We show that $A^\times = \varinjlim A_n^\times$ as a topological group and (provided A is Hausdorff) also as a Lie group (Proposition 12.1). Non-unital Banach algebras are discussed as well (Proposition 12.2). In the case where each inclusion map is an isometry, the topological result has been obtained earlier by Edamatsu [10, Theorem 1].

4. Lie groups of germs of analytic mappings. We also discuss the direct limit properties of the groups $\Gamma(K,H)$ of germs of H-valued analytic functions on open neighbourhoods of a non-empty compact set $K \subseteq X$, where H is a (real or complex) Banach–Lie group and X a (real or complex) metrizable locally convex space. Such groups are interesting in this context because they are prototypical examples of direct limits $G = \bigcup_{n\in\mathbb{N}} G_n$ of a direct system $(G_n)_{n\in\mathbb{N}}$ which is not strict (nor $(L(G_n))_{n\in\mathbb{N}}$). For X and H finite-dimensional, $\Gamma(K,H)$ is modelled on a Silva space, whence $\Gamma(K,H)$ has all desired direct limit properties by general results concerning Silva–Lie groups prepared in Section 9 (see Proposition 10.6). For infinite-dimensional X (or H), we can still show that $\Gamma(K,H) = \varinjlim G_n$ as a topological group and as a Lie group (Corollary 13.3).

Finally, we obtain results concerning *locally convex direct limits*. Consider a Hausdorff locally convex space $E = \bigcup_{n\in\mathbb{N}} E_n$ which is the locally convex direct limit $\varinjlim E_n$ of Hausdorff locally convex spaces E_n. Then $E = \varinjlim E_n$ as a topological group (see [35, Proposition 3.1]). Our results imply that $E = \varinjlim E_n$ also as a Lie group (Example 2.8). It is well known that the direct limit topology on E can be properly finer than the locally convex direct limit topology. We provide concrete criteria ensuring that $E \neq \varinjlim E_n$ as a topological space and smooth manifold (Lemma 3.5). These criteria are also useful for the study of general Lie groups because of Theorem 3.3 (already mentioned). Similarly, weak direct products of Lie groups (and topological groups) are useful tools for the study of general direct limits (see Section 11).

Further results

In Section 14, we construct a Lie group structure on a union $G = \bigcup_{n\in\mathbb{N}} G_n$ of infinite-dimensional Lie groups, under suitable hypotheses. A variant of the construction can be used to turn the group of germs of analytic diffeomorphisms around a non-empty compact subset K of $\mathbb{C}^d$ (or $\mathbb{R}^d$) into an analytic Lie group (Section 15). Here $\mathrm{GermDiff}(K,\mathbb{C}^d) = \bigcup_{n\in\mathbb{N}} M_n$, where M_n is the Banach manifold of all germs having a representative γ which is a bounded $\mathbb{C}^d$-valued holomorphic map on U_n with $\gamma|_K = \mathrm{id}_K$, for a given fundamental sequence

$U_1 \supseteq U_2 \supseteq \cdots$ of open neighbourhoods of K. We show that $\mathrm{GermDiff}(K, \mathbb{C}^d) = \varinjlim M_n$ as a topological space, as a complex analytic manifold, and as a smooth manifold (Proposition 15.9).

1. Preliminaries and notation

We fix notation and terminology concerning differential calculus, direct limits and properties of locally convex spaces (C^r-regularity and related concepts).

Calculus in locally convex spaces and Lie groups

We recall some basic definitions of Keller's C^r_c-theory and the theory of analytic mappings (see [13,31,42,43] for the proofs and further details). Throughout the article, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 1.1. Let E and F be Hausdorff locally convex topological $\mathbb{K}$-vector spaces, $U \subseteq E$ be open, $f: U \to F$ be a map, and $r \in \mathbb{N}_0 \cup \{\infty\}$. We say that f is $C^r_{\mathbb{K}}$ (or simply C^r) if it is continuous and, for all $k \in \mathbb{N}$ such that $k \leqslant r$, the iterated directional derivatives $d^k f(x, y_1, \ldots, y_k) := D_{y_1} \ldots D_{y_k} f(x)$ exist for all $x \in U$ and $y_1, \ldots, y_k \in E$, and define a continuous map $d^k f: U \times E^k \to F$. The $C^\infty_{\mathbb{R}}$-maps are also called *smooth*.

Occasionally, we shall also encounter analytic maps.

Definition 1.2. If $\mathbb{K} = \mathbb{C}$, then f as before is called *complex analytic* if it is continuous and given locally by a pointwise convergent series of continuous homogeneous polynomials (see [6, Definition 5.6] for details). If $\mathbb{K} = \mathbb{R}$, we call f *real analytic* if it extends to a complex analytic map between open subsets of the complexifications of E and F.

It is well known that f is $C^\infty_{\mathbb{C}}$ if and only if it is complex analytic (see, e.g., [31]). Since compositions of $C^r_{\mathbb{K}}$-maps (respectively, $\mathbb{K}$-analytic maps) are $C^r_{\mathbb{K}}$ (respectively, $\mathbb{K}$-analytic), $C^r_{\mathbb{K}}$-manifolds and $\mathbb{K}$-analytic Lie groups (modelled on Hausdorff locally convex spaces) can be defined as usual (see [13,31]; cf. [43]). As in [13,31], we shall not presume that the modelling spaces are complete.[1] Unlike [43], we do not require that manifolds are regular topological spaces.

We warn the reader that topological spaces and locally convex spaces are not assumed Hausdorff in this article. However, the topological spaces underlying manifolds and Lie groups are assumed Hausdorff. We remark that switching to the categories of Hausdorff topological groups (or Hausdorff topological spaces) would not affect the validity (or failure) of direct limit properties of Lie groups in such categories (as Lie groups are Hausdorff anyway).

General conventions. The word "Lie group" (without further specification) refers to a smooth Lie group modelled on a real locally convex space. A $C^\infty_{\mathbb{K}}$-Lie group is a smooth real Lie group (if $\mathbb{K} = \mathbb{R}$), respectively, a complex analytic Lie group (if $\mathbb{K} = \mathbb{C}$). A $C^\omega_{\mathbb{K}}$-Lie group means a

[1] Some readers may prefer to work with categories of Lie groups and manifold modelled on complete, sequentially complete, or Mackey complete locally convex spaces. This only causes minor changes of our results.

$\mathbb{K}$-analytic Lie group. Likewise for manifolds. If $(E, \|\cdot\|)$ is a normed space, $x \in E$ and $r > 0$, we set

$$B_r^E(x) := \{y \in E\colon \|y - x\| < r\} \quad \text{and} \quad \overline{B}_r^E(x) := \{y \in E\colon \|y - x\| \leqslant r\}.$$

See [31] for the following useful fact (or [4, Lemma 10.1], if $r \in \mathbb{N}_0 \cup \{\infty\}$).

Lemma 1.3. *Let $f : U \to F$ be as in Definition* 1.1, *and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$. If $f(U) \subseteq F_0$ for a closed vector subspace $F_0 \subseteq F$, then f is $C^r_{\mathbb{K}}$ if and only if its co-restriction $f|^{F_0} : U \to F_0$ is $C^r_{\mathbb{K}}$.*

Conventions and basic facts concerning direct limits

We recall some basic definitions and facts concerning direct limits.

1.4. A *direct system* in a category $\mathbb{A}$ is a pair $\mathcal{S} = ((X_i)_{i \in I}, (\phi_{i,j})_{i \geqslant j})$, where $(I, \leqslant)$ is a directed set, each X_i an object of $\mathbb{A}$, and each $\phi_{i,j} : X_j \to X_i$ a morphism ("bonding map") such that $\phi_{i,i} = \mathrm{id}_{X_i}$ and $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}$ if $i \geqslant j \geqslant k$. A *cone over* $\mathcal{S}$ is a pair $(X, (\phi_i)_{i \in I})$, where $X \in \operatorname{ob} \mathbb{A}$ and $\phi_i : X_i \to X$ is a morphism for $i \in I$ such that $\phi_i \circ \phi_{i,j} = \phi_j$ if $i \geqslant j$. A cone $(X, (\phi_i)_{i \in I})$ is a *direct limit cone* over $\mathcal{S}$ in the category $\mathbb{A}$ if, for every cone $(Y, (\psi_i)_{i \in I})$ over $\mathcal{S}$, there exists a unique morphism $\psi : X \to Y$ such that $\psi \circ \phi_i = \psi_i$ for each i. We then write $(X, (\phi_i)_{i \in I}) = \varinjlim \mathcal{S}$. If the bonding maps and "limit maps" ϕ_i are understood, we simply call X the *direct limit* of $\mathcal{S}$ and write $X = \varinjlim X_i$. If also $\mathcal{T} = ((Y_i)_{i \in I}, (\psi_{i,j})_{i \leqslant j})$ is a direct system over I and $(Y, (\psi_i)_{i \in I})$ a cone over $\mathcal{T}$, we call a family $(\eta_i)_{i \in I}$ of morphisms $\eta_i : X_i \to Y_i$ *compatible* if $\eta_i \circ \phi_{i,j} = \psi_{i,j} \circ \eta_j$ for $i \geqslant j$. Then $(Y, (\psi_i \circ \eta_i)_{i \in I})$ is a cone over $\mathcal{S}$; we write $\varinjlim \eta_i := \eta$ for the morphism $\eta : X \to Y$ such that $\eta \circ \phi_i = \psi_i \circ \eta_i$.

1.5. For all direct systems $\mathcal{S} = ((X_i)_{i \in I}, (\phi_{i,j})_{i \geqslant j})$ encountered in the article, I will contain a co-final subsequence. It therefore suffices to state all results for the case where the directed set is $(\mathbb{N}, \leqslant)$, i.e., for direct sequences.

1.6. *Direct limits of sets, topological spaces, and groups.* For basic facts concerning direct limits of topological spaces and topological groups, the reader is referred to [18,33,35,53] (where also many of the pitfalls and subtleties of the topic are described). In particular, we shall frequently use that the set underlying a direct limit of groups is the corresponding direct limit in the category of sets, and that the direct limit of a direct system of topological spaces in the category of (not necessarily Hausdorff) topological spaces is its direct limit in the category of sets, equipped with the final topology with respect to the limit maps. A direct system $((X_i)_{i \in I}, (\phi_{i,j})_{i \geqslant j})$ of topological spaces is called *strict* if each bonding map $\phi_{i,j} : X_j \to X_i$ is a topological embedding.

The following simple fact will be used:

Lemma 1.7. *Consider a direct sequence $((X_n)_{n \in \mathbb{N}}, (i_{n,m})_{n \geqslant m})$ of topological spaces X_n and continuous maps $i_{n,m} : X_m \to X_n$, with direct limit $(X, (i_n)_{n \in \mathbb{N}})$ in the category of topological spaces. Let $U_n \subseteq X_n$ be open subsets such that $i_{n,m}(U_m) \subseteq U_n$ whenever $m \leqslant n$, and $U := \bigcup_{n \in \mathbb{N}} i_n(U_n)$. Then U is open in X, and $(U, (i_n|_{U_n}^U)_{n \in \mathbb{N}}) = \varinjlim((U_n)_{n \in \mathbb{N}}, (i_{n,m}|_{U_m}^{U_n})_{n \geqslant m})$ in the category of topological spaces.*

Proof. It is clear that $U = \varinjlim U_n$ as a set, and that the inclusion map $\varinjlim U_n \to \varinjlim X_n$ is continuous (being continuous on each U_n). If $V \subseteq \varinjlim U_n$ is open (e.g., $V = U$), then $V_n := (i_n|_{U_n})^{-1}(V)$ is open in U_n and hence in X_n, for each $n \in \mathbb{N}$. If $x \in i_n^{-1}(V)$, there exists $k \geqslant n$ such that $i_n(x) = i_k(y)$ for some $y \in V_k$. Hence, there is $\ell \geqslant k$ such that $i_{\ell,n}(x) = i_{\ell,k}(y) \in V_\ell$. We deduce that $i_n^{-1}(V) = \bigcup_{\ell \geqslant n} i_{\ell,n}^{-1}(V_\ell)$ is open in X_n. Hence V open in X. □

All necessary background concerning direct limits of locally convex spaces can be found in [7,37,52,54].

General conventions. If we write $G = \bigcup_{n\in\mathbb{N}} G_n$ for a topological group (respectively, $C^\infty_{\mathbb{K}}$-Lie group) G, we always presuppose that each G_n is a topological group (respectively, $C^\infty_{\mathbb{K}}$-Lie group), $G_n \subseteq G_{n+1}$ for each $n \in \mathbb{N}$, and that all of the inclusion maps $G_n \to G_{n+1}$ and $G_n \to G$ are continuous (respectively, $C^\infty_{\mathbb{K}}$-) homomorphisms. Analogous conventions apply to topological spaces $X = \bigcup_{n\in\mathbb{N}} X_n$ and manifolds $M = \bigcup_{n\in\mathbb{N}} M_n$.

Smooth regularity and related concepts

Definition 1.8. Given $r \in \mathbb{N}_0 \cup \{\infty\}$, a Hausdorff real locally convex space E is called *C^r-regular*, if for each 0-neighbourhood $U \subseteq E$, there exists a C^r-function $f : E \to \mathbb{R}$ such that $f(0) = 1$ and $f|_{E\setminus U} = 0$. If E is C^∞-regular, we also say that E is *smoothly regular*.

After composing a suitable smooth self-map of $\mathbb{R}$ with f, we may assume that $f|_V = 1$ for some 0-neighbourhood $V \subseteq U$, $f(E) \subseteq [0, 1]$, and $\operatorname{supp}(f) \subseteq U$.

Remark 1.9. Note that every Hausdorff locally convex space is C^0-regular, being a completely regular topological space (cf. [34, Theorem 8.4]). It is easy to see that every Hilbert space is smoothly regular. Furthermore, vector subspaces and (finite or infinite) direct products of C^r-regular locally convex spaces are C^r-regular. This implies that every nuclear locally convex space is smoothly regular, because it can be realized as a vector subspace of a direct product of Hilbert spaces (cf. [52, Section 7.3, Corollary 2], also [50,54]).

To prove that certain locally convex spaces have pathological properties, at some point we shall find it useful to use ideas from Convenient Differential Calculus (see [12,39]). We recall that the final topology on a locally convex space E with respect to the set $C^\infty(\mathbb{R}, E)$ of smooth curves is called the *c^∞-topology*. We write $c^\infty(E)$ for E, equipped with the c^∞-topology; open subsets of $c^\infty(E)$ are called *c^∞-open*. Given locally convex spaces E and F, a map $f : U \to F$ on a c^∞-open set $U \subseteq E$ will be called a *c^∞-map* if $f \circ \gamma : \mathbb{R} \to F$ is smooth for each smooth curve $\gamma : \mathbb{R} \to E$ with $\gamma(\mathbb{R}) \subseteq U$.

Definition 1.10. We say that a locally convex space E is *c^∞-regular* if, for each 0-neighbourhood $U \subseteq c^\infty(E)$, there exists a c^∞-function $f : E \to \mathbb{R}$ such that $f(0) = 1$ and $f|_{E\setminus U} = 0$.

2. Tools to identify direct limits of Lie groups

Consider a Lie group $G = \bigcup_{n\in\mathbb{N}} G_n$. If $G = \varinjlim G_n$ as a topological group, then automatically also $G = \varinjlim G_n$ as a Lie group, provided that G admits a "direct limit chart" in a sense defined presently. The existence of a direct limit chart is a very natural requirement, which ties together the direct system of Lie groups and its associated direct system of locally convex topological Lie algebras. The concept can be defined more generally for direct systems of manifolds modelled on locally convex spaces.

Definition 2.1. Let $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ and M be a $C^r_{\mathbb{K}}$-manifold such that $M = \bigcup_{n\in\mathbb{N}} M_n$ for a sequence $M_1 \subseteq M_2 \subseteq \cdots$ of $C^r_{\mathbb{K}}$-manifolds, such that all inclusion maps $i_{n,m} : M_m \to M_n$ $(n \geqslant m)$, and $i_n : M_n \to M$ are $C^r_{\mathbb{K}}$. Let E and E_n be the modelling spaces of M and M_n, respectively. A chart $\phi : U \to V \subseteq E$ of M is called a *weak direct limit chart* if (a) and (b) hold for some $n_0 \in \mathbb{N}$:

(a) There exist continuous linear maps $j_{n,m} : E_m \to E_n$ for $n \geqslant m \geqslant n_0$ and $j_n : E_n \to E$ such that $\mathcal{S} := ((E_n)_{n\geqslant n_0}, (j_{n,m})_{n\geqslant m\geqslant n_0})$ is a direct system of locally convex spaces, $(E, (j_n)_{n\geqslant n_0})$ is a cone of locally convex spaces over $\mathcal{S}$, and $E = \bigcup_{n\geqslant n_0} j_n(E_n)$.
(b) There exist charts $\phi_n : U_n \to V_n \subseteq E_n$ of M_n such that $U_m \subseteq U_n$ and $j_{n,m}(V_m) \subseteq V_n$ if $n \geqslant m \geqslant n_0$, $U = \bigcup_{n\geqslant n_0} U_n$, $V = \bigcup_{n\geqslant n_0} j_n(V_n)$ and

$$\phi \circ i_n|_{U_n} = j_n|_{V_n} \circ \phi_n \quad \text{for each } n \geqslant n_0. \tag{1}$$

If, furthermore, $(E, (j_n)_{n\geqslant n_0}) = \varinjlim \mathcal{S}$ as a locally convex space, then ϕ is called a *direct limit chart*. If ϕ is a direct limit chart and $U \cap M_n = U_n$ for each $n \geqslant n_0$, we call ϕ a *strict direct limit chart*. We say that a Lie group G admits a direct limit chart if it has a direct limit chart around 1 (and hence also a direct limit chart around any $x \in G$).

In general, there is no relationship between existence of a strict direct limit chart and strictness of the direct sequence $((E_n), (j_{n,m}))$ (although both properties frequently occur simultaneously).

Remark 2.2. With notation as in Definition 2.1, we have:

(a) Each of the linear maps $j_n : E_n \to E$ (and hence also each $j_{n,m}$) is injective because $j_n|_{V_n} = \phi \circ i_n \circ \phi_n^{-1}$ by (1), which is injective. Identifying E_n with a subspace of E via j_n, we may assume that j_n (and each $j_{n,m}$) simply is the inclusion map. Then (1) becomes $\phi|_{U_n} = \phi_n$, and we have $\phi_n|_{U_m} = \phi_m$ if $n \geqslant m$.
(b) If $M = \bigcup_{n\in\mathbb{N}} M_n$ admits a weak direct limit chart $\phi : U \to V$ (as above) around $x \in M$, we may assume that $x \in M_{n_0}$ and $\phi(x) = 0$ (after a translation). We shall usually assume this in the following.
(c) Let ϕ be a weak direct limit chart around $x \in M$. If $r \geqslant 1$, we can identify $T_x M_n$ with E_n and $T_x M$ with E, using the linear automorphisms $d\phi_n(x)$, respectively, $d\phi(x)$. Then $j_{n,m} = T_x(i_{n,m})$ and $j_n = T_x(i_n)$ for all integers $n \geqslant m \geqslant n_0$.
(d) If $\phi : U \to V$ is a (weak) direct limit chart around x and $W \subseteq U$ an open neighbourhood of x, then also $\phi|_W : W \to \phi(W)$ is a (weak) direct limit chart, because $W = \bigcup_{n\in\mathbb{N}} (W \cap U_n)$ with $W \cap U_n$ open in U_n.

Example 2.3. We shall see later that countable weak direct products of Lie groups, groups of compactly supported diffeomorphisms and test function groups admit a (strict) direct limit chart (see Remarks 4.3, 5.2 and 7.2, respectively). The Lie groups of germs encountered in Section 10 admit a direct limit chart (albeit not a strict one).

In the absence of additional conditions (like existence of a direct limit chart), we cannot hope to establish direct limit properties. The following examples illustrate some of the possible pathologies.

Example 2.4. Let G be the additive topological group of the locally convex space $\mathbb{R}^{(\mathbb{N})}$. Give $G_n := \mathbb{R}^n$ and $D := \mathbb{R}^{(\mathbb{N})}$ the discrete topology. Then $G = \bigcup_{n\in\mathbb{N}} G_n$, but the topologies on the subgroups G_n are just too fine compared to the topology on G to be of any use, and the discontinuity of the homomorphism $\mathrm{id}: G \to D$ (which is smooth on each G_n) shows that $G \neq \varinjlim G_n$ as a Lie group, topological group, smooth manifold, and as a topological space.

The next example shows that the existence of a weak direct limit chart on a Lie group $G = \bigcup_{n\in\mathbb{N}} G_n$ is not enough for the discussion of direct limit properties of G. For other purposes, it suffices (e.g., for Proposition 2.9).

Example 2.5. If we give $G := C^\infty_{\mathrm{c}}(\mathbb{R},\mathbb{R}) = \bigcap_{k\in\mathbb{N}_0} C^k_{\mathrm{c}}(\mathbb{R},\mathbb{R})$ the (unusually coarse!) topology of the projective limit of LB-spaces $\varprojlim_{k\in\mathbb{N}_0} C^k_{\mathrm{c}}(\mathbb{R},\mathbb{R})$, then $G_n := C^\infty_{[-n,n]}(\mathbb{R},\mathbb{R})$ is a closed vector subspace (and hence a Lie subgroup) of G, and $\mathrm{id}_G: G \to G$ is a weak direct limit chart for $G = \bigcup_{n\in\mathbb{N}} G_n$. Let $H := C^\infty_{\mathrm{c}}(\mathbb{R},\mathbb{R})$, with the usual LF-topology. The discontinuity of the homomorphism $\mathrm{id}: G \to H$ (which is smooth on each G_n) shows that $G \neq \varinjlim G_n$ as a Lie group, topological group, topological space, and smooth manifold.

Recall a well-known fact: if $f: G \to H$ is $C^\infty_{\mathbb{K}}$-homomorphism, and $x \in G$, then $\lambda^H_{f(x)} \circ f = f \circ \lambda^G_x$ holds with left translations as indicated, whence

$$T_x(f) = T_1\big(\lambda^H_{f(x)}\big) \circ T_1(f) \circ T_x\big(\lambda^G_{x^{-1}}\big). \tag{2}$$

Theorem 2.6 *(Reduction to topological groups). Consider a $C^\infty_{\mathbb{K}}$-Lie group $G = \bigcup_{n\in\mathbb{N}} G_n$. If $G = \varinjlim G_n$ as a topological group and G admits a direct limit chart, then $G = \varinjlim G_n$ as a $C^\infty_{\mathbb{K}}$-Lie group.*

Proof. In view of Remark 2.2(a) and (c), we can identify $L(G_m)$ with a subalgebra of $L(G_n)$ (if $n \geqslant m$) and $L(G)$. Let $f: G \to H$ be a homomorphism to a $C^\infty_{\mathbb{K}}$-Lie group H such that $f_n := f|_{G_n}$ is $C^\infty_{\mathbb{K}}$, for each $n \in \mathbb{N}$. Then f is continuous, by the direct limit property of G as a topological group. Pick a chart $\psi: P \to Q \subseteq L(H)$ of H around 1 such that $\psi(1) = 0$. Let $\phi = \varinjlim \phi_n: U \to V$ be a direct limit chart of G around 1 with $f(U) \subseteq P$, where $\phi_n: U_n \to V_n$ and $\phi(0) = 0$. To see that f is $C^1_{\mathbb{K}}$, we pass to local coordinates: we define $h := \psi \circ f|_U \circ \phi^{-1}: V \to Q$ and $h_n := \psi \circ f|_U \circ \phi_n^{-1}: V_n \to Q$. If $x \in V$, then $x \in V_{n_0}$ for some n_0. Given $n \geqslant n_0$ and $y \in L(G_n)$, the limit

$$dh(x,y) = \frac{d}{dt}\bigg|_{t=0} h(x+ty) = \frac{d}{dt}\bigg|_{t=0} h_n(x+ty) = dh_n(x,y) \tag{3}$$

exists in $L(H)$. We abbreviate $\theta := dh(0,\bullet): L(G) \to L(H)$ and $h_n'(0) := dh_n(0,\bullet): L(G_n) \to L(H)$. Since $\theta|_{L(G_n)} = h_n'(0)$ for each $n \geqslant n_0$ by (3), which is a continuous linear map, and $L(G) = \varinjlim L(G_n)$ as a locally convex space, we deduce that θ is continuous linear. Since f_n is a

$C^\infty_{\mathbb{K}}$-homomorphism, (2) implies that $dh_n(x,y) = d\lambda^H_{h_n(x)}(0, h_n'(0).d\lambda^{G_n}_{x^{-1}}(x,y))$ for all x and y as before (using the respective locally defined left translation maps in local coordinates). Hence

$$dh(x,y) = d\lambda^H_{h(x)}\big(0, \theta.d\lambda^G_{x^{-1}}(x,y)\big),$$

entailing that $dh : V \times L(G) \to L(H)$ is continuous. Thus h is $C^1_{\mathbb{K}}$ and hence also $f|_U$ is $C^1_{\mathbb{K}}$. Since f is a homomorphism, it readily follows that f is $C^1_{\mathbb{K}}$ (see [24, Lemma 3.1]) and hence $C^\infty_{\mathbb{K}}$, by [22, Lemma 2.1]. □

We recall a simple fact (cf. [35, Proposition 3.1] and [7, Chapter II, Exercise 14 to Section 4]).

Lemma 2.7. *Let $((E_n)_{n\in\mathbb{N}}, (f_{n,m})_{n\geqslant m})$ be a direct sequence of topological $\mathbb{K}$-vector spaces, with direct limit $(E, (f_n)_{n\in\mathbb{N}})$ in the category of topological $\mathbb{K}$-vector spaces. Then $E = \varinjlim E_n$ as a topological group. If each E_n is locally convex, then $E = \varinjlim E_n$ also as a locally convex space.*

Proof. It is well known that the box topology on $S := \bigoplus_{n\in\mathbb{N}} E_n$ is a vector topology which makes S the direct sum in the category of topological vector spaces (cf. [37, Section 4.1, Proposition 4]) and in the category of topological abelian groups (cf. also Lemma 4.4). If each E_n is locally convex, then also S, and the box topology coincides with the locally convex direct sum topology (cf. [7, Chapter II, Exercise 14 to Section 4]). Let $i_n : E_n \to E$ be the canonical embedding. Then the subgroup $R := \langle \bigcup_{n\geqslant m} \operatorname{im}((i_n \circ f_{n,m}) - i_m)\rangle \subseteq S$ is a vector subspace of S. The direct limit E in each of the categories described in (a) can be realized as S/R. □

Example 2.8. Let $E = \bigcup_{n\in\mathbb{N}} E_n$ be a Hausdorff locally convex space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ such that $E = \varinjlim E_n$ as a locally convex space. Then the identity map $E \to E$ is a direct limit chart. Since $E = \varinjlim E_n$ as a topological group (by Lemma 2.7), Theorem 2.6 shows that $E = \varinjlim E_n$ also as a $C^\infty_{\mathbb{K}}$-Lie group.

The following application of weak direct limit charts is the main result of [29]. Information on homotopy groups (notably on $\pi_1(G)$ and $\pi_2(G)$) is important for the extension theory of infinite-dimensional groups (see [47–49]).

Proposition 2.9. *If a Lie group $G = \bigcup_{n\in\mathbb{N}} G_n$ admits a weak direct limit chart, then its connected component of the identity is $G_0 = \bigcup_{n\in\mathbb{N}} (G_n)_0$. Furthermore, $\pi_k(G) = \varinjlim \pi_k(G_n)$ for each $k \in \mathbb{N}$.*

Further applications of direct limit charts can be found in [26].

3. Tools to identify direct limits of topological spaces and manifolds

Given a Lie group $G = \bigcup_{n\in\mathbb{N}} G_n$, it is natural to hope that $G = \varinjlim G_n$ as a topological space if and only if $L(G) = \varinjlim L(G_n)$ as a topological space. In this section, we show that this hope is justified if G has a direct limit chart. The analogous problem for the category of smooth manifolds is also addressed. At the end of the section, we consider a locally convex direct limit $E = \varinjlim E_n$ and compile conditions ensuring that $E \neq \varinjlim E_n$ as a topological space (respectively, smooth manifold).

Lemma 3.1 *(Localization Lemma). Let M be a C^r-manifold modelled on a real locally convex space E, where $r \in \mathbb{N}_0 \cup \{\infty\}$, such that M is a regular topological space (e.g., M is a Lie group). If $r \geqslant 1$, we assume that E is C^r-regular. Let $P \subseteq M$ be open and $x \in P$. Then there exists a C^r-map $\rho : M \to P$ with the following properties*:

(a) *$\rho(y) = y$ for all y in an open neighbourhood $Q \subseteq P$ of x*;
(b) *the closure of $\{y \in M\colon\ \rho(y) \neq x\}$ in M is a subset of P.*

Furthermore, the following can be achieved:

(c) *If $M = \bigcup_{n\in\mathbb{N}} M_n$ and M admits a direct limit chart around x, then for every $n \in \mathbb{N}$ and $y \in M_n$ there exists $m \geqslant n$ and a neighbourhood W of y in M_n such that $\rho(W) \subseteq M_m$ and $\rho|_W : M_n \supseteq W \to M_m$ is C^r.*
(d) *If $M = \bigcup_{n\in\mathbb{N}} M_n$ and M admits a strict direct limit chart around x, then it can be achieved that $\rho(M_n) \subseteq M_n$ for all $n \geqslant n_0$ and that $\rho|_{M_n} : M_n \to M_n$ is a C^r-map, for a suitable $n_0 \in \mathbb{N}$.*

Proof. If $r = 0$, then E is C^0-regular (see Remark 1.9).

(a), (b) After shrinking P if necessary, we may assume that there exists a chart $\phi : P \to V \subseteq E$ of M such that $\phi(x) = 0$ and $[0, 1]V = V$. Since M is a regular topological space, we find a 0-neighbourhood $B \subseteq V$ such that $A := \phi^{-1}(B)$ is closed in M. Since E is a C^r-regular, there exists a C^r-function $\beta : E \to \mathbb{R}$ such that $\operatorname{supp}(\beta) \subseteq B$, $\operatorname{im}(\beta) \subseteq [0, 1]$, and such that $\beta|_R = 1$ for some 0-neighbourhood $R \subseteq V$. Set $Q := \phi^{-1}(R)$. Then

$$\psi : P \to V, \quad \psi(x) = \beta\big(\phi(x)\big) \cdot \phi(x)$$

is a C^r-mapping such that $\psi|_Q = \phi|_Q$ and $\psi(x) = 0$ for each $x \in P \setminus A$. Extending ψ by 0, we obtain a C^r-map $\tilde{\psi} : M \to V$. Then $\rho := \phi^{-1} \circ \tilde{\psi}$ is a C^r-map $M \to M$ such that $\rho(M) \subseteq P$, $\rho|_Q = \mathrm{id}_Q$ and $\rho|_{M\setminus A} = x$.

(c) Because we can always pass to a cofinal subsequence $(M_{n+n_0})_{n\in\mathbb{N}}$, we may assume that $x \in M_1$. After shrinking P if necessary, we may assume that $\phi : P \to V$ is a direct limit chart, say $\phi = \bigcup_{n\in\mathbb{N}} \phi_n$ with $P = \bigcup_{n\in\mathbb{N}} P_n$ and $V = \bigcup_{n\in\mathbb{N}} V_n$ for certain compatible charts $\phi_n : P_n \to V_n \subseteq E_n$ around x, where E_n is the modelling space of M_n. Let $y \in M$, say $y \in M_n$. If $y \notin A$, then $W := M_n \setminus A$ is an open neighbourhood of y in M_n such that $\rho|_W$ is constant (with value x) and hence a C^r-map into M_n. If $y \in P$, let $z := \phi(y) \in V$. Then $z \in V_k$ for some $k \geqslant n$ and $\beta(z)z \in V_m$ for some $m \geqslant k$. By continuity of β, scalar multiplication in E_m and the inclusion map $E_k \to E_m$, there exists an open neighbourhood Z of z in V_k such that $\beta(v)v \in V_m$ for all $v \in Z$. Then $W := M_n \cap \phi_k^{-1}(Z)$ is an open neighbourhood of y in M_n such that $\rho|_W$ is a C^r-map into $U_m \subseteq M_m$.

(d) If M admits a strict direct limit chart at x, then we may assume that $V \cap E_n = V_n$ for each n. Given y and n as in the proof of (c), we can now take $m := k := n$, from which (d) follows. □

Lemma 3.2. *Let $r \in \mathbb{N} \cup \{\infty\}$ and $M = \bigcup_{n\in\mathbb{N}} M_n$ be a C^r-manifold which is a regular topological space and admits a direct limit chart around some $x \in M$.*

(a) *If $T_x M \neq \varinjlim T_x M_n$ as a topological space, then also $M \neq \varinjlim M_n$ as a topological space.*

(b) *If $T_x M \neq \varinjlim T_x M_n$ as a $C^r_{\mathbb{R}}$-manifold and M is modelled on a C^r-regular locally convex space, then $M \neq \varinjlim M_n$ as a $C^r_{\mathbb{R}}$-manifold.*

Proof. We may assume that $x \in M_1$. To prove (a), set $s := 0$; for (b), set $s := r$. Let $\phi : P \to V \subseteq T_x M$ be a direct limit chart around x and $\rho : M \to P$ be a C^s-map as in Lemma 3.1(c). Since $T_x M \neq \varinjlim T_x M_n$ as a topological space (respectively, C^r-manifold), there exists a map $h : T_x M \to X$ to a topological space (respectively, C^r-manifold) X that is not C^s, although $h|_{M_n}$ is C^s for each n. Hence, there is $z \in T_x M$ such that h is not C^s on any open neighbourhood of z. We may assume that $z \in T_x M_1$; after replacing h with $h(\bullet - z)$, we may assume that $z = 0$. Then $f := h \circ \phi \circ \rho$ is not C^s. We claim that $f|_{M_n}$ is C^s for each $n \in \mathbb{N}$; thus M is not the direct limit topological space (respectively, C^r-manifold). To prove the claim, let $A \subseteq P$ be a closed subset of M such that $\rho|_{M \setminus A} = x$, and $y \in M_n$. If $y \notin A$, then $W := M_n \setminus A$ is an open neighbourhood of y in M_n such that $f(w) = h(0)$ for each $w \in W$; thus $f|_W$ is C^s. If $y \in P$, let m and W be as in the proof of Lemma 3.1(c). Then $\rho(W) \subseteq U_n$ and $f|_W = h|_{V_n} \circ \phi_n \circ \rho|_W^{U_n}$ is C^s, as claimed. □

Replacing the topological space X by a C^0-manifold in the proof of (a), we see that $M \neq \varinjlim M_n$ as a C^0-manifold if $T_x M \neq \varinjlim T_x M_n$ as a C^0-manifold.

Theorem 3.3 *(Reduction to the Lie algebra level). Let $G = \bigcup_{n \in \mathbb{N}} G_n$ be a real Lie group admitting a direct limit chart.*

(a) *Then $L(G) = \varinjlim L(G_n)$ as a topological space if and only if $G = \varinjlim G_n$ as a topological space.*
(b) *If $L(G)$ is C^r-regular for $r \in \mathbb{N}_0 \cup \{\infty\}$, then $L(G) = \varinjlim L(G_n)$ as a C^r-manifold if and only if $G = \varinjlim G_n$ as a C^r-manifold.*

Proof. We set $r := 0$ in the situation of (a). If $L(G) \neq \varinjlim L(G_n)$ as a topological space (respectively, as a C^r-manifold, in the situation of (b)), then $G \neq \varinjlim G_n$ as a topological space (respectively, C^r-manifold), by Lemma 3.2.

Conversely, assume that $G \neq \varinjlim G_n$ as a topological space (respectively, as a C^r-manifold, in the situation of (b)). Then there exists a map $f : G \to X$ to a topological space (respectively, C^r-manifold) X which is not C^r, although $f|_{G_n}$ is C^r for each $n \in \mathbb{N}$. There is $x \in G$ such that f is not C^r on any open neighbourhood of x. After replacing f with $y \mapsto f(x^{-1}y)$, we may assume that $x = 1$. Let $\phi : U \to V \subseteq L(G)$ be a direct limit chart of G around 1 such that $\phi(1) = 0$, with $\phi = \bigcup_{n \in \mathbb{N}} \phi_n$, $U = \bigcup_{n \in \mathbb{N}} U_n$ and $V = \bigcup_{n \in \mathbb{N}} V_n$ for charts $\phi_n : U_n \to V_n \subseteq L(G_n)$ such that $\phi_{n+1}|_{U_n} = \phi_n$. By Lemma 3.1, there exists a C^r-map $\rho : L(G) \to V$ such that $\rho|_Q = \mathrm{id}_Q$ for an open 0-neighbourhood $Q \subseteq L(G)$ such that $\rho|_{V_n}$ locally is a C^r-map into some V_m, $m \geqslant n$. Then $f \circ \phi^{-1} \circ \rho$ is not C^r, although $(f \circ \phi^{-1} \circ \rho)|_{L(G_n)}$ is C^r for each $n \in \mathbb{N}$. □

Remark 3.4. Theorem 3.3 complements Yamasaki's Theorem [56, Theorem 4]:

Consider a group $G = \bigcup_{n \in \mathbb{N}} G_n$, where each G_n is a metrizable topological group and each inclusion map $G_n \to G_{n+1}$ a topological embedding. Assume that neither (a) *nor* (b) *holds*:
(a) *for each $m \in \mathbb{N}$, there exists $n \geqslant m$ and an identity neighbourhood $U \subseteq G_m$ whose closure in G_n is compact*;
(b) *there exists $m \in \mathbb{N}$ such that G_m is open in G_n for each $n \geqslant m$.*
Then the direct limit topology does not make G a topological group.

Stimulated by Theorem 3.3, we turn to locally convex direct limits and their direct limit properties as topological spaces and manifolds.

Lemma 3.5. *Let $E_1 \subseteq E_2 \subseteq \cdots$ be an ascending sequence of Hausdorff locally convex spaces which is a strict direct sequence and does not become stationary. Let $E := \bigcup_{n\in\mathbb{N}} E_n$, equipped with the locally convex direct limit topology.*

(a) *If each E_n is infinite-dimensional and metrizable, then $E \neq \varinjlim E_n$ as a topological space.*
(b) *If each E_n is an infinite-dimensional nuclear Fréchet space, then $E \neq \varinjlim E_n$ as a C^r-manifold, for each $r \in \mathbb{N}_0 \cup \{\infty\}$. Furthermore, E is smoothly regular.*

Proof. (a) (Cf. [39, Theorem 4.11(3) and Proposition 4.26(ii)] if each E_n is a Fréchet space.) Let $E_1 \subset E_2 \subset \cdots$ be a strict direct sequence of infinite-dimensional metrizable topological vector spaces which is strictly increasing. Then E_m is not open in E_n for any integers $m < n$, and the closure of a 0-neighbourhood U of E_m in E_n cannot be compact because then U would be pre-compact and thus $\dim(E_m) < \infty$. Now Yamasaki's Theorem (see Remark 3.4) shows that the direct limit topology does not make $E = \bigcup_{n\in\mathbb{N}} E_n$ a topological group. The assertion follows.

(b) It is well known that the c^∞-topology on E coincides with the direct limit topology (cf. [39, Theorem 4.11(3)]). By part (a) just established (or [39, Proposition 4.26(ii)]), the latter is properly finer than the locally convex direct limit topology. Therefore, there exists a 0-neighbourhood $U \subseteq c^\infty(E)$ which is not a 0-neighbourhood of E. Since E is c^∞-regular (see [39, Theorem 16.10]), there exists a c^∞-function $f : E \to \mathbb{R}$ such that $f(0) = 1$ and $f|_{E\setminus U} = 0$. Then $f|_{E_n}$ is a c^∞-map and hence smooth (as E_n is metrizable), for each $n \in \mathbb{N}$. However, f is discontinuous (and hence not C^r for any $r \in \mathbb{N}_0 \cup \{\infty\}$). In fact, if f was continuous, then $f^{-1}(\mathbb{R}^\times) \subseteq U$ would be a 0-neighbourhood in E and hence also U, contradicting our choice of U. Like any countable locally convex direct limit of nuclear spaces, E is nuclear [52, Section 7.4, Corollary] and hence smoothly regular (see Remark 1.9). □

We close this section with a variant of Yamasaki's Theorem for Lie groups.

Proposition 3.6. *Let $G = \bigcup_{n\in\mathbb{N}} G_n$ be a Lie group, where $L(G_n)$ is metrizable for each $n \in \mathbb{N}$. Assume that condition* (i) *or* (ii) *is satisfied*:

(i) *G has a direct limit chart, and the direct sequence $(L(G_n))_{n\in\mathbb{N}}$ is strict*;
(ii) *for each $n \in \mathbb{N}$, the Lie group G_n has an exponential function which is a local homeomorphism at* 0, *and the direct sequence $(G_n)_{n\in\mathbb{N}}$ is strict.*

If $G = \varinjlim G_n$ as a topological space, then (a) *or* (b) *holds*:

(a) *G_n is a finite-dimensional Lie group, for each $n \in \mathbb{N}$*; *or*
(b) *there exists $m \in \mathbb{N}$ such that G_m is open in G_n for each $n \geqslant m$.*

Proof. If (i) holds but neither (a) nor (b), then we find $n_0 \in \mathbb{N}$ such that $L(G_{n_0}) \subseteq L(G_{n_0+1}) \subseteq \cdots$ is a strict direct sequence of infinite-dimensional Fréchet spaces which does not become stationary, whence $G \neq \varinjlim G_n$ as a topological space by Theorem 3.3(a) and Lemma 3.5(a).

If (ii) holds, then $(L(G_n))_{n\in\mathbb{N}}$ is strict. To see this, given $n \geqslant m$ define $j := L(i_{n,m})$ and let $V \subseteq L(G_m)$ and $W \subseteq L(G_n)$ be open 0-neighbourhoods such that $\phi := \exp_{G_m}|_V$ is a homeomorphism onto an open identity neighbourhood $\tilde{V} \subseteq G_m$, $\psi := \exp_{G_n}|_W$ is a homeomorphism onto an open identity neighbourhood $\tilde{W} \subseteq G_n$, and $j(V) \subseteq W$. Since $i_{n,m}$ is a topological embedding, there exists an open identity neighbourhood $\tilde{X} \subseteq \tilde{W}$ such that $\tilde{X} \cap G_m = \tilde{V}$; define $X := \psi^{-1}(\tilde{X}) \subseteq W$. Since $\exp_{G_n} \circ j|_V = i_{n,m} \circ \exp_{G_m}|_V$ is a topological embedding, also $j|_V$ is an embedding, whence the continuous linear map j is injective. To see that j is a topological embedding, let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $L(G_m)$ such that $j(x_k) \to 0$ in $L(G_n)$. After omitting finitely many terms, we may assume that $j(x_k) \in X$ for each k. Then $\exp_{G_n}(j(x_k)) = i_{n,m}(\exp_{G_m}(x_k)) \in \tilde{X} \cap G_m = \tilde{V}$ and $y_k := \phi^{-1}(\exp_{G_n}(j(x_k))) \to 0$ in $L(G_m)$. Since $y_k \in V$, we have that $j(y_k) \in W$. Now $\psi(j(x_k)) = \exp_{G_n}(j(x_k)) = i_{n,m}(\exp_{G_m}(y_k)) = \exp_{G_n}(j(y_k)) = \psi(j(y_k))$ by the definition of y_k and naturality of exp. Since ψ is injective, we deduce that $j(x_k) = j(y_k)$ and hence $x_k = y_k \to 0$. Thus j is an embedding.

Now assume that (ii) holds but neither (a) nor (b). After passing to a suitable subsequence $(G_{n_k})_{k\in\mathbb{N}}$, we may assume that each G_m is infinite-dimensional and G_m is not open in G_n whenever $n > m$. Then $L(G_m)$ (identified with $L(i_{n,m}).L(G_m)$) is a proper vector subspace of $L(G_n)$, because otherwise $i_{n,m}(G_m) \supseteq i_{n,m}(\exp_{G_m}(L(G_m))) = \exp_{G_n}(L(i_{n,m}).L(G_m)) = \exp_{G_n}(L(G_n))$ would contain an open identity neighbourhood and hence be an open subgroup (which we just ruled out). For any identity neighbourhood $U \subseteq G_m$, we now show that its closure $K := \overline{U}$ in G_n cannot be compact. To see this, suppose to the contrary that K was compact. Let V, W, ϕ and ψ be as earlier in the proof. Since G_n is a regular topological space, it has a closed identity neighbourhood $A \subseteq G_n$ such that $A \subseteq \tilde{W}$. Then $Q := \phi^{-1}(G_m \cap A \cap K)$ is a 0-neighbourhood in $L(G_m)$ such that $\psi(Q)$ has compact closure $\overline{\psi(Q)} \subseteq A \cap K$, whence Q has compact closure $\psi^{-1}(\overline{\psi(Q)})$ in $L(G_n)$. The inclusion map $L(G_m) \to L(G_n)$ being an embedding, this entails that Q is precompact and thus $\dim(L(G_m)) < \infty$ (which is absurd).

Now Yamasaki's Theorem shows that the direct limit topology does not make G a topological group. It therefore differs from given topology on G. □

4. Example: Weak direct products of Lie groups

In this section, we recall the definition of weak direct products of Lie groups and analyze their direct limit properties. In 4.1 and 4.2, I is an arbitrary set (which need not be countable).

4.1. If $(G_i)_{i\in I}$ is a family of topological groups, we let $\prod^*_{i\in I} G_i \leqslant \prod_{i\in I} G_i$ be the subgroup of all families $(g_i)_{i\in I}$ such that $g_i = 1$ for all but finitely many i. A *box* is a set of the form $\prod^*_{i\in I} U_i := (\prod^*_{i\in I} G_i) \cap \prod_{i\in I} U_i$, where $U_i \subseteq G_i$ is open and $1 \in U_i$ for all but finitely many i. It is well known that the set of boxes is a basis for a topology on $\prod^*_{i\in I} G_i$ making it a topological group.

4.2. If $(G_i)_{i\in I}$ is a family of $C^r_{\mathbb{K}}$-Lie groups, where $r \in \{\infty, \omega\}$, then $\prod^*_{i\in I} G_i$ can be made a $C^r_{\mathbb{K}}$-Lie group, modelled on the locally convex direct sum $\bigoplus_{i\in I} L(G_i)$ (see [19]). The Lie group structure is characterized by the following property (cf. [19, proof of Proposition 7.3]):

For each $i \in I$, let $\phi_i : \tilde{U}_i \to \tilde{V}_i \subseteq L(G_i)$ be a chart of G_i around 1 such that $\phi_i(1) = 0$. Let $U_i \subseteq \tilde{U}_i$ be an open, symmetric identity neighbourhood such that $U_i U_i \subseteq \tilde{U}_i$, $V_i := \tilde{\phi}(U_i)$, and $\phi_i := \tilde{\phi}_i|^{V_i}_{U_i}$. Then

$$\kappa := \bigoplus_{i\in I} \phi_i : \prod_{i\in I}^* U_i \to \bigoplus_{i\in I} V_i, \quad (x_i)_{i\in I} \mapsto \big(\phi_i(x_i)\big)_{i\in I}$$

is a chart for $\prod_{i\in I}^* G_i$.

If I is countable, then $\bigoplus_{i\in I} L(G_i)$ carries the box topology, entailing that the topology underlying the Lie group $\prod_{i\in I}^* G_i$ is the box topology from 4.1 (because the sets of identity neighbourhoods coincide).

Remark 4.3. Assume that $I = \mathbb{N}$ in the preceding situation. Then, as is clear, $\prod_{n\in\mathbb{N}}^* G_n = \varinjlim_{n\in\mathbb{N}} \prod_{k=1}^n G_k$ as an abstract group. Since κ restricts to the chart $\prod_{k=1}^n \phi_k : \prod_{k=1}^n U_k \to \prod_{k=1}^n V_k$ of $\prod_{k=1}^n G_k$, we see that $\prod_{n\in\mathbb{N}}^* G_n = \bigcup_{n\in\mathbb{N}} \prod_{k=1}^n G_k$ admits a strict direct limit chart.

Lemma 4.4. *Let* $(G_n)_{n\in\mathbb{N}}$ *be a sequence of topological groups,* H *be a topological group and* $f_n : G_n \to H$ *for* $n \in \mathbb{N}$ *be a map which is continuous at* 1 *and such that* $f_n(1) = 1$. *Then the map* $f : \prod_{n\in\mathbb{N}}^* G_n \to H$ *taking* $x = (x_n)_{n\in\mathbb{N}}$ *to*

$$f(x) := f_1(x_1) f_2(x_2) \cdots f_N(x_N) \quad \text{if } x_n = 1 \text{ for all } n > N$$

is continuous at 1. *In particular,* $\prod_{n\in\mathbb{N}}^* G_n = \varinjlim_{n\in\mathbb{N}} \prod_{k=1}^n G_k$ *in the category of topological groups.*

Proof. Given an identity neighbourhood $V_0 \subseteq H$, there is a sequence $(V_n)_{n\in\mathbb{N}}$ of identity neighbourhoods of H such that $V_n V_n \subseteq V_{n-1}$ for each $n \in \mathbb{N}$. Then $V_1 V_2 \cdots V_n \subseteq V_0$, for each $n \in \mathbb{N}$. Since f_n is continuous at 1, the preimage $U_n := f_n^{-1}(V_n)$ is an identity neighbourhood in G_n. Then $U := \prod_{n\in\mathbb{N}}^* U_n$ is an identity neighbourhood in $G := \prod_{n\in\mathbb{N}}^* G_n$ such that $f(U) \subseteq V_0$. In fact, if $x \in U$ and $x_n = 1$ for all $n > N$, then $f(x) = f_1(x_1) \cdots f_N(x_N) \in V_1 \cdots V_N \subseteq V_0$. Hence f is continuous at 1. To prove the final assertion, let $f : G \to H$ be a homomorphism to a topological group H such that $f_n := f|_{G_n} : G_n \to H$ is continuous for each $n \in \mathbb{N}$. Since f is a homomorphism, given $x \in G$ with $x_n = 1$ for $n > N$ we have

$$f(x) = f(x_1 \cdots x_N) = f(x_1) \cdots f(x_N) = f_1(x_1) \cdots f_N(x_N).$$

Thus f is a mapping of the form just discussed. Therefore f is continuous at 1 and hence continuous, being a homomorphism. □

Proposition 4.5. *Let* $(G_n)_{n\in\mathbb{N}}$ *be a sequence of* $C^\infty_\mathbb{K}$*-Lie groups modelled on locally convex spaces. Then*

$$\prod_{n\in\mathbb{N}}^* G_n = \varinjlim_{n\in\mathbb{N}} \prod_{k=1}^n G_k \tag{4}$$

holds as a topological group, and as a $C^\infty_\mathbb{K}$*-Lie group.*

Proof. By Lemma 4.4, $G := \prod_{n\in\mathbb{N}}^* G_n$ has the desired direct limit property in the category of topological groups. Since G admits a direct limit chart (see Remark 4.3), it also is the desired direct limit in the category of $C^\infty_\mathbb{K}$-Lie groups, by Theorem 2.6. □

Remark 4.6. There is no uniform answer concerning the validity of (4) in the categories of topological spaces respectively, smooth manifolds.

(a) If each G_n is modelled on a Silva space (or on a k_ω-space, as in 9.2), then (4) holds in the category of topological spaces and the category of $C^r_{\mathbb{K}}$-manifolds, for each $r \in \mathbb{N}_0 \cup \{\infty\}$ (see Proposition 9.8(ii)). In particular, this is the case if each G_n is finite-dimensional.
(b) If each G_n is modelled on an infinite-dimensional Fréchet space, then $\prod^*_{n\in\mathbb{N}} G_n \neq \varinjlim \prod_{k=1}^n G_k$ as a topological space, by Theorem 3.3(a) and Lemma 3.5(a). If $L(G_n)$ is an infinite-dimensional nuclear Fréchet space for each $n \in \mathbb{N}$, then $\prod^*_{n\in\mathbb{N}} G_n \neq \varinjlim \prod_{k=1}^n G_k$ as a $C^r_{\mathbb{R}}$-manifold for each $r \in \mathbb{N}_0 \cup \{\infty\}$, by Theorem 3.3(b) and Lemma 3.5(b).

5. Example: Groups of compactly supported diffeomorphisms

In this section, we outline the proofs of the results concerning diffeomorphism groups described in the introduction.

5.1. Recall that the Lie group $\mathrm{Diff}_c(M)$ is modelled on the space $C^\infty_c(M, TM)$ of compactly supported smooth vector fields. To obtain a chart around id_M, one chooses a smooth Riemannian metric g on M, with associated exponential map $\exp_g$. Then there is an open 0-neighbourhood $V \subseteq C^\infty_c(M, TM)$ with the following properties: for each $\gamma \in V$, the composition $\psi(\gamma) := \exp_g \circ \gamma$ makes sense and is a C^∞-diffeomorphism of M; $\psi : V \to \mathrm{Diff}_c(M)$ is injective; and $\phi := \psi^{-1} : U \to V$ (with $U := \psi(V)$) is a chart for $\mathrm{Diff}_c(M)$. Furthermore, it can be achieved that, for each compact subset $K \subseteq M$, $\phi(U \cap \mathrm{Diff}_K(M)) = V \cap C^\infty_K(M, TM)$ and the restriction of ϕ to a map $U \cap \mathrm{Diff}_K(M) \to V \cap C^\infty_K(M, TM)$ is a chart for $\mathrm{Diff}_K(M)$ (see [17]; cf. [42]).

Remark 5.2. Let $K_1 \subseteq K_2 \subseteq \cdots$ be an exhaustion of M by compact sets, i.e., $M = \bigcup_{n\in\mathbb{N}} K_n$ and $K_n \subseteq K^\circ_{n+1}$ for each $n \in \mathbb{N}$, where K°_{n+1} is the interior of K_{n+1}. Then $(K_n)_{n\in\mathbb{N}}$ is a cofinal subsequence of the directed set of all compact subsets of M. It is clear that any chart ϕ of $\mathrm{Diff}_c(M)$ of the form just described is a strict direct limit chart of $\mathrm{Diff}_c(M) = \bigcup_{n\in\mathbb{N}} \mathrm{Diff}_{K_n}(M)$.

We begin with the negative results.

Proposition 5.3. *Let M be a σ-compact, non-compact, finite-dimensional smooth manifold of positive dimension. Then there exists a discontinuous function $f : \mathrm{Diff}_c(M) \to \mathbb{R}$ whose restriction to $\mathrm{Diff}_K(M)$ is smooth, for each compact set $K \subseteq M$. Hence $\mathrm{Diff}_c(M) \neq \varinjlim \mathrm{Diff}_K(M)$ as a topological space and as a C^r-manifold, for each $r \in \mathbb{N}_0 \cup \{\infty\}$.*

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be an exhaustion of M by compact sets. Then the space $C^\infty_{K_n}(M, TM)$ of smooth vector fields on M supported in K_n is a nuclear Fréchet space (cf. [50,52,54]), and the direct sequence $(C^\infty_{K_n}(M, TM))_{n\in\mathbb{N}}$ is strict (each topology being induced by $C^\infty(M, TM)$) and does not become stationary. Hence Lemma 3.5(b) and its proof show that the locally convex direct limit $C^\infty_c(M, TM) = \varinjlim C^\infty_{K_n}(M, TM)$ is smoothly regular, and that there is a discontinuous map $h : C^\infty_c(M, TM) \to \mathbb{R}$ which is smooth on $C^\infty_{K_n}(M, TM)$ for each $n \in \mathbb{N}$. After composing with a translation, we may assume that h is discontinuous at 0. Since $\mathrm{Diff}_c(M) = \bigcup_{n\in\mathbb{N}} \mathrm{Diff}_{K_n}(M)$ has a direct limit chart (Remark 5.2), the proof of Lemma 3.2(b) enables us to manufacture a function $f : \mathrm{Diff}_c(M) \to \mathbb{R}$ which is discontinuous at id_M, although its restriction to $\mathrm{Diff}_{K_n}(M)$ is smooth for each $n \in \mathbb{N}$. □

Proposition 5.4. *Let M be a non-compact, σ-compact finite-dimensional smooth manifold. Then $\mathrm{Diff}_c(M) = \varinjlim \mathrm{Diff}_K(M)$ in the category of topological groups, and in the category of Lie groups.*

The proof hinges on the technique of *fragmentation*. The idea of fragmentation is to write a compactly supported diffeomorphism as a composition of diffeomorphisms supported in given sets (cf. [3, Section 2.1] and the references therein; cf. also [32] for fragmentation in the convenient setting of analysis). The following lemma (proved in Section 6) establishes a link between fragmentation and weak direct products; it asserts that, close to id_M, diffeomorphisms can be decomposed smoothly into pieces supported in some locally finite cover of compact sets.

Lemma 5.5 *(Fragmentation Lemma for diffeomorphism groups). For any finite-dimensional, σ-compact C^∞-manifold M, the following holds*:

(a) *There exists a locally finite cover $(K_n)_{n\in\mathbb{N}}$ of M by compact sets, an open identity neighbourhood $\Omega \subseteq \mathrm{Diff}_c(M)$ and a smooth map*

$$\Phi : \Omega \to \prod_{n\in\mathbb{N}}^{*} \mathrm{Diff}_{K_n}(M), \quad \gamma \mapsto \Phi(\gamma) =: (\gamma_n)_{n\in\mathbb{N}}$$

such that $\Phi(1) = 1$ and $\gamma = \gamma_1 \circ \cdots \circ \gamma_n$ for each $\gamma \in \Omega$ and each sufficiently large n.

(b) *If $(U_n)_{n\in\mathbb{N}}$ is a locally finite cover of M by relatively compact, open sets, then $(K_n)_{n\in\mathbb{N}}$ in* (a) *can be chosen such that $K_n \subseteq U_n$ for all $n \in \mathbb{N}$.*

Proof of Proposition 5.4. Because the hypotheses of Theorem 2.6 are satisfied by $\mathrm{Diff}_c(M)$ and a cofinal subsequence of its Lie subgroups $\mathrm{Diff}_K(M)$ (see Remark 5.2), we only need to show that $\mathrm{Diff}_c(M) = \varinjlim \mathrm{Diff}_K(M)$ as a topological group. To this end, let $f : \mathrm{Diff}_c(M) \to H$ be a homomorphism to a topological group H whose restriction $f_K : \mathrm{Diff}_K(M) \to H$ to $\mathrm{Diff}_K(M)$ is continuous, for each compact subset $K \subseteq M$. We have to show that f is continuous. Let $(K_n)_{n\in\mathbb{N}}$ and Φ be as in the Fragmentation Lemma and consider the auxiliary function $h : \prod_{n\in\mathbb{N}}^{*} \mathrm{Diff}_{K_n}(M) \to H$,

$$h\big((\gamma_n)_{n\in\mathbb{N}}\big) := f_{K_1}(\gamma_1) \cdot \cdots \cdot f_{K_N}(\gamma_N) \quad \text{if } \gamma_n = 1 \text{ for all } n > N,$$

which is continuous at 1 by Lemma 4.4. Then $f|_\Omega = h \circ \Phi$, since f is a homomorphism. Thus also f is continuous at 1 and hence continuous. □

6. Proof of the Fragmentation Lemma for diffeomorphism groups

In this section, we prove the Fragmentation Lemma for diffeomorphism groups (Lemma 5.5). We start with a preparatory lemma.

Lemma 6.1. *Let $r \in \mathbb{N}_0 \cup \{\infty\}$ and $\pi : E \to M$ be a C^r-vector bundle over a σ-compact finite-dimensional C^r-manifold M. Let $P \subseteq C^r_c(M, E)$ be a 0-neighbourhood in the space of compactly supported C^r-sections, and $(s_n)_{n\in\mathbb{N}}$ be a sequence in $C^r(M, \mathbb{R})$ such that, for each compact set $K \subseteq M$, there exists $N \in \mathbb{N}$ such that $s_n|_K = s_m|_K$ for all $n, m \geqslant N$. Then there is a 0-neighbourhood $Q \subseteq P$ such that $s_n \cdot Q \subseteq P$ for all $n \in \mathbb{N}$.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a locally finite cover of M by relatively compact, open sets. Then the linear map

$$p: C^r_c(M,E) \to \bigoplus_{n\in\mathbb{N}} C^r(U_n, E|_{U_n}), \quad \gamma \mapsto (\gamma|_{U_n})_{n\in\mathbb{N}}$$

is a topological embedding (see [16] or [27, Proposition F.19]). Hence, there are open 0-neighbourhoods $V_n \subseteq C^r(U_n, E|_{U_n})$ such that $p^{-1}(\bigoplus_{n\in\mathbb{N}} V_n) \subseteq P$. The hypothesis entails that $F_n := \{s_k|_{U_n}: k\in\mathbb{N}\}$ is a finite set, for each $n\in\mathbb{N}$. Fix $n\in\mathbb{N}$. Since $C^r(U_n, E|_{U_n})$ is a topological $C^r(U_n,\mathbb{R})$-module (see [16] or [27, Corollary F.13]), for each $s\in F_n$ the multiplication operator $C^r(U_n, E|_{U_n}) \to C^r(U_n, E|_{U_n})$, $\gamma\mapsto s\cdot\gamma$ is continuous. Hence, there exists an open 0-neighbourhood $W_n \subseteq V_n$ such that $s\cdot W_n \subseteq V_n$ for all $s\in F_n$. Then $Q := p^{-1}(\bigoplus_{n\in\mathbb{N}} W_n) \subseteq P$ is an open 0-neighbourhood such that $s_n\cdot Q\subseteq P$ for each $n\in\mathbb{N}$. The proof is complete. □

6.2. To prove Lemma 5.5, let $(U_n)_{n\in\mathbb{N}}$ be a locally finite cover of M by relatively compact, open subsets $U_n\subseteq M$, and $(h_n)_{n\in\mathbb{N}}$ be a smooth partition of unity of M such that $K_n := \operatorname{supp}(h_n) \subseteq U_n$ for each $n\in\mathbb{N}$. For each $n\in\mathbb{N}_0$, we set $s_n := \sum_{i=1}^n h_i$. We define $U_0 := K_0 := \emptyset$. For each $n\in\mathbb{N}$, we set $W_n := U_n \cup U_{n-1}$ and choose $\xi_n \in C^\infty_c(W_n,\mathbb{R})$ such that $\xi_n|_{K_n\cup K_{n-1}} = 1$. We abbreviate $L_n := \operatorname{supp}(\xi_n)$.

6.3. Let $\tilde\phi:\tilde U \to \tilde V \subseteq C^\infty_c(M,TM)$ be a chart of $\mathrm{Diff}_c(M)$ around id_M such that $\tilde\phi(\mathrm{id}_M)=0$ and $\tilde\phi$ restricts to a chart $\tilde U \cap \mathrm{Diff}_K(M) \to \tilde V\cap C^\infty_K(M,TM)$ of $\mathrm{Diff}_K(M)$, for each compact subset $K\subseteq M$ (see 5.1). There exists an open, symmetric identity neighbourhood $U\subseteq\tilde U$ such that $UU\subseteq\tilde U$. Set $V := \tilde\phi(U)$, $\phi := \tilde\phi|_U^V$, and let $\phi_n: U\cap \mathrm{Diff}_{K_n}(M) \to V\cap C^\infty_{K_n}(M,TM)$ be the restriction of ϕ to a chart of $\mathrm{Diff}_{K_n}(M)$. By 4.2, the map

$$\kappa := \bigoplus_{n\in\mathbb{N}} \phi_n : \prod_{n\in\mathbb{N}}^{*} \bigl(U\cap \mathrm{Diff}_{K_n}(M)\bigr) \to \bigoplus_{n\in\mathbb{N}} \bigl(V\cap C^\infty_{K_n}(M,TM)\bigr) \tag{5}$$

sending $(\eta_n)_{n\in\mathbb{N}}$ to $(\phi_n(\eta_n))_{n\in\mathbb{N}}$ is a chart of $\prod^*_{n\in\mathbb{N}} \mathrm{Diff}_{K_n}(M)$ around 1.

6.4. Pick an open, symmetric identity neighbourhood $P\subseteq U$ such that $PP\subseteq U$. Let $Q := \phi(P)$. By Lemma 6.1, there is an open 0-neighbourhood $S\subseteq Q$ such that $s_n\cdot S\subseteq Q$ for all $n\in\mathbb{N}$. We set $R := \phi^{-1}(S)$. In local coordinates, the group multiplication of $\mathrm{Diff}_c(M)$ corresponds to the smooth map $\mu: Q\times Q\to V$, $\mu(\sigma,\tau) := \sigma * \tau := \phi(\phi^{-1}(\sigma)\circ\phi^{-1}(\tau))$. The group inversion corresponds to the smooth map $Q\to Q$, $\sigma\mapsto\sigma^{-1} := \phi(\phi^{-1}(\sigma)^{-1})$. For each compact set $K\subseteq M$, the local multiplication restricts to a smooth map $(Q\cap C^\infty_K(M,TM))^2 \to V\cap C^\infty_K(M,TM)$, and the local inversion to a smooth map $Q\cap C^\infty_K(M,TM) \to Q\cap C^\infty_K(M,TM)$.

6.5. Given $\gamma\in R$ and $n\in\mathbb{N}_0$, we define $s_n\odot\gamma\in P$ via $s_n\odot\gamma := \phi^{-1}(s_n\cdot\phi(\gamma))$. For each $n\in\mathbb{N}$, we let $\gamma_n := (s_{n-1}\odot\gamma)^{-1}\circ(s_n\odot\gamma)\in U$. Since $(s_n\odot\gamma)(x) = (s_{n-1}\odot\gamma)(x)$ for all $x\in M\setminus K_n$, we have $\gamma_n(x)=x$ for such x and thus $\gamma_n\in C^\infty_{K_n}(M,TM)\cap U$. Given γ, there is $N\in\mathbb{N}$ such that $U_n\cap\operatorname{supp}(\gamma)=\emptyset$ for all $n>N$. Then $s_n|_{\operatorname{supp}(\gamma)}=1$ and thus $s_n\odot\gamma=\gamma$ for all $n\geqslant N$, whence $\gamma_1\circ\cdots\circ\gamma_n = s_n\odot\gamma=\gamma$ for all $n\geqslant N$. Thus

$$\Phi: R\to \prod_{n\in\mathbb{N}}^{*} \mathrm{Diff}_{K_n}(M), \quad \Phi(\gamma) := (\gamma_n)_{n\in\mathbb{N}}$$

has the desired properties, except for smoothness. To complete the proof, we show that $\Phi|_\Omega$ is smooth for some open identity neighbourhood $\Omega \subseteq R$.

6.6. Since $(W_n)_{n\in\mathbb{N}}$ is a locally finite cover of M by relatively compact, open sets, the map

$$p : C^\infty_c(M, TM) \to \bigoplus_{n\in\mathbb{N}} C^\infty(W_n, TW_n), \quad \gamma \mapsto (\gamma|_{W_n})_{n\in\mathbb{N}}$$

is continuous linear (and in fact an embedding onto a closed vector subspace, see [16] or [27, Proposition F.19]). Because $C^\infty(W_n, TW_n)$ is a topological $C^\infty(W_n, \mathbb{R})$-module (see [16] or [27, Corollary F.13]), the multiplication operators $\mu_n : C^\infty(W_n, TW_n) \to C^\infty_{L_n}(W_n, TW_n)$, $\gamma \mapsto \xi_n \cdot s_n|_{W_n} \cdot \gamma$ and $\lambda_n : C^\infty(W_n, TW_n) \to C^\infty_{L_n}(W_n, TW_n)$, $\gamma \mapsto \xi_n \cdot s_{n-1}|_{W_n} \cdot \gamma$ are continuous linear. Then

$$\lambda := \bigoplus_{n\in\mathbb{N}} (\lambda_n, \mu_n) : \bigoplus_{n\in\mathbb{N}} C^\infty(W_n, TW_n) \to \bigoplus_{n\in\mathbb{N}} C^\infty_{L_n}(W_n, TW_n) \times C^\infty_{L_n}(W_n, TW_n)$$

is continuous linear. The restriction map $\rho_n : C^\infty_{L_n}(M, TM) \to C^\infty_{L_n}(W_n, TW_n)$ is an isomorphism of topological vector spaces for each $n \in \mathbb{N}$ (see [16] or [27, Lemma F15(b)]), whence so is

$$\rho := \bigoplus_{n\in\mathbb{N}} \left(\rho_n^{-1} \times \rho_n^{-1}\right) : \bigoplus_{n\in\mathbb{N}} C^\infty_{L_n}(W_n, TW_n)^2 \to \bigoplus_{n\in\mathbb{N}} C^\infty_{L_n}(M, TM)^2.$$

6.7. Then $Z := \{\gamma \in V\colon (\rho \circ \lambda \circ p)(\gamma) \in \bigoplus_{n\in\mathbb{N}} (Q \cap C^\infty_{L_n}(M, TM))^2\}$ is an open identity neighbourhood in V. For each $n \in \mathbb{N}$, the map

$$g_n : \left(Q \cap C^\infty_{L_n}(M, TM)\right)^2 \to V \cap C^\infty_{L_n}(M, TM), \quad g_n(\sigma, \tau) := \sigma^{-1} * \tau$$

is smooth. Hence $g := \bigoplus_{n\in\mathbb{N}} g_n : \bigoplus_{n\in\mathbb{N}} (Q \cap C^\infty_{L_n}(M, TM))^2 \to \bigoplus_{n\in\mathbb{N}} (V \cap C^\infty_{L_n}(M, TM))$ is smooth, by [19, Proposition 7.1]. Then $g \circ \rho \circ \lambda \circ p|_Z$ is a smooth map with values in the closed vector subspace $\bigoplus_{n\in\mathbb{N}} C^\infty_{K_n}(M, TM)$ of $\bigoplus_{n\in\mathbb{N}} C^\infty_{L_n}(M, TM)$, and hence is C^∞ also as a map into this vector subspace (see Lemma 1.3). We now consider $g \circ \rho \circ \lambda \circ p|_Z$ as a C^∞-map into the open 0-neighbourhood $\bigoplus_{n\in\mathbb{N}} (V \cap C^\infty_{K_n}(M, TM)) \subseteq \bigoplus_{n\in\mathbb{N}} C^\infty_{K_n}(M, TM)$. Then $\Omega := \phi^{-1}(Z)$ is an open identity neighbourhood in $\mathrm{Diff}_c(M)$, and the formula

$$\Phi|_\Omega = \kappa^{-1} \circ g \circ \rho \circ \lambda \circ p|_Z \circ \phi|_\Omega$$

shows that Φ is smooth. This completes the proof of Lemma 5.5.

Remark 6.8. Closer inspection shows that Ω can be chosen of the form $\Omega = \mathrm{Diff}_c(M) \cap \Omega_1$ for some open identity neighbourhood $\Omega_1 \subseteq \mathrm{Diff}^1_c(M)$ in the topological group of compactly supported C^1-diffeomorphisms.

The function Φ in the Fragmentation Lemma is not unique, and in fact various constructions give rise to such functions. The simple construction used in this section has been adapted from [32].

7. Example: Test function groups

In this section, we prove the results concerning direct limit properties of test function groups described in the Introduction. More generally, we discuss $C^r_c(M, H)$ for $r \in \mathbb{N}_0 \cup \{\infty\}$ and H an arbitrary (not necessarily finite-dimensional) smooth or $\mathbb{K}$-analytic Lie group.

7.1. We recall: if $r \in \mathbb{N}_0 \cup \{\infty\}$ and $s \in \{\infty, \omega\}$, M is a σ-compact finite-dimensional $C^r_{\mathbb{R}}$-manifold and H a $C^s_{\mathbb{K}}$-Lie group, then the group $C^r_c(M, H)$ of all compactly supported H-valued $C^r_{\mathbb{R}}$-maps on M is a $C^s_{\mathbb{K}}$-Lie group, modelled on the locally convex direct limit $C^r_c(M, L(H)) = \varinjlim C^r_K(M, L(H))$. Its Lie group structure is characterized by the following property:

Let $\tilde{\phi}: \tilde{U} \to \tilde{V} \subseteq L(H)$ be a chart of H around 1 such that $\tilde{\phi}(1) = 0$, and $U \subseteq \tilde{U}$ be an open, symmetric identity neighbourhood such that $UU \subseteq \tilde{U}$. Set $V := \tilde{\phi}(U)$ and $\phi := \tilde{\phi}|_U^V$. Then $C^r_c(M, U) := \{\gamma \in C^r_c(M, H)\colon \gamma(M) \subseteq U\}$ is open in $C^r_c(M, H)$ and

$$C^r_c(M, \phi): C^r_c(M, U) \to C^r_c(M, V), \quad \gamma \mapsto \phi \circ \gamma$$

is a chart for $C^r_c(M, H)$ (cf. [14, Section 4.2]).

Remark 7.2. Let $K_1 \subseteq K_2 \subseteq \cdots$ be an exhaustion of M by compact sets K_n. Since $C^r_{K_n}(M, \phi)$: $C^r_{K_n}(M, U) \to C^r_{K_n}(M, V) \subseteq C^r_{K_n}(M, L(H))$ is a chart of $C^r_{K_n}(M, H)$ for each $n \in \mathbb{N}$ (see [14, Section 3.2]), where $C^r_{K_n}(M, U) = C^r_c(M, U) \cap C^r_{K_n}(M, H)$, we deduce that $C^r_c(M, \phi)$ is a strict direct limit chart for $C^r_c(M, H) = \bigcup_{n \in \mathbb{N}} C^r_{K_n}(M, H)$.

We begin our discussion of direct limit properties with the negative results.

Proposition 7.3. *Let M be a σ-compact, non-compact, finite-dimensional smooth manifold of positive dimension and H be a non-discrete Lie group whose locally convex modelling space is smoothly regular* (*for instance, a finite-dimensional Lie group*). *Then there exists a discontinuous map*

$$f: C^\infty_c(M, H) \to C^\infty_c(M, \mathbb{R})$$

whose restriction to $C^\infty_K(M, H)$ is $C^\infty_{\mathbb{R}}$, for each compact set $K \subseteq M$. Hence $C^\infty_K(M, H) \neq \varinjlim C^\infty_K(M, H)$ as a topological space and as a $C^s_{\mathbb{R}}$-manifold, for any $s \in \mathbb{N}_0 \cup \{\infty\}$.

The proof uses the following variant of Lemma 3.1.

Lemma 7.4. *Given $r \in \mathbb{N}_0 \cup \{\infty\}$, let M be a σ-compact finite-dimensional C^r-manifold, H be a Lie group modelled on a locally convex space which is smoothly regular, and $P \subseteq H$ be an open identity neighbourhood. Then there exists a smooth map $\rho: C^r_c(M, H) \to C^r_c(M, H)$ with the following properties*:

(a) *the image of ρ is contained in $C^r_c(M, P)$*;
(b) *there exists an open identity neighbourhood $Q \subseteq P$ such that $\rho(\gamma) = \gamma$ for each $\gamma \in C^r_c(M, Q)$*; *and*

(c) *ρ restricts to a smooth map from $C^r_K(M,H)$ to $C^r_K(M,P) \subseteq C^r_K(M,H)$, for each compact subset $K \subseteq M$.*

Proof. Using Lemma 3.1, we find a smooth map $f : H \to P$ and an open identity neighbourhood $Q \subseteq P$ such that $f|_Q = \mathrm{id}_Q$. By [14, Propositions 3.20 and 4.20], the map

$$\rho := C^r_{\mathrm{c}}(M,f) : C^r_{\mathrm{c}}(M,H) \to C^r_{\mathrm{c}}(M,H), \quad \gamma \mapsto f \circ \gamma$$

is smooth and induces smooth self-maps of $C^r_K(M,H)$ for each compact subset $K \subseteq M$. By construction, it also has all other desired properties. □

Proof of Proposition 7.3. By [23, Proposition 3.1], there exists a mapping $h : C^\infty(M, L(H)) \to C^\infty_{\mathrm{c}}(M,\mathbb{R})$ which is discontinuous at 0, and such that $h|_{C^\infty_K(M,L(H))}$ is smooth for each compact subset $K \subseteq M$. Choose a chart $\phi : U \to V \subseteq L(H)$ of H around 1 such that $\phi(1) = 0$ and such that $\Psi := C^\infty_{\mathrm{c}}(M,\phi) : C^\infty_{\mathrm{c}}(M,U) \to C^\infty_{\mathrm{c}}(M,V)$ is a chart of $C^\infty_{\mathrm{c}}(M,H)$ and restricts to a chart of $C^\infty_K(M,H)$ with domain $C^\infty_K(M,U)$, for each compact subset $K \subseteq M$. Let $P \subseteq U$ and $\rho : C^\infty_{\mathrm{c}}(M,H) \to C^\infty_{\mathrm{c}}(M,P)$ be as in Lemma 7.4. Then the map

$$f := h \circ \Psi \circ \rho : C^\infty_{\mathrm{c}}(M,H) \to C^\infty_{\mathrm{c}}(M,\mathbb{R})$$

is discontinuous at 1. Since ρ restricts to a smooth map from $C^\infty_K(M,H)$ to $C^\infty_K(M,U)$ and Ψ to a smooth map from $C^\infty_K(M,U)$ to $C^\infty_K(M,V) \subseteq C^\infty_K(M,L(H))$, it follows that h is smooth on each $C^\infty_K(M,H)$. □

Remark 7.5. Given $r \in \mathbb{N}_0 \cup \{\infty\}$, let M be a σ-compact, non-compact, finite-dimensional C^r-manifold of positive dimension, and H be a Lie group modelled on a metrizable locally convex space $\neq \{0\}$. Then $C^r_K(M,H)$ is metrizable (cf. [27, Proposition 4.19(c) and (d)]). Using Proposition 3.6(i), we deduce that $C^r_{\mathrm{c}}(M,H) \neq \varinjlim C^r_K(M,H)$ as a topological space.

Next, we establish the positive results.

Proposition 7.6. *Let M be a σ-compact, finite-dimensional $C^r_{\mathbb{R}}$-manifold, where $r \in \mathbb{N}_0 \cup \{\infty\}$, and H be a $C^\infty_{\mathbb{K}}$-Lie group modelled on a locally convex space. Then $C^r_{\mathrm{c}}(M,H) = \varinjlim C^r_K(M,H)$ as a topological group and as a $C^\infty_{\mathbb{K}}$-Lie group.*

The next lemma (proved in Section 8) helps us to prove Proposition 7.6.

Lemma 7.7 *(Fragmentation Lemma for test function groups). For $r \in \mathbb{N}_0 \cup \{\infty\}$ and $s \in \{\infty, \omega\}$, let H be a $C^s_{\mathbb{K}}$-Lie group modelled on a locally convex space and M be a σ-compact, finite-dimensional $C^r_{\mathbb{R}}$-manifold.*

(a) *Then there exists a locally finite cover $(K_n)_{n\in\mathbb{N}}$ of M by compact sets, an open identity neighbourhood $\Omega \subseteq C^r_{\mathrm{c}}(M,H)$ and a $C^s_{\mathbb{K}}$-map*

$$\Phi : \Omega \to \prod_{n\in\mathbb{N}}^{*} C^r_{K_n}(M,H), \quad \gamma \mapsto \Phi(\gamma) =: (\gamma_n)_{n\in\mathbb{N}}$$

such that $\Phi(1) = 1$ and $\gamma = \gamma_1\gamma_2\cdots\gamma_n$ for each $\gamma \in \Omega$ and each sufficiently large n.

(b) *If* $(U_n)_{n\in\mathbb{N}}$ *is a locally finite cover of* M *by relatively compact, open sets, then it can be achieved in* (a) *that* $K_n \subseteq U_n$ *for each* $n \in \mathbb{N}$.

Remark 7.8. Considering a finite-dimensional Lie group as a real analytic Lie group, Lemma 7.7 provides a *real analytic* fragmentation map in this case.

Proof of Proposition 7.6. By Remark 7.2, the hypotheses of Proposition 2.6 are satisfied by $C^r_c(M,H)$ and any cofinal subsequence of its Lie subgroups $C^r_K(M,H)$. Therefore, we only need to show that $C^r_c(M,H) = \varinjlim C^r_K(M,H)$ as a topological group. Using Lemma 7.7 instead of Lemma 5.5, we can show this exactly as in the proof of Proposition 5.4. □

Remark 7.9. While the case of a complex Lie group H is included in Proposition 7.6, we had to exclude it from Proposition 7.3, and the direct limit properties of $C^r_c(M,H)$ in the category of complex manifolds remain elusive (because localization arguments do not work in the complex case). The following example makes it clear that the direct limit property can fail in some cases (but there is no argument for the general case). We consider the map

$$f : C^\infty_c(M,\mathbb{C}) \to C^\infty_c(M\times M,\mathbb{C}), \quad \gamma \mapsto \gamma\otimes\gamma$$

with $(\gamma\otimes\gamma)(x,y) := \gamma(x)\gamma(y)$, which is a homogeneous polynomial of degree 2. It is clear from [15, Proposition 7.1] and [14, Lemma 3.7] that the restriction of f to $C^\infty_K(M,\mathbb{C})$ is a continuous homogeneous polynomial of degree 2 and hence complex analytic. However, f is discontinuous because the symmetric bilinear map $C^\infty_c(M,\mathbb{C})^2 \to C^\infty_c(M\times M,\mathbb{C})$ associated to f via polarization is discontinuous (cf. [53, Theorem 2.4]).

Remark 7.10. If H is a real analytic Lie group, then also $C^r_c(M,H)$ and $C^r_K(M,H)$ are real analytic Lie groups, and one may ask whether $C^r_c(M,H) = \varinjlim C^r_K(M,H)$ as a $C^\omega_{\mathbb{R}}$-Lie group or as a $C^\omega_{\mathbb{R}}$-manifold. However, since real analyticity is an even more delicate property than complex analyticity, one cannot expect results except in special situations. We therefore refrain from any attempt in this direction, and merely remind the reader of a notorious pathology: already on $\mathbb{R}^{(\mathbb{N})} = C^r_c(\mathbb{N},\mathbb{R})$, non-analytic real-valued functions exist which are $C^\omega_{\mathbb{R}}$ on $\mathbb{R}^n = C^r_{\{1,\ldots,n\}}(\mathbb{N},\mathbb{R})$ for each $n\in\mathbb{N}$ [39, Example 10.8].

8. Proof of the Fragmentation Lemma for test function groups

This section is devoted to the proof of the Fragmentation Lemma for test function groups (Lemma 7.7). We proceed in steps.

8.1. Let $(U_n)_{n\in\mathbb{N}}$ be a locally finite cover of M be relatively compact, open subsets $U_n\subseteq M$, and $(h_n)_{n\in\mathbb{N}}$ be a C^r-partition of unity of M such that $K_n := \operatorname{supp}(h_n)\subseteq U_n$ for each $n\in\mathbb{N}$. For each $n\in\mathbb{N}_0$, we set $s_n := \sum_{i=1}^n h_i$. We define $U_0 := K_0 := \emptyset$. For each $n\in\mathbb{N}$, we set $W_n := U_n\cup U_{n-1}$ and choose $\xi_n\in C^r_c(W_n,\mathbb{R})$ such that $\xi_n(W_n)\subseteq[0,1]$ and $\xi_n|_{K_n\cup K_{n-1}} = 1$. We abbreviate $L_n := \operatorname{supp}(\xi_n)$.

8.2. Pick a chart $\phi'' : U'' \to V'' \subseteq L(H)$ of H around 1 such that $\phi''(1) = 0$. Let $U'\subseteq U$ be an open, symmetric identity neighbourhood with $U'U'\subseteq U''$. Set $V' := \phi''(U')$ and $\phi' := \phi''|^{V'}_{U'}$. Then $C^r_c(M,\phi') : C^r_c(M,U') \to C^r_c(M,V') \subseteq C^r_c(M,L(H))$ is a chart of $C^r_c(M,H)$ and

$C^r_K(M,\phi')\colon C^r_K(M,U') \to C^r_K(M,V') \subseteq C^r_K(M,L(H))$ is a chart of $C^r_K(M,H)$, for each compact subset $K \subseteq M$. Let $U \subseteq U'$ be an open, symmetric identity neighbourhood with $UU \subseteq U'$, and set $V := \phi'(U)$ and $\phi := \phi'|_U^V$. Then the map

$$\kappa := \bigoplus_{n\in\mathbb{N}} C^r_{K_n}(M,\phi)\colon \prod_{n\in\mathbb{N}}^{*} C^r_{K_n}(M,U) \to \bigoplus_{n\in\mathbb{N}} C^r_{K_n}(M,V), \tag{6}$$

$(\eta_n)_{n\in\mathbb{N}} \mapsto (\phi\circ\eta_n)_{n\in\mathbb{N}}$ is a chart of $\prod^*_{n\in\mathbb{N}} C^r_{K_n}(M,H)$ around 1 (see 4.2).

8.3. There exists an open, symmetric identity neighbourhood $P \subseteq U$ such that $PP \subseteq U$. We set $Q := \phi(P)$, let $S \subseteq Q$ be an open 0-neighbourhood such that $[0,1]S = S$, and define $R := \phi^{-1}(S)$. In local coordinates, the group multiplication of $C^r_c(M,H)$ corresponds to the map $\mu\colon Q\times Q \to V$, $\mu(\sigma,\tau) := \sigma * \tau := \phi(\phi^{-1}(\sigma)\phi^{-1}(\tau))$. We write $\tau^{-1} := \phi(\phi^{-1}(\tau)^{-1})$ for $\tau \in Q$.

8.4. Given $\gamma \in C^r_c(M,R)$ and $h \in C^r_c(M,\mathbb{R})$ such that $h(M) \subseteq [0,1]$, we define $h \odot \gamma \in C^r_c(M,R)$ via $(h\odot\gamma)(x) := \phi^{-1}(h(x)\cdot\phi(\gamma(x)))$. For each $n\in\mathbb{N}$, set $\gamma_n := (s_{n-1}\odot\gamma)^{-1}(s_n\odot\gamma) \in C^r_c(M,U)$. Since $(s_n\odot\gamma)(x) = (s_{n-1}\odot\gamma)(x)$ for all $x \in M\setminus K_n$, we have $\gamma_n \in C^r_{K_n}(M,U)$. Given γ, there is $N\in\mathbb{N}$ such that $U_n \cap \operatorname{supp}(\gamma) = \emptyset$ for all $n > N$. Then $s_n \odot \gamma = \gamma$ for all $n \geqslant N$, whence $\gamma_1\cdots\gamma_n = \gamma$ for all $n \geqslant N$. Thus

$$\Phi\colon C^r_c(M,R) \to \prod_{n\in\mathbb{N}}^{*} C^r_{K_n}(M,H), \quad \Phi(\gamma) := (\gamma_n)_{n\in\mathbb{N}}$$

will have the desired properties, if we can show that this map is $C^s_{\mathbb{K}}$.

8.5. Since $(W_n)_{n\in\mathbb{N}}$ is a locally finite cover of M by relatively compact, open sets, the map

$$p\colon C^r_c\big(M,L(H)\big) \to \bigoplus_{n\in\mathbb{N}} C^r\big(W_n,L(H)\big), \quad \gamma \mapsto (\gamma|_{W_n})_{n\in\mathbb{N}}$$

is continuous linear (and in fact an embedding onto a closed vector subspace, see [16] or [27, Proposition 8.13]). Because $C^r(W_n,L(H))$ is a topological $C^r(W_n,\mathbb{R})$-module (see [16] or [27, Proposition 9.1(b)]), the multiplication operators $\mu_n\colon C^r(W_n,L(H)) \to C^r_{L_n}(W_n,L(H))$, $\gamma \mapsto \xi_n\cdot s_n|_{W_n}\cdot\gamma$ and $\lambda_n\colon C^r(W_n,L(H)) \to C^r_{L_n}(W_n,L(H))$, $\gamma\mapsto \xi_n\cdot s_{n-1}|_{W_n}\cdot\gamma$ are continuous linear. Identifying $C^r_{L_n}(W_n,L(H))^2$ with $C^r_{L_n}(W_n,L(H)\times L(H))$ in the natural way (cf. [14, Lemma 3.4]), we can consider (λ_n,μ_n) as a continuous linear map into $C^r_{L_n}(W_n,L(H)^2)$. Then

$$\lambda := \bigoplus_{n\in\mathbb{N}} (\lambda_n,\mu_n)\colon \bigoplus_{n\in\mathbb{N}} C^r\big(W_n,L(H)\big) \to \bigoplus_{n\in\mathbb{N}} C^r_{L_n}\big(W_n,L(H)^2\big)$$

is continuous linear. Now $\lambda\circ p$ is a continuous linear map which restricts to a $C^s_{\mathbb{K}}$-map $f\colon C^r_c(M,S) \to \bigoplus_{n\in\mathbb{N}} C^r_{L_n}(W_n,S\times S)$. Now define $g_n\colon C^r_{L_n}(W_n,S\times S) \to C^r_{L_n}(W_n,V)$ via

$$g_n(\tau,\sigma) := \tau^{-1} * \sigma,$$

using local inversion, respectively, the local multiplication $*$ pointwise. Then each g_n is $C^s_{\mathbb{K}}$, as a consequence of [14, Corollaries 3.11 and 3.12], and thus also

$$g := \bigoplus_{n\in\mathbb{N}} g_n : \bigoplus_{n\in\mathbb{N}} C^r_{L_n}(W_n, S) \to \bigoplus_{n\in\mathbb{N}} C^r_{L_n}(W_n, V)$$

is $C^s_{\mathbb{K}}$ by [19, Proposition 7.1 and Corollary 7.2]. Note that $g \circ \lambda \circ p$ has image in the closed vector subspace $\bigoplus_{n\in\mathbb{N}} C^r_{K_n}(W_n, L(H))$ of $\bigoplus_{n\in\mathbb{N}} C^r_{L_n}(W_n, L(H))$; we may therefore consider $g \circ f$ as a $C^s_{\mathbb{K}}$-map into $\bigoplus_{n\in\mathbb{N}} C^r_{K_n}(W_n, V)$ now (by Lemma 1.3). For each $n \in \mathbb{N}$, the map $\rho_n : C^r_{K_n}(M, L(H)) \to C^r_{K_n}(W_n, L(H))$, $\gamma \mapsto \gamma|_{W_n}$ is an isomorphism of locally convex spaces (see [27, Lemma 4.24] or [16]), whence also

$$\psi := \bigoplus_{n\in\mathbb{N}} \rho_n^{-1} : \bigoplus_{n\in\mathbb{N}} C^r_{K_n}\big(W_n, L(H)\big) \to \bigoplus_{n\in\mathbb{N}} C^r_{K_n}\big(M, L(H)\big)$$

is an isomorphism of locally convex spaces. Consequently, the composition

$$\Phi = \kappa^{-1} \circ \psi \circ g \circ f \circ C^r_c\big(M, \phi|^S_R\big)$$

is a $C^s_{\mathbb{K}}$-map from $C^r_c(M, R)$ to $\prod^*_{n\in\mathbb{N}} C^r_{K_n}(M, H)$, where κ is as in (6). This completes the proof of Lemma 7.7.

Remark 8.6. The proof shows that $(K_n)_{n\in\mathbb{N}}$ can be chosen independently of r and H, and that Ω can be chosen of the form $C^r_c(M, U)$.

9. Direct limit properties of Lie groups modelled on Silva spaces or k_ω-spaces

We describe conditions ensuring that a Lie group $G = \bigcup_{n\in\mathbb{N}} G_n$ carries the direct limit topology and is the direct limit in all categories of interest. In particular, the result applies to many typical examples of Lie groups modelled on Silva spaces. We also obtain information on certain Lie groups modelled on k_ω-spaces.

9.1. Recall that a locally convex space E is called a *Silva space* (or (*LS*)-space) if it is the locally convex direct limit $E = \bigcup_{n\in\mathbb{N}} E_n = \varinjlim E_n$ of a sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces such that each inclusion map $E_n \to E_{n+1}$ is a compact linear operator. Then E is Hausdorff (cf. [11, Section 7.3, Satz]) and $E = \varinjlim E_n$ as a topological space [11, Section 7.1, Satz]. The ascending sequence $(E_n)_{n\in\mathbb{N}}$ can always be chosen such that, for a suitable norm on E_n defining its topology, all closed balls $\overline{B}^{E_n}_r(x)$, $x \in E_n$, $r > 0$, are compact in E_{n+1} (cf. [36, Section 7.3, Proposition 1]). It is clear from the definition that finite direct products of Silva spaces are Silva spaces; this will be useful later.

9.2. A Hausdorff topological space X is called a *k_ω-space* if there exists an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets of X such that $X = \bigcup_{n\in\mathbb{N}} K_n$ and $U \subseteq X$ is open if and only if $U \cap K_n$ is open in K_n, for each $n \in \mathbb{N}$ (i.e., $X = \varinjlim K_n$ as a topological space). Then $(K_n)_{n\in\mathbb{N}}$ is called a *k_ω-sequence* for X. For background information concerning k_ω-spaces with a view towards direct limit constructions, see [30].

Example 9.3. The dual space E' of every metrizable locally convex space E is a k_ω-space when equipped with the topology of compact convergence (cf. [1, Corollary 4.7 and Proposition 5.5]). We write E'_{c} if this topology is used.

Example 9.4. Every Silva space E is a k_ω-space. In fact, E is reflexive by [11, Section 9, Satz 6] and thus $E \cong (E'_{\mathrm{b}})'_{\mathrm{b}}$, using the topology of bounded convergence. Since E'_{b} is a Fréchet–Schwartz space [11, Section 9, Satz 6], bounded subsets in E'_{b} are relatively compact and hence $(E'_{\mathrm{b}})'_{\mathrm{b}} = (E'_{\mathrm{b}})'_{\mathrm{c}}$. But $(E'_{\mathrm{b}})'_{\mathrm{c}}$ is k_ω, by Example 9.3.

Example 9.5. If E is an infinite-dimensional Banach space (or, more generally, a Fréchet space which is not a Schwartz space), then E'_{c} is a locally convex space which is a k_ω-space but not a Silva space. In fact, if E'_{c} was Silva, then $(E'_{\mathrm{c}})'_{\mathrm{b}}$ would be a Fréchet–Schwartz space [11, Section 9, Satz 6]. Here $(E'_{\mathrm{c}})'_{\mathrm{b}} = (E'_{\mathrm{c}})'_{\mathrm{c}}$ since bounded subsets of Silva spaces are relatively compact [11, Section 7.6]. Since $E \cong (E'_{\mathrm{c}})'_{\mathrm{c}}$ holds for every Fréchet space [2, Proposition 15.2], we deduce that E is a Fréchet–Schwartz space, contradicting our hypotheses.

The following facts concerning k_ω-spaces will be used.

Lemma 9.6.

(a) *If X and Y are k_ω-spaces, then also $X \times Y$.*
(b) *Let $((X_n)_{n\in\mathbb{N}}, (i_{n,m})_{n\geqslant m})$ be a direct sequence of k_ω-spaces and continuous maps $i_{n,m} : X_m \to X_n$, with direct limit Hausdorff topological space X. Then X is a k_ω-space. If each $i_{n,m}$ is injective, then the direct limit topological space $\varinjlim X_n$ is Hausdorff.*
(c) *Let $((E_n)_{n\in\mathbb{N}}, (i_{n,m})_{n\geqslant m})$ be a direct sequence of locally convex spaces which are k_ω-spaces, and continuous linear maps $i_{n,m} : E_m \to E_n$. Then the Hausdorff locally convex direct limit E coincides with the direct limit Hausdorff topological space* (*as discussed in* (b)).

Proof. (a) See, e.g., [30, Proposition 4.2(c)].

(b) The case of injective direct sequences is covered by [30, Proposition 4.5]. In the general case, let $i_n : X_n \to X$ be the limit map, and $\overline{X}_n := i_n(X_n)$, equipped with the quotient topology, which is Hausdorff and hence k_ω by [30, Proposition 4.2]. Then $X = \varinjlim \overline{X}_n$ as a topological space, and so X is k_ω by [30, Proposition 4.5].

(c) As in (b), after passing to Hausdorff quotients we may assume that each i_n (and $i_{n,m}$) is injective. But this case is [30, Proposition 7.12]. □

The following situation arises frequently. We are given a map $f : U \to F$, where E and F are Hausdorff locally convex topological $\mathbb{K}$-vector spaces and U a subset of E; we would like to show that U is open and f is $C^r_{\mathbb{K}}$ for some $r \in \mathbb{N}_0 \cup \{\infty\}$. We are given the following information: E is the Hausdorff locally convex direct limit of a sequence $((E_n)_{n\in\mathbb{N}}, (i_{n,m})_{n\geqslant m})$ of Hausdorff locally convex spaces E_n and continuous homomorphisms $i_{n,m} : E_m \to E_n$, with limit maps $i_n : E_n \to E$. Also, $U = \bigcup_{n\in\mathbb{N}} i_n(U_n)$, where $U_n \subseteq E_n$ is an open subset such that $i_{n,m}(U_m) \subseteq U_n$ if $n \geqslant m$. Finally, we assume that $f_n := f \circ i_n|_{U_n} : U_n \to F$ is $C^r_{\mathbb{K}}$ for each $n \in \mathbb{N}$.

Lemma 9.7. *In the preceding situation, suppose that* (a) *or* (b) *holds*:

(a) *Each E_n is a Banach space and each of the linear maps $i_{n,m}$ is compact.*

(b) *Each E_n is a Silva space or, more generally, a k_ω-space.*

Then U is open in E and f is $C^r_{\mathbb{K}}$.

Proof. Let N_n be the kernel of the limit map $E_n \to E$. Then $\overline{E}_n := E_n/N_n$ is a Banach space (respectively, k_ω-space). Let $q_n : E_n \to \overline{E}_n$ be the quotient map, and $\bar{i}_{n,m} : \overline{E}_m \to \overline{E}_n$ be the continuous linear map determined by $\bar{i}_{n,m} \circ q_m = q_n \circ i_{n,m}$ (which is again a compact operator in case (a)). Then $E = \varinjlim \overline{E}_n$ as a locally convex space, together with the continuous linear maps $\bar{i}_n : \overline{E}_n \to E$ determined by $\bar{i}_n \circ q_n = i_n$. The set $\overline{U}_n := q_n(U_n)$ is open in $\overline{E}_n$ and f_n factors to a map $g_n := f \circ \bar{i}_n : \overline{U}_n \to F$ determined by $g_n \circ q_n|_{U_n} = f_n$, which is $C^r_{\mathbb{K}}$ by [4, Lemma 10.4]. Furthermore, $U = \bigcup_{n\in\mathbb{N}} \bar{i}_n(\overline{U}_n)$. After replacing E_n by $\overline{E}_n$, we may thus assume now that each $i_{n,m}$ is injective.

Recall that $E = \varinjlim E_n$ as a topological space (see 9.1 in the situation of (a), respectively, Lemma 9.6(b) and (c) in the situation of (b)). Hence U is open in E and $U = \varinjlim U_n$, by Lemma 1.7.

To see that f is $C^r_{\mathbb{K}}$, we may assume that $r \in \mathbb{N}_0$, and proceed by induction. If $r = 0$, then f is continuous since $U = \varinjlim U_n$ as a topological space. Now assume that the assertion holds for r and that each $f \circ i_n$ is $C^{r+1}_{\mathbb{K}}$. Then f is continuous. Given $x' \in U$ and $y' \in E$, there exists $n \in \mathbb{N}$ and $x \in U_n$, $y \in E_n$ such that $x' = i_n(x)$, $y' = i_n(y)$. Then the directional derivative $d(f \circ i_n)(x, y) = \frac{d}{dt}|_{t=0} f(i_n(x + ty)) = \frac{d}{dt}|_{t=0} f(i_n(x) + t i_n(y)) = \frac{d}{dt}|_{t=0} f(x' + ty') = df(x', y')$ exists. The preceding calculation shows that

$$df \circ (i_n \times i_n)|_{U_n \times E_n} = d(f \circ i_n) \tag{7}$$

for each $n \in \mathbb{N}$, which is a $C^r_{\mathbb{K}}$-map. Since $E \times E = \varinjlim E_n \times E_n$ is a locally convex direct limit of the form described in (a), respectively, (b) (see 9.1 respectively, Lemma 9.6(a)), the map df is $C^r_{\mathbb{K}}$ by induction and hence f is $C^{r+1}_{\mathbb{K}}$. □

Note that E in Lemma 9.7 is a k_ω-space, by 9.4, respectively, Lemma 9.6(b) and (c).

Proposition 9.8. *Let $G = \bigcup_{n\in\mathbb{N}} G_n$ be a $C^\infty_{\mathbb{K}}$-Lie group admitting a direct limit chart and assume that at least one of the following conditions is satisfied:*

(i) *G_n is a Banach-Lie group for each $n \in \mathbb{N}$, and the inclusion map $L(G_m) \to L(G_n)$ is a compact linear operator, for all $m < n$.*
(ii) *$L(G_n)$ is a k_ω-space, for each $n \in \mathbb{N}$.*

Then $G = \varinjlim G_n$ as a topological space, topological group, $C^\infty_{\mathbb{K}}$-Lie group, and as a $C^r_{\mathbb{K}}$-manifold, for each $r \in \mathbb{N}_0 \cup \{\infty\}$.

Proof. Let $\phi : U \to V \subseteq L(G)$ be a direct limit chart around 1, where $U = \bigcup_{n\in\mathbb{N}} U_n$, $V = \bigcup_{n\in\mathbb{N}} V_n$ and $\phi = \bigcup_{n\in\mathbb{N}} \phi_n$ for charts $\phi_n : U_n \to V_n \subseteq L(G_n)$ of G_n around 1. Suppose that $f : G \to X$ is a map to a topological space (respectively, $C^r_{\mathbb{K}}$-manifold) X, such that $f|_{G_n}$ is continuous (respectively, $C^r_{\mathbb{K}}$), for each $n \in \mathbb{N}$. Then $(f \circ \phi)|_{V_n}$ is continuous (respectively, $C^r_{\mathbb{K}}$) for each $n \in \mathbb{N}$ and hence $f \circ \phi$ is continuous (respectively, $C^r_{\mathbb{K}}$) and thus also $f|_U$, by Lemma 9.7. Given $x \in G$, applying the same argument to $h : G \to X$, $h(y) := f(xy)$, we see that $h|_U$ (and hence also $f|_{xU}$) is continuous, respectively, $C^r_{\mathbb{K}}$. Hence f is continuous (respectively, $C^r_{\mathbb{K}}$). We have shown that $G = \varinjlim G_n$ as a topological space and as a $C^r_{\mathbb{K}}$-manifold. The remaining direct limit properties follow. $\square$

10. Example: Groups of germs of Lie group-valued analytic maps

In this section, we begin our discussion of the Lie group $\Gamma(K, H)$ of germs of analytic mappings with values in a Banach–Lie group H, where K is a non-empty compact subset of a metrizable locally convex space X. For X and H finite-dimensional, $\Gamma(K, H)$ is modelled on a Silva space, and we obtain a prime example for the type of direct limit groups just discussed in Proposition 9.8(i). This facilitates a complete clarification of the direct limit properties of $\Gamma(K, H)$ (Proposition 10.6). In Section 11, we develop tools to tackle $\Gamma(K, H)$ also for infinite-dimensional X and H (see Section 13).

10.1. Let H be a Banach–Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $K \neq \emptyset$ a compact subset of a metrizable locally convex topological $\mathbb{K}$-vector space X. Then the group $G := \Gamma(K, H)$ of germs $[\gamma]$ of $\mathbb{K}$-analytic maps $\gamma : U \to H$ on open neighbourhoods $U \subseteq X$ of K is a $C^\omega_{\mathbb{K}}$-Lie group in a natural way, with the multiplication of germs induced by pointwise multiplication of functions (see [20]). We now recall the relevant aspects of the construction of the Lie group structure, starting with the case $\mathbb{K} = \mathbb{C}$. In this case, $\Gamma(K, H)$ is modelled on the locally convex direct limit $\Gamma(K, L(H)) = \varinjlim \mathrm{Hol}_{\mathrm{b}}(U_n, L(H))$, where $U_1 \supseteq U_2 \supseteq \cdots$ is a fundamental sequence of open neighbourhoods of K. Here $\mathrm{Hol}_{\mathrm{b}}(U_n, L(H)) =: A_n$ is the Banach space of bounded holomorphic functions from U_n to $L(H)$, equipped with the supremum norm $\|\cdot\|_{A_n}$. The space $\Gamma(K, H)$ is Hausdorff [20, Section 2]. We can (and will always) assume that each connected component of U_n meets K; then all bonding maps $j_{n,m} : \mathrm{Hol}_{\mathrm{b}}(U_m, L(H)) \to \mathrm{Hol}_{\mathrm{b}}(U_n, L(H))$, $\gamma \mapsto \gamma|_{U_n}$, $n \geqslant m$, are injective and hence also all limit maps $j_n : \mathrm{Hol}_{\mathrm{b}}(U_n, L(H)) \to \Gamma(K, L(H))$, $\gamma \mapsto [\gamma]$. We occasionally identify $\gamma \in \mathrm{Hol}_{\mathrm{b}}(U_n, L(H))$ with $j_n(\gamma) = [\gamma]$.

10.2. If $\mathbb{K} = \mathbb{R}$, choose open neighbourhoods U_n as before and set $\tilde{U}_n := U_n + iV_n \subseteq X_{\mathbb{C}}$, where $(V_n)_{n\in\mathbb{N}}$ is a basis of open, balanced 0-neighbourhoods in X. We set $C_n := \{\gamma \in \mathrm{Hol}_{\mathrm{b}}(\tilde{U}_n, L(H)_{\mathbb{C}}): \gamma(U_n) \subseteq L(H)\}$ and

$$A_n := \{\gamma|_{U_n}: \gamma \in C_n\}.$$

Then C_n is a closed real vector subspace of $\mathrm{Hol}_{\mathrm{b}}(\tilde{U}_n, L(H)_{\mathbb{C}})$. Because $\gamma \in \mathrm{Hol}_{\mathrm{b}}(\tilde{U}_n, L(H)_{\mathbb{C}})$ is uniquely determined by $\gamma|_{U_n}$, we see that $\|\gamma|_{U_n}\|_{A_n} := \|\gamma\|_\infty$ (supremum-norm) for γ as before with $\gamma(U_n) \subseteq L(H)$ defines a norm $\|\cdot\|_{A_n}$ on A_n making it a Banach space isomorphic to C_n. We give $\Gamma(K, L(H))$ the vector topology making it the locally convex direct limit $\varinjlim A_n$.

10.3. We may assume that the norm $\|.\|$ on $\mathfrak{h} := L(H)$ defining its topology has been chosen such that $\|[x, y]\| \leqslant \|x\| \cdot \|y\|$ for all $x, y \in \mathfrak{h}$. Choose $\varepsilon \in]0, \frac{1}{2}\log 2]$ such that $\exp_H|_{B^{\mathfrak{h}}_\varepsilon(0)}$ is a diffeomorphism onto an open identity neighbourhood in H. Then the Baker–Campbell–Hausdorff series converges to a $\mathbb{K}$-analytic function $* : B^{\mathfrak{h}}_\varepsilon(0) \times B^{\mathfrak{h}}_\varepsilon(0) \to \mathfrak{h}$ (see [8, Chapter II, Section 7, No. 2]). We choose $\delta \in]0, \varepsilon]$ such that $B^{\mathfrak{h}}_\delta(0) * B^{\mathfrak{h}}_\delta(0) \subseteq B^{\mathfrak{h}}_\varepsilon(0)$. Define

$$\mathrm{Exp} : \Gamma\big(K, L(H)\big) \to \Gamma(K, H), \quad \mathrm{Exp}\big([\gamma]\big) := [\exp_H \circ \gamma].$$

Then $\Gamma(K,H)$ can be given a $\mathbb{K}$-analytic Lie group structure such that $\Psi := \mathrm{Exp}|_Q$ is a $C^\omega_\mathbb{K}$-diffeomorphism onto an open identity neighbourhood in the group $\Gamma(K,H)$, where $Q := \{[\gamma] \in \Gamma(K, L(H))\colon \gamma(K) \subseteq B^{\mathfrak{h}}_\delta(0)\}$ (see [20, Section 5]). We set $P := \Psi(Q)$ and $\phi := \Psi^{-1}$.

10.4. We now show that $\Gamma(K,H) = \bigcup_{n\in\mathbb{N}} G_n$ for some Banach–Lie groups G_n, and that $\phi\colon P \to Q$ is a direct limit chart. To this end, let $C^\omega(U_n,H)$ be the group of all $\mathbb{K}$-analytic H-valued maps on U_n. The BCH-series defines a $\mathbb{K}$-analytic map $*\colon B^{A_n}_\varepsilon(0) \times B^{A_n}_\varepsilon(0) \to A_n$, such that $B^{A_n}_\delta(0) * B^{A_n}_\delta(0) \subseteq B^{A_n}_\varepsilon(0)$. The mapping $\mathrm{Exp}_n\colon A_n \to C^\omega(U_n,H)$, $\mathrm{Exp}_n(\gamma) := \exp_H \circ \gamma$ is injective on $B^{A_n}_\varepsilon(0)$, and application of point evaluations shows that $\mathrm{Exp}_n(\gamma * \eta) = \mathrm{Exp}_n(\gamma)\,\mathrm{Exp}_n(\eta)$ for all $\gamma,\eta \in B^{A_n}_\varepsilon(0)$. Set $Q_n := B^{A_n}_\delta(0)$ and $P_n := \mathrm{Exp}_n(Q_n)$. Now standard arguments show that the subgroup G^0_n of $C^\omega(U_n,H)$ generated by $\mathrm{Exp}_n(A_n)$ can be made a Banach–Lie group with Lie algebra A_n and such that $\mathrm{Exp}_n|_{Q_n}$ is a $C^\omega_\mathbb{K}$-diffeomorphism onto P_n, which is open in G^0_n (cf. [8, Chapter III, Section 1, No. 9, Proposition 18]). Thus $\phi_n := (\mathrm{Exp}_n|^{P_n}_{Q_n})^{-1}\colon P_n \to Q_n$ is a chart for G^0_n. If $\mathbb{K} = \mathbb{C}$, let G_n be the group of all $\gamma \in C^\omega(U_n,H)$ such that $\sup\{\|\mathrm{Ad}^H_{\gamma(x)}\|\colon x \in U_n\} < \infty$, a condition which ensures that the Lie algebra homomorphism $A_n \to A_n$, $\eta \mapsto (x \mapsto \mathrm{Ad}^H_{\gamma(x)}(\eta(x)))$, is continuous linear. If $\mathbb{K} = \mathbb{R}$, let G_n be the group of all $\gamma \in C^\omega(U_n,H)$ such that $U_n \to \mathrm{Aut}(\mathfrak{h}_\mathbb{C})$, $x \mapsto (\mathrm{Ad}^H_{\gamma(x)})_\mathbb{C}$ has a complex analytic extension $\tilde{U}_n \to \mathrm{Aut}(\mathfrak{h}_\mathbb{C})$ which is bounded. Then $G^0_n \subseteq G_n$, and standard arguments provide a unique $\mathbb{K}$-analytic manifold structure on G_n making it a Banach–Lie group with G^0_n as an open subgroup (cf. [8, Chapter III, Section 1, No. 9, Proposition 18]). The restriction map $i_{n,m}\colon G_m \to G_n$, $\gamma \mapsto \gamma|_{U_n}$ is an injective homomorphism for $n \geqslant m$, which is $\mathbb{K}$-analytic because $\mathrm{Exp}_n \circ j_{n,m} = i_{n,m} \circ \mathrm{Exp}_m$ with $j_{n,m}\colon A_m \to A_n$ continuous linear. Likewise, $i_n\colon G_n \to \Gamma(K,H)$, $\gamma \mapsto [\gamma]$ is an injective homomorphism and $\mathbb{K}$-analytic because $\mathrm{Exp} \circ j_n = i_n \circ \mathrm{Exp}_n$. We identify G_n with its image $i_n(G_n)$ in $\Gamma(K,H)$. Then $\Gamma(K,H) = \bigcup_{n\in\mathbb{N}} G_n$. To see this, let $[\gamma] \in \Gamma(K,H)$. If $\mathbb{K} = \mathbb{C}$, then $U_n \to \mathrm{Aut}(\mathfrak{h})$, $x \mapsto \mathrm{Ad}^H_{\gamma(x)}$ is bounded for some $n \in \mathbb{N}$. If $\mathbb{K} = \mathbb{R}$, then $U_n \to \mathrm{Aut}(\mathfrak{h}_\mathbb{C})$, $x \mapsto (\mathrm{Ad}^H_{\gamma(x)})_\mathbb{C}$ has a bounded complex analytic extension to $\tilde{U}_n$ for some n. Since $Q = \bigcup_{n\in\mathbb{N}} Q_n$, $P = \bigcup_{n\in\mathbb{N}} P_n$ and $\phi = \bigcup_{n\in\mathbb{N}} \phi_n$, we see that ϕ is a direct limit chart.

10.5. If $\dim(X) < \infty$, we assume that U_{n+1} is relatively compact in U_n, for each $n \in \mathbb{N}$ (and V_{n+1} relatively compact in V_n, if $\mathbb{K} = \mathbb{R}$). If, furthermore, $\dim(H) < \infty$, then $L(i_{n,m}) = j_{n,m}\colon A_m \to A_n$, $\gamma \mapsto \gamma|_{U_n}$ is a compact operator whenever $n > m$. If $\mathbb{K} = \mathbb{C}$, this is a simple consequence of Montel's Theorem; if $\mathbb{K} = \mathbb{R}$, it follows from the compactness of the corresponding restriction map $\mathrm{Hol}_\mathrm{b}(\tilde{U}_m, L(H)_\mathbb{C}) \to \mathrm{Hol}_\mathrm{b}(\tilde{U}_n, L(H)_\mathbb{C})$.

Proposition 10.6. *For $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, consider a Lie group of germs $\Gamma(K,H)$ as in* 10.1–10.3 *and let G_n be as in* 10.4. *If X and H are finite-dimensional, then $\Gamma(K,H) = \varinjlim G_n$ in the categories of $C^\infty_\mathbb{K}$-Lie groups, topological groups, topological spaces and $C^r_\mathbb{K}$-manifolds, for each $r \in \mathbb{N}_0 \cup \{\infty\}$.*

Proof. 10.4 and 10.5 guarantee the conditions of Proposition 9.8(i). □

Remark 10.7. If $\dim(H) < \infty$ and X is an infinite-dimensional Fréchet–Schwartz space, then $\Gamma(K, L(H))$ still is a Silva space (cf. [5, Theorem 7]). The preceding proposition extends to this situation.

Remark 10.8. By Lemma 2.9, we have $\Gamma(K,H)_0 = \bigcup_{n\in\mathbb{N}} G_n^0$, with G_n^0 as in 10.4. Replacing $\Gamma(K,H)$ by its connected component $\Gamma(K,H)_0$ and G_n by G_n^0, all of the results of Proposition 10.6 (and likewise those of Proposition 13.1 and Corollary 13.3) remain valid, by trivial modifications of the proofs. In many cases, $\Gamma(K,H)$ is connected (e.g., if H is connected and K a singleton); then simply $\Gamma(K,H) = \varinjlim G_n^0$ in all relevant categories.

11. Tools to identify direct limits of topological groups

We describe a criterion ensuring that a topological group $G = \bigcup_{n\in\mathbb{N}} G_n$ is the direct limit topological group $\varinjlim G_n$. In combination with Theorem 2.6, this facilitates to identify Lie groups as direct limits in the category of Lie groups, under quite weak hypotheses. The criterion, requiring that "product sets are large," is satisfied in all situations known to the author.

Definition 11.1. Let G be a topological group which is a union $G = \bigcup_{n\in\mathbb{N}} G_n$ of a sequence $G_1 \subseteq G_2 \subseteq \cdots$ of topological groups such that all of the inclusion maps $G_m \to G_n$ and $G_n \to G$ are continuous homomorphisms. We say that *product sets are large in G* if the *product map*

$$\pi : \prod_{n\in\mathbb{N}}^{*} G_n \to G, \quad (g_n)_{n\in\mathbb{N}} \mapsto g_1 g_2 \cdots g_N \quad \text{if } g_n = 1 \text{ for all } n > N,$$

takes identity neighbourhoods in the weak direct product to identity neighbourhoods in G. If the *product map*

$$\tilde{\pi} : \prod_{n\in\mathbb{N}\cup(-\mathbb{N})}^{*} G_n \to G, \quad (g_n)_{n\in\mathbb{N}\cup(-\mathbb{N})} \mapsto g_{-N} \cdots g_{-1} g_1 g_2 \cdots g_N$$

(with N so large that $g_n = 1$ whenever $|n| > N$) takes identity neighbourhoods to such, then we say that *two-sided product sets are large in G*.

Remark 11.2. Thus product sets are large in $G = \bigcup_{n\in\mathbb{N}} G_n$ if and only if $\bigcup_{n\in\mathbb{N}} U_1 U_2 \cdots U_n$ is an identity neighbourhood in G, for each choice of identity neighbourhoods $U_k \subseteq G_k$, $k \in \mathbb{N}$. If product sets are large in G, then also two-sided product sets are large.

The following observation provides first examples with large product sets.

Proposition 11.3. *Consider a topological group $G = \bigcup_{n\in\mathbb{N}} G_n$ such that $G = \varinjlim G_n$ as a topological space. Then product sets are large in G.*

Proof. Consider a product set $U = \bigcup_{n\in\mathbb{N}} U_1 U_2 \cdots U_n$ with U_n an open identity neighbourhood in G_n. Then $U_1 \cdots U_n$ is open in G_n and thus U is open in G, by Lemma 1.7. □

Example 11.4. The Lie groups $G = \bigcup_{n\in\mathbb{N}} G_n$ considered in Proposition 9.8 satisfy the hypothesis of Proposition 11.3, whence product sets are large in G.

The following observation provides more interesting examples.

Remark 11.5. If the product map $\pi : \prod^*_{n\in\mathbb{N}} G_n \to G$ admits a local section $\sigma : U \to \prod^*_{n\in\mathbb{N}} G_n$ on an identity neighbourhood $U \subseteq G$ which is continuous at 1 and takes 1 to 1, then product sets are large in G. This condition is satisfied in particular if π admits a continuous (or smooth) local section σ around $1 \in G$, such that $\sigma(1) = 1$.

Example 11.6. Consider a test function group $C^r_c(M, G)$, where M is a σ-compact finite-dimensional C^r-manifold and G a Lie group modelled on a locally convex space. Let $(A_n)_{n\in\mathbb{N}}$ be any exhaustion of M by compact sets. Then the product map π of $C^r_c(M, G) = \bigcup_{n\in\mathbb{N}} C^r_{A_n}(M, G)$ admits a smooth local section around 1 taking 1 to 1, and hence product sets are large in $C^r_c(M, G) = \bigcup_{n\in\mathbb{N}} C^r_{A_n}(M, G)$. To see this, we use a map $\Phi : C^r_c(M, G) \supseteq \Omega \to \prod_{n\in\mathbb{N}} C^r_{K_n}(M, G)$ as described in Fragmentation Lemma 7.7. Pick a sequence $m_1 < m_2 < \cdots$ of positive integers such that $K_n \subseteq A_{m_n}$ for each $n \in \mathbb{N}$, and let $\psi_n : C^r_{K_n}(M, G) \to C^r_{A_{m_n}}(M, G)$ be the inclusion map, which is a smooth homomorphism. Consider the map $\psi : \prod^*_{n\in\mathbb{N}} C^r_{K_n}(M, G) \to \prod^*_{n\in\mathbb{N}} C^r_{A_n}(M, G)$ sending $\gamma = (\gamma_n)_{n\in\mathbb{N}}$ to $(\eta_k)_{k\in\mathbb{N}}$, where $\eta_k := \gamma_n$ if $k = m_n$ for some (necessarily unique) $n \in \mathbb{N}$ and $\eta_k := 1$ otherwise. Then ψ is smooth (cf. [19, Proposition 7.1]) and $\pi \circ \psi \circ \Phi = \mathrm{id}_\Omega$. Thus $\sigma := \psi \circ \Phi$ is the desired smooth section for π.

Example 11.7. Using Lemma 5.5, the same argument shows that the product map π of $\mathrm{Diff}_c(M) = \bigcup_{n\in\mathbb{N}} \mathrm{Diff}_{A_n}(M)$ admits a smooth local section around 1 which takes 1 to 1, for each σ-compact smooth manifold M and exhaustion $(A_n)_{n\in\mathbb{N}}$ of M by compact sets. Thus product sets are large in $\mathrm{Diff}_c(M) = \bigcup_{n\in\mathbb{N}} \mathrm{Diff}_{A_n}(M)$.

Proposition 11.8. *If two-sided product sets are large in a topological group $G = \bigcup_{n\in\mathbb{N}} G_n$ (in particular, if product sets are large in G), then $G = \varinjlim G_n$ in the category of topological groups.*

Proof. Let $f : G \to H$ be a homomorphism to a topological group H such that $f_n := f|_{G_n}$ is continuous for all $n \in \mathbb{N}$. Consider $h : \prod^*_{n\in\mathbb{N}\cup(-\mathbb{N})} G_{|n|} \to H$,

$$(x_n)_{n\in\mathbb{N}\cup(-\mathbb{N})} \mapsto f_N(x_{-N}) \cdots f_1(x_{-1}) f_1(x_1) \cdots f_N(x_N),$$

with N so large that $x_n = 1$ for all $n \in \mathbb{N} \cup (-\mathbb{N})$ such that $|n| > N$. A simple modification of the proof of Lemma 4.4 shows that h is continuous at 1. Therefore, for each identity neighbourhood $V \subseteq H$ there exists a family $(U_n)_{n\in\mathbb{N}\cup(-\mathbb{N})}$ of identity neighbourhoods $U_n \subseteq G_{|n|}$ such that $h(U) \subseteq V$, for $U := \prod^*_{n\in\mathbb{N}\cup(-\mathbb{N})} U_n$. Let $\tilde{\pi} : \prod^*_{n\in\mathbb{N}\cup(-\mathbb{N})} G_n \to G$ be the product map. Then $h(U) = f(\tilde{\pi}(U))$ where $\tilde{\pi}(U) \subseteq G$ is an identity neighbourhood because two-sided product sets are large in G. As a consequence, f is continuous. □

Remark 11.9. Following [35, Section 3.1], an ascending sequence $G_1 \leqslant G_2 \leqslant \cdots$ of topological groups with continuous inclusion maps is said to satisfy the "passing through assumption" (PTA, for short), if each G_n has a basis of symmetric identity neighbourhoods U such that, for each $m > n$ and identity neighbourhood $V \subseteq G_m$, there exists an identity neighbourhood $W \subseteq G_m$ such that $WU \subseteq UV$ (cf. also [53] for a slightly different, earlier concept). If condition PTA is satisfied, then the two-sided product sets (or "bamboo-shoot neighbourhoods") $\bigcup_{n\in\mathbb{N}} U_n \cdots U_1 U_1 \cdots U_n$ form a basis of identity neighbourhoods for the topology $\mathcal{O}$ on $G = \bigcup_{n\in\mathbb{N}} G_n$ making G the direct limit topological group (cf. [53, Proposition 2.3]). In this case, $\mathcal{O}$ is called the "bamboo-shoot topology" in [53]. Hence *two-sided product sets are large in the direct limit topological group G if condition PTA is satisfied.*

Remark 11.10. Let $G = \bigcup_{n\in\mathbb{N}} G_n$ be a topological group such that $(G_n)_{n\in\mathbb{N}}$ satisfies condition PTA. A priori, this only provides information concerning the direct limit group topology $\mathcal{O}$; it does not help us to see that the *given* topology on G coincides with $\mathcal{O}$.

Proposition 11.11. *Assume that $G_1 \subseteq G_2 \subseteq \cdots$ is an ascending sequence of Banach–Lie groups, such that each inclusion map $G_n \to G_{n+1}$ is a smooth homomorphism. Then $(G_n)_{n\in\mathbb{N}}$ satisfies the PTA.*

Proof. For each $n \in \mathbb{N}$, fix a norm $\|.\|_n$ on $L(G_n)$ defining its topology and such that $\|[x, y]\|_n \leqslant \|x\| \cdot \|y\|$ for all $x, y \in L(G_n)$. The sets $U_\varepsilon^{(n)} := \exp_{G_n}(B_\varepsilon^{L(G_n)}(0))$, $\varepsilon > 0$, form a basis of identity neighbourhoods in G_n. For each $m > n$, the set $B_\varepsilon^{L(G_n)}(0)$ is bounded in $L(G_m)$ and thus $M_m := \sup \|B_\varepsilon^{L(G_n)}(0)\|_m < \infty$. Given $\delta > 0$, set $\tau := e^{-M_m}\delta$. Then $\|\mathrm{Ad}_u^{G_m}(y)\|_m = \|e^{\mathrm{ad}_x^{L(G_m)}}.y\|_m \leqslant e^{M_m}\|y\|_m$ for each $u \in U_\varepsilon^{(n)}$ and $y \in L(G_m)$, where $u = \exp_{G_n}(x)$ with $x \in B_\varepsilon^{L(G_n)}(0)$, say. Thus

$$\mathrm{Ad}_u^{G_m}\big(B_\tau^{L(G_m)}(0)\big) \subseteq B_\delta^{L(G_m)}(0) \quad \text{for each } u \in U_\varepsilon^{(n)}. \tag{8}$$

Given $w \in U_\tau^{(m)}$ and $u \in U_\varepsilon^{(n)}$, say $w = \exp_{G_m}(y)$ with $y \in B_\tau^{L(G_m)}(0)$, we see that $wu = uu^{-1}wu = uu^{-1}\exp_{G_m}(y)u = u\exp_{G_m}(\mathrm{Ad}_{u^{-1}}^{G_m}(y)) \in U_\varepsilon^{(n)}U_\delta^{(m)}$, using (8). Hence $U_\tau^{(m)}U_\varepsilon^{(n)} \subseteq U_\varepsilon^{(n)}U_\delta^{(m)}$. We have verified the PTA. □

12. Example: Unit groups of direct limit algebras

The following proposition generalizes [10, Theorem 1] (where all inclusion maps are isometries) and complements it by a Lie theoretic perspective.

Proposition 12.1. *Let $A_1 \subseteq A_2 \subseteq \cdots$ be an ascending sequence of unital Banach algebras A_n over $\mathbb{K}$, such that each inclusion map $A_n \to A_{n+1}$ is a continuous homomorphism of unital algebras. Then the following holds:*

(a) *The locally convex direct limit topology makes $A := \bigcup_{n\in\mathbb{N}} A_n$ a locally m-convex topological algebra. Its unit group $A^\times = \bigcup_{n\in\mathbb{N}} A_n^\times$ is open, and is a topological group when equipped with the topology induced by A.*
(b) *$A^\times = \varinjlim A_n^\times$ as a topological group.*
(c) *Product sets are large in $A^\times = \bigcup_{n\in\mathbb{N}} A_n^\times$, and $(A_n^\times)_{n\in\mathbb{N}}$ satisfies the PTA.*

If A is Hausdorff (which is automatic if the direct sequence is strict), then $A^\times$ is a $C^\omega_\mathbb{K}$-Lie group and $A^\times = \varinjlim A_n^\times$ as a $C^\infty_\mathbb{K}$-Lie group.

Proof. The PTA holds by Proposition 11.11. By [9, Theorem 1], A is a locally m-convex topological algebra, i.e., the vector topology of A can be defined by a family of sub-multiplicative seminorms (see [41]). Thus $A^\times$ is a topological group. It is known that $A^\times$ is open (Wengenroth communicated a proof [55]), but we need not use this fact here, as an alternative proof is part of the following arguments. Consider a product set $P := \bigcup_{n\in\mathbb{N}} U_1 \cdots U_n$, with identity neighbourhoods $U_n \subseteq A_n^\times$. After shrinking U_n, we may assume that $U_n = \mathbf{1} + B_{\varepsilon_n}^{A_n}(0)$ for some $\varepsilon_n \in]0, \frac{1}{2}]$.

Then $\|x^{-1}\| \leqslant \sum_{k=0}^{\infty} 2^{-k} \leqslant 2$ for each $x \in U_n$, whence U_n^{-1} is bounded in A_n and hence also in A_m for each $m \geqslant n$. Thus $M_{m,n} := \sup\{\|x^{-1}\|_m \colon x \in U_n\} < \infty$.

We claim that

$$\mathbf{1} + \bigcup_{n\in\mathbb{N}} \sum_{k=1}^{n} B_{\delta_k}^{A_k}(0) \subseteq P,$$

where $\delta_1 := \varepsilon_1$ *and* $\delta_n := M_{n,1}^{-1} \cdots M_{n,n-1}^{-1}\varepsilon_n$ *for integers* $n \geqslant 2$.

If this claim is true, then P is a neighbourhood of $\mathbf{1}$ in A, whence $A^\times$ an open subset of A (cf. [15, Lemma 2.6]) and product sets are large in $A^\times$. Therefore $A^\times = \varinjlim A_n^\times$ as a topological group (Proposition 11.8). Since $A^\times$ is open in A and inversion is continuous, $A^\times$ is a $C_{\mathbb{K}}^{\omega}$-Lie group provided A is Hausdorff [15, Proposition 3.2, respectively, 3.4]. The identity map $A^\times \to A^\times$ being a direct limit chart, Theorem 2.6 shows that $A^\times = \varinjlim A_n^\times$ also as a Lie group.

Proof of the claim. We show that $\mathbf{1} + \sum_{k=1}^{n} B_{\delta_k}^{A_k}(0) \subseteq U_1 \cdots U_n \subseteq P$, by induction on n. If $n = 1$, then $\mathbf{1} + B_{\delta_1}^{A_1}(0) = U_1 \subseteq P$. Let $n \geqslant 2$ now and suppose that $\mathbf{1} + \sum_{k=1}^{n-1} B_{\delta_k}^{A_k}(0) \subseteq U_1 \cdots U_{n-1}$. Let $y_k \in B_{\delta_k}^{A_k}(0)$ for $k \in \{1, \dots, n\}$. There are $x_j \in U_j$ for $j \in \{1, \dots, n-1\}$ such that

$$y := \mathbf{1} + y_1 + \cdots + y_{n-1} = x_1 \cdots x_{n-1}.$$

Set $x_n := y^{-1}(y + y_n) = \mathbf{1} + y^{-1}y_n$. Then $\|y^{-1}y_n\|_n = \|x_{n-1}^{-1} \cdots x_1^{-1} y_n\|_n < M_{n,n-1} \cdots M_{n,1}\delta_n = \varepsilon_n$ and thus $x_n \in U_n$. By construction, $\mathbf{1} + \sum_{k=1}^{n} y_k = y + y_n = yx_n = x_1 \cdots x_n$. □

If the direct sequence $A_1 \subseteq A_2 \subseteq \cdots$ in Proposition 12.1 is strict, then $A^\times \neq \varinjlim A_n^\times$ as a topological space unless each A_n is finite-dimensional or the sequence A_n becomes stationary (by Yamasaki's Theorem, see Remark 3.4).

We also have a variant for not necessarily unital associative algebras and algebra homomorphisms which need not take units to units (if units do exist). Recall that if A is an associative $\mathbb{K}$-algebra (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$), then $A_e := \mathbb{K}e \oplus A$ is a unital algebra via $(re + a)(se + b) = rse + (rb + sa + ab)$. Then $(A, \diamond)$ with $a \diamond b := a + b - ab$ is a monoid with neutral element 0, whose unit group is denoted $Q(A)$. The inverse of $a \in Q(A)$ is called the quasi-inverse of a and denoted $q(a)$. The map $(A, \diamond) \to (A_e, \cdot)$, $a \mapsto e - x$ is a homomorphism of monoids (see, e.g., [15, Section 2] for all of this).

Proposition 12.2. *Let $A_1 \subseteq A_2 \subseteq \cdots$ be a sequence of (not necessarily unital) associative Banach algebras over $\mathbb{K}$, such that each inclusion map $A_n \to A_{n+1}$ is a continuous algebra homomorphism. Then we have*:

(a) *The locally convex direct limit topology makes $A := \bigcup_{n\in\mathbb{N}} A_n$ a locally m-convex associative topological algebra with an open group $Q(A)$ of quasi-invertible elements and a continuous quasi-inversion map $q : Q(A) \to A$. Thus $Q(A)$ is a topological group.*
(b) *$Q(A) = \varinjlim Q(A_n)$ as a topological group.*
(c) *Product sets are large in $Q(A) = \bigcup_{n\in\mathbb{N}} Q(A_n)$, and $(Q(A_n))_{n\in\mathbb{N}}$ satisfies the PTA.*

If the locally convex direct limit topology on A is Hausdorff, then $Q(A)$ is a $C^\omega_{\mathbb{K}}$-Lie group and $Q(A) = \varinjlim Q(A_n)$ in the category of $C^\infty_{\mathbb{K}}$-Lie groups.

Proof. (a) The locally convex direct limit topology on $A_e = \varinjlim(\mathbb{K} \oplus A_n)$ makes A_e the direct product $\mathbb{K} \times A$, where $A = \varinjlim A_n$ carries the locally convex direct limit topology. Since A_e is a topological algebra with open unit group and continuous inversion, it follows that $Q(A)$ is open in A and q is continuous [15, Lemma 2.8]. Since A_e is locally m-convex, so is A.

(b), (c) Consider a product set $P := \bigcup_{n\in\mathbb{N}} B^{A_1}_{\varepsilon_1}(0) \diamond \cdots \diamond B^{A_n}_{\varepsilon_n}(0)$, with $\varepsilon_n \in]0, \frac{1}{2}]$. In A_e, we then have $e - P = \bigcup_{n\in\mathbb{N}} U_1 \cdots U_n$ with $U_n = e - B^{A_n}_{\varepsilon_n}(0) = e + B^{A_n}_{\varepsilon_n}(0)$. Extend the norm on A_n to $(A_n)_e$ via $\|re + a\| := |r| + \|a\|$, and define $M_{m,n}$ and δ_m as in the proof of Proposition 12.1. Re-using the arguments from the proof just cited, we see that $e + \bigcup_{n\in\mathbb{N}} \sum_{k=1}^n B^{A_k}_{\delta_k}(0) \subseteq e - P$. Hence P is a neighbourhood of 0 in A and thus product sets are large in $Q(A)$. Therefore $Q(A) = \varinjlim Q(A_n)$ as a topological group, by Proposition 11.8. The PTA holds by Proposition 11.11.

Now assume that A is Hausdorff. Then $(A_e)^\times \cong Q(A_e)$ is a $C^\omega_{\mathbb{K}}$-Lie group and $Q(A) = Q(A_e) \cap A$ (see [15, Lemma 2.5]) is a subgroup and submanifold of $Q(A_e)$ and therefore a $C^\omega_{\mathbb{K}}$-Lie group as well. The identity map $Q(A) \to Q(A)$ being a direct limit chart, Theorem 2.6 shows that $Q(A) = \varinjlim Q(A_n)$ as a $C^\infty_{\mathbb{K}}$-Lie group. □

13. Example: Lie groups of germs beyond the Silva case

For all X, K, H as in 10.1 and $(G_n)_{n\in\mathbb{N}}$ as in 10.4, we show:

Proposition 13.1. *Product sets are large in $\Gamma(K, H) = \bigcup_{n\in\mathbb{N}} G_n$.*

Lemma 13.2. *Let $(\mathfrak{h}, \|\cdot\|)$ be a Banach–Lie algebra over $\mathbb{K}$ and $R > 0$ such that the BCH-series converges to a $\mathbb{K}$-analytic mapping $B^{\mathfrak{h}}_R(0) \times B^{\mathfrak{h}}_R(0) \to \mathfrak{h}$, $(x, y) \mapsto x * y$. Then there exist $r \in]0, R]$, a $\mathbb{K}$-analytic mapping $F : B^{\mathfrak{h}}_r(0) \times B^{\mathfrak{h}}_r(0) \to B^{\mathfrak{h}}_R(0)$ and $C > 0$ such that*

$$x + y = x * F(x, y) \quad \text{and} \tag{9}$$

$$\|F(x, y) - y\| \leqslant C\|x\|\|y\|, \tag{10}$$

for all $x, y \in B^{\mathfrak{h}}_r(0)$.

Proof. Choosing $r \in]0, R]$ sufficiently small, we can achieve that $F(x, y) := (-x) * (x + y)$ is defined, $F(x, y) \in B^{\mathfrak{h}}_R(0)$ and that (9) holds, for all $x, y \in B^{\mathfrak{h}}_r(0)$. Since $F(x, 0) = 0$ and $F(0, y) = y$ for all $x, y \in B^{\mathfrak{h}}_r(0)$, the second order Taylor expansion of F entails (10), after shrinking r further if necessary (see [24, Lemma 1.7]). □

Multiple products with respect to the BCH-multiplication $*$ are formed recursively in the order $x_1 * \cdots * x_n := (x_1 * \cdots * x_{n-1}) * x_n$ (provided that all partial products are defined).

Proof of Proposition 13.1. Let $R > 0$, r, C and $*$ be as in Lemma 13.2, applied with $\mathfrak{h} := L(H)$ if $\mathbb{K} = \mathbb{C}$ (respectively, $\mathfrak{h} := L(H)_{\mathbb{C}}$ if $\mathbb{K} = \mathbb{R}$). After shrinking r, we may assume that $Cr \leqslant \frac{1}{2}$. Now let $(W_n)_{n\in\mathbb{N}}$ be a sequence of identity neighbourhoods $W_n \subseteq G_n$. After shrinking W_n, we may assume that $W_n = \operatorname{Exp}_n(B^{A_n}_{\varepsilon_n}(0))$ for some $\varepsilon_n > 0$, with A_n as in 10.1 (respectively, 10.2).

Let $\delta_n := \min\{r2^{-n}, \varepsilon_n/2\}$ for $n \in \mathbb{N}$. Then $S := \bigcup_{n\in\mathbb{N}} \sum_{k=1}^{n} B_{\delta_n}^{A_n}(0)$ is a 0-neighbourhood in $\Gamma(K, L(H))$. We claim that

$$\pi\left(\prod_{n\in\mathbb{N}}^{*} W_n\right) \supseteq \operatorname{Exp}(S), \tag{11}$$

where $\pi : \prod_{n\in\mathbb{N}}^{*} G_n \to \Gamma(K, H)$ is the product map. Therefore $\pi(\prod_{n\in\mathbb{N}}^{*} W_n)$ is an identity neighbourhood in $\Gamma(K, H)$, and hence product sets are large in $\Gamma(K, H)$. To prove the claim, let $z \in \operatorname{Exp}(S)$. Thus $z = \operatorname{Exp}(\sum_{n=1}^{\infty}[\gamma_n])$ for some sequence $(\gamma_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} B_{\delta_n}^{A_n}(0)$. Choose $N \in \mathbb{N}$ such that $[\gamma_n] = 0$ for all $n > N$. If $\mathbb{K} = \mathbb{C}$, we set $\eta_1 := \gamma_1$. If $n \in \{2, \ldots, N\}$, then $\eta_n(x) := F(\gamma_1(x) + \cdots + \gamma_{n-1}(x), \gamma_n(x))$ makes sense for each $x \in U_n$ since $\|\gamma_n(x)\| < r$ and $\|\gamma_1(x) + \cdots + \gamma_{n-1}(x)\| < r$, and defines a bounded holomorphic function $\eta_n : U_n \to L(H)$. We have $\|\eta_n\|_\infty \leqslant (1 + Cr)\|\gamma_n\|_\infty \leqslant \frac{3}{2}\delta_n < \varepsilon_n$ by (10), whence $\eta_n \in B_{\varepsilon_n}^{A_n}(0)$ and thus $\operatorname{Exp}_n(\eta_n) \in W_n$. Furthermore, $\gamma_1(x) + \cdots + \gamma_n(x) = (\gamma_1(x) + \cdots + \gamma_{n-1}(x)) * \eta_n(x)$ and hence $\gamma_1(x) + \cdots + \gamma_n(x) = \eta_1(x) * \eta_2(x) * \cdots * \eta_n(x)$ for each $n \in \{1, \ldots, N\}$ and $x \in U_n$, by induction. Therefore,

$$\gamma_1(x) + \cdots + \gamma_N(x) = \eta_1(x) * \eta_2(x) * \cdots * \eta_N(x) \quad \text{for each } x \in U_N,$$

entailing that $\sum_{n=1}^{\infty}[\gamma_n] = [x \mapsto \eta_1(x) * \cdots * \eta_N(x)]$ and $z = \operatorname{Exp}(\sum_{n=1}^{\infty}[\gamma_n]) = [x \mapsto \exp_H(\eta_1(x) * \cdots * \eta_N(x))] = [x \mapsto \exp_H(\eta_1(x)) \cdots \exp_H(\eta_N(x))] = \operatorname{Exp}([\eta_1]) \cdots \operatorname{Exp}([\eta_N]) \in \pi(\prod_{n\in\mathbb{N}}^{*} W_n)$. Thus (11) holds. If $\mathbb{K} = \mathbb{R}$, we use the unique bounded holomorphic extension $\tilde{\gamma}_n : \tilde{U}_n \to \mathfrak{h}$ of each γ_n in place of γ_n to define bounded holomorphic maps $\tilde{\eta}_n : \tilde{U}_n \to \mathfrak{h}$ along the lines of the construction of η_n. Set $\eta_n := \tilde{\eta}_n|_{U_n}$. Then $\|\eta_n\|_{A_n} = \|\tilde{\eta}_n\|_\infty < \varepsilon_n$ for each n and we see as above that $z = \operatorname{Exp}([\eta_1]) \cdots \operatorname{Exp}([\eta_N]) \in \pi(\prod_{n\in\mathbb{N}}^{*} W_n)$. □

Combining Proposition 11.8 with Theorem 2.6, we obtain:

Corollary 13.3. *$\Gamma(K, H) = \varinjlim G_n$ holds in the category of $C^\infty_{\mathbb{K}}$-Lie groups, and in the category of topological groups.*

14. Construction of Lie group structures on direct limit groups

Consider an abstract group $G = \bigcup_{n\in\mathbb{N}} G_n$ which is the union of an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of $C^\infty_{\mathbb{K}}$-Lie groups G_n, such that the inclusion maps $i_{n,m} : G_m \to G_n$ (for $m \leqslant n$) are $C^\infty_{\mathbb{K}}$-homomorphisms and each G_n is a subgroup of G. In this section, we describe conditions which facilitate to construct a $C^\infty_{\mathbb{K}}$-Lie group structure on G such that $G = \varinjlim G_n$ as a $C^\infty_{\mathbb{K}}$-Lie group. For finite-dimensional Lie groups G_n, such a Lie group structure has been constructed in [21] (cf. [44,45], [39, Theorem 47.9] and [18] for special cases). The conditions formulated in this section apply just as well to suitable infinite-dimensional Lie groups G_n.

14.1. We shall always assume that G has a *candidate for a direct limit chart*, viz. we assume that there exist charts $\phi_n : G_n \supseteq U_n \to V_n \subseteq L(G_n)$ of G_n around 1 for $n \in \mathbb{N}$ such that $U_m \subseteq U_n$ and $\phi_n|_{U_m} = L(i_{n,m}) \circ \phi_m$ if $m \leqslant n$, and $V := \bigcup_{n\in\mathbb{N}} V_n$ is open in the locally convex direct limit $E := \varinjlim L(G_n)$, which we assume Hausdorff. Here, we identify $L(G_m)$ with the image of $L(i_{n,m})$ in $L(G_n)$; this is possible because $L(i_{n,m})$ is injective by an argument as in Remark 2.2(a). We define $U := \bigcup_{n\in\mathbb{N}} U_n$ and $\phi := \varinjlim \phi_n : U \to V \subseteq E$.

It is natural to wonder whether ϕ (or its restriction to a smaller identity neighbourhood in U) can always be used as a chart around 1 for a Lie group structure on G. Unfortunately, the answer is *negative* (without extra hypotheses): even if ϕ is globally defined on all of G, it need not make $G = \bigcup_n G_n$ a Lie group.

Example 14.2. Let $A_1 \subseteq A_2 \subseteq \cdots$ be any ascending sequence of locally convex unital associative topological algebras such that:

1. The inclusion maps are homomorphisms of unital algebras and topological embeddings;
2. The locally convex direct limit topology renders the algebra multiplication on the union $A := \bigcup_{n\in\mathbb{N}} A_n$ discontinuous at $(1, 1)$;
3. The unit group $A^\times$ is open in A.

(See [15, Section 10] for such algebras.) Then $A^\times = \bigcup_{n\in\mathbb{N}} A_n^\times$ is a union of Lie groups and $A^\times$ admits the global chart $\phi := \mathrm{id}_{A^\times}$, which is a candidate for a direct limit chart around 1. However, $A^\times$ is not a Lie group because the group multiplication is discontinuous at $(1, 1)$.

We now describe additional requirements ensuring that the question just posed has an affirmative answer. They are satisfied in many situations.

Proposition 14.3. *Consider an abstract group $G = \bigcup_{n\in\mathbb{N}} G_n$ which is the union of an ascending sequence of $C^\infty_\mathbb{K}$-Lie groups. Assume that G admits a candidate $\phi : U \to V \subseteq E := \varinjlim L(G_n)$ for a direct limit chart, and assume that condition* (i) *or* (ii) *from Proposition* 9.8 *is satisfied. Then there exists a unique $C^\infty_\mathbb{K}$-Lie group structure on G making $\phi|_W$ a direct limit chart for G around* 1, *for an open identity neighbourhood $W \subseteq U$.*

Remark 14.4. By Proposition 9.8, the Lie group structure described in Proposition 14.3 makes G the direct limit $\varinjlim G_n$ as a $C^\infty_\mathbb{K}$-Lie group, topological group, topological space, and as a $C^r_\mathbb{K}$-manifold, for each $r \in \mathbb{N}_0 \cup \{\infty\}$.

We shall deduce Proposition 14.3 from a technical lemma. It requires further terminology. Consider a set M which is an ascending union $M = \bigcup_{n\in\mathbb{N}} M_n$ of $C^\infty_\mathbb{K}$-manifolds, and $x \in M$. Changing 14.1 in the obvious way,[2] we obtain the definition of a *candidate for a direct limit chart around x*.

Lemma 14.5. *Let G be an abstract group which is the union $G = \bigcup_{n\in\mathbb{N}} M_n$ of an ascending sequence $M_1 \subseteq M_2 \subseteq \cdots$ of $C^\infty_\mathbb{K}$-manifolds, such that $1 \in M_1$ and* (a)–(d) *hold*:

(a) *the inclusion maps $M_m \to M_n$ are $C^\infty_\mathbb{K}$ for all $m \leqslant n$;*
(b) *G admits a candidate $\phi : U \to V \subseteq E := \varinjlim T_1(M_n)$ for a direct limit chart around* 1;
(c) *for each $n \in \mathbb{N}$ and $x, y \in M_n$, there exists $k \geqslant n$ and open neighbourhoods $A, B \subseteq M_n$ of x, respectively, y such that $AB \subseteq M_k$ and the group multiplication $A \times B \to M_k$ is $C^\infty_\mathbb{K}$; and*
(d) *for each $n \in \mathbb{N}$, there exists $k \geqslant n$ and an open identity neighbourhood $A \subseteq M_n$ such that $A^{-1} \subseteq M_k$ and the group inversion $A \to M_k$, $x \mapsto x^{-1}$ is $C^\infty_\mathbb{K}$.*

[2] Replace G by M, $L(G_n)$ by $T_x M_n$, and $L(i_{n,m})$ by $T_x(i_{n,m})$.

Furthermore, we assume that (i) *or* (ii) *is satisfied*:

(i) *M_n is modelled on a Banach space for each $n \in \mathbb{N}$, and the inclusion map $T_1(M_m) \to T_1(M_n)$ is a compact operator for all $m < n$.*
(ii) *The modelling locally convex space of each M_n is a k_ω-space.*

Then there is a unique $C^\infty_{\mathbb{K}}$-Lie group structure on G making ϕ a chart for G around 1. *Furthermore, $G = \varinjlim M_n$ as a topological space and as a $C^r_{\mathbb{K}}$-manifold, for each $r \in \mathbb{N}_0 \cup \{\infty\}$.*

Proof. Suppose that $\phi = \varinjlim \phi_n$ with $\phi_n : U_n \to V_n$. Equip G with the topology $\mathcal{T}$ turning it into the direct limit topological space $\varinjlim M_n$. Given $x \in M_n$, consider $\lambda_x : G \to G$, $\lambda_x(y) := xy$. Hypothesis (c) implies that $\lambda_x|_{M_m}$ is continuous for each $m \in \mathbb{N}$. Hence λ_x is continuous, and hence a homeomorphism. Likewise, all right translations are homeomorphisms. Let $\mathcal{S}$ be the topology on $G \times G$ making it the direct limit $\varinjlim(M_n \times M_n)$. By Lemma 1.7, the topology induced by $\mathcal{T}$ on U makes U the direct limit topological space $U = \varinjlim U_n$, and $\mathcal{S}$ induces on $U \times U$ the topology making it the direct limit $\varinjlim(U_n \times U_n)$. This topology is the product topology on $U \times U$; this follows from the fact that the product topology on $E \times E$ coincides with the locally convex direct limit topology on $E \times E = \varinjlim T_1 M_n \times T_1 M_n$, which makes $E \times E$ the direct limit topological space $\varinjlim T_1 M_n \times T_1 M_n$ by 9.1 (respectively, Lemma 9.6(b) and (c)). As a consequence of (c), the group multiplication μ restricts to a continuous map $U \times U = \varinjlim(U_n \times U_n) \to G$. Since μ is continuous on the identity neighbourhood $U \times U$ and all left and right translations are homeomorphisms, it follows that μ is continuous. Let $\iota : G \to G$ be the inversion map. Since $y^{-1} = (x^{-1}y)^{-1}x^{-1}$ for $x \in M_n$ and $y \in M_n$, combining (c) and (d) we see that $\iota|_{M_n}$ is continuous on a neighbourhood of x and hence continuous. Hence ι is continuous and hence G is a topological group, which is Hausdorff because the intersection of all identity neighbourhoods is $\{1\}$. For $x \in G$, define $\phi_x : xU \to V$, $\phi_x(y) = \phi(x^{-1}y)$. Given $x, y \in G$, the map $\phi_x \circ \phi_y^{-1}$ is defined on the open set $\phi(U \cap y^{-1}xU)$ and takes z to $\phi(x^{-1}y\phi^{-1}(z))$. In view of (c), we easily deduce from Lemma 9.7 that $\phi_x \circ \phi_y^{-1}$ is $C^\infty_{\mathbb{K}}$. Hence the charts are compatible and thus G is a $C^\infty_{\mathbb{K}}$-manifold. Since $\phi_{xy} \circ \lambda_x \circ \phi_y^{-1} = \mathrm{id}_V$ is $C^\infty_{\mathbb{K}}$, each left translation map λ_x is $C^\infty_{\mathbb{K}}$ and hence a $C^\infty_{\mathbb{K}}$-diffeomorphism. Let $W \subseteq U$ be an open, symmetric identity neighbourhood such that $WW \subseteq U$. Replacing continuity by smoothness in the above arguments, we see (with the help of Lemma 9.7) that the group multiplication $W \times W \to U$ and inversion $W \to W$ are $C^\infty_{\mathbb{K}}$. Similarly, (c) implies that each inner automorphism $c_x : G \to G$ takes some identity neighbourhood smoothly into U. Now standard arguments provide a unique $C^\infty_{\mathbb{K}}$-Lie group structure on G making W an open smooth submanifold (see, e.g., [14, Proposition 1.13]). Since λ_x is a diffeomorphism from W onto xW for each $x \in G$, both for the manifold structure making G a Lie group and the manifold structure constructed before, we deduce that the two manifold structures coincide. □

Remark 14.6. If each M_n is a finite-dimensional $C^\infty_{\mathbb{K}}$-manifold in the situation of Lemma 14.5 and each inclusion map $M_m \to M_n$, $m \leqslant n$, a smooth immersion, then a direct limit chart around 1 exists by [21, Theorem 3.1] and thus condition (b) of Lemma 14.5 is automatically satisfied.

Proof of Proposition 14.3. This is a special case of Lemma 14.5, applied with $M_n := G_n$ (we use $k := n$, $A := B := G_n$ in (c) and $k := n$, $A := G_n$ in (d)). □

15. Example: Lie groups of germs of analytic diffeomorphisms

Let K be a non-empty compact subset of $X := \mathbb{K}^d$, where $d \in \mathbb{N}$, and let $\mathrm{GermDiff}(K, X)$ be the group of all germs $[\gamma]$ around K of $\mathbb{K}$-analytic maps $\gamma : U \to X$ on an open neighbourhood U of K such that $\gamma|_K = \mathrm{id}_K$, $\gamma(U)$ is open in X and $\gamma : U \to \gamma(U)$ is a $\mathbb{K}$-analytic diffeomorphism. Then $\mathrm{GermDiff}(K, X)$ is a group in a natural way, with group operation $[\gamma][\eta] := [\gamma \circ \eta|_{\eta^{-1}(U)}]$ (for $\gamma : U \to X$). To illustrate the usefulness of Lemmas 9.7 and 14.5, we apply them to turn $\mathrm{GermDiff}(K, X)$ into a $\mathbb{K}$-analytic Lie group, modelled on the space $\Gamma(K, X)_K$ of germs $[\gamma]$ around K of X-valued $\mathbb{K}$-analytic maps $\gamma : U \to X$ such that $\gamma|_K = 0$.

Monoids of germs of complex analytic self-maps

It simplifies the construction (and provides additional information) to consider in a first step the monoid $\mathrm{GermEnd}(K, X)$ of all germs $[\gamma]$ around K of $\mathbb{K}$-analytic maps $\gamma : U \to X$ on an open neighbourhood of K such that $\gamma|_K = \mathrm{id}_K$ (with multiplication given by composition of representatives). We equip $\mathrm{GermEnd}(K, X)$ with a $\mathbb{K}$-analytic manifold structure which makes the monoid multiplication a $\mathbb{K}$-analytic map. In a second step, we show that the unit group $\mathrm{GermDiff}(K, X)$ of $\mathrm{GermEnd}(K, X)$ is open and has a $\mathbb{K}$-analytic inversion map. Until Remark 15.10, we let $\mathbb{K} = \mathbb{C}$.

15.1. Choose a norm on X. For $n \in \mathbb{N}$, the sets $U_n := K + B^X_{1/n}(0)$ form a fundamental sequence of open neighbourhoods of K in X. The supremum norm makes the space $\mathrm{Hol_b}(U_n, X)$ of bounded X-valued $C^\omega_\mathbb{C}$-maps on U_n a complex Banach space, and $\mathrm{Hol_b}(U_n, X)_K := \{\gamma \in \mathrm{Hol_b}(U_n, X)\colon \gamma|_K = 0\}$ is a closed vector subspace. Let $j_{n,m} : \mathrm{Hol_b}(U_m, X)_K \to \mathrm{Hol_b}(U_n, X)_K$ be the restriction map, for $n \geqslant m$. Given an open neighbourhood U of K, let $\mathrm{Hol}(U, X)$ be the Fréchet space of all X-valued $C^\omega_\mathbb{C}$-maps on U (equipped with the compact-open topology) and $\mathrm{Hol}(U, X)_K$ be its closed subspace of functions vanishing on K. Since $\overline{U_{n+1}} = K + \overline{B}^X_{1/(n+1)}(0)$ is a compact subset of U_n, we then have continuous linear restriction maps

$$\mathrm{Hol_b}(U_n, X)_K \to \mathrm{Hol}(U_n, X)_K \to \mathrm{Hol_b}(U_{n+1}, X)_K, \tag{12}$$

whose composition $j_{n+1,n}$ is a compact operator due to Montel's Theorem. Thus, the locally convex direct limit $\Gamma(K, X)_K = \varinjlim \mathrm{Hol_b}(U_n, X)_K$ is a Silva space. Since each connected component of U_n meets K and hence meets U_{n+1}, the Identity Theorem implies that each bonding map $j_{n,m}$ is injective and hence also each limit map $j_n : \mathrm{Hol_b}(U_n, X)_K \to \Gamma(K, X)_K$, $\gamma \mapsto [\gamma]$.

15.2. It is useful to note that the map

$$\rho : \Gamma(K, X)_K \to C\big(K, \mathcal{L}(X)\big), \quad [\gamma] \mapsto \gamma'|_K \tag{13}$$

is continuous, where $\mathcal{L}(X)$ is the Banach algebra of continuous endomorphisms of X with the operator norm, and $C(K, \mathcal{L}(X))$ is given the supremum norm. Since $\Gamma(K, X)_K = \varinjlim \mathrm{Hol_b}(U_n, X)_K$ as a locally convex space, this follows from the fact that the inclusion maps $\mathrm{Hol_b}(U_n, X) \to \mathrm{Hol}(U_n, X)$ and $\mathrm{Hol}(U_n, X) \to C^\infty_\mathbb{C}(U_n, X)$ are continuous (see [46, Proposition III.15]).

15.3. It is clear from the definition that the map

$$\Phi : \Gamma(K, X)_K \to \mathrm{GermEnd}(K, X), \quad \Phi\big([\gamma]\big) := [\mathrm{id}_X + \gamma]$$

is a bijection. We use Φ to give GermEnd(K, X) a complex manifold structure with Φ^{-1} as a global chart. For $\gamma, \eta \in \Gamma(K, X)_K$, we then have

$$\Phi^{-1}\big(\Phi\big([\gamma]\big)\Phi\big([\eta]\big)\big) = [\eta] + \big[\gamma \circ (\mathrm{id}_X + \eta)\big]. \tag{14}$$

Proposition 15.4. GermEnd(K, X) *is a complex analytic monoid, i.e., the multiplication map* GermEnd$(K, X) \times$ GermEnd$(K, X) \to$ GermEnd(K, X)*,* $([\gamma], [\eta]) \mapsto [\gamma][\eta] = [\gamma \circ \eta]$ *is complex analytic.*

Proof. For $k \in \mathbb{N}$, let Ω_k be the set of all $[\gamma] \in \Gamma(K, X)_K$ such that $\sup\{\|\gamma'(x)\|\colon x \in K\} < k$. Then $\Gamma(K, X)_K = \bigcup_{k\in\mathbb{N}} \Omega_k$, and each of the sets Ω_k is an open 0-neighbourhood in $\Gamma(K, X)_K$, by continuity of ρ (from (13). Hence, in view of (14), the multiplication map will be $C^\infty_{\mathbb{C}}$ if the map

$$f : \Gamma(K, X)_K \times \Omega_k \to \Gamma(K, X)_K, \quad f\big([\gamma], [\eta]\big) := \big[\gamma \circ (\mathrm{id}_X + \eta)\big]$$

is $C^\infty_{\mathbb{C}}$ for each $k \in \mathbb{N}$. Fix k. For $n \in \mathbb{N}$, set $\ell_n := (n + 1)(k + 1)$, $m_n := \ell_n + 1$, $P_n := \mathrm{Hol_b}(U_n, X)_K$, and let Q_n be the set of all $\gamma \in \mathrm{Hol_b}(U_n, X)_K$ with $\sup\{\|\gamma'(x)\|\colon x \in U_{n+1}\} < k$. Then Q_n is open in $\mathrm{Hol_b}(U_n, X)_K$. Hence

$$\Gamma(K, X)_K \times \Omega_k = \bigcup_{n\in\mathbb{N}} j_n(P_n) \times j_n(Q_n),$$

here $P_n \times Q_n$ is open in $\mathrm{Hol_b}(U_n, X)_K \times \mathrm{Hol_b}(U_n, X)_K$, $n \in \mathbb{N}$, and $(j_{n,m} \times j_{n,m})(P_m \times Q_m) \subseteq P_n \times Q_n$ if $m \leqslant n$. Using Lemma 9.7(a), we see that f will be $C^\infty_{\mathbb{C}}$ if $f|_{P_n\times Q_n}$ is $C^\infty_{\mathbb{C}}$ for each $n \in \mathbb{N}$ (identifying $\mathrm{Hol_b}(U_n, X)_K$ with its image in $\Gamma(K, X)_K$ using j_n). Note that if $\eta \in Q_n$ and $y \in \overline{U_{\ell_n}}$, say $y \in \overline{B}^X_{\ell_n^{-1}}(x)$ with $x \in K$, then

$$\begin{aligned}\big\|\big(y + \eta(y)\big) - x\big\| &= \big\|y - x + \eta(y) - \eta(x)\big\| \leqslant \|x - y\| + \left\|\int_0^1 \eta'\big(x + t(y - x)\big).(y - x)\, dt\right\| \\ &\leqslant (1 + k)\|y - x\| \leqslant \frac{1 + k}{\ell_n} < \frac{1}{n}\end{aligned}$$

and thus $y + \eta(y) \in B^X_{1/n}(x)$. Hence $(\mathrm{id}_X + \eta)(\overline{U_{\ell_n}}) \subseteq U_n$, enabling us to define

$$g_n : P_n \times Q_n \to \mathrm{Hol_b}(U_{m_n}, X)_K, \quad g_n(\gamma, \eta) := \gamma \circ (\mathrm{id}_X + \eta)|_{U_{m_n}}.$$

Since $f|_{P_n\times Q_n} = j_{m_n} \circ g_n$, it remains to show that g_n is $C^\infty_{\mathbb{C}}$. The set $S_n := \{\eta \in \mathrm{Hol}(U_n, X)_K\colon (\mathrm{id}_X + \eta)(\overline{U_{\ell_n}}) \subseteq U_n\}$ is open in $R_n := \mathrm{Hol}(U_n, X)_K$. Using continuous linear inclusion and restriction maps and the mapping

$$h_n : R_n \times S_n \to \mathrm{Hol}(U_{\ell_n}, X)_K, \quad h_n(\gamma, \eta) := \gamma \circ (\mathrm{id}_X + \eta)|_{U_{\ell_n}},$$

we can write g_n as a composition

$$P_n \times Q_n \to R_n \times S_n \xrightarrow{h_n} \mathrm{Hol}(U_{\ell_n}, X)_K \to \mathrm{Hol}_{\mathrm{b}}(U_{m_n}, X)_K.$$

Therefore, g_n will be $C^\infty_{\mathbb{C}}$ if we can show that h_n is $C^\infty_{\mathbb{C}}$. To this end, we exploit that $\mathrm{Hol}(U_n, X) = C^\infty_{\mathbb{C}}(U_n, X)$, equipped with the compact-open C^∞-topology (cf. [46, Proposition III.15]). By [27, Lemma 11.4], the map

$$C^\infty_{\mathbb{C}}(U_n, X) \times \lfloor \overline{U_{\ell_n}}, U_n \rfloor_\infty \to C^\infty_{\mathbb{C}}(U_{\ell_n}, X), \quad (\gamma, \eta) \mapsto \gamma \circ \eta|_{U_{\ell_n}}$$

is $C^\infty_{\mathbb{C}}$, where $\lfloor \overline{U_{\ell_n}}, U_n \rfloor_\infty := \{\eta \in C^\infty_{\mathbb{C}}(U_n, X)\colon\ \eta(\overline{U_{\ell_n}}) \subseteq U_n\}$. It easily follows that h_n is $C^\infty_{\mathbb{C}}$ as a map into $\mathrm{Hol}(U_{\ell_n}, X)$ and hence also as a map into the closed vector subspace $\mathrm{Hol}(U_{\ell_n}, X)_K$ (by Lemma 1.3). □

Lie groups of germs of complex analytic diffeomorphisms

Lemma 15.5. *The complex analytic monoid* $\mathrm{GermEnd}(K, X)$ *has an open unit group* $\mathrm{GermEnd}(K, X)^\times$. *It is given by*

$$\begin{aligned}\mathrm{GermEnd}(K, X)^\times &= \mathrm{GermDiff}(K, X)\\ &= \big\{[\gamma] \in \mathrm{GermEnd}(K, X)\colon\ \gamma'(K) \subseteq \mathrm{GL}(X)\big\}.\end{aligned} \tag{15}$$

Proof. If $\gamma : U \to \gamma(U) \subseteq X$ is a diffeomorphism between open neighbourhoods of K, then $[\gamma] \in \mathrm{GermEnd}(K, X)^\times$ and thus $\mathrm{GermDiff}(K, X) \subseteq \mathrm{GermEnd}(K, X)^\times$. If $[\gamma], [\eta] \in \mathrm{GermEnd}(K, X)^\times$ with $[\eta] = [\gamma]^{-1}$, then $\eta \circ \gamma|_W = \mathrm{id}_W$ on some open neighbourhood W of K, whence $\gamma|_W$ is injective and $\gamma'(x) \in \mathrm{GL}(X)$ for each $x \in W$. Hence $\gamma|_W$ is a diffeomorphism onto an open neighbourhood of K and thus $[\gamma] \in \mathrm{GermDiff}(K, X)$.

If $[\gamma] \in \mathrm{GermDiff}(K, X)$, then $\gamma'(K) \subseteq \mathrm{GL}(X)$. As the converse follows from the next lemma, we see that (15) holds. Since $C(K, \mathrm{GL}(X)) = C(K, \mathcal{L}(X))^\times$ is open in the Banach algebra $C(K, \mathcal{L}(X))$ and ρ from (13) is continuous, we deduce that $\{[\gamma] \in \mathrm{GermEnd}(K, X)\colon \gamma'(K) \subseteq \mathrm{GL}(X)\} = \rho^{-1}(C(K, \mathrm{GL}(X)))$ is open in $\mathrm{GermEnd}(K, X)$. □

Lemma 15.6. *Let X be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $K \subseteq X$ be a non-empty compact set, $r \in \mathbb{N} \cup \{\infty, \omega\}$ and $f : U \to E$ be a $C^r_{\mathbb{K}}$-map on an open neighbourhood U of K such that $f|_K$ is injective and $f'(K) \subseteq \mathrm{GL}(X)$. If $r = 1$, assume* $\dim(X) < \infty$. *Then there is an open neighbourhood $V \subseteq U$ of K such that $f(V) \subseteq X$ is open and $f|_V$ is a $C^r_{\mathbb{K}}$-diffeomorphism onto $f(V)$.*

Proof. Define $U_n := K + B^X_{1/n}(0)$. There is $n_0 \in \mathbb{N}$ such that $U_n \subseteq U$ and $\gamma'(U_{n_0}) \subseteq \mathrm{GL}(X)$. Then $\gamma|_{U_{n_0}}$ is a local $C^r_{\mathbb{K}}$-diffeomorphism by the Inverse Function Theorem (see [28, Theorem 5.1] if $r \neq \omega$; the analytic case is well known). In particular, $\gamma(U_n)$ is open in X, for each $n \geqslant n_0$. If $\gamma|_{U_n}$ fails to be injective for each $n \geqslant n_0$, we find $v_n \neq w_n \in U_n$ such that $f(v_n) = f(w_n)$. Then $v_n \in B^X_{1/n}(x_n)$ and $w_n \in B^X_{1/n}(y_n)$ for certain $x_n, y_n \in K$. There exist $k_1 < k_2 < \cdots$ such that $x_{n_k} \to x$ and $y_{n_k} \to y$ for certain $x, y \in K$. Then $v_{n_k} \to x$ and $w_{n_k} \to y$, whence $f(x) = f(y)$ by continuity and hence $x = y$. Let W be a neighbourhood of x on which f is injective. Then $v_{n_k} \in W$ and $w_{n_k} \in W$ for large k. Since $v_{n_k} \neq w_{n_k}$ and $f(v_{n_k}) = f(w_{n_k})$, this contradicts the injectivity of $f|_W$. Hence, there exists $n \geqslant n_0$ such that f is injective on $V := U_n$. Then $f|_V : V \to f(V)$ is a $C^r_{\mathbb{K}}$-diffeomorphism. □

We return to the default notations of this section. The following quantitative variant of Lemma 15.6 will be needed.

Lemma 15.7. *Given $\gamma \in \mathrm{Hol}_{\mathrm{b}}(U_n, X)_K$ with $C := \sup\{\|\gamma'(x)\|: x \in U_n\} < 1$, set $\eta := \mathrm{id}_X + \gamma : U_n \to X$. Then $\eta|_{U_{6n}}$ is injective.*

Proof. Let $v, w \in U_{6n}$ such that $\eta(v) = \eta(w)$. There are $x, y \in K$ with $v \in B^X_{1/(6n)}(x)$ and $w \in B^X_{1/(6n)}(y)$. Let $[x, v]$ be the line segment joining x and v. Since $[x, v] \subseteq B^X_{1/(6n)}(x) \subseteq U_n$, the Mean Value Theorem yields $\|\eta(v) - \eta(x)\| \leqslant \|v - x\| \cdot \max\{\|\eta'(z)\|: z \in [x, v]\} \leqslant 2\|v - x\| < 1/(3n)$. Likewise, $\|\eta(w) - \eta(y)\| < 1/(3n)$ and hence $\|y - x\| = \|\eta(y) - \eta(x)\| < 2/(3n)$, entailing that $[v, w] \subseteq B^X_{1/n}(x) \subseteq U_n$ and therefore $0 = \|\eta(v) - \eta(w)\| \geqslant \|v - w\| - \|(\eta - \mathrm{id}_X)(v) - (\eta - \mathrm{id}_X)(w)\| \geqslant \|v - w\| - C \cdot \|v - w\|$. If $v \neq w$, we get the contradiction $0 \geqslant (1 - C)\|v - w\| > 0$. Hence $v = w$ and thus $\eta|_{U_{6n}}$ is injective. □

15.8. Lemma 15.5 enables us to consider $\mathrm{GermDiff}(K, X)$ as an open $C^\infty_{\mathbb{C}}$-submanifold of $\mathrm{GermEnd}(K, X)$. Then $D := \Phi^{-1}(\mathrm{GermDiff}(K, X))$ is an open 0-neighbourhood in $\Gamma(K, X)_K$, and the restriction Ψ of Φ^{-1} to a map $\Psi : \mathrm{GermDiff}(K, X) \to D$ is a global chart for $\mathrm{GermDiff}(K, X)$. We set $M_n := \Psi^{-1}(D \cap \mathrm{Hol}_{\mathrm{b}}(U_n, X)_K)$ and give M_n the complex Banach manifold structure with $\Psi|_{M_n} : M_n \to D \cap \mathrm{Hol}_{\mathrm{b}}(U_n, X)_K$ as a global chart.

Proposition 15.9. $\mathrm{GermDiff}(K, X)$ *is a complex Lie group. Furthermore,* $\mathrm{GermDiff}(K, X) = \varinjlim M_n$ *holds as a topological space, as a $C^r_{\mathbb{C}}$-manifold, and as a $C^r_{\mathbb{R}}$-manifold, for each $r \in \mathbb{N}_0 \cup \{\infty\}$.*

Proof. We check the hypotheses of Lemma 14.5. Here (a) is clear and also (b), since Ψ is a candidate for a direct limit chart. Condition (i) holds because the restriction maps $\mathrm{Hol}_{\mathrm{b}}(U_n, X)_K \to \mathrm{Hol}_{\mathrm{b}}(U_{n+1}, X)_K$ are compact. The validity of condition (c) is clear from the proof of Proposition 15.4.

Condition (d) *concerning the inversion map.* It suffices to show that, for each $n \in \mathbb{N}$, there exist $k > \ell > n$ and an open 0-neighbourhood $P \subseteq \mathrm{Hol}_{\mathrm{b}}(U_n, X)_K$ such that $\eta_\gamma := (\mathrm{id}_X + \gamma)|_{U_\ell}$ is injective for each $\gamma \in P$, $U_k \subseteq \eta_\gamma(U_\ell)$, $g(\gamma) := \eta_\gamma^{-1}|_{U_k} - \mathrm{id}_X \in \mathrm{Hol}_{\mathrm{b}}(U_k, X)_K$, and that $g : P \to \mathrm{Hol}_{\mathrm{b}}(U_k, X)_K$ is $C^\infty_{\mathbb{C}}$. Because we can build in continuous linear restriction maps, it suffices to find an open 0-neighbourhood $Q \subseteq \mathrm{Hol}(U_n, X)_K$ and $m > \ell > n$ such that $\eta_\gamma := (\mathrm{id}_X + \gamma)|_{U_\ell}$ is injective for each $\gamma \in Q$, $U_m \subseteq \eta_\gamma(U_\ell)$, $h(\gamma) := (\eta_\gamma^{-1} - \mathrm{id}_X)|_{U_m} \in \mathrm{Hol}(U_m, X)_K$, and that $h : P \to \mathrm{Hol}(U_m, X)_K$ is $C^\infty_{\mathbb{C}}$ (then take $k := m + 1$). We set $m := 12(n + 1)$, $\ell := 6(n + 1)$ and let Q be the set of all $\gamma \in \mathrm{Hol}(U_n, X)_K$ such that $\sup\{\|\gamma'(x)\|: x \in \overline{U_{n+1}}\} < \frac{1}{2}$. Then Q is open in $\mathrm{Hol}(U_n, X)_K$ and $\eta_\gamma := (\mathrm{id}_X + \gamma)|_{U_\ell}$ is injective for each $\gamma \in Q$, by 15.7. Furthermore, $B^X_{1/(2\ell)}(x) \subseteq \eta_\gamma(B^X_{1/\ell}(x))$ for each $x \in K$ by [28, Theorem 5.3(d)] (applied with $A := \mathrm{id}_X$) and thus $U_m \subseteq \eta_\gamma(U_\ell)$. The map

$$f : Q \times U_\ell \to X, \quad f(\gamma, x) := (\mathrm{id}_X + \gamma)(x)$$

is $C^\infty_\mathbb{C}$ since the evaluation map is $C^\infty_\mathbb{C}$ (see [27, Proposition 11.1]). Also, the mapping $f_\gamma := f(\gamma, \bullet): U_\ell \to X$ is injective by the preceding, and is a local $C^\infty_\mathbb{C}$-diffeomorphism (by the Inverse Function Theorem). Furthermore, the map

$$\psi : Q \times U_m \to X, \quad \psi(\gamma, x) := f_\gamma^{-1}(x) = (\mathrm{id}_X + \gamma)^{-1}(x) = \eta_\gamma^{-1}(x)$$

is $C^\infty_\mathbb{C}$ by the Inverse Function Theorem with parameters (see [25, Theorem 2.3(c)] or [28, Theorem 5.13(b)]). Then, by the exponential law [27, Lemma 12.1(a)]), also the map

$$\psi^\vee : Q \to C^\infty_\mathbb{C}(U_m, X) = \mathrm{Hol}(U_m, X), \quad \gamma \mapsto \psi(\gamma, \bullet) = \eta_\gamma^{-1}\big|_{U_m}$$

is $C^\infty_\mathbb{C}$, whence also $Q \to \mathrm{Hol}(U_m, X)$, $\gamma \mapsto \psi^\vee(\gamma) - \mathrm{id}_X = h(\gamma)$ is $C^\infty_\mathbb{C}$. As this map takes values in the closed vector subspace $\mathrm{Hol}(U_m, X)_K$ of $\mathrm{Hol}(U_m, X)$, Lemma 1.3 shows that also its co-restriction h is $C^\infty_\mathbb{C}$, as required. $\square$

Remark 15.10. For K a singleton (say, the origin), it is well known that $\mathrm{GermDiff}(\{0\}, \mathbb{C}^d)$ is a Lie group. The Lie group structure has first been constructed in [51], where this group is denoted by $\mathrm{Gh}(d, \mathbb{C})$.

Lie groups of germs of real analytic diffeomorphisms

Let $\mathbb{K} = \mathbb{R}$ now. It is clear that $\Gamma(K, X)_K$ can be identified with the set of germs $[\gamma] \in \Gamma(K, X_\mathbb{C})_K$ such that $\gamma(U) \subseteq X$ for some neighbourhood U of K in X. In the same way, we identify $\mathrm{GermEnd}(K, X)$ with a subset of $\mathrm{GermEnd}(K, X_\mathbb{C})$. We give $\Gamma(K, X)_K$ the topology induced by $\Gamma(K, X_\mathbb{C})_K$. Then $\Gamma(K, X)_K$ is a closed real vector subspace of $\Gamma(K, X_\mathbb{C})$ and $\Gamma(K, X_\mathbb{C})_K = (\Gamma(K, X)_K)_\mathbb{C}$ as a locally convex space (cf. [20, Sections 4.2–4.4]). Then $\Phi^{-1}: \mathrm{GermEnd}(K, X_\mathbb{C}) \to \Gamma(K, X_\mathbb{C})_K$ (defined as before) is a global chart of $\mathrm{GermEnd}(K, X_\mathbb{C})$ such that $\Phi(\mathrm{GermEnd}(K, X)) = \Gamma(K, X)_K$, showing that $\mathrm{GermEnd}(K, X)$ is a real analytic submanifold of $\mathrm{GermEnd}(K, X_\mathbb{C})$. As a consequence, $\mathrm{GermDiff}(K, X) = \mathrm{GermDiff}(K, X_\mathbb{C}) \cap \mathrm{GermEnd}(K, X)$ is open in $\mathrm{GermEnd}(K, X)$. The real analyticity of the monoid multiplication and group inversion is inherited by the submanifolds. Summing up:

Corollary 15.11. *If* $\mathbb{K} = \mathbb{R}$, *then* $\mathrm{GermDiff}(K, X)$ *is a real analytic Lie group modelled on* $\Gamma(K, X)_K$. *Furthermore,* $\mathrm{GermEnd}(K, X)$ *is a real analytic monoid with open unit group* $\mathrm{GermEnd}(K, X)^\times = \mathrm{GermDiff}(K, X)$.

Remark 15.12. It is clear that $C_n := \{\gamma \in \mathrm{Hol}_\mathrm{b}(U_n, X_\mathbb{C})\colon \gamma(U_n \cap X) \subseteq X\}$ is a closed vector subspace, enabling us to make $A_n := \{\gamma|_{U_n \cap X}\colon \gamma \in C_n\}$ a real Banach space isometrically isomorphic to C_n. It is easy to see that $\Gamma(K, X)_K = \varinjlim A_n$ as a locally convex space. Setting $M_n := \Phi^{-1}(A_n \cap D)$, we easily deduce from the proofs of Lemma 15.5 and Proposition 15.9 that the conditions of Lemma 14.5 are satisfied. We deduce:

$\mathrm{GermDiff}(K, X) = \varinjlim M_n$ *as a topological space and as a* $C^r_\mathbb{R}$*-manifold, for each* $r \in \mathbb{N}_0 \cup \{\infty\}$.

Remark 15.13. Note that, since ρ in (13) is continuous linear, the set $E := \{[\gamma] \in \Gamma(K,X)_K\colon \gamma'|_K = 0\}$ is a closed $\mathbb{K}$-vector subspace of $\Gamma(K,X)_K$ (in both cases, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$). Then

$$\mathrm{GermDiff}(K,X)^* := \bigl\{[\gamma] \in \mathrm{GermDiff}(K,X)\colon\ \gamma'|_K = \mathrm{id}_X\bigr\}$$

is a Lie subgroup of GermDiff(K,X), since the chart Φ from above (respectively, its restriction the group of germs of $C^\omega_\mathbb{R}$-diffeomorphisms) takes GermDiff$(K,X)^*$ onto $D \cap E = E$. Because ρ is continuous linear, the homomorphism

$$\theta : \mathrm{GermDiff}(K,X) \to C\bigl(K, \mathrm{GL}(X)\bigr), \quad \theta\bigl([\gamma]\bigr) := \gamma'|_K$$

is $\mathbb{K}$-analytic. We therefore have an exact sequence of $C^\omega_\mathbb{K}$-Lie groups

$$\mathbf{1} \to \mathrm{GermDiff}(K,X)^* \hookrightarrow \mathrm{GermDiff}(K,X) \xrightarrow{\theta} C\bigl(K, \mathrm{GL}(X)\bigr), \tag{16}$$

where GermDiff$(K,X)^*$ is $C^\omega_\mathbb{K}$-diffeomorphic to E and hence contractible.

Remark 15.14. The Lie group GermDiff$(\{0\}, \mathbb{R})^*$ has also been discussed in [40], using different notation. Lie groups of germs of real analytic (and more general) diffeomorphisms around a point have been studied in [38].

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