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Dynamics of the infinitely-thin kink

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ABSTRACT

We consider the dynamics of the domain-wall kink soliton, in particular we study the zero mode of translation. In the infinitely-thin kink limit, we show that the zero mode is almost completely frozen out, the only remnant being a dynamically constrained four-dimensional mode of a single but arbitrary frequency. In relation to this result, we show that the usual mode expansion for dealing with zero modes – implicit collective coordinates – is *not* in fact a completely general expansion, and that one must use instead a traditional generalised Fourier analysis.

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1. Introduction

The classical kink soliton solution of the $\lambda\phi^4$ theory has found many applications. One such use has been in models of extra-dimensions, where a background scalar field assumes the kink solution and becomes a domain-wall brane, a specific realisation of the generic idea of a brane world. From the point of view of a model builder, the kink can be used to localise fermions [1,2], gauge fields [3], Higgs fields [4] and gravity [5,6] (building on [7]). Giving the kink a non-trivial representation under some internal symmetry allows for exciting symmetry breaking opportunities, such as GUT breaking [8–11] and supersymmetry breaking [12]. All these ingredients are able to play together in a comprehensive model of extra-dimensions, and a domain-wall-localised standard model can be implemented [13].

Even though the kink has played a central role in domain-wall models for many decades now, there are some interesting and important technical properties of the kink that have been overlooked. These loose ends were alluded to in a previous work by the authors [4], and relate to the precise nature of the zero mode of translation of the kink, the thin-kink limit, and the implicit collective coordinate treatment.¹ In this paper we shall resolve these issues and make clear the following two facts: first, that the kink zero mode corresponding to translations is almost completely frozen out in the thin-kink limit, and, second, that the

implicit collective coordinate expansion (ICCE) does not capture all physically-acceptable classical field configurations. Both of these results appear to be contrary to common understanding, they impact the conclusions of previous work, and they must be taken into account in future studies of kinks.

Our analysis is chiefly mathematical and the results are valid for any application of the kink solution, not just domain-wall brane theories. But to aid in physical understanding and help the flow of our argument, we have in mind the specific scenario of a five-dimensional theory with a bulk scalar field forming a kink. We are interested in integrating out the extra-dimension to determine the equivalent four-dimensional theory, and we shall elucidate the scalar degrees of freedom present in this reduced spacetime. The thin-kink limit is an important phenomenological limit for such a model, as the masses of the Kaluza–Klein (KK) modes are pushed to infinity. In addition, the action for a thin kink can be compared with the Nambu–Goto action for a fundamental brane. For the case of the infinitely-thin kink, we show that the Nambu–Goldstone boson, related to the spontaneous breaking of the translation symmetry, is not fully dynamical: the only remnant of translation invariance in the four-dimensional theory is the allowance of a single frequency massless mode. When dealing with translation invariance, one usually employs the ICCE; see Rajaraman [16] and references therein. We shall demonstrate that such an expansion must be used with caution, as it is not able to adequately encode all field configurations of the original five-dimensional field and does not properly handle the non-linear interactions of the zero-mode at high order.

The paper is organised as follows. In Section 2 we review the kink solution, its energy density and its behaviour in the thin-kink limit. We demonstrate the existence of a ‘wavy kink’ solution of a fixed frequency, which persists in the thin-kink limit. We then

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¹ For earlier analyses of the thin kink limit, see [14,15].

argue that, in such a limit, this fixed frequency wave is the only remaining dynamical behaviour and hence the four-dimensional zero mode – the Nambu–Goldstone boson corresponding to translations of the kink – is almost completely frozen out. In Section 3 we analyse the modes of the kink, showing that the ICCE is not completely general, and we use the fully-general Fourier expansion to show that the zero mode is truly frozen out. We make some further remarks regarding dimensional reduction and then conclude in Section 4.

2. The ‘wavy kink’ and the frozen zero mode

The set-up of the problem is quite simple; we consider five-dimensional Minkowski spacetime, and a single scalar field with a quartic potential. The action is

$$S = \int d^5x \left[\frac{1}{2} \partial^M \Phi \partial_M \Phi - V(\Phi) \right], \quad (1)$$

where Φ is the scalar field, and the potential is

$$V(\Phi) = \frac{\lambda}{4} (\Phi^2 - v^2)^2. \quad (2)$$

Indices M, N run over the spacetime coordinates (t, x, y, z, w) , the Minkowski metric is $\eta_{MN} = \text{diag}(+1, -1, -1, -1, -1)$, and λ and v are real parameters. The equation of motion for Φ is

$$\partial^M \partial_M \Phi - v^2 \lambda \Phi + \lambda \Phi^3 = 0. \quad (3)$$

The well-known classical kink solution to Eq. (3) is

$$\phi_c(w) = v \tanh(kw), \quad (4)$$

where $k = v\sqrt{\lambda/2}$ is the inverse width of the kink. Here, we have chosen the kink profile to depend on the extra-dimensional coordinate w , as this is the dimension we want to eliminate when constructing the equivalent four-dimensional theory. Integrating over w , one obtains the energy density per unit four-volume of the kink:

$$\varepsilon = \int dw \left[\frac{1}{2} \phi_c'^2 + V(\phi_c) \right] = \frac{2}{3} v^3 \sqrt{2\lambda}. \quad (5)$$

The thin-kink limit has the width of the hyperbolic tangent profile tending to zero, and is defined by $k \rightarrow \infty$ while ε is kept finite. For the two parameters of the model, this limit translates to $\lambda \rightarrow \infty$ and $v \rightarrow 0$, with $v^6 \lambda$ finite.

We would now like to make a less restrictive ansatz for the solution to the five-dimensional Euler–Lagrange equation, an ansatz which can describe degrees of freedom on top of the static kink profile. Due to the Poincaré invariance of the action, any w -translated form of Eq. (4) is also a valid solution for Φ . Using this fact as a hint, we try the more general translated ansatz

$$\Phi(x^M) = \phi_c(w - Z(x^\mu)) \quad (6)$$

$$= v \tanh[k(w - Z(x^\mu))]. \quad (7)$$

Here, the index μ runs over the four-dimensional subspace (t, x, y, z) . $Z(x^\mu)$ is a real scalar field which acts to translate the kink by an x^μ -dependent amount, and includes as a particular case any constant shift of the kink. This ansatz is in fact of the same form as the first term in the ICCE approach to re-describing the five-dimensional scalar field as an infinite tower of four-dimensional KK scalar fields. In terms of the ICCE, we have here taken the solutions for all massive KK four-dimensional fields to be zero. We shall examine the general expansion, which retains all KK modes, in the next section.

The fundamental theory is that of a five-dimensional scalar field. To ensure that all of the physics is retained, the correct approach to finding solutions for $Z(x^\mu)$ is therefore to substitute the ansatz into the five-dimensional Euler–Lagrange equation (3). Doing this gives

$$-\phi_c'(w - Z) \partial^\mu \partial_\mu Z + \phi_c''(w - Z) \partial^\mu Z \partial_\mu Z = 0, \quad (8)$$

where prime denotes derivative with respect to w . Since ϕ_c'' is an odd function, integrating this equation over w eliminates the second term,² and so the most general solution obeys

$$\partial^\mu \partial_\mu Z = \partial^\mu Z \partial_\mu Z = 0. \quad (9)$$

Solutions for $Z(x^\mu)$ are massless plane waves of a single frequency only. The usual equation of motion for a zero mode, $\partial^\mu \partial_\mu Z = 0$, now has an auxiliary constraint, $\partial^\mu Z \partial_\mu Z = 0$, and one can no longer take a Fourier sum of all frequencies. The most general solution to both of these equations is

$$Z(x^\mu) = A \cos(p_\mu x^\mu) + B \sin(p_\mu x^\mu) + C, \quad (10)$$

with A, B and C arbitrary real numbers and $p_\mu p^\mu = 0$. Notice that this solution solves the five-dimensional equation of motion (8) irrespective of the values of the parameters λ and v ; in particular, it remains valid in the thin kink limit. The auxiliary constraint means that, as an effective four-dimensional field, Z does not manifest as a standard dynamical scalar field in the four-dimensional theory.

Let us now compute the energy density per unit four-volume for the more general kink solution given by Eqs. (7) and (9). It is³

$$E = \int dw \left[\frac{1}{2} \dot{\Phi}^2 + \frac{1}{2} (\nabla \Phi \cdot \nabla \Phi) + \frac{1}{2} \Phi'^2 + V(\Phi) \right] \quad (11)$$

$$= \varepsilon + \frac{1}{2} \varepsilon \dot{Z}^2 + \frac{1}{2} \varepsilon (\nabla Z \cdot \nabla Z). \quad (12)$$

Here, an over-dot denotes derivative with respect to t . E is the energy density of the original kink background, ε , plus kinetic and gradient energy of Z , with larger energy for higher frequency Z solutions. The energy density is not sensitive to the individual parameters v and λ , only their combination ε . Importantly, in the infinitely-thin kink limit, we are allowed a non-zero form for Z , as its contribution to the total energy density remains finite (assuming the spacetime derivatives of Z are finite).

Summarising, we have found a slightly more general kink solution, given by Eq. (7), which is an *exact* solution of the five-dimensional Euler–Lagrange equation so long as Z takes the form of Eq. (10). Due to the fact that this solution for Z must be of a fixed frequency (with arbitrary phase and amplitude), we shall call the resulting solution the ‘wavy kink’ solution. The ‘wave’ appears along the length of the kink such that the hyperbolic tangent profile is shifted in the w -direction by an amount that varies sinusoidally in the three-space (x, y, z) . This wave oscillates in time at a fixed frequency, and, from the point of view of a four-dimensional observer, is the only dynamical behaviour that can be observed given the ansatz (7). Consequently, Z cannot be called a proper four-dimensional mode, as, from a momentum-space perspective, its degrees of freedom consist of a set of measure zero: a single frequency.

Now, it may seem that we have been too restrictive in our ansatz for the scalar field. After all, conventional wisdom has it

² Note that in doing the integration over w we have terms such as $\int \phi_c'(w - Z) dw$ which look like functions of x^μ . These terms actually yield the same value for each point in the four-space (which can be seen by changing the integration variable independently at each x^μ), and so the integral results in an x^μ -independent answer.

³ As before, one will encounter terms such as $\int \phi_c'^2(w - Z) dw$ which are actually x^μ -independent.

that the kink spontaneously breaks translation invariance, and so there should be a massless Nambu–Goldstone boson at the effective four-dimensional level. This boson would correspond to translations of the kink. In fact, at first order, this Nambu–Goldstone boson is exactly Z : for small Z , where we ignore Z^2 and higher terms, the auxiliary constraint in Eq. (9) is eliminated. In this approximation, we are left only with the usual massless wave equation describing the behaviour of Z , and so the system admits a fully-dynamical scalar mode at the four-dimensional level.

One may think that this Nambu–Goldstone zero mode should persist, even if we move outside the regime of the approximation and keep higher order terms for Z . We shall show that this is not actually the case, and that the necessary coupling of Z to higher-mass modes constrains its dynamics. What follows is a brief intuitive argument for such behaviour. In the next section we provide a more rigorous mathematical analysis.

Consider what happens if one excites the scalar field Φ to a configuration $\Phi(w - Z)$ where the form of Z consists of multiple frequencies. Obviously, such an excitation does not satisfy the five-dimensional Euler–Lagrange equation (3) with our restricted ansatz (7). What will happen is that, as the system evolves in time, the KK fields set to zero in our ansatz will be excited. It is important to realise that if we prohibit the excitation of such modes, then we can only have solutions of the form $\Phi(w - Z)$ with Z a single frequency massless plane wave. Now we are in a position to state one of the main results of this paper: in the infinitely-thin kink limit, such extra modes are infinitely heavy (they are frozen out), and so the zero mode Z is dynamically constrained to a single frequency. As a consequence, from the effective four-dimensional point of view, Z does not have enough degrees of freedom to look anything like a traditional scalar field, and so this Nambu–Goldstone boson is not present in the four-dimensional spectrum. In fact, in the infinitely-thin kink limit, the four-dimensional spectrum contains no propagating degrees of freedom at all.

3. Collective coordinates and the general expansion

In this section we proceed to analyse the full spectrum of modes of the kink, and demonstrate that all the propagating degrees of freedom are frozen out in the thin-kink limit. To do this, we shall expand the five-dimensional field Φ in a set of complete four-dimensional modes – a generalised Fourier transformation, or Kaluza–Klein decomposition. The extra dimension can then be integrated out to obtain a four-dimensional action, giving an equivalent, but alternative, description of the original theory. The appropriate expansion is written as

$$\Phi(x^\mu, w) = \phi_c(w) + \sum_i \phi_i(x^\mu) \eta_i(w), \quad (13)$$

where the sum over i includes two discrete modes ($i = 0, 1$) and an integral over a continuum ($i = q$, where $q \in \mathbb{R}$). The profiles $\eta_i(w)$ form a complete basis (in the sense that any physically-acceptable five-dimensional field configuration $\Phi(x^\mu, w)$ can be represented by suitable choice of $\phi_i(x^\mu)$) and are determined by linearising the five-dimensional Euler–Lagrange equation about the kink background; see reference [4] for explicit forms of the basis functions η_i . Note that even though the η_i were determined after linearising, they still form a complete basis in the exact regime, and can be used for a general expansion with no loss of information. The fields ϕ_i are a tower of scalar fields, and serve to faithfully represent, at the four-dimensional level, all degrees of freedom inherent in Φ . The tower consists of a zero-mass mode, followed by a discrete massive mode, followed by a massive continuum.

Before using this expansion, we shall discuss a slightly different version of Eq. (13), the aforementioned ICCE [16]. Since any translated version of the background kink ϕ_c is just as good as any other, there exists an entire class of basis functions η_i which are also translated by an equivalent amount. The first basis function η_0 is proportional to the first derivative of ϕ_c and corresponds to infinitesimal (first order) translations of the static kink profile. The mode η_0 therefore plays a unique role, and it should perhaps be treated differently from the other η_i modes. The ICCE is motivated by this observation, and removes the zero mode from the tower of modes, placing it in a more ‘obvious’ spot:

$$\Phi(x^\mu, w) = \phi_c(w - Z(x^\mu)) + \sum_{i \neq 0} \tilde{\phi}_i(x^\mu) \eta_i(w - Z(x^\mu)). \quad (14)$$

The idea now is that the four-dimensional scalar fields $Z(x^\mu)$ and $\tilde{\phi}_{1,q}(x^\mu)$ can faithfully encode all degrees of freedom of Φ . Note that, in this expansion, ϕ_c and $\eta_{1,q}$ have the same form as they do in Eq. (13), but now the sum excludes $i = 0$. The ICCE has seen numerous applications to problems where continuous symmetries and zero modes are present. For example, in a perturbative quantum field theory analysis, the zero mode can potentially lead to divergent energy contributions in higher-order terms [16]. The ICCE allows one to treat the zero mode separately and avoid such difficulties.

Although Eq. (14) looks quite reasonable, it is actually not general enough to expand an arbitrary field $\Phi(x^\mu, w)$. For example, there are no (finite) choices of $Z(x^\mu)$ and $\tilde{\phi}_{1,q}(x^\mu)$ which yield $\Phi(x^\mu, w) = \omega(x^\mu) \eta_0(w)$ for any non-zero choice for $\omega(x^\mu)$.⁴ If there were, then we could write

$$\omega(x^\mu) \eta_0(w) = \phi_c(w - Z) + \sum_{i \neq 0} \tilde{\phi}_i(x^\mu) \eta_i(w - Z). \quad (15)$$

Keep in mind that Z may depend on x^μ , we have just neglected to write this explicitly to keep the equation clear. Now, multiply through by $\eta_0(w - Z)$ and integrate over w :

$$\begin{aligned} \omega(x^\mu) \int \eta_0(w) \eta_0(w - Z) dw \\ = \int \phi_c(w - Z) \eta_0(w - Z) dw \\ + \sum_{i \neq 0} \tilde{\phi}_i(x^\mu) \int \eta_i(w - Z) \eta_0(w - Z) dw. \end{aligned} \quad (16)$$

There is the freedom to shift the integrals on the right-hand side by Z , and then, because η_0 is orthogonal to ϕ_c and $\eta_{1,q}$, we have

$$\omega(x^\mu) \int \eta_0(w) \eta_0(w - Z(x^\mu)) dw = 0. \quad (17)$$

Since η_0 is strictly positive (or strictly negative, depending on the normalisation convention) the integral in this equation will always be positive, regardless of the form of $Z(x^\mu)$, and so it must be that $\omega(x^\mu) = 0$. (We shall discuss shortly the possibility that Z is infinite.) Hence we have shown that the implicit collective coordinate expansion (14) cannot faithfully represent all possible configurations of Φ , and so is less general than the mode expansion (13).

We should make clear what we mean by an expansion being general enough to represent any (physically-acceptable) classical

⁴ Note that it is not necessary for the configuration $\omega(x^\mu) \eta_0(w)$ to be a classical solution. It is enough that it exists in the space of all possible configurations. At the level of the action, the field Φ is, of course, taken to be a variable and this configuration is one possible ‘value’ this variable can take. At the quantum level, the path integral must include this configuration in the domain of functional integration.

field configuration. In one-dimensional, non-relativistic quantum mechanics, one looks for the eigenfunctions of a time-independent Schrödinger equation, and builds a set out of those eigenfunctions which are *bounded at infinity*. Relying on Sturm–Liouville theory, one can make the statement that this set forms a complete set of modes, and any function that is also bounded at infinity can be expanded as a linear combination of the eigenfunctions. It is this idea of completeness that we have in mind throughout the current paper. Our argument above demonstrates that the ICCE is not general enough to represent an arbitrary configuration which is bounded at infinity. In contrast, the mode expansion given by Eq. (13) is determined from a Schrödinger-like equation as the set of eigenfunctions which are bounded at infinity, and so is able to represent a more general, and in fact adequate, class of configurations than the ICCE.⁵

In an attempt to satisfy Eq. (17), one may try and take $Z(x^\mu) \rightarrow \infty$, in which case the overlap of the two η_0 profiles becomes infinitesimally small and the integral vanishes [17]. If we allow Z to be infinite, our argument above (that the ICCE is not general) breaks down because multiplying Eq. (15) through by $\eta_0(w - Z)$ is essentially multiplying through by zero. To understand what is happening, we must look back at the actual definition of the ICCE, Eq. (14), and consider the effect of taking $Z \rightarrow \infty$. Mathematically, the $\phi_c(w - Z)$ term becomes a constant ($+v$ or $-v$, depending on whether $Z \rightarrow -\infty$ or $Z \rightarrow \infty$, respectively), the discrete mode $\eta_1(w - Z)$ vanishes, and the continuum modes $\eta_q(w - Z)$ become plane waves with frequency beginning at zero. In such a limit, the ICCE thus reduces to the standard Fourier transform of sines and cosines. Physically, one has translated the kink away, off to infinity, leaving a homogeneous vacuum (which is of course an allowed solution of the theory). Using the ICCE with Z of infinite magnitude is therefore equivalent to doing a mode decomposition around a homogeneous vacuum background, rather than around the kink background. Although the underlying five-dimensional theory can be equally well recast into equivalent four-dimensional forms using either decomposition (around the kink background, or around a homogeneous vacuum), for a given application one re-description will be more convenient than the other. For the application of concern to us here, the kink-background approach is clearly the more convenient. If one were to try to do the analysis using the homogeneous-vacuum mode (standard Fourier) basis, one would first have to understand how to choose the Fourier coefficients to produce the kink background plus excitations. This can be done, of course, but it is an awkward way to proceed. So, with a finite Z the ICCE is not fully general, while for infinite Z one recovers a mode basis that is not convenient for studying kink-related physics.

There are certain regimes of analysis where the ICCE is adequate. This includes the case where one restricts oneself to look only at small perturbations of the kink background, as there are no troubles expanding a perturbed kink using the ICCE. The configuration used in the above argument – the one which cannot be represented by the ICCE, $\Phi(x^\mu, w) = \omega(x^\mu)\eta_0(w)$ – is not a small perturbation of the kink since its asymptotic behaviour differs from that of $\phi_c(w)$. In this paper we are interested in determining the full, non-linear behaviour of the kink *exactly*, and must allow for the possibility that the kink background configuration is significantly modified. The ICCE approach is therefore unsuitable. Instead, using the more general expansion (13) allows us to trans-

form the five-dimensional kink into an equivalent description in terms of four-dimensional scalar modes. Analysing these modes is then straightforward because the resulting Lagrangian contains just massive interacting fields, with usual Klein–Gordon equations of motion (as opposed to the ICCE which yields difficult-to-interpret derivative couplings). This is due to the proper choice of basis functions η_i .⁶

Substituting Eq. (13) in the five-dimensional action (1) and integrating out the extra dimension yields the equivalent four-dimensional action:

$$\mathcal{S}_\Phi = \int d^4x [-\varepsilon_{\phi_c} + \mathcal{L}_\Phi], \quad (18)$$

where the kinetic, mass and self-coupling terms for the scalar modes are (see [4])

$$\begin{aligned} \mathcal{L}_\Phi = & \frac{1}{2} \partial^\mu \phi_0 \partial_\mu \phi_0 + \frac{1}{2} \partial^\mu \phi_1 \partial_\mu \phi_1 - \frac{3}{4} v^2 \lambda \phi_1^2 \\ & + \int_{-\infty}^{\infty} dq \left[\frac{1}{2} \partial^\mu \phi_q^* \partial_\mu \phi_q - \frac{1}{4} (q^2 + 4) v^2 \lambda \phi_q^* \phi_q \right] \\ & - \kappa_{ijk}^{(3)} \phi_i \phi_j \phi_k - \kappa_{ijkl}^{(4)} \phi_i \phi_j \phi_k \phi_l. \end{aligned} \quad (19)$$

In the last line here there are implicit sums over discrete modes, and integrals over continuum modes, for each of the indices i, j, k and l . For these terms, the cubic and quartic coupling coefficients are, respectively,

$$\kappa_{ijk}^{(3)} = \lambda \int_{-\infty}^{\infty} \phi_c \eta_i \eta_j \eta_k dw, \quad (20)$$

$$\kappa_{ijkl}^{(4)} = \frac{\lambda}{4} \int_{-\infty}^{\infty} \eta_i \eta_j \eta_k \eta_l dw. \quad (21)$$

The four-dimensional equivalent theory described by Eq. (19) contains a massless scalar field ϕ_0 , a massive scalar ϕ_1 , and a continuum of massive fields ϕ_q . There exist cubic and quartic couplings among these fields, and, importantly, a quartic *self-coupling* term for ϕ_0 ; this is due to the non-zero value of $\kappa_{0000}^{(4)}$:

$$\kappa_{0000}^{(4)} = \frac{9}{70} \left(\frac{3\varepsilon}{8} \right)^{1/3} \lambda^{4/3}. \quad (22)$$

Determining the Euler–Lagrange equations for each of the fields is a straightforward task. For our purposes, it suffices to examine the two discrete modes:

$$\begin{aligned} \partial^\mu \partial_\mu \phi_0 + 6\kappa_{001}^{(3)} \phi_0 \phi_1 + 4\kappa_{0000}^{(4)} \phi_0^3 + 12\kappa_{0011}^{(4)} \phi_0 \phi_1^2 \\ + (\text{terms involving continuum modes}) = 0, \end{aligned} \quad (23)$$

$$\begin{aligned} \partial^\mu \partial_\mu \phi_1 + \frac{3}{2} v^2 \lambda \phi_1 + 3\kappa_{001}^{(3)} \phi_0^2 + 3\kappa_{111}^{(3)} \phi_1^2 \\ + 12\kappa_{0011}^{(4)} \phi_0^2 \phi_1 + 4\kappa_{1111}^{(4)} \phi_1^3 \\ + (\text{terms involving continuum modes}) = 0. \end{aligned} \quad (24)$$

Given this rather neat, and exact, dimensional reduction of the original Φ model, we can now make a rigorous conclusion regarding the thin-kink limit. In this limit, $v^2 \lambda \rightarrow \infty$ and so the mass

⁵ We have checked explicitly that the expansion (13) can represent the configuration $\Phi(x^\mu, w) = \omega(x^\mu)\eta_0(w)$ for any choice of $\omega(x^\mu)$. Essentially, there exists a linear combination of the massive modes $\eta_{1,q}(w)$ which cancel the kink configuration $\phi_c(w)$, leaving the zero mode $\eta_0(w)$.

⁶ At the very least, regardless of the regime of validity of the ICCE, the mode expansion given by Eq. (13) is general enough to describe all configurations which remain bounded at infinity, and we can be confident in using it to obtain an equivalent four-dimensional action. If the ICCE is equally valid, it should produce the same results.

terms in the Lagrangian, Eq. (19), become infinitely large. From the point of view of the Euler–Lagrange equations for ϕ_1 and ϕ_q , the mass terms for these fields have an infinite coefficient, and these equations of motion can only be generally satisfied if the associated fields are identically zero. We therefore conclude that, in the infinitely-thin kink limit, the massive modes ϕ_1 and ϕ_q are frozen out.

Since ϕ_1 and the continuum modes must be zero, Eq. (24) reduces to $3\kappa_{001}^{(3)}\phi_0^2 = 0$, implying that ϕ_0 must also be zero.⁷ This is the central part of the argument, and supports our earlier claim that Z is constrained due to its coupling to massive, frozen modes. Here, the dynamics dictate that ϕ_0 must excite ϕ_1 (if ϕ_1 begins as zero) and so if ϕ_1 is forbidden (for example, if it is infinitely heavy), then ϕ_0 cannot be excited at all. Similar statements can be made regarding the coupling of ϕ_0 to the massive continuum modes. Furthermore, the quartic coupling of ϕ_0 to itself also prevents it from being excited: in the thin kink limit, $\kappa_{0000}^{(4)} \rightarrow \infty$, and, in order to satisfy Eq. (23), ϕ_0 is driven to zero.

From a slightly different point of view, consider all fields ϕ_i to be identically zero to begin with, and attempt to excite them individually. In the thin kink limit, all of the Euler–Lagrange equations contain potential terms that are infinite if any one of the fields are independently excited. In the equation for ϕ_0 , this term has coefficient $4\kappa_{0000}^{(4)}$, for ϕ_1 it has $\frac{3}{2}v^2\lambda$, and for ϕ_q it has $\frac{1}{2}(q^2 + 4)v^2\lambda$. Thus, each field is individually frozen.

We are essentially arguing that, in the Euler–Lagrange equations for the four-dimensional fields (and also in the four-dimensional action), there are coefficients which become infinite in the thin kink limit, and so the fields that make up such terms must be zero at the solution level. The reader may wonder if there exists some special combination of these fields which conspire to cancel the infinities. This is actually true. The special combination of ϕ_0 and the massive modes that persists in the thin-kink limit is the fixed frequency, wavy kink solution that we found in Section 2. But this is not a true four-dimensional dynamical field. In reference [4] it is shown that there is no other special combination that manifests as a proper four-dimensional scalar field with a canonical kinetic term.

The mode expansion given by Eq. (13) retains all degrees of freedom of Φ , and our analysis shows that these degrees of freedom are all driven to zero in the thin kink limit. Furthermore, there is no special combination of the modes which yields a field with a proper kinetic term. We have therefore shown that there are no observable dynamics, at the four-dimensional level, of the infinitely-thin kink. It is perhaps best, then, to consider a thin kink as also being a rigid kink; that is, it cannot be perturbed. For a thin kink (but not infinitely thin), one can set up a finite-energy-density configuration $\phi_c(w - Z(x^\mu))$ with arbitrary form for $Z(x^\mu)$. But, if the kink is made thinner, and hence more rigid, the same configuration will have a greater energy cost and will dissipate more rapidly to a wavy kink of fixed frequency (possibly zero frequency: the usual, static kink). For the case of the infinitely-thin kink limit, the initial configuration must *begin* as a fixed frequency wavy kink.

4. Discussion and conclusion

It must be stressed again that a five-dimensional theory is ruled by the five-dimensional Euler–Lagrange equations. Four-dimensional Euler–Lagrange equations provide an equivalent description only when all five-dimensional fields have been expanded

in a full, or general, set of modes. To go to an equivalent four-dimensional theory, one should *not* make a non-general ansatz for the five-dimensional fields, then integrate out the extra dimension. To find a four-dimensional theory which is equivalent to the original five-dimensional one, all degrees of freedom must be kept to begin with, and then the irrelevant ones eliminated at the four-dimensional level.

If one does not begin with a general expansion of the five-dimensional fields, then one may miss some important low-energy dynamics, dynamics which influence the behaviour of other low-energy degrees of freedom that have been included. For a concrete example of this statement, consider the non-general expansion $\Phi = \phi_c(w - Z(x^\mu))$. There is nothing wrong with employing such an expression as a solution ansatz, but, since it ignores a great number of degrees of freedom, one must use the five-dimensional Euler–Lagrange equation to determine the behaviour of Z . This is what we did in Section 2, where we found that Z must be a massless plane wave of a fixed frequency. Now, to contrast this method, we try and substitute the non-general expansion into the original five-dimensional action, integrate out the extra dimension, and obtain the effective four-dimensional action:

$$S = \int d^5x \left[\frac{1}{2} \phi_c'^2 (w - Z) \partial^\mu Z \partial_\mu Z - \frac{1}{2} \phi_c'^2 (w - Z) - V(\phi_c(w - Z)) \right] \quad (25)$$

$$= \int d^4x \left[\frac{1}{2} \varepsilon \partial^\mu Z \partial_\mu Z - \varepsilon \right]. \quad (26)$$

This procedure gives the four-dimensional Euler–Lagrange equation $\partial^\mu \partial_\mu Z = 0$, which is not correct, as it is missing the auxiliary constraint that fixes Z to a single frequency. The first method we used is the correct method, as the solution respects the full five-dimensional theory. Reduction to lower dimensions can only proceed if one uses a full, general mode expansion.

In light of this argument, there is no sense in using the ICCE to redescribe a five-dimensional theory as a completely equivalent four-dimensional theory. As we have shown, the ICCE is not general, and, in going to a four-dimensional description, one will potentially miss out on degrees of freedom which are pertinent to the low-energy dynamics.

Having said this, the ICCE is useful in certain contexts; for example, where one is only interested in expanding a model up to a given order in perturbation theory. This is actually the case for the discussions in Rajaraman [16], where modes are quantised around a classical ground state (like the kink), and perturbation theory is used to analyse the quantum excitations. As pointed out in Section 2, if one works in the regime where Z is small and $Z^2 \sim 0$, the auxiliary constraint $\partial^\mu Z \partial_\mu Z = 0$ is automatically satisfied at this order, and one is allowed a fully dynamical field $Z(x^\mu)$ at the four-dimensional level. Physically, this means that Z is so small that it does not excite the higher-mass modes.

As a relevant aside, we shall make some brief comments regarding fundamental branes. Such branes may originate from string theory, and are modelled by an effective action – the Nambu–Goto action – which treats them as infinitely thin, delta-distribution sources. These branes are assumed to support a proper translation zero mode which couples only through derivative terms to other fields (via the metric). We can accept this behaviour by understanding that branes modelled by the Nambu–Goto action are flexible, even though they are infinitely thin. Their degree of flexibility is dictated by their tension, which is equivalent to their energy density. In contrast, modelling a brane by a thin domain-wall kink solution yields different effective four-dimensional dynamics;

⁷ Analysis of the Euler–Lagrange equations for the continuum modes reveals similar constraints, such as $\kappa_{00q}^{(3)}\phi_0^2 = 0$ (q corresponding to an odd mode) and $\kappa_{000p}^{(4)}\phi_0^3 = 0$ (p corresponding to an even mode).

the domain wall does not exhibit a dynamical zero mode, as it is extremely rigid.

Our conclusion that the ICCE is not a general expansion may have an impact on previous work that relied on this method. For example, Burnier and Zuleta [18] compared fundamental branes and kinks using the ICCE for two scalar fields (the kink and an additional coupled scalar). The low-energy expansion of the domain-wall model was compared with the low-energy expansion of the Nambu–Goto action. Their use of the ICCE to describe *perturbations* of the kink is well justified, but it is not clear to us that their conclusions would remain unchanged using the more general mode expansion given by Eq. (13). Another interesting analysis to revisit is that where gravity is included [19]. Here, the zero mode of the kink mixes with gravitational degrees of freedom. It would be important to understand which effects are more important: particle physics modifications to the zero mode due to its quartic self-coupling, or the mixing with gravity.

In conclusion, we have established two facts that have been overlooked during the study of domain walls and of kinks. First, that the zero mode of translation is almost completely frozen out in the thin-kink limit. The only remnant is a four-dimensional entity which must assume a single frequency, yielding a wavy kink solution. This entity does not manifest as a proper mode in the effective four-dimensional theory; almost all degrees of freedom are frozen out. Second, that the implicit collective coordinate expansion is not completely general. It should only be used with caution and in certain approximations.

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