On the second largest Laplacian eigenvalue of trees

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Abstract

Very little is known about lower bounds and upper bounds for the second largest Laplacian eigenvalue of trees. This paper mainly gives a sharp upper bound for the second largest Laplacian eigenvalue of trees with perfect matchings. We also provide the smallest three values of the second largest Laplacian eigenvalue for any tree, and characterize the trees attaining those values.

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1. Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. We will frequently abuse the language by writing $v \in G$ and $uv \in G$, rather than $v \in V$ and $uv \in E$, to indicate that $v$ is a vertex of $G$ and $uv$ is an edge of $G$, respectively. Denote the degree of vertex $v_i$ by $d(v_i)$. The
Laplacian matrix $L(G) = D(G) - A(G)$ is the difference of $D(G) = \text{diag}(d(v_1), d(v_2), \ldots, d(v_n))$, the diagonal matrix of vertex degrees, and the adjacency matrix. It is well known that $L(G)$ is positive semidefinite symmetric and singular. Moreover, since $G$ is connected, $L(G)$ is irreducible. Denote its eigenvalues by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) = 0,$$

always enumerated in non-increasing order and repeated according to their multiplicity. We shall use the notation $\lambda_k(G)$ to denote the $k$th largest Laplacian eigenvalue of the graph $G$. If $v \in G$, let $L_v(G)$ be the principal submatrix of $L(G)$ formed by deleting the row and column corresponding to vertex $v$.

Two distinct edges in a graph $G$ are independent if they are not incident with a common vertex in $G$. A set of pairwise independent edges of $G$ is called a matching in $G$. A matching of maximum cardinality is a maximum matching in $G$. The cardinality of a maximum matching of $G$ is commonly known as its matching number, denoted by $\beta(G)$. A matching $M$ that satisfies $2|M| = n = |V(G)|$, is called a perfect matching.

Let

$$\omega(G) = \min \left\{ \frac{\varepsilon(S) + e(S, \bar{S})}{|S|} : S \subset V \text{ and } 0 < |S| < n \right\},$$

where $\varepsilon(S)$ denotes the edge connectivity of the subgraph induced by $S$, and $e(S, \bar{S})$ the number of edges whose one vertex is in $S$ and the other one in $\bar{S}$. It is obvious that $\omega(G) = 0$ if and only if $G$ has at least three connected components. In [7], Pan et al. proved that for a graph $G$ with $n$ vertices,

$$\omega(G) \leq \sqrt{2d_1(G) - \lambda_{n-2}(G)} \lambda_{n-2}(G).$$

In [8], Pati explored the relationship between the third smallest Laplacian eigenvalue and the graph structure. Let $\bar{G}$ denote the complement of the graph $G$. Since $L(G) + L(\bar{G}) = nI - J$, where $I$ and $J$ denote the identity matrix and the matrix whose each entry is 1, respectively. We have $\lambda_2(G) + \lambda_{n-2}(\bar{G}) = n$. Thus, while studying the third smallest Laplacian eigenvalue of a graph $G$ some information about the second largest Laplacian eigenvalue of its complement $\bar{G}$ might prove to be helpful. There is another motivation for this work. In [6], the authors discussed the maximal diagonal entry in the group(generalized) inverse of the Laplacian matrix corresponding to a tree with a perfect matching.

Throughout this paper, we shall denote by $\Phi(B) = \Phi(B; x) = \det(xI - B)$ the characteristic polynomial of $B$.

### 2. Lemmas and results

Let $G$ be a graph and let $G' = G + e$ be the graph obtained from $G$ by inserting a new edge $e$ into $G$. It follows by the well-known Courant–Weyl inequalities (see, e.g., [1, Theorem 2.1]) that the following is true.
Lemma 1. The Laplacian eigenvalues of $G$ and $G'$ interlace, that is,

$$
\lambda_1(G') \geq \lambda_1(G) \geq \lambda_2(G') \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G') = \lambda_n(G) = 0.
$$

The following inequalities are known as Cauchy’s inequalities and the whole theorem is also known as interlacing theorem.

Lemma 2 [1]. Let $A$ be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $B$ be a principal submatrix of order $m$; let $B$ have eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Then the inequalities

$$
\lambda_n - m + i \leq \mu_i \leq \lambda_i \quad (i = 1, 2, \ldots, m)
$$

hold.

Lemma 3 [1]. The maximum eigenvalue $\tilde{r}$ of every proper principal submatrix of a non-negative matrix $A$ does not exceed the maximum eigenvalue $r$ of $A$. If $A$ is irreducible, then $\tilde{r} < r$ always holds. If $A$ is reducible, then $\tilde{r} = r$ holds for at least one principal submatrix.

If $A$ is a matrix, denote by $|A|$ the matrix obtained by replacing each entry of $A$ by its absolute value.

Lemma 4 [2]. If $A$ is irreducible, and $\lambda$ is an eigenvalue of $A$, then $|\lambda|$ does not exceed the maximum eigenvalue $r$ of $|A|$.

Lemma 5 [3]. Let $G$ be a bipartite graph. Then $B(G) = D(G) + A(G)$ and $L(G)$ are unitarily similar; in particular, the maximum eigenvalue of $L(G)$ is simple provided $G$ is connected.

Lemma 6 [9]. Let $T$ be a tree on $n$ vertices. Then for any positive integer $a$, there exists a vertex $v \in T$ such that there is one component of $T - v$ with order not exceeding $\max\{n - 1 - a, a\}$ and all the other components of $T - v$ have order not exceeding $a$.

Let $T_i^n (2i \leq n)$ denote the tree graph obtained from a star $K_{1,n-i}$ by joining $i - 1$ pendant vertices of $K_{1,n-i}$ to $i - 1$ isolated vertices by $i - 1$ edges.

Lemma 7 [5]. Let $T$ be a tree on $n$ vertices with matching number $\beta = \beta(T)$. Then $\lambda_1(T) \leq r$, where $r$ is the maximum root of the equation

$$
x^3 - (n - \beta + 4)x^2 + (3n - 3\beta + 4)x - n = 0
$$

and equality holds if and only if $T = T^n_\beta$.

Corollary 1 [5]. Let $T$ be a tree on $n = 2t$ vertices with a perfect matching. Then

$$
\lambda_1(T) \leq \frac{t + 2 + \sqrt{t^2 + 4}}{2}
$$

and equality holds if and only if $T = T^n_t$. 
If $G = v$, then suppose that $\Phi(L_v(G)) = 1$. We have the following.

**Lemma 8.** Let $G = G_1u : vG_2$ be the graph obtained by joining the vertex $u$ of the graph $G_1$ to the vertex $v$ of the graph $G_2$ by an edge. Then

$$\Phi(L(G)) = \Phi(L(G_1))\Phi(L(G_2)) - \Phi(L(G_1))\Phi(L_v(G_2)) - \Phi(L(G_2))\Phi(L_u(G_1)).$$

**Proof.** Let $L(G_1^*) (L(G_2^*))$ be the principal submatrix obtained by deleting the row and column corresponding to vertex $v(u)$ from $L(G_1 : v) (L(G_2 : u))$, where $G_1u : v$ is the graph formed from $G_1$ by joining a new pendant vertex $v$ to $u$. Without loss of generality, we may assume that $L(G) = \begin{bmatrix} L(G_1^*) & -E_{11} \\ -E_{11}^T & L(G_2^*) \end{bmatrix}$, where $E_{11}$ is the $|V(G_1)|$-by-$|V(G_2)|$ matrix whose only non-zero entry is a 1 in position $(1,1)$.

By the Laplace Theorem, we have

$$\Phi(L(G)) = \Phi(L(G_1^*))\Phi(L(G_2^*)) - \Phi(L_u(G_1))\Phi(L_v(G_2)).$$

Since

$$\Phi(L(G_1^*)) = \Phi(L(G_1)) - \Phi(L_u(G_1)),$$
$$\Phi(L(G_2^*)) = \Phi(L(G_2)) - \Phi(L_v(G_2)),$$

Combined with $(*)$, we have

$$\Phi(L(G)) = \Phi(L(G_1))\Phi(L(G_2)) - \Phi(L(G_1))\Phi(L_v(G_2)) - \Phi(L(G_2))\Phi(L_u(G_1)).$$

Let $T(s, t)$ denote trees of diameter 3, these trees have exactly two non-pendant vertices and they are adjacent, where one of these vertices is connected to $s$ pendants (degree 1 vertices) and the other one to $t$ pendants. In particular $n = s + t + 2$. Then, we have

**Corollary 2** [4]. $\Phi(L(T(s, t))) = x(x - 1)^{n-4}[x^3 - (n + 2)x^2 + (2n + st + 1)x - n].$

**Theorem 1.** Let $T$ be a tree on $n \geq 3$ vertices. Then $\lambda_2(T) \geq 1$, equality holds if and only if $T = K_{1,n-1}$.

**Proof.** Note that $L(T)$ has 0 as an eigenvalue, and that $\lambda_1(T) \leq n$. Since $T$ has $n - 1$ edges, we have

$$2(n - 1) = \text{trace}(L(T)) \leq 0 + (n - 2)\lambda_2(T) + \lambda_1(T) \leq (n - 2)\lambda_2(T) + n.$$
Hence, $1 \leq \lambda_2(T)$, with equality if and only if the eigenvalues of $L(T)$ are $0$, $n$ and $1$, the latter with multiplicity $n - 2$. This last holds if and only if $T = K_{1,n-1}$. □

**Theorem 2.** Let $T$ be a tree on $n \geq 4$ vertices. If $T \neq K_{1,n-1}$, then $\lambda_2(T) \geq r$, where $r$ is the second largest root of the equation

$$x^3 - (n + 2)x^2 + (3n - 2)x - n = 0$$

and equality holds if and only if $T = T_n^2 = T(n - 3, 1)$.

Moreover, $r > 2$ with equality if and only if $n = 4$, $r$ is strictly increasing as a function of $n$, and converges to $\frac{3 + \sqrt{5}}{2}$ as $n \to +\infty$.

**Proof.** By Corollary 2, we have

$$\Phi(L(T(n - 3, 1))) = x(x - 1)^{n-4}[x^3 - (n + 2)x^2 + (3n - 2)x - n].$$

Let

$$f(x) = x^3 - (n + 2)x^2 + (3n - 2)x - n.$$

We have

$$f(n) = n^2 - 3n > 0 \quad (n \geq 4),$$

$$f(\frac{n + \sqrt{5}}{2}) = -1 < 0,$$

$$f(2) = n - 4 \geq 0 \quad (n \geq 4),$$

$$f(1) = n - 3 > 0 \quad (n \geq 4),$$

$$f(0) = -n < 0.$$

Hence,

$$2 \leq \lambda_2(T_n^2) = r < \frac{3 + \sqrt{5}}{2}$$

and the left equality holds if and only if $n = 4$.

From Lemma 1, $r$ is an increasing function of $n$.

If $\lambda_2(T_n^2) = \lambda_2(T_{n+1}^2)$, then

$$f(\lambda_2(T_n^2)) = -\lambda_2^3(T_n^2) + 3\lambda_2(T_n^2) - 1 = 0.$$

Thus, $\lambda_2(T_n^2) = \frac{3 + \sqrt{5}}{2}$, a contradiction with $\lambda_2(T_n^2) < \frac{3 + \sqrt{5}}{2}$.

So, $r$ is a strictly increasing function of $n$.

Since

$$f(\lambda_2(T_n^2)) = \lambda_2^3(T_n^2) - (n + 2)\lambda_2^2(T_n^2) + (3n - 2)\lambda_2(T_n^2) - n = 0,$$

we have

$$\frac{\lambda_2^3(T_n^2)}{n} = \frac{n + 2}{n} \lambda_2^2(T_n^2) - \frac{3n + 2}{n} \lambda_2(T_n^2) + 1$$

and the left-hand side converges to 0 as $n \to +\infty$. Then

$$\lim_{n \to +\infty} \lambda_2(T_n^2) = \frac{3 + \sqrt{5}}{2}.$$
If \( T \) is neither \( K_{1, n-1} \) nor \( T^2_n \), then \( T \) contains \( P_5 \cup (n-5)K_1 \) or \( T(2, 2) \cup (n-6)K_1 \) as a spanning subgraph, where \( P_5 \cup (n-5)K_1 \) denotes the disconnected graph formed from a path \( P_5 \) on 5 vertices and \( n-5 \) isolated vertices. Since

\[
\lambda_2(P_5) = \frac{3 + \sqrt{5}}{2},
\]

\[
\lambda_2(T(2, 2)) = 3 > \frac{3 + \sqrt{5}}{2},
\]

we have from Lemma 1 that

\[
\lambda_2(T) \geq \min\{\lambda_2(P_5), \lambda_2(T(2, 2))\}
\]

\[
= \frac{3 + \sqrt{5}}{2}
\]

producing the desired result. \( \square \)

Let \( \text{diam}(G) \) denote the diameter of the graph \( G \). We have the following.

**Theorem 3.** Let \( T \) be a tree on \( n \geq 5 \) vertices. If \( T \) is neither \( K_{1, n-1} \) nor \( T^2_n \), then

\[
\lambda_2(T) \geq \frac{3 + \sqrt{5}}{2},
\]

and equality holds if and only if \( T = T^i_n \) \( (i \geq 3) \).

**Proof.** First, we prove that \( \lambda_2(T^i_n) = \frac{3 + \sqrt{5}}{2} \) \( (n \geq 3) \). Since \( i \geq 3 \), then there is a unique vertex \( v \in T^i_n \) such that

\[
L_v(T^i_n) = \begin{bmatrix}
L_1 & 0 & \cdots & 0 & 0 \\
0 & L_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & L_{i-1} & 0 \\
0 & 0 & \cdots & 0 & I_{n-2i+1}
\end{bmatrix},
\]

where

\[
L_1 = L_2 = \cdots = L_{i-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.
\]

Since

\[
\det(xI - L_1) = x^2 - 3x + 1
\]

and the maximum root of the equation \( x^2 - 3x + 1 = 0 \) is \( \frac{3 + \sqrt{5}}{2} \), we have from Lemma 2 that

\[
\lambda_2(T^i_n) = \frac{3 + \sqrt{5}}{2} \quad (i \geq 3).
\]
We distinguish the following three cases:

**Case 1.** Suppose that \( \text{diam}(T) \geq 5 \). Then \( T \) contains \( P_6 \cup (n-6)K_1 \) as a spanning subgraph. We have from Lemma 1 that
\[
\lambda_2(T) \geq \lambda_2(P_6) = 3 > \frac{3 + \sqrt{5}}{2}.
\]

**Case 2.** Suppose that \( \text{diam}(T) \leq 3 \). Since \( T \) is neither \( K_1,n-1 \) nor \( T_n^2 \), \( T \) contains \( T(2, 2) \cup (n-6)K_1 \) as a spanning subgraph. We have from Lemma 1 that
\[
\lambda_2(T) \geq \lambda_2(T(2, 2)) = 3 > \frac{3 + \sqrt{5}}{2}.
\]

**Case 3.** Suppose that \( \text{diam}(T) = 4 \) and \( T/\text{in.} = T_{in} \). Then \( T \) contains \( T_1 \cup (n-6)K_1 \) as a spanning subgraph, where \( T_1 \) denotes the tree graph obtained from a star \( K_{1,3} \) by joining a pendant vertex of \( K_{1,3} \) to a vertex of \( P_2 \) by an edge. We have from Lemma 1 that
\[
\lambda_2(T) \geq \lambda_2(T_1) = 3 > \frac{3 + \sqrt{5}}{2}.
\]

Combining \( \lambda_2(T_i^n) = \frac{3 + \sqrt{5}}{2} \) \( (i \geq 3) \) and Cases 1–3, we obtain the desired result. □

**Corollary 3.** Let \( T \) be a tree on \( n = 2k \geq 6 \) vertices with a perfect matching. Then
\[
\lambda_2(T) \geq \frac{3 + \sqrt{5}}{2},
\]
and equality holds if and only if \( T = T_n^k \).

In the following, we give our main result of this paper.

**Theorem 4.** Let \( T \) be a tree on \( n = 2k \) vertices with a perfect matching. We have

1. If \( k = 2t \), then
   \[
   \lambda_2(T) \leq \frac{t + 2 + \sqrt{t^2 + 4}}{2}
   \]
   and equality holds if and only if there exists an edge \( e \) of \( T \) such that \( T - e = 2T_{2t}^t \).

2. If \( k = 2t + 1 \), then \( \lambda_2(T) \leq r \), where \( r \) is the maximum root of the equation
   \[
   x^3 - (t + 5)x^2 + (3t + 7)x - 2t - 1 = 0
   \]
   and equality holds if and only if there exists an edge \( e \) of \( T \) such that \( T - e = 2T_{2t+1}^t \).

Moreover, \( t + 2 \leq r < \frac{t + 4 + \sqrt{t^2 + 4}}{2} \), \( r \) is strictly increasing as a function of \( t \) and the left equality holds if and only if \( t = 1 \).

**Proof.** First, we prove that (1) holds. Take \( a = 2t - 1 \) in Lemma 6. Then there exists a vertex \( v \in T \) such that there is one component \( T_0 \) of \( T - v \) of order \( n_0 = |V(T_0)| \leq 

\[ n - a - 1 = 2t, \text{ and all the other components of } T - v, \text{ say } T_i (i = 1, 2, \ldots, s), \text{ have order not exceeding } 2t - 1. \] Suppose that \( v_0, v_1, \ldots, v_s \) are vertices of \( T \) and \( v_i \in T_i, \forall v_i \in T (i = 0, 1, 2, \ldots, s) \). For every \( i \) (\( i = 0, 1, 2, \ldots, s \)), let \( T'_i \) be the tree graph obtained from \( T_i \) by attaching to a new pendant vertex \( v'_i \) to vertex \( v_i \). Then \( |V(T'_i)| = |V(T_i)| + 1 \). Without loss of generality, we may assume that

\[
L_v(T) = \begin{bmatrix}
L(T_0) + E_0 & 0 & \ldots & 0 \\
0 & L(T_1) + E_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & L(T_s) + E_s
\end{bmatrix},
\]

where \( E_i \) is the \(|V(T_i)|\)-by-\(|V(T_i)|\) matrix whose only non-zero entry is a 1 in position \((1, 1)\) \((i = 0, 1, 2, \ldots, s)\).

We distinguish the following two cases:

**Case 1.** Suppose that \( |V(T_0)| \leq 2t - 1 \). Then \( |V(T_i)| \leq 2t - 1, \ i = 0, 1, 2, \ldots, s \). Since \( T \) has a perfect matching, then for some \( i(i = 0, 1, 2, \ldots, s) \), the edge \( vv_i \) is in the perfect matching of \( T \). Consequently, \( |V(T_i)| \) is odd, \( T'_i \) has a perfect matching and \( |V(T'_i)| \leq 2t \). We have from Lemmas 3, 5 and Corollary 1 that

\[
\lambda_1(D(T_i) + A(T_i) + E_i) < \lambda_1(D(T'_i) + A(T'_i)) = \lambda_1(T'_i) \leq \lambda_1(T_{2i}) = t + 2 + \sqrt{t^2 + 4}.
\]

also, if \( j \neq i \), then \( |V(T_j)| \) is even, and we conclude that \( T_j \) has a perfect matching. Let \( T''_j \) be the tree graph obtained from \( T'_j \) by attaching a new pendant vertex \( u' \) to the vertex \( v'_j \), then \( |V(T''_j)| \leq 2t \) and \( T''_j \) also has a perfect matching. We have from Lemmas 3, 5 and Corollary 1 that

\[
\lambda_1(D(T_j) + A(T_j) + E_j) < \lambda_1(D(T''_j) + A(T''_j)) = \lambda_1(T''_j) \leq \lambda_1(T_{2j}) = t + 2 + \sqrt{t^2 + 4}.
\]

Thus, we have from Lemmas 2, 5 and the above discussions that

\[
\lambda_2(T) = \lambda_2(D(T) + A(T)) \leq \max_{i=0,1,\ldots,s} \{ \lambda_1(D(T_i) + A(T_i) + E_i) \}
\]
Case 2. Suppose that |V(T₀)| = 2t. Then there exists an edge e = v₀v of T such that

\[ T - v₀v = T₀ ∪ H \quad \text{and} \quad |V(T₀)| = |V(H)| = 2t. \]

Since T has a perfect matching, then T₀ and H have a perfect matching, respectively. We have from Lemma 1 and Corollary 1 that

\[ λ₂(T) ≤ \max\{λ₁(T₀), λ₁(H)\} = λ₁(T₂) = \frac{t + 2 + √t² + 4}{2}. \]

In the following, we only need to prove that the equality holds in (1) if and only if there exists an edge e of T such that \( T - e = 2T₂ \). We now consider the following two subcases:

Subcase 1. Suppose that neither T₀ = T₂ nor H = T₂. Then we have from Lemma 1 and Corollary 1 that

\[ λ₂(T) ≤ \max\{λ₁(T₀), λ₁(H)\} < λ₁(T₂) = \frac{t + 2 + √t² + 4}{2}. \]

Subcase 2. Suppose that either T₀ = T₂ or H = T₂. Without loss of generality, we may assume that T₀ = T₂. By Corollary 1, we have

\[ \Phi\left( L(T₀); \frac{t + 2 + √t² + 4}{2} \right) = 0. \]

(a) If H ≠ T₂, then by Corollary 1, we have \( λ₁(H) < \frac{t + 2 + √t² + 4}{2} \). Hence,

\[ \Phi\left( L(H); \frac{t + 2 + √t² + 4}{2} \right) > 0. \]

We have from Lemma 8 that

\[ \Phi(L(T)) = \Phi(L(T₀))\Phi(L(H)) - \Phi(L(T₀))\Phi(L_0(H)) - \Phi(L(H))\Phi(L_0(T₀)). \]

So, we have

\[ \Phi\left( L(T); \frac{t + 2 + √t² + 4}{2} \right) = -\Phi\left( L(H); \frac{t + 2 + √t² + 4}{2} \right) \times \Phi\left( L_0(T₀); \frac{t + 2 + √t² + 4}{2} \right). \]
By Lemmas 3–5 and Corollary 1, we have
\[ \lambda_1(L_{v_0}(T_0)) \leq \lambda_1(L(T_0)) \]
\[ < \lambda_1(D(T_0) + A(T_0)) \]
\[ = \lambda_1(L(T_0)) \]
\[ = t + 2 + \sqrt{t^2 + 4} \]
\[ = \frac{t + 2 + \sqrt{t^2 + 4}}{2}. \]

Then,
\[ \phi \left( L_{v_0}(T_0); \frac{t + 2 + \sqrt{t^2 + 4}}{2} \right) > 0. \]

Hence, we have
\[ \phi \left( L(T); \frac{t + 2 + \sqrt{t^2 + 4}}{2} \right) < 0. \]

Thus,
\[ \lambda_2(T) \neq \frac{t + 2 + \sqrt{t^2 + 4}}{2}. \]

Combined with (***), we have
\[ \lambda_2(T) < \frac{t + 2 + \sqrt{t^2 + 4}}{2}. \]

(b) If \( H = T_{2j} \), we have from Lemma 1 that
\[ \lambda_2(T) = \frac{t + 2 + \sqrt{t^2 + 4}}{2}. \]

By the above discussions, we complete the proof of (1).

Finally, we outline the proof of (2). By a proof similar to that of Theorem 2, \( r = \lambda_1(T_{2j+1}) \) is strictly increasing as a function of \( t \). Let
\[ f(x) = x^3 - (t + 5)x^2 + (3t + 7)x - 2t - 1. \]

Since
\[ f(0) = -2t - 1 < 0, \]
\[ f(1) = 1 - (t + 5) + 3t + 7 - 2t - 1 = 2 > 0, \]
\[ f(x) = (x - 3)(x^2 - 3x + 1) \quad (t = 1), \]
\[ f(t + 2) = 1 - t < 0 \quad (t \geq 2), \]
\[ f \left( \frac{t + 4 + \sqrt{t^2 + 4}}{2} \right) = 2 > 0, \]

we have \( t + 2 \leq r < \frac{t + 4 + \sqrt{t^2 + 4}}{2} \), the left equality holds if and only if \( t = 1 \).
Take \( a = 2t \) in Lemma 6. Then there exists a vertex \( v \in T \) such that there is one component \( \tilde{T}_0 \) of \( T - v \) of order \( n_0 \leq 2t + 1 \), and all the other components of \( T - v \), say \( \tilde{T}_i \) (\( i = 1, 2, \ldots, k \)), have order not exceeding \( 2t \). Apply Lemma 7 but not Corollary 1 and replace \( \lambda_1(T_{2t}^2) = t^2 + \sqrt{t^2 + 4} \) with \( \lambda_1(T_{2t+1}^2) = r \) in the proof of (1). By reasoning similar to that in the proof of (1), we can prove that (2) holds.

\[ \square \]

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