Annals of Mathematical Logic 11 (1977) 57–103

COUNTABLE APPROXIMATIONS AND LÖWENHEIM–SKOLEM THEOREMS

David W. KUEKER*
Department of Mathematics, University of Maryland, Maryland 20742, U.S.A.

Received 20 October 1975

0. Introduction

In this paper we present our work on various Löwenheim–Skolem-type properties of infinitary logic. Fundamental to this work are the concept of a *countable approximation* to a model or a formula of $L_{\infty\omega}$ and a filter on the countable subsets of an arbitrary set which provides a natural notion of *almost all* countable approximations. After some preliminaries in Section 1, these basic ideas are introduced in Section 2. In Section 3 we prove various Löwenheim–Skolem results for $L_{\infty\omega}$ which show how $L_{\infty\omega}$ properties of models are determined by properties of their countable approximations. In particular we prove (Theorem 3.1) that a sentence σ of $L_{\infty\omega}$ is true on a model $\mathfrak{A}$ if and only if almost all countable approximations to σ are true on the corresponding countable approximations to $\mathfrak{A}$. This is used, for example, in obtaining a biconditional strengthening of the usual downward Löwenheim–Skolem theorem for $L_{\omega_1\omega}$ (Corollary 3.2(a)) and a characterisation of $L_{\infty\omega}$-elementary equivalence in terms of isomorphisms between countable submodels (Theorem 3.5). Section 4 concerns reduced products modulo the special filter we introduce, using which we give another characterisation of $L_{\infty\omega}$-elementary equivalence (Theorem 4.2); these results are not needed for the later sections. In Section 5 we study classes of models satisfying certain abstract Löwenheim–Skolem conditions suggested by the property sentences of $L_{\omega_1\omega}$ were shown to have in Section 3. We introduce an interesting extension of $L_{\omega_1\omega}$ with some game quantification, called $L^P(\omega)$, which suffices to axiomatise such classes (Theorem 5.5). This is exploited in proving, for example, that $L^P(\omega)$ satisfies interpolation (Theorem 5.8). The ideas of Section 5 are applied in Section 6 to yield a variety of Löwenheim–Skolem (or "transfer") results for certain properties which can be expressed using $L^P(\omega)$. Section 7 concerns uncountable approximations and contains generalisations of many of the results of Section 3 (to $L_{\infty\kappa}$) and Section 5 (involving larger logics with some game quantification).

The main results of Sections 2–5 were presented without proof in [15], and some

* The preparation of this paper has been supported in part by the National Science Foundation under grant GP–43701.

of those in Section 6 were announced in [16]. Our presentation here has benefited greatly from reading Barwise's paper [2].

1. Preliminaries

Throughout this paper L denotes a language all of whose relation and function symbols have just finitely many arguments. Except where explicitly noted to the contrary, L is assumed to be countable. Models for L are denoted by $\mathfrak{A}$, $\mathfrak{B}$, etc. and their universes by the corresponding capital letters A, B, etc.

We will consider various logics built from the language L using equality and an unlimited supply of individual variables. If κ and λ are infinite cardinals then $L_{\kappa\lambda}$ is the logic whose formulas are defined as follows:

(i) every atomic formula of L is a formula of $L_{\kappa\lambda}$,

(ii) if φ is a formula of $L_{\kappa\lambda}$ then so is $\neg\varphi$,

(iii) if φ_i is a formula of $L_{\kappa\lambda}$ for each $i \in I$ and $|I| < \kappa$ then $\bigwedge_{i\in I}\varphi_i$ and $\bigvee_{i\in I}\varphi_i$ are formulas of $L_{\kappa\lambda}$,

(iv) if φ is a formula of $L_{\kappa\lambda}$ and $\boldsymbol{x}$ is a sequence of less than λ individual variables then $\forall\boldsymbol{x}\varphi$ and $\exists\boldsymbol{x}\varphi$ are formulas of $L_{\kappa\lambda}$.

Note that $L_{\omega\omega}$ is essentially the usual first-order logic on L, which we also denote simply by L. We write $L_{\infty\lambda}$ for $\bigcup_\kappa L_{\kappa\lambda}$. Notice that a formula of $L_{\infty\lambda}$ can contain any number of free variables but that only those formulas with fewer than λ free variables can be subformulas of sentences of $L_{\infty\lambda}$.

We assume the reader is familiar with the usual syntactic and model-theoretic notions and notations of the logics $L_{\kappa\lambda}$. For example, see [1], [4], [11], [12]. Some more special, or less familiar, material follows.

We frequently (as in the definition of $L_{\kappa\lambda}$) use vectors as abbreviations for sequences. For example, $\boldsymbol{x}$ and $\boldsymbol{a}$ would abbreviate sequences $x_0, x_1, \ldots$ and $a_0, a_1, \ldots$ respectively, the precise lengths of the sequences (finite or infinite) to be determined by context.

Often we write a formula φ to exhibit its free variables, for example as $\varphi(\boldsymbol{x})$. By convention, we assume that *all* the free variables of φ are listed, but not that all the listed variables occur free.

A formula is said to be in *negation-normal form* if it only contains negations in front of atomic subformulas. Every formula of $L_{\kappa\lambda}$ is equivalent to a formula of $L_{\kappa\lambda}$ in negation-normal form. We define the *canonical negation-normal form* of φ, φ^n, by induction as follows:

(i) φ^n is φ if φ is atomic;

(ii) $(\bigwedge_{i\in I}\varphi_i)^n$ is $\bigwedge_{i\in I}\varphi_i^n$, and $(\bigvee_{i\in I}\varphi_i)^n$ is $\bigvee_{i\in I}\varphi_i^n$;

(iii) $(\exists\boldsymbol{x}\varphi)^n$ is $\exists\boldsymbol{x}\varphi^n$, and $(\forall\boldsymbol{x}\varphi)^n$ is $\forall\boldsymbol{x}\varphi^n$;

(iv) $(\neg\varphi)^n$ is defined by cases on φ as follows:

$(\neg\varphi)^n$ is $\neg\varphi$ if φ is atomic;

$(\neg\bigwedge_{i\in I}\varphi_i)^n$ is $\bigvee_{i\in I}(\neg\varphi_i)^n$ and $(\neg\bigvee_{i\in I}\varphi_i)^n$ is $\bigwedge_{i\in I}(\neg\varphi_i)^n$;

$(\neg\forall\boldsymbol{x}\psi)^n$ is $\exists\boldsymbol{x}(\neg\psi)^n$ and $(\neg\exists\boldsymbol{x}\psi)^n$ is $\forall\boldsymbol{x}(\neg\psi)^n$;

$(\neg\neg\psi)^n$ is ψ^n.

It is then clear that φ^n is in negation-normal form and that $\models \varphi \leftrightarrow \varphi^n$.

An *existential* formula of $L_{\kappa\lambda}$ is a formula of $L_{\kappa\lambda}$ in negation-normal form which contains no universal quantifiers. We use $\exists_{\kappa\lambda}$ and $\exists_{\infty\lambda}$ for the classes of all existential formulas of $L_{\kappa\lambda}$ and $L_{\infty\lambda}$, respectively. The *universal* formulas of $L_{\kappa\lambda}$ are those in negation-normal form without existential quantifiers. The *positive* formulas of $L_{\kappa\lambda}$ are those in negation-normal form without negations; the classes of positive formulas of $L_{\kappa\lambda}$ and $L_{\infty\lambda}$ are denoted by $P_{\kappa\lambda}$ and $P_{\infty\lambda}$ respectively.

If Γ is some class of formulas we write $\mathfrak{A}\, \Gamma\, \mathfrak{B}$ to mean that every sentence in Γ true on $\mathfrak{A}$ is also true on $\mathfrak{B}$. We write $\mathfrak{A} \equiv \mathfrak{B}(\Gamma)$ to mean that $\mathfrak{A}\, \Gamma\, \mathfrak{B}$ and $\mathfrak{B}\, \Gamma\, \mathfrak{A}$.

If L′ is some other language, then $(L')_{\kappa\lambda}$ is the logic built from the non-logical symbols in L′. If L′ contains L and $\mathfrak{A}'$ is an L′-model then $\mathfrak{A}' \restriction L$ is the reduct of $\mathfrak{A}'$ to the language L. If $\boldsymbol{K}'$ is a class of L′-models then we define

$$\boldsymbol{K}' \restriction L = \{\mathfrak{A}' \restriction L : \mathfrak{A}' \in \boldsymbol{K}'\}.$$

If $\mathfrak{A}$ is any model and $s \subseteq A$ then $\mathfrak{A} \restriction s$ is the submodel of $\mathfrak{A}$ generated by s (that is, the least submodel of $\mathfrak{A}$ containing s).

The cardinality of a set X is denoted by $|X|$. If κ is any infinite cardinal we define

$$\mathcal{P}_\kappa(C) = \{s \subseteq C : |s| < \kappa\}.$$

The classical infinitary languages $L_{\kappa\lambda}$ are well-known to have intimate connections with "back-and-forth" relations between models (see [1], [4], [10]). We sketch the connection between $L_{\infty\omega}$ and "one-at-a-time" back-and-forths, which is all we shall require until the last section of this paper.

Definition. A *back-and-forth* relation between models $\mathfrak{A}$ and $\mathfrak{B}$ is a relation $\sim\, \subseteq \bigcup_{n\in\omega}(A^n \times B^n)$ such that

(i) $0 \sim 0$;

(ii) if $\boldsymbol{a} \sim \boldsymbol{b}$ and $\alpha(\boldsymbol{x})$ is an atomic formula, then $\mathfrak{A} \models \alpha[\boldsymbol{a}]$ if and only if $\mathfrak{B} \models \alpha[\boldsymbol{b}]$;

(iii) if $(a_0, \ldots, a_{n-1}) \sim (b_0, \ldots, b_{n-1})$ then given any $a_n \in A$ (or, $b_n \in B$) there is some $b_n \in B$ (or, $a_n \in A$) such that $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$.

We write $\mathfrak{A} \sim \mathfrak{B}$ if there is a back-and-forth relation between $\mathfrak{A}$ and $\mathfrak{B}$.

Theorem 1.1 (Karp [10]). *$\mathfrak{A} \sim \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$.*

More precisely, the proof of Theorem 1.1 tells us the following: if $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ then the relation $\sim$ defined by $\boldsymbol{a} \sim \boldsymbol{b}$ iff $(\mathfrak{A}, \boldsymbol{a}) \equiv_{\infty\omega} (\mathfrak{B}, \boldsymbol{b})$ is a back-and-forth relation; if $\sim$ is a back-and-forth relation then $\boldsymbol{a} \sim \boldsymbol{b}$ implies $(\mathfrak{A}, \boldsymbol{a}) \equiv_{\infty\omega} (\mathfrak{B}, \boldsymbol{b})$.

As a consequence we obtain Scott's isomorphism theorem [26].

Corollary 1.2. (a) *If $\mathfrak{A}$ and $\mathfrak{B}$ are countable and $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ then $\mathfrak{A} \cong \mathfrak{B}$.*

(b) *If $\mathfrak{A}$ is countable then there is a sentence $\sigma_{\mathfrak{A}}$ of $L_{\omega_1\omega}$ such that $\mathfrak{B} \models \sigma_{\mathfrak{A}}$ iff $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$.*

Such a sentence $\sigma_{\mathfrak{A}}$ is often called a Scott sentence for $\mathfrak{A}$.

We shall make considerable use of strings of quantifiers which alternate infinitely often between $\forall$ and $\exists$, like

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 \cdots \forall x_{2n} \exists x_{2n+1} \cdots . \tag{1}$$

Until the last section we will only need to consider such strings of length ω; the most general such quantifier-string, called a *game quantifier*, is

$$Q_0 x_0 Q_1 x_1 \cdots Q_n x_n \cdots , \tag{2}$$

where each Q_i is either $\exists$ or $\forall$.

The intuitive meaning of the quantifier string (1) is that given x_0 you can choose x_1 such that given x_2 you can choose x_3 (depending on x_0, x_2) etc. so that the assertion following the string is true. Formally the choosing is done by Skolem functions choosing x_{2k+1} in terms of previous x_i's. That is

$$\mathfrak{A} \models \forall x_0 \exists x_1 \cdots \forall x_{2n} \exists x_{2n+1} \cdots \varphi(x_0, x_1, \ldots)$$

means, by definition, that there are functions $f_{k+1} : A^{k+1} \to A$ for all $k \in \omega$ such that

$$\mathfrak{A} \models \varphi[a_0, f_1(a_0), a_2, f_3(a_0, a_2), \ldots]$$

for any $a_0, a_2, a_4, \ldots$ in A. Such f_i's will be said to be *winning functions* for the sentence, or to define a *winning strategy* for the sentence. A similar definition can be given for the general quantifier in (2) (see [6], [11]).

Infinite quantifier-strings will be abbreviated in fairly obvious ways. Thus (2) will be abbreviated as $(Q_n x_n)_{n<\omega}$, and (1) will be abbreviated as $(\forall x_{2n} \exists x_{2n+1})_{n<\omega}$.

We will also use such quantifier-strings meta-theoretically, and their meanings would be explained in the same way. The meaning of a string of restricted quantifiers, such as

$$\forall x_0 \in C \exists x_1 \in C \cdots \forall x_{2n} \in C \exists x_{2n+1} \in C \cdots ,$$

is also clear.

Keisler in [11] introduced the logic $L(\omega)$, an extension of $L_{\infty\omega}$ allowing game quantification. The formulas of $L(\omega)$ are defined as follows:

(i) every atomic formula is a formula of $L(\omega)$,

(ii) if φ is a formula of $L(\omega)$ then so is $\neg\varphi$,

(iii) the conjunction or disjunction of any set of formulas of $L(\omega)$ is a formula of $L(\omega)$,

(iv) if φ is a formula of $L(\omega)$ then so are $\forall x\varphi$ and $\exists x\varphi$ provided they have only finitely many free variables,

(v) if φ is a formula of $L(\omega)$ then so is $(Q_n x_n)_{n<\omega}\varphi$, provided it has only finitely many free variables.

Then every sentence of $L_{\infty\omega}$ is a sentence of $L(\omega)$, but not, of course, conversely. $L(\omega)$ is not quite as strong as it might seem, however, due to the following result.

Theorem 1.3 (Keisler [11]). *If* $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ *then* $\mathfrak{A} \equiv \mathfrak{B}(L(\omega))$.

As is well-known, games are not always determined. That is, the negation of a game-quantified formula is not found by "pushing the negation through" the quantifier-string since

$$\neg(\forall x_{2n} \exists x_{2n+1})_{n<\omega}\varphi \leftrightarrow (\exists x_{2n} \forall x_{2n+1})_{n<\omega} \neg\varphi$$

is *not* usually true, although the implication from right to left does always hold. One useful criterion for the biconditional to hold is the following, which is essentially the Gale–Stewart theorem that open games are determined (see [24]).

Theorem 1.4 (Gale–Stewart). *Assume each α_i has only finitely many free variables. Then*

$$\models \neg(\forall x_{2n} \exists x_{2n+1})_{n<\omega} \bigvee_{i\in I} \alpha_i \leftrightarrow (\exists x_{2n} \forall x_{2n+1})_{n<\omega} \bigwedge_{i\in I} \neg\alpha_i.$$

Proof. The implication from right to left is easy, and so omitted. By replacing any free variables of the biconditional by new individual constants, we may assume no α_i contains any variable other than the x_k's free. By associating together all α_i's with the same free variables we may assume that $I = \omega$ and that α_i is $\alpha_i(x_0, \ldots, x_{2i+1})$ for each $i \in \omega$. For each $k \in \omega$ let θ_k be the formula

$$\neg(\forall x_{2n} \exists x_{2n+1})_{k\leq n<\omega} \bigvee_{k\leq i<\omega} \alpha_i.$$

Then θ_k is $\theta_k(x_0, \ldots, x_{2k-1})$, and the following are all logically equivalent:

$$\theta_k,$$

$$\exists x_{2k} \forall x_{2k+1} \neg(\forall x_{2n} \exists x_{2n+1})_{k<n<\omega} \bigvee_{k\leq i<\omega} \alpha_i,$$

$$\exists x_{2k} \forall x_{2k+1}(\neg\alpha_k \wedge \theta_{k+1}).$$

So assume $\mathfrak{A} \models \theta_0$, and consider the following way to choose $x_0, x_2, x_4, \ldots$. Let $x_0 = f_0 \in A$ be such that

$$\mathfrak{A} \models \neg\alpha_0[f_0, a_1] \wedge \theta_1[f_0, a_1]$$

for every $a_1 \in A$. Given a_1 let $x_2 = f_2(a_1)$ be such that

$$\mathfrak{A} \models \neg\alpha_1[f_0, a_1, f_2(a_1), a_3] \wedge \theta_2[f_0, a_1, f_2(a_1), a_3]$$

for every $a_3 \in A$. Continue in this way to make every θ_k true. Then these are winning functions for the sentence $(\exists x_{2n} \forall x_{2n+1})_{n<\omega} \bigwedge_{i\in\omega} \neg\alpha_i$, as desired. ⊣

By taking negations of both sides we derive the same statement with $\wedge$ and $\vee$ interchanged.

We use ⊣ to mark the end of a proof.

2. The filter and countable approximations

We first define a filter on the countable subsets of an arbitrary set C which will provide us with our concept of "almost all countable subsets" of C, and derive the basic properties of the filter.

Definition. For any set C the filter $D(C)$ on $\mathcal{P}_{\omega_1}(C)$ is defined as follows: $X \in D(C)$ if and only if $X \subseteq \mathcal{P}_{\omega_1}(C)$ and there is some $X' \subseteq X$ such that
(i) for every $s \in \mathcal{P}_{\omega_1}(C)$ there is some $s' \in X'$ with $s \subseteq s'$,
(ii) X' is closed under unions of countable chains.

Any set X' with properties (i) and (ii) will be called a *closed unbounded* subset of $\mathcal{P}_{\omega_1}(C)$.

The following result gives the basic properties of the filter which we will use throughout the paper.

Proposition 2.1. *Let D be $D(C)$ for some C.*
(a) *D is a countably complete filter.*
(b) *D is closed under diagonalisation — that is, if $X_a \in D$ for all $a \in C_0$ then*

$$\bar{X} = \{s \in \mathcal{P}_{\omega_1}(C) : s \in X_a \quad \text{for all} \quad a \in s \cap C_0\} \in D.$$

(c) *$X \in D$ if and only if*

$$(\forall x_{2n} \in C\, \exists x_{2n+1} \in C)_{n<\omega}[\{x_n : n \in \omega\} \in X].$$

(d) *If C is uncountable then D is non-principal and not an ultrafilter.*

Proof. (a). It is easily verified that the intersection of countably many closed unbounded subsets of $\mathcal{P}_{\omega_1}(C)$ is also closed unbounded, hence D is closed under countable intersections.

(b) We may assume that each X_a is closed unbounded. It is then easily verified that $\bar{X}$ is closed under unions of countable chains. So we will have $\bar{X} \in D$ once we show that for every $s \in \mathcal{P}_{\omega_1}(C)$ there is some $s' \supseteq s$ with $s' \in \bar{X}$. Let $s = s_0 \in \mathcal{P}_{\omega_1}(C)$. Then $Y_0 = \bigcap\{X_a : a \in s_0 \cap C_0\} \in D$ since D is countably complete. Hence there is some $s_1 \in Y_0$ with $s_0 \subseteq s_1$. In general, we get $s_{n+1} \supseteq s_n$ with $s_{n+1} \in Y_n = \bigcap\{X_a : a \in s_n \cap C_0\}$ for each $n \in \omega$. Let $s' = \bigcup_{n\in\omega} s_n$. Let $a \in s' \cap C_0$. Then $a \in s_n \cap C_0$ for all $n \geqslant n_0$ for some n_0, hence $s_n \in X_a$ for all $n > n_0$, and so $s' = \bigcup_{n>n_0} s_n \in X_a$ since X_a is closed under unions of countable chains. Therefore $s' \in \bar{X}$ as desired.

(c) We first show the condition is sufficient for $X \in D$. Assume the condition holds; by definition this means there are functions $f_{2n} : C^{n+1} \to C$ such that

$$\{a_n : n < \omega\} \cup \{f_{2n}(a_0, a_1, \ldots, a_n) : n < \omega\} \in X$$

whenever $a_n \in C$ for each n. Let X' be the set of all $s \in \mathcal{P}_{\omega_1}(C)$ such that s is closed under f_{2n} for all $n \in \omega$. X' is clearly closed unbounded, hence $X' \in D$. Let $s \in X'$, say $s = \{a_n : n \in \omega\}$. Then $f_{2n}(a_0, \ldots, a_n) \in s$ for each n, hence

$$s = \{a_n : n < \omega\} \cup \{f_{2n}(a_0, \ldots, a_n) : n < \omega\} \in X.$$

So $X' \subseteq X$ and therefore $X \in D$.

Now let $X \in D$. We may suppose X is closed unbounded. For any $a_0 \in C$ let $s_{a_0} \in X$, $a_0 \in s_{a_0}$. In general for each n and any $a_0, \ldots, a_n \in C$ let $s_{a_0, \ldots, a_n} \in X$ be such that $s_{a_0, \ldots, a_n} \supseteq s_{a_0, \ldots, a_{n-1}} \cup \{a_n\}$. Then given any countable sequence $a_0, a_1, \ldots$ from C we know that $\bar{s} = \bigcup\{s_{a_0, \ldots, a_n} : n \in \omega\} \in X$ since X is closed under unions of countable chains. So, just define $f_{2n} : C^{n+1} \to C$ so that

$$\{f_{2n}(a_0, \ldots, a_n) : n < \omega\} = \bar{s}.$$

This can be done, for example, by having

$$\{f_{2k}(a_0, \ldots, a_k) : 2 \mid k\} = s_{a_0},$$

$$\{f_{2k}(a_0, \ldots, a_k) : 2 \nmid k, 3 \mid k\} = s_{a_0 a_1}, \text{etc.}$$

Then these functions are such that given any sequence of a_n's from C

$$\{a_n : n < \omega\} \cup \{f_{2n}(a_0, \ldots, a_n) : n \in \omega\} = \bar{s} \in X.$$

Therefore the condition holds.

(d) Let $|C| > \omega$. Then D is clearly non-principal since for any $s \in \mathcal{P}_{\omega_1}(C)$, $\{s' : s \neq s'\} \in D$. For a proof that D is not an ultrafilter, see [9]. All we will need later is that $D(\omega_1)$ is not an ultrafilter, which is clear since there are no countably complete non-principal ultrafilters on a set of cardinality $2^\omega = |\mathcal{P}_{\omega_1}(\omega_1)|$. $\dashv$

This filter was also studied in a different context by Jech [9], who independently obtained (a) and (b) (but not (c)) of Proposition 2.1.

Part (c) of 2.1 is not just an interesting characterisation of the filter but will become central to some developments in sections 5 and 6. Let us note the following useful characterisation of the filter which follows from the proof of 2.1(c).

Corollary 2.2. *For $X \subseteq \mathcal{P}_{\omega_1}(C)$, $X \in D$ if and only if there are functions f_n on C, each of finitely many arguments, such that $s \in X$ whenever $s \in \mathcal{P}_{\omega_1}(C)$ and s is closed under every f_n, $n \in \omega$.*

We will later need the following generalisation of 2.1(b), which could be proved in the same way or derived from it.

Corollary 2.3. *If $X_{a_1, \ldots, a_n} \in D$ for all $n \in \omega$ and all $a_1, \ldots, a_n \in C_0$, then*

$$\bar{X} = \{s \in \mathcal{P}_{\omega_1}(C) : s \in X_{a_1, \ldots, a_n} \text{ whenever } a_1, \ldots, a_n \in s \cap C_0, n \in \omega\} \in D.$$

Intuitively, a countable approximation to a model $\mathfrak{A}$ should be a countable submodel of $\mathfrak{A}$. Similarly, a countable approximation to a formula φ of $L_{\infty\omega}$ should be a formula obtained from φ by replacing the uncountable conjunctions and disjunctions in φ by certain countable ones. The next definition states precisely how this is done and the result indexed by countable sets.

Definition. If s is any countable set, then
(a) $\mathfrak{A}^s = \mathfrak{A} \upharpoonright (s \cap A)$ for any model $\mathfrak{A}$,
(b) if φ is a formula of $L_{\infty\omega}$, φ^s is inductively defined as follows:
(i) φ^s is φ if φ is atomic;
(ii) $(\neg\varphi)^s$ is $\neg(\varphi^s)$, $(\forall x\varphi)^s$ is $\forall x(\varphi^s)$, and $(\exists x\varphi)^s$ is $\exists x(\varphi^s)$;
(iii) $(\bigwedge_{i\in I}\varphi_i)^s$ is $\bigwedge_{i\in s\cap I}\varphi_i^s$ and $(\bigvee_{i\in I}\varphi_i)^s$ is $\bigvee_{i\in s\cap I}\varphi_i^s$.

We want $\mathfrak{A}^s$ and φ^s to be defined for all countable s, so two clarifications of the above definitions are needed. First, we do not allow the empty model, so by convention $\mathfrak{A}^s$ is $\mathfrak{A}^{\{a\}}$ for some $a \in A$ if otherwise $\mathfrak{A}^s$ would be empty. Secondly, we allow empty conjunctions and disjunctions under the convention that an empty conjunction is logically true and an empty disjunction logically false. We are thus able to say without restriction that $\mathfrak{A}^s$ is a countable submodel of $\mathfrak{A}$ and φ^s is a formula of $L_{\omega_1\omega}$ for all countable s.

Definition. A set C is *large enough* to approximate $\mathfrak{A}$ if $A \subseteq C$; C is *large enough* to approximate φ if every conjunction and disjunction in φ is indexed by elements of C.

We temporarily use $\mathbf{a}$, $\mathbf{a}_1$, etc. ambiguously to refer to either models or formulas of $L_{\infty\omega}$. If C is large enough to approximate $\mathbf{a}$ then $\mathbf{a}^s = \mathbf{a}^{s\cap C}$ for every countable s. If $\mathbf{a}$ is a countable model or a formula of $L_{\omega_1\omega}$ some countable C is large enough to approximate $\mathbf{a}$. The following definition states what it means for a predicate P to hold of almost all countable approximations to given $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition. $P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s)$ holds for *almost all countable* s iff

$$\{s \in \mathcal{P}_{\omega_1}(C) : P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s) \text{ holds}\} \in D(C) \tag{$*$}$$

for some C large enough to approximate $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

The definition is actually independent of the choice of C.

Proposition 2.4. *If $P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s)$ holds for almost all countable s then $(*)$ holds for every C large enough to approximate $\mathbf{a}_1, \ldots, \mathbf{a}_n$. This happens if and only if*

$$(\forall x_{2n}\, \exists x_{2n+1})_{n<\omega}\, [P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s) \textit{ holds for } s = \{x_n : n \in \omega\}]. \tag{$\dagger$}$$

Proof. By Proposition 2.1(c) condition $(*)$ is equivalent to

$$(\forall x_{2n} \in C\, \exists x_{2n+1} \in C)_{n<\omega}\, [P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s) \text{ holds for } s = \{x_n : n \in \omega\}]. \tag{$**$}$$

If C is large enough to approximate $\mathbf{a}_1, \ldots, \mathbf{a}_n$, then $\mathbf{a}_i^s = \mathbf{a}_i^{s\cap C}$ for every countable s and so $(\dagger)$ is equivalent to

$$(\forall x_{2n} \exists x_{2n+1})_{n<\omega} [P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s) \text{ holds for } s = \{x_n \in C : n < \omega\}]. \qquad (\dagger\dagger)$$

But it is easy to see that (**) is equivalent to (††), since the truth of (††) does not depend on any x_n's not in C. Therefore, if (*) holds for some C large enough to approximate $\mathbf{a}_1, \ldots, \mathbf{a}_n$ then (†) holds; and if (†) holds then (*) holds for every C large enough to approximate $\mathbf{a}_1, \ldots, \mathbf{a}_n$. ⊣

We will often use *a.e.* as an abbreviation for "for almost all countable *s*." In addition we will often say that $P(\mathfrak{A}_0)$ holds *for almost all countable submodels* $\mathfrak{A}_0$ *of* $\mathfrak{A}$ to mean that $P(\mathfrak{A}^s)$ holds a.e.

As simple examples, notice that the universe of $\mathfrak{A}^s$ is $s \cap A$ a.e. (that is, $s \cap A$ is closed under the functions of $\mathfrak{A}$ for almost all countable s); if $\mathfrak{A}$ is countable then $\mathfrak{A}^s = \mathfrak{A}$ a.e., and if φ is a formula of $L_{\omega_1\omega}$ then φ^s is φ a.e.

The following consequence of Proposition 2.4 will often be used without explicit mention.

Corollary 2.5. *For given* $\mathbf{a}_1, \ldots, \mathbf{a}_n$ *the properties P such that* $P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s)$ *holds a.e. are closed under countable conjunction.*

Proof. Let P_k, $k \in \omega$, be any countable collection of such properties, and let C be large enough to approximate $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Then, by 2.4,

$$X_k = \{s \in \mathcal{P}_{\omega_1}(C) : P_k(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s) \text{ holds}\} \in D(C)$$

for all $k \in \omega$, so $\bigcap_{k\in\omega} X_k \in D(C)$ by 2.1(?). But

$$\bigcap_{k\in\omega} X_k = \{s \in \mathcal{P}_{\omega_1}(C) : \bigwedge_{k\in\omega} P_k(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s) \text{ holds}\},$$

so the conjunction of all P_k, $k \in \omega$, is also such a property.

We will also speak of statements being true for *almost all countable subsets of C* even when they are not predicates of approximations. What we will mean, of course, is that the set of all $s \in \mathcal{P}_{\omega_1}(C)$ for which the statement is true belongs to $D(C)$. The truth of such assertions usually depends heavily on C. For example, "s is an ordinal" is true for almost all countable $s \subseteq \omega_1$, but not for almost all countable $s \subseteq \omega_2$. When C is clear from context, however, we will say simply "for almost all countable s" or even "a.e.".

Barwise in [2] has generalised our notion of countable approximation by defining the countable approximations $\mathbf{a}^s$ to any set $\mathbf{a}$ in such a manner that if $\mathbf{a}$ is a model or formula of $L_{\infty\omega}$ then his definition agrees with ours almost everywhere. In addition he proves a strong general result concerning which properties of sets are inherited by almost all of their countable approximations.

To work best, however, his definition must be given in set theory with a proper class of ur-elements (individuals). He also requires the universe of every model to consist only of ur-elements, and all symbols of L to be ur-elements. The reason for this is that in approximating a model $\mathfrak{A}$ we wish to leave the elements of the model alone, and only approximate the set-theoretic structure of $\mathfrak{A}$ as built up from those elements, which are therefore best treated as objects which are not themselves approximated.

His definition, then, is the following.

Definition (Barwise [2]). For any set s the approximation $\mathbf{a}^s$ is defined by $\in$-recursion on $\mathbf{a}$:

$$\mathbf{a}^s = \mathbf{a} \text{ if } \mathbf{a} \text{ is an ur-element,}$$

$$\mathbf{a}^s = \{x^s : x \in \mathbf{a} \cap s\} \text{ if } \mathbf{a} \text{ is a set.}$$

The definition of "almost all" proceeds as above, and he then proves the following theorem, which generalises our Theorems 3.1, 3.5, and 3.7 given below.

Theorem 2.5 (Barwise [2]). *If P is a Σ-predicate of set theory and if $P(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ holds, then $P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s)$ holds a.e.*

Since in this paper we are only interested in approximating models and formulas, we decided to keep to our earliest approach and also to prove 3.1, 3.5, and 3.7 directly rather than appeal to Barwise's generalisation.

3. The Löwenheim–Skolem theorem $L_{\infty\omega}$

In this section we use the material from section 2 to prove two generalisations of the Löwenheim–Skolem theorem for $L_{\infty\omega}$. These make precise the way in which $L_{\infty\omega}$ properties of models are determined by properties of their countable submodels.

Theorem 3.1. *If σ is a sentence of $L_{\infty\omega}$ then $\mathfrak{A} \models \sigma$ iff $\mathfrak{A}^s \models \sigma^s$ a.e.*

Proof. First notice that it is enough to prove the implication from left to right, since: if we know that implication for every sentence and if $\mathfrak{A} \not\models \sigma$ then $\mathfrak{A} \models \neg\sigma$, so by assumption $\mathfrak{A}^s \models (\neg\sigma)^s$ a.e., that is $\mathfrak{A}^s \models \neg(\sigma^s)$ a.e., and so we cannot have $\mathfrak{A}^s \models \sigma^s$ a.e. (since there is no s such that $\mathfrak{A}^s \models \sigma^s$ and $\mathfrak{A}^s \models \neg\sigma^s$).

Next notice that $(\sigma^n)^s$ is $(\sigma^s)^n$ for all σ and s (where φ^n is the canonical negation-normal-form of φ), and so it is sufficient to consider only sentences in negation-normal form.

Therefore the proof of the theorem will be completed once we show for every formula $\varphi(x_0, \ldots, x_n)$ of $L_{\infty\omega}$ in negation-normal form and for every $a_0, \ldots, a_n \in A$ that

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_n] \text{ implies } \mathfrak{A}^s \models \varphi^s[a_0, \ldots, a_n] \text{ a.e.}$$

We do this by induction on φ.

So, assume that $\mathfrak{A} \models \varphi[a]$ and that the result is known for subformulas of φ. Let $D = D(C)$ for some C large enough to approximate $\mathfrak{A}$ and φ, and let

$$X(\varphi, a) = \{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}^s \models \varphi^s[a]\}.$$

We will show $X(\varphi, a) \in D$.

If φ is atomic or negated atomic then $X(\varphi, a)$ contains $\{s \in \mathcal{P}_{\omega_1}(C) : a_1, \ldots, a_n \in s\}$ and so belongs to D.

Let φ be $\bigvee_{i \in I} \psi_i$. Then $\mathfrak{A} \models \psi_{i_0}[a]$ for some $i_0 \in I$, so by our inductive hypothesis $X(\psi_{i_0}, a) \in D$. But φ^s is $\bigvee_{i \in I \cap s} \psi_i^s$, so $X(\varphi, a) \supseteq X(\psi_{i_0}, a) \cap \{s : i_0 \in s\}$ must certainly be in D.

The case where φ is $\exists x \psi$ proceeds just like the case of a disjunction.

Let φ be $\bigwedge_{i \in I} \psi_i$. Then $\mathfrak{A} \models \psi_i[a]$ for every $i \in I$, so the inductive hypothesis implies that $X_i = X(\psi_i, a) \in D$ for all $i \in I$. Hence by diagonalisation, Proposition 2.1(b), we know

$$\bar{X} = \{s \in \mathcal{P}_{\omega_1}(C) : s \in X_i \text{ whenever } i \in s \cap I\} \in D.$$

But

$$\begin{aligned}\bar{X} &= \{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}^s \models \psi_i^s[a] \text{ all } i \in s \cap I\} \\ &= \left\{ s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}^s \models \left(\bigwedge_{i \in I} \psi_i \right)^s [a] \right\} \\ &= X(\varphi, a),\end{aligned}$$

so this case is finished.

The case where φ is $\forall x \psi$ proceeds just like that of a conjunction, since it is just a conjunction over the elements of A, and $A \subseteq C$. $\dashv$

Theorem 3.1 has two immediate corollaries for the cases in which $\mathfrak{A}^s = \mathfrak{A}$ a.e. or $\sigma^s = \sigma$ a.e.

Corollary 3.2. (a) *If σ is a sentence of $L_{\omega_1\omega}$ then $\mathfrak{A} \models \sigma$ iff $\mathfrak{A}_0 \models \sigma$ for almost all countable submodels $\mathfrak{A}_0$ of $\mathfrak{A}$.*

(b) *If $\mathfrak{A}$ is countable and σ is a sentence of $L_{\infty\omega}$ then $\mathfrak{A} \models \sigma$ iff $\mathfrak{A} \models \sigma^s$ a.e.*

Theorem 3.1 implies that if σ has a model then so does σ^s for almost all countable s. The converse is easily seen to fail in general, but under certain circumstances we do get the equivalence.

Corollary 3.3. (a) *Assume that σ of $L_{\infty\omega}$ is in negation-normal form and has no uncountable disjunctions. Then $\models \sigma$ iff σ^s* a.e.

(b) *If σ and $\psi_i(x)$ belong to $L_{\omega_1\omega}$ for all $i \in I$, then $\models \sigma \to \exists x \bigwedge_{i \in I} \psi_i$ iff $\models \sigma \to \exists x \bigwedge_{i \in I_0} \psi_i$ for every countable $I_0 \subseteq I$.*

Proof. (a) Theorem 3.1 implies that if $\neg\sigma$ has a model then so do $\neg\sigma^s$ a.e.

Therefore $\models \sigma$ if $\models \sigma^s$ a.e. For the other direction, it is easy to establish by induction on formulas φ in negation-normal form with no uncountable disjunctions that $\models \varphi \rightarrow \varphi^s$ a.e.

(b) is an obvious consequence of (a). ⊣

For any class Γ of formulas of $L_{\infty\omega}$, let Γ_{ω_1} be the class of all formulas of Γ that belong to $L_{\omega_1\omega}$. For some choices of Γ (for example $\exists_{\infty\omega}$ and $P_{\infty\omega}$, see [4]) it has been noticed that if $\mathfrak{A}$ and $\mathfrak{B}$ are countable and $\mathfrak{A}\Gamma_{\omega_1}\mathfrak{B}$ holds, then in fact $\mathfrak{A}\Gamma\mathfrak{B}$ holds. Corollary 3.3(b) easily yields a general condition for this to be true.

Corollary 3.4. *Assume that if $\varphi \in \Gamma$ then $\varphi^s \in \Gamma$ a.e. Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable. Then*

$$\mathfrak{A}\Gamma_{\omega_1}\mathfrak{B} \text{ implies } \mathfrak{A}\Gamma\mathfrak{B}.$$

Proof. Assume $\mathfrak{A}\Gamma_{\omega_1}\mathfrak{B}$ holds, and let σ be a sentence of Γ such that $\mathfrak{A} \models \sigma$. Then $\mathfrak{A} \models \sigma^s$ a.e. by 3.2(b), and $\sigma^s \in \Gamma_{\omega_1}$ a.e. by hypothesis, so $\mathfrak{B} \models \sigma^s$ a.e. Thus $\mathfrak{B} \models \sigma$ by 3.2(b) again. ⊣

This has a natural generalisation to models of power at most κ which could be proved either using Lemma 3.8 below or the uncountable approximations of section 7.

The following result is also a generalisation of Scott's isomorphism theorem, Corollary 1.2(a). It is proved using Theorem 3.1 and a back-and-forth argument. Notice that (b) is not an immediate consequence of (a) since the almost-all filter is not an ultrafilter.

Theorem 3.5. (a) *$\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ iff $\mathfrak{A}^s \cong \mathfrak{B}^s$ a.e.*

(b) *$\mathfrak{A} \not\equiv_{\infty\omega} \mathfrak{B}$ iff $\mathfrak{A}^s \not\cong \mathfrak{B}^s$ a.e.*

Proof. It suffices to show the implications from left to right in both (a) and (b) since the reverse implications then follow immediately.

(b) Assume $\mathfrak{A} \not\equiv_{\infty\omega} \mathfrak{B}$. Then there is a sentence σ of $L_{\infty\omega}$ such that $\mathfrak{A} \models \sigma$ but $\mathfrak{B} \models \neg\sigma$. By 3.1 $\mathfrak{A}^s \models \sigma^s$ a.e. and $\mathfrak{B}^s \models \neg\sigma^s$ a.e. Therefore $\mathfrak{A}^s \not\equiv_{\omega_1\omega} \mathfrak{B}^s$ a.e., so in particular $\mathfrak{A}^s \not\cong \mathfrak{B}^s$ a.e.

(a) Assume $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$. By the back-and-forth properties of $L_{\infty\omega}$-elementary equivalence (Theorem 1.1 and the comments following it), there are functions $f_n : A^{n+1} \times B^n \rightarrow B$ and $g_n : A^n \times B^{n+1} \rightarrow A$ for all $n \in \omega$ such that if $(\mathfrak{A}, a_1, \ldots, a_n) \equiv_{\infty\omega} (\mathfrak{B}, b_1, \ldots, b_n)$ then for any $\boldsymbol{a} \in A$ and any $\boldsymbol{b} \in B$ we have

$$(\mathfrak{A}, a_1, \ldots, a_n, a) \equiv_{\infty\omega} (\mathfrak{B}, b_1, \ldots, b_n, f_n(\boldsymbol{a}, a, \boldsymbol{b}))$$

and

$$(\mathfrak{A}, a_1, \ldots, a_n, g_n(\boldsymbol{a}, \boldsymbol{b}, b)) \equiv_{\infty\omega} (\mathfrak{B}, b_1, \ldots, b_n, b).$$

Let $C \supseteq A \cup B$, and let

$$X = \{s \in \mathscr{P}_{\omega_1}(C) : s \text{ is closed under } f_n, g_n \text{ for all } n\}.$$

Then, by Corollary 2.2, $X \in D(C)$. It is easy to see that if $s \in X$ then you can use f_n and g_n to go back-and-forth to construct an isomorphism of $\mathfrak{A}^s$ and $\mathfrak{B}^s$. Thus $\mathfrak{A}^s \cong \mathfrak{B}^s$ a.e. ⊣

Theorem 3.5 can be applied to particular cases. For example we easily obtain the following result (which we had proved earlier in a different way; see [1], [17]).

Corollary 3.6. *$\mathfrak{A}$ is $L_{\infty\omega}$-elementarily equivalent to a free group if and only if every countable subgroup of $\mathfrak{A}$ is free.*

Proof. Let $\mathfrak{B}$ be a free group and let $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$. Then $\mathfrak{A}^s \cong \mathfrak{B}^s$ a.e. by 3.5, so $\mathfrak{A}^s$ is free a.e. and therefore every countable subgroup of $\mathfrak{A}$ is free, since subgroups of free groups are free. On the other hand, assume every countable subgroup of $\mathfrak{A}$ is free. If $\mathfrak{A}$ is countable we are through; if $\mathfrak{A}$ is not countable then, in particular, $\mathfrak{A}^s$ is not finitely generated a.e., and so $\mathfrak{A}^s \cong \mathfrak{B}$ a.e. where $\mathfrak{B}$ is the free group on ω generators. Hence, by 3.5, $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$. ⊣

The proof of the following is similar to that of Theorem 3.5 and is therefore omitted.

Theorem 3.7. (a) *$\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$ iff $\mathfrak{A}^s$ is embeddable in $\mathfrak{B}^s$ a.e.*
(b) *not ($\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$) iff $\mathfrak{A}^s$ is not embeddable in $\mathfrak{B}^s$ a.e.*

It is not sufficient in 3.7(a) to say only that $\mathfrak{A}^s$ is embeddable in $\mathfrak{B}$ a.e. The following example shows that this does not imply $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$.

Example 1. Let $\mathfrak{B} = \langle \omega_1, < \rangle$ and let $\mathfrak{A} = \langle \omega_1 + 1, < \rangle$. Then $\mathfrak{A}^s$ is embeddable in $\mathfrak{B}$ for all countable s. If $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$ then there is some $\mu \in \omega_1$ such that $(\mathfrak{A}, \omega_1) \exists_{\infty\omega} (\mathfrak{B}, \mu)$, and so by 3.7(a) $(\mathfrak{A}^s, \omega_1)$ can be embedded in $(\mathfrak{B}^s, \mu)$ a.e., which is clearly impossible for any $s \supseteq \mu + 1$. So $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$ is false.

It is well-known that the conditions in 3.7(a) do not imply that $\mathfrak{A}$ can be embedded in $\mathfrak{B}$. The following example shows that they do not even imply that $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{A}'$ and $\mathfrak{B} \equiv_{\infty\omega} \mathfrak{B}'$ for some models $\mathfrak{A}'$ and $\mathfrak{B}'$ such that $\mathfrak{A}'$ can be embedded in $\mathfrak{B}'$.

Example 2. Let $\mathfrak{B} = \langle B, < \rangle$ for $B = \omega \cup Q$, where Q is the set of rationals between 0 and 1. Let Q be enumerated as $\{r_n : n \in \omega\}$ without repetition; then $<$ on B is defined so that $<$ on ω and $<$ on Q are the usual orderings of ω and Q, and if $n \in \omega$ and $r \in Q$ then $r < n$ does not hold and $n < r$ holds if and only if $r = r_n$. Then every element of B is definable in $\mathfrak{B}$, so $\mathfrak{B} \equiv_{\infty\omega} \mathfrak{B}'$ implies $\mathfrak{B} \cong \mathfrak{B}'$. But if $\mathfrak{A} = \langle A, < \rangle$ is any linearly ordered model then $\mathfrak{A}^s$ is embeddable in $\langle Q, < \rangle$, and hence in $\mathfrak{B}^s$ a.e. In particular this is true of $\langle \omega_1, < \rangle$ which is not $L_{\infty\omega}$-elementarily equivalent to any model which could be embedded in $\mathfrak{B}$.

We digress briefly to put this example into a different context. Let σ be a sentence of $L_{\omega_1\omega}$, let $\boldsymbol{K}_0$ be the class of all submodels of models of σ, let Σ be the set of all sentences of $L_{\omega_1\omega}$ true on every model of $\boldsymbol{K}_0$, and let Σ^* be the set of universal $L_{\omega_1\omega}$ consequences of σ. By Malitz's interpolation theorem, $\mathrm{Mod}(\Sigma) = \mathrm{Mod}(\Sigma^*)$. Malitz [21] gave an example showing that in general $\boldsymbol{K}_0 \subsetneq \mathrm{Mod}(\Sigma)$, and asked whether the class $\boldsymbol{K}_1$ of all models which are $L_{\omega_1\omega}$-elementarily equivalent to models in $\boldsymbol{K}_0$ was equal to $\mathrm{Mod}(\Sigma)$. The example just given shows this to be false in general — let σ be the Scott sentence of $\mathfrak{B}$; then $\mathfrak{A} = \langle \omega_1, < \rangle$ is a model of Σ but not in $\boldsymbol{K}_1$. It is, of course, true for any σ of $L_{\omega_1\omega}$ that $\mathfrak{A}$ is a model of Σ iff every countable submodel of $\mathfrak{A}$ belongs to $\boldsymbol{K}_0$.

We stated above that Theorem 3.1 generalises the (downward) Löwenheim–Skolem theorem for $L_{\infty\omega}$. Corollary 3.2(a) shows that it implies an improved version of it for $L_{\omega_1\omega}$. Now we point out how 3.1 actually implies the usual form of the Löwenheim–Skolem theorem for each $L_{\kappa^+\omega}$, Corollary 3.9 below. A stronger generalisation, for $L_{\kappa\lambda}$, will be proved in Section 7 as Corollary 7.4.

Lemma 3.8. *Let $X \in D(C)$, let $\omega \leq \kappa < |C|$, and let $I \subseteq C$, $|I| \leq \kappa$. Then there is C_0 such that $I \subseteq C_0 \subseteq C$, $|C_0| = \kappa$ and $X \cap \mathcal{P}_{\omega_1}(C_0) \in D(C_0)$.*

Proof. Using Corollary 2.2 let f_n be functions on C such that $s \in X$ whenever $s \in \mathcal{P}_{\omega_1}(C)$ is closed under every f_n, $n \in \omega$. Let C_0 be such that $I \subseteq C_0 \subseteq C$, $|C_0| = \kappa$ and C_0 is closed under every f_n. Then $s \in X \cap \mathcal{P}_{\omega_1}(C_0)$ whenever $s \in \mathcal{P}_{\omega_1}(C_0)$ is closed under every f_n, so $X \cap \mathcal{P}_{\omega_1}(C_0) \in D(C_0)$ by 2.2 again. ⊣

Corollary 3.9. *Let σ be a sentence of $L_{\kappa^+\omega}$ and let $\mathfrak{A} \models \sigma$. Then whenever $A_0 \subseteq A$, $|A_0| \leq \kappa$, there is some $\mathfrak{B} \subseteq \mathfrak{A}$ such that $A_0 \subseteq B$, $|B| = \kappa$ and $\mathfrak{B} \models \sigma$.*

Proof. Let C be large enough to approximate $\mathfrak{A}$ and σ. Let $X = \{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}^s \models \sigma^s \text{ and } A^s = A \cap s\}$. Then there is some $I \subseteq C$, $|I| \leq \kappa$, which is large enough to approximate σ. We may also take $A_0 \subseteq I$. Take C_0 as given by Lemma 3.8 and let $\mathfrak{B} = \mathfrak{A} \restriction C_0$. Then C_0 is large enough to approximate $\mathfrak{B}$ and σ, and

$$\{s \in \mathcal{P}_{\omega_1}(C_0) : \mathfrak{B}^s \models \sigma^s\} = X \cap \mathcal{P}_{\omega_1}(C_0) \in D(C_0).$$

Hence $\mathfrak{B}^s \models \sigma^s$ a.e., and so $\mathfrak{B} \models \sigma$ by 2.1. ⊣

Using Lemma 3.8 in conjunction with Theorem 3.5 we obtain the following.

Corollary 3.10. *Assume that $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ and $|A| = \kappa < |B|$. Then there is $\mathfrak{B}_0 \subseteq \mathfrak{B}$, $|B_0| = \kappa$, such that $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}_0$ — in fact, such that $\mathfrak{B}_0 <_{\infty\omega} \mathfrak{B}$.*

Looking at Theorem 3.5 it is natural to consider what happens if $\mathfrak{A}^s \cong \mathfrak{B}^s$ for every countable s. This certainly can only happen if $A = B$, but easy examples show that we need not have $\mathfrak{A} = \mathfrak{B}$. Some time ago we formulated the following conjecture, for purely relational models:

$$\text{If } \mathfrak{A}^s \cong \mathfrak{B}^s \text{ for every countable } s \text{ then } \mathfrak{A} \cong \mathfrak{B}. \qquad (*)$$

Since then, this conjecture has been verified in several cases, but defeated in its complete generality. The results are as follows.

Theorem 3.11. (a) (McKenzie; Galvin) (*) *is true if* $\mathfrak{A}$ *and* $\mathfrak{B}$ *involve only unary and binary relations.*

(b) (McKenzie) (*) *is true if* $\mathfrak{A}$ *is a group* (*with the group operation considered as a ternary relation*).

(c) (McKenzie) *There are* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of cardinality* ω_1 *such that* $\mathfrak{A}^s \cong \mathfrak{B}^s$ *for every countable* s *but* $\mathfrak{A} \not\cong \mathfrak{B}$.

The restriction to countable languages in this section is not really essential. The best way to deal with uncountable languages is probably to make the countable approximation to a model be a model for a countable approximation to the language. That is, we say C is large enough to approximate $\mathfrak{A}$ if $A \subseteq C$ and $\mathrm{L} \subseteq C$, and define $\mathfrak{A}^s = \mathfrak{A} \upharpoonright s \cap \mathrm{L} \upharpoonright s \cap A$. We can now proceed with essentially the same proofs as before. Further details can be left to the interested reader.

4. Reduced products

The obvious thing to do with an indexed family of models and a filter on the index set is to form the reduced product. Given models $\mathfrak{A}$ and $\mathfrak{B}$, if D is $D(C)$ for some C large enough to approximate both $\mathfrak{A}$ and $\mathfrak{B}$ then an immediate consequence of Theorem 3.5(a) is:

$$\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B} \text{ implies } \prod \mathfrak{A}^s/D \cong \prod \mathfrak{B}^s/D.$$

The converse is easily seen to fail. But we will see that we do get the converse provided we first expand the language of the models so that in the expansions every negation of an atomic formula is equivalent to an atomic formula.

Definition. (a) L^* is the language L together with a new k-place predicate $P_\neg$ for every k-place predicate P (including $=$) of L, for all $k \in \omega$.

(b) If $\mathfrak{A}$ is an L-model then $\mathfrak{A}^*$ is its expansion to L^* satisfying $\forall x[P_\neg(x) \leftrightarrow \neg P(x)]$ for every predicate P of L. An L^*-model of the form $\mathfrak{A}^*$ is called *standard.*

(c) If $\mathfrak{A}'$ is an L^*-model and $\mathfrak{B}' \subseteq \mathfrak{A}'$ then $\mathfrak{B}'$ is *strongly maximal* in $\mathfrak{A}'$ if for every $a \in A'$ there is exactly one $b \in B'$ such that $\mathfrak{A}' \models \neg(a =_\neg b)$.

Notice that if $\mathfrak{B}^*$ is standard and strongly maximal in $\mathfrak{A}'$, then $\mathfrak{B}^*$ is a maximal standard submodel of $\mathfrak{A}'$. The converse, as we shall see, is not true.

The following characterisations of the relations $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$ and $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ use Theorems 3.7 and 3.5. In addition, the proof of 4.1 is used in the proof of 4.2.

Theorem 4.1. *Let D be $D(C)$ where C is large enough to approximate $\mathfrak{A}$ and $\mathfrak{B}$. Then the following are equivalent:*
(i) $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$.
(ii) $\Pi\mathfrak{A}^{*s}/D$ *is embeddable in* $\Pi\mathfrak{B}^{*s}/D$.
(iii) $\mathfrak{A}^*$ *is embeddable in* $\Pi\mathfrak{B}^{*s}/D$.

Proof. If f is a function in $\Pi\{A^s : s \in \mathcal{P}_{\omega_1}(C)\}$ we use $f/\sim$ to denote its equivalence class with respect to D, which is then an element of the universe of $\Pi\mathfrak{A}^{*s}/D$.

(i) $\Longrightarrow$ (ii). Assume $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$. Then, by 3.7, $\mathfrak{A}^s$ is embeddable in $\mathfrak{B}^s$ a.e., say by a mapping h^s. Define $\bar{h}$ of $\Pi\mathfrak{A}^{*s}/D$ into $\Pi\mathfrak{B}^{*s}/D$ by $\bar{h}(f/\sim) = g/\sim$ where $g(s) = h^s(f(s))$ a.e. Then $\bar{h}$ is easily verified to be the desired embedding.

(ii) $\Longrightarrow$ (iii). It suffices to show that $\mathfrak{A}^*$ can be embedded in $\Pi\mathfrak{A}^{*s}/D$. The mapping h defined by $h(a) = g/\sim$ where $g(s) = a$ a.e. is clearly such an embedding.

(iii) $\Longrightarrow$ (i). For simplicity we assume L has only predicate symbols. Let $\bar{h}$ be an embedding of $\mathfrak{A}^*$ into $\Pi\mathfrak{B}^{*s}/D$, say that $\bar{h}(a) = f_a/\sim$ for each $a \in A$. Now for $a \in s \cap A$, define $h^s(a) = f_a(s)$. Then h^s is a mapping of A^s into B^s a.e. since $A^s = s \cap A$ a.e. We will show that h^s is an isomorphism of $\mathfrak{A}^s$ into $\mathfrak{B}^s$ for almost all countable s, and hence that $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$.

Assume that $\mathfrak{A}^{*s} \models P[a_0, \ldots, a_k]$. Then $\Pi\mathfrak{B}^{*s}/D \models P[\bar{h}(a_0), \ldots, \bar{h}(a_k)]$ and so $\mathfrak{B}^{*s} \models P[f_{a_0}(s), \ldots, f_{a_k}(s)]$ a.e., that is $\mathfrak{B}^{*s} \models P[h^s(a_0), \ldots, h^s(a_k)]$ a.e. The same implications hold for $P_\neg$, which is equivalent to $\neg P$ on $\mathfrak{A}^{*s}$ and $\mathfrak{B}^{*s}$, and so

$$\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}^s \models P[a_0, \ldots, a_k] \text{ iff } \mathfrak{B}^s \models P[h^s(a_0), \ldots, h^s(a_k)]\} \in D.$$

Call this set $X(P, a_0, \ldots, a_k)$. Diagonalising, using 2.3, we obtain

$$X(P) = \{s \in \mathcal{P}_{\omega_1}(C) : s \in X(P, a_0, \ldots, a_k) \text{ whenever } a_0, \ldots, a_k \in s \cap A\} \in D.$$

So by 2.1(a), $X = \bigcap\{X(P) : \text{predicates } P \text{ of L}\} \in D$. Since h^s is an isomorphism for all $s \in X$ we are through. $\dashv$

Later in this section we will see that we can add another equivalent condition to 4.1.

Theorem 4.2. *Let D be $D(C)$ where C is large enough to approximate $\mathfrak{A}$ and $\mathfrak{B}$. Then the following are equivalent:*
(i) $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$,
(ii) $\Pi\mathfrak{A}^{*s}/D \cong \Pi\mathfrak{B}^{*s}/D$,
(iii) $\mathfrak{A}^*$ *is isomorphic to a strongly maximal submodel of* $\Pi\mathfrak{B}^{*s}/D$.

Proof. (i) $\Longrightarrow$ (ii). This is like the corresponding step in 4.1.

(ii) $\Longrightarrow$ (iii). It suffices to show that $\mathfrak{A}^*$ is isomorphic to a strongly maximal submodel of $\Pi\mathfrak{A}^{*s}/D$. Consider the mapping h defined by $h(a) = g_a/\sim$ where

$g_a(s) = a$ a.e. As in 4.1 this is an isomorphism. Since the image of $\mathfrak{A}^*$ under h is standard we need only show that given any $f/\sim$ in $\prod\mathfrak{A}^{*s}/D$ there is some $a \in A$ such that $\prod\mathfrak{A}^{*s}/D \models \neg(f/\sim =_{\neg} g_a/\sim)$ to be able to conclude that the image is strongly maximal. So, let $f/\sim$ be given. If there is no such a, then $\prod\mathfrak{A}^{*s}/D \models (f/\sim =_{\neg} g_a/\sim)$ for every $a \in A$, that is

$$\{s \in \mathcal{P}_{\omega_1}(C) : f(s) \neq g_a(s)\} \in D,$$

so

$$X_a = \{s \in \mathcal{P}_{\omega_1}(C) : f(s) \neq a\} \in D.$$

Diagonalising using 2.1(b) we see that

$$X = \{s \in \mathcal{P}_{\omega_1}(C) : f(s) \neq a \text{ for all } a \in s \cap A\} \in D$$

which is impossible since $f(s) \in A^s = s \cap A$ a.e.

(iii) $\Longrightarrow$ (i). Assume (iii) holds, and let the isomorphism be $\bar{h}$. Then, in particular, $\mathfrak{A}^*$ is embedded by $\bar{h}$ into $\prod\mathfrak{B}^{*s}/D$, so the functions h^s defined in the proof of 4.1 are isomorphisms of $\mathfrak{A}^s$ into $\mathfrak{B}^s$ for almost all countable $s \subseteq C$. Define

$$Z = \{s \in \mathcal{P}_{\omega_1}(C) : h^s \text{ is } \textit{not} \text{ onto } B^s\}.$$

We first show that $Z \notin D$. Define $g \in \prod B^s$ so that $g(s) \in B^s - Ra(h^s)$ for all $s \in Z$. Then for $s \in Z$ and $a \in s \cap A$ we know $g(s) \neq h^s(a) = f_a(s)$. Assume that $Z \in D$. Then for every $a \in A$

$$\{s \in \mathcal{P}_{\omega_1}(C) : g(s) \neq f_a(s)\} \in D$$

since this set contains $Z \cap \{s \in \mathcal{P}_{\omega_1}(C) : a \in s\}$. Therefore $\prod\mathfrak{B}^{*s}/D \models g/\sim =_{\neg} f_a/\sim$ for all $a \in A$, that is $\prod\mathfrak{B}^{*s}/D \models g/\sim =_{\neg} \bar{h}(a)$ for all $a \in A$ which contradicts the strong maximality of the range of $\bar{h}$ in $\prod\mathfrak{B}^{*s}/D$. So $Z \notin D$. But

$$\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}^s \not\cong \mathfrak{B}^s\} \subseteq Z,$$

so this set is not in D either. Therefore it is not true that $\mathfrak{A}^s \not\cong \mathfrak{B}^s$ a.e. and Theorem 3.5(b) thus implies that $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$. $\dashv$

Given $\mathfrak{B}$ and a cardinal κ, let D be $D(C)$ where C is large enough to approximate $\mathfrak{B}$ and $|C| \geqslant \kappa$. Then every model of cardinality at most κ is isomorphic to a model which can be approximated by C. Therefore we may say that given $\mathfrak{B}$ and κ there is a filter D such that for every $\mathfrak{A}$ of cardinality at most κ (i) is equivalent to (iii) in both 4.1 and 4.2. Thus $\prod\mathfrak{B}^{*s}/D$ serves as a sort of κ^+-universal model for the $L_{\infty\omega}$-theory of $\mathfrak{B}$ — that is, the models $\mathfrak{A}$ of cardinality at most κ such that $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$ are precisely those which are isomorphic to standard submodels of $\prod\mathfrak{B}^{*s}/D$, and those such that $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ are precisely those which are isomorphic to standard strongly maximal submodels of $\prod\mathfrak{B}^{*s}/D$. Of course, this reduced product is not $L_{\infty\omega}$-elementarily equivalent to $\mathfrak{B}$ but it is as good as we can hope for, due to the following examples.

Example 1 (Malitz): There are $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ which cannot be simultaneously embedded in any model $L_{\infty\omega}$-elementarily equivalent to them. This is contained in the theorem on page 180 in [20].

Example 2. There are $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$ such that no model $L_{\infty\omega}$-elementarily equivalent to $\mathfrak{A}$ is embeddable in any model $L_{\infty\omega}$-elementarily equivalent to $\mathfrak{B}$. This is given in example two of the previous section. This implies that there are standard submodels of $\Pi\mathfrak{B}^{*s}/D$ (namely isomorphic images of $\mathfrak{A}^*$) which are not contained in any strongly maximal standard submodel. Therefore, condition (iii) of 4.2 cannot be weakened to require only that $\mathfrak{A}^*$ is isomorphic to a maximal standard submodel of $\Pi\mathfrak{B}^{*s}/D$.

Theorems 4.1 and 4.2 are capable of some generalisation. In the proofs we never used the fact that the models $\mathfrak{B}^s$ were the countable approximations to some given model $\mathfrak{B}$, and in fact any suitably indexed family of countable models would do. The resulting theorem is as follows.

Theorem 4.3. *Let D be $D(C)$ for some C, let $\{\mathfrak{B}_s : s \in \mathcal{P}_{\omega_1}(C)\}$ be a family of countable models, and let $\mathfrak{B}^* = \Pi\mathfrak{B}_s^*/D$.*
(a) *For any $\mathfrak{A}$ with $A \subseteq C$ the following are equivalent:*
(i) *$\mathfrak{A}^*$ can be embedded in $\mathfrak{B}^*$.*
(ii) *$\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}^s$ can be embedded in $\mathfrak{B}_s\} \in D(C)$.*
(b) *For any $\mathfrak{A}$ with $A \subseteq C$ the following are equivalent:*
(i) *$\mathfrak{A}^*$ is isomorphic to a strongly maximal submodel of $\mathfrak{B}^*$.*
(ii) *$\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}^s \cong \mathfrak{B}_s\} \in D(C)$.*

As a consequence notice that if $\mathfrak{B}^*$ has any (non-empty) standard submodel then this standard submodel is a model of

$$\{\sigma : \sigma \text{ is a universal sentence of } L_{\infty\omega} \text{ and } \mathfrak{B}_s \models \sigma^s \text{ a.e.}\}.$$

For example, let C be a set of universal $L_{\omega_1\omega}$-sentences every countable subset of which has a model. If, for every $s \in \mathcal{P}_{\omega_1}(C)$, a model $\mathfrak{B}_s$ of s can be chosen so that the resulting product $\mathfrak{B}^*$ has a standard submodel, then C has a model. If L is a finite language with no function symbols, then it is easy to see that we can so choose $\mathfrak{B}_s$. We consequently obtain the following result (of Čudnovskiĭ [5]) which is, of course, easier to prove without recourse to reduced products.

Corollary 4.4. (Čudnovskiĭ). *Let L be a finite language with no function symbols, and let Σ be a set of universal sentences of $L_{\omega_1\omega}$. Then Σ has a model if every countable subset of Σ does.*

We can also consider sentences preserved by reduced products modulo our filters $D(C)$ and obtain compactness and related results for such sentences.

Definition. $H_{\kappa+\omega}$, the class of *Horn formulas* of $L_{\kappa+\omega}$, is the least class containing all atomic and negated atomic formulas, closed under conjunctions of length at most κ, closed under both quantifiers, and such that $(\varphi \rightarrow \psi)$ is in $H_{\kappa+\omega}$ whenever $\varphi \in P_{\kappa+\omega}$ and $\psi \in H_{\kappa+\omega}$.

We first prove the following lemma about the class $P_{\kappa+\omega}$ of positive formulas.

Lemma 4.5. *Let D be $D(C)$ for some C, let $\{\mathfrak{A}_s : s \in \mathcal{P}_{\omega_1}(C)\}$ be a family of arbitrary models, and let $\varphi(x_1, \ldots, x_n) \in P_{\kappa+\omega}$ be such that C is large enough to approximate φ. Assume that $\prod\mathfrak{A}_s/D \models \varphi[f_1/\sim, \ldots, f_n/\sim]$. Then*

$$\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}_s \models \varphi^s[f_1(s), \ldots, f_n(s)]\} \in D.$$

Proof. The assertion is proved by induction on φ. Assume all hypotheses and assume the lemma true for all subformulas of φ. If φ is atomic, the conclusion is just the definition of truth in a reduced product. Let φ be $\bigwedge_{i \in I} \psi_i$. Then $\prod\mathfrak{A}_s/D \models \psi_i[f_1/\sim, \ldots, f_n/\sim]$ for all $i \in I$, so by our inductive hypothesis

$$\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}_s \models \psi_i^s[f_1(s), \ldots, f_n(s)]\} \in D$$

for every $i \in I$. Diagonalising we obtain

$$\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}_s \models \bigwedge_{i \in s \cap I} \psi_i^s[f_1(s), \ldots, f_n(s)]\} \in D,$$

which is exactly the conclusion desired. The case where φ is $\forall x \psi$ is easy; the cases of $\bigvee_{i \in I} \psi_i$ and $\exists x \psi$ are easier. ⊣

Let C be any set such that $|C| = \kappa$ and let φ be any formula of $L_{\kappa+\omega}$. Then φ is equivalent to a formula φ' in which all conjunctions and disjunctions are indexed by elements of C, and so C is large enough to approximate φ'. In the statement of the next theorem, we tacitly identify σ with some such σ'.

Theorem 4.6. *Let D be $D(C)$ for some C with $|C| = \kappa$ and let $\{\mathfrak{A}_s : s \in \mathcal{P}_{\omega_1}(C)\}$ be a family of arbitrary models. Then for any sentence σ in $H_{\kappa+\omega}$, if $\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}_s \models \sigma^s\} \in D$, then $\prod\mathfrak{A}_s/D \models \sigma$.*

Proof. Assuming the hypotheses we show, by induction on $\theta(x_1, \ldots, x_n)$ of $H_{\kappa+\omega}$, that if

$$\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}_s \models \theta^s[f_1(s), \ldots, f_n(s)]\} \in D$$

then $\prod\mathfrak{A}_s/D \models \theta[f_1/\sim, \ldots, f_n/\sim]$. This is clear if θ is atomic or negated atomic. If θ is a conjunction or a quantification of Horn formulas for which the implication is known then it is easily seen to be true of θ. So let θ be $\varphi \rightarrow \psi$ where $\varphi \in P_{\kappa+\omega}$ and the implication is true of ψ. Assume that

$$\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}_s \models (\varphi \rightarrow \psi)^s[f_1(s), \ldots, f_n(s)]\} \in D.$$

Say that $\prod\mathfrak{A}_s/D \models \varphi[f_1/\sim, \ldots, f_n/\sim]$. Then, by Lemma 4.5,

$$\{s \in \mathscr{P}_{\omega_1}(C) : \mathfrak{A}_s \models \varphi^s[f_1(s), \ldots, f_n(s)]\} \in D;$$

hence

$$\{s \in \mathscr{P}_{\omega_1}(C) : \mathfrak{A}_s \models \psi^s[f_1(s), \ldots, f_n(s)]\} \in D,$$

and so by inductive hypothesis $\prod\mathfrak{A}_s/D \models \psi[f_1/\sim, \ldots, f_n/\sim]$ as desired. ⊣

As an easy consequence of 4.6 we have the following result on embeddings.

Corollary 4.7. (Embedding for $H_{\kappa+\omega}$). *Let Σ be a set of sentences of $H_{\kappa+\omega}$. Then for any $\mathfrak{B}$ the following are equivalent:*

(i) *$\mathfrak{B}$ can be embedded in some model of Σ;*

(ii) *$\mathfrak{B} \exists_{\infty\omega} \mathfrak{A}$ for some model $\mathfrak{A}$ of Σ.*

Proof. That (i) implies (ii) is obvious. For the other direction, assume $\mathfrak{B} \exists_{\infty\omega} \mathfrak{A}$ where $\mathfrak{A} \models \Sigma$. Let σ be $\bigwedge_{\theta\in\Sigma}\theta$ and let D be $D(C)$ for some C large enough to approximate $\mathfrak{A}$ and σ. Then $\mathfrak{B}$ can be embedded in $\prod\mathfrak{A}^s/D$ by 4.1. But $\mathfrak{A}^s \models \sigma^s$ a.e. by 3.1, and so $\prod\mathfrak{A}^s/D \models \sigma$ by 4.6. ⊣

We have already shown in example two of Section 3 that 4.7. is false for non-Horn sentences, even for $\Sigma = \{\sigma\}$ with σ in $L_{\omega_1\omega}$. If Σ is a set of sentences of $H_{\omega_1\omega}$ then the following condition is equivalent to those in 4.7:

(iii) Every countable submodel of $\mathfrak{B}$ can be embedded in a model of Σ.

But this condition cannot be added for arbitrary $H_{\kappa+\omega}$.

We could use 4.6 to obtain compactness and upward Löwenheim–Skolem results for $H_{\kappa+\omega}$, but we obtain stronger results more easily by simply using κ^+-complete filters and ignoring approximations.

We omit the verification of the following easy result.

Proposition 4.8. *Let E be any κ^+-complete filter on I. Then:*

(a) *If $\varphi(x_1, \ldots, x_n) \in P_{\kappa+\omega}$ and $\prod\mathfrak{A}_i/E \models \varphi[f_1/\sim, \ldots, f_n/\sim]$ then $\{i \in I : \mathfrak{A}_i \models \varphi[f_1(i), \ldots, f_n(i)]\} \in E$.*

(b) *If σ is a sentence in $H_{\kappa+\omega}$ and $\{i \in I : \mathfrak{A}_i \models \sigma\} \in E$ then $\prod\mathfrak{A}_i/E \models \sigma$.*

Notice that $\{I\}$ is κ^+-complete for every κ, and that $\{I\}$-reduced products are just direct products. Hence sentences of every $H_{\kappa+\omega}$ are preserved by arbitrary direct products. We therefore immediately obtain:

Corollary 4.9. (Upward Löwenheim–Skolem for $H_{\kappa+\omega}$). *Let Σ be a set of sentences of $H_{\kappa+\omega}$ which has a model with at least two elements. Then Σ has models of arbitrarily large cardinality.*

The next corollary uses a non-trivial filter.

Corollary 4.10. (Compactness for $H_{\kappa^+\omega}$). *Let Σ be a set of sentences of $H_{\kappa^+\omega}$ and assume that every $\Sigma_0 \subseteq \Sigma$ of cardinality at most κ has a model. Then Σ has a model.*

Proof. Let E be a κ^+-complete filter on $\mathcal{P}_{\kappa^+}(\Sigma)$ such that $\{\Sigma_1 \in \mathcal{P}_{\kappa^+}(\Sigma) : \Sigma_0 \subseteq \Sigma_1\} \in E$ for every $\Sigma_0 \in \mathcal{P}_{\kappa^+}(\Sigma)$. Take $\mathfrak{A}_{\Sigma_0} \models \Sigma_0$ for each $\Sigma_0 \in \mathcal{P}_{\kappa^+}(\Sigma)$. Then 4.8(b) implies that $\prod \mathfrak{A}_{\Sigma_0}/E \models \Sigma$. ⊣

Using Lemma 4.5 we can add a further equivalent condition to Theorem 4.1.

Corollary 4.11. *Under the hypotheses of Theorem* 4.1 *the following condition is equivalent to those listed*:

(iv) $\mathfrak{A}^* \exists_{\infty\omega} \prod \mathfrak{B}^{*s}/D$.

Proof. It suffices to show that (iv) implies that $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$. Let $\kappa = |C|$. Then $|B| \leq \kappa$. Let σ be an existential positive sentence of $L^*_{\kappa^+\omega}$ and assume that $\mathfrak{A}^* \models \sigma$. Then $\prod \mathfrak{B}^{*s}/D \models \sigma$ by hypothesis, and so $\mathfrak{B}^{*s} \models \sigma^s$ a.e. by Lemma 4.5, hence $\mathfrak{B}^* \models \sigma$ by 3.1. Now, any existential sentence of $L_{\kappa^+\omega}$ is equivalent to an existential positive sentence of $L^*_{\kappa^+\omega}$ on all standard models. Therefore we have shown that $\mathfrak{A} \exists_{\kappa^+\omega} \mathfrak{B}$. Since $|B| \leq \kappa$ this actually implies $\mathfrak{A} \exists_{\infty\omega} \mathfrak{B}$ (see [4]). ⊣

There are several interesting problems concerning the results in this section. For example, we do not know if $H_{\kappa^+\omega}$ is the largest class of formulas (up to logical equivalence) satisfying Theorem 4.6. We would also like to know if Theorem 4.3, or similar considerations, could be used to yield any "compactness" or model-existence theorems other than the rather weak application in 4.4.

5. Closed classes and $L^P(\omega)$

In this section we investigate classes of models satisfying certain Löwenheim–Skolem closure conditions, of which the two following are the most important.

Definition. Let $\boldsymbol{K}$ be a class of models closed under isomorphism.

(a) $\boldsymbol{K}$ is *closed downward* if: $\mathfrak{A} \in \boldsymbol{K}$ implies $\mathfrak{A}^s \in \boldsymbol{K}$ a.e.

(b) $\boldsymbol{K}$ is *closed* if: $\mathfrak{A} \in \boldsymbol{K}$ iff $\mathfrak{A}^s \in \boldsymbol{K}$ a.e.

A class which is closed downward satisfies an abstract form of downward Löwenheim–Skolem theorem. A closed class also satisfies a restricted sort of upward Löwenheim–Skolem property. Corollary 3.2(a) says precisely that if σ is a sentence of $L_{\omega_1\omega}$ then $\mathrm{Mod}(\sigma)$ is closed. We will see shortly that there are closed classes which are not axiomatisable in any $L_{\kappa\lambda}$ but that a fairly natural logic, $L^P(\omega)$, suffices to axiomatise them all.

To show that $\boldsymbol{K}$ is closed downward it is sufficient to show the following for every $\mathfrak{A} \in \boldsymbol{K}$:

$$\{s \in \mathcal{P}_{\omega_1}(A) : \mathfrak{A}^s \in \boldsymbol{K}\} \in D(A).$$

In other words, one shows that for every $\mathfrak{A} \in \boldsymbol{K}$

$$\{\mathfrak{A}_0 \subseteq \mathfrak{A} : |A_0| \leq \omega \quad \text{and} \quad \mathfrak{A}_0 \in \boldsymbol{K}\}$$

contains a subset S' closed under unions of countable chains and containing an extension of each countable submodel of $\mathfrak{A}$. Such an S' will be called a *closed unbounded* set of countable submodels of $\mathfrak{A}$.

The following omnibus proposition lists most of the elementary facts about classes which are closed or closed downward.

Proposition 5.1. (a) *If $\boldsymbol{K}$ is closed then $\boldsymbol{K}$ is closed under $L_{\infty\omega}$-elementary equivalence.*
(b) *If $\boldsymbol{K}$ and its complement $-\boldsymbol{K}$ are both closed downward then they are both closed.*
(c) *Each class $\boldsymbol{S}$ of countable models closed under isomorphism determines a unique closed class whose countable models are precisely those in $\boldsymbol{S}$. Hence there are exactly $2^{2^{\omega}}$ different closed classes.*
(d) *If $\boldsymbol{K}'$ is a class of L'-models which is closed downward, where L' is countable, then $\boldsymbol{K}' \restriction L$ is also closed downward.*
(e) *A countable intersection of closed classes (classes closed downward) is also closed (closed downward).*
(f) *A union of any number of classes closed downward is also closed downward.*

Proof. (a) This is clear from Theorem 3.5 and the definition of closed.

(b) This is clear from the definitions since if $\mathfrak{A}^s \in \boldsymbol{K}$ a.e. then it cannot be true that $\mathfrak{A}^s \in -\boldsymbol{K}$ a.e.

(c) The unique closed class $\boldsymbol{K}$ determined by $\boldsymbol{S}$ is defined by: $\mathfrak{A} \in \boldsymbol{K}$ iff $\mathfrak{A}^s \in \boldsymbol{S}$ a.e. The cardinality assertion follows since there are exactly 2^{ω} non-isomorphic countable models.

(d) This is clear since, for any L'-model $\mathfrak{A}'$ we have $(\mathfrak{A}')^s \restriction L = (\mathfrak{A}' \restriction L)^s$ a.e.

(e) Let $\boldsymbol{K}_n$ be closed downward for all $n \in \omega$ and let $\mathfrak{A} \in \bigcap_{n\in\omega} \boldsymbol{K}_n$. Then $\mathfrak{A}^s \in \boldsymbol{K}_n$ a.e. for each n, hence $\mathfrak{A}^s \in \boldsymbol{K}_n$ for all $n \in \omega$ a.e. by 2.5. So $\bigcap_{n\in\omega} \boldsymbol{K}_n$ is closed downward. If each $\boldsymbol{K}_n$ is closed and $\mathfrak{A}^s \in \bigcap_{n\in\omega} \boldsymbol{K}_n$ a.e. then $\mathfrak{A}^s \in \boldsymbol{K}_n$ a.e. for all n, so $\mathfrak{A} \in \boldsymbol{K}_n$ for all n.

(f) This is obvious. $\dashv$

Corollary 5.2. *If $\boldsymbol{K}$ contains only countably many non-isomorphic countable models, then $\boldsymbol{K}$ is closed iff $\boldsymbol{K} = \mathrm{Mod}(\sigma)$ for some sentence σ of $L_{\omega_1\omega}$.*

Proof. Let $\boldsymbol{K}$ be closed and contain only countably many non-isomorphic countable models, say $\mathfrak{A}_k$ for $k \in \omega$. Let σ_k of $L_{\omega_1\omega}$ be the Scott sentence of $\mathfrak{A}_k$ for each k. Let σ be $\bigvee_{k\in\omega} \sigma_k$. Then σ is a sentence of $L_{\omega_1\omega}$, so $\mathrm{Mod}(\sigma)$ is closed. $\mathrm{Mod}(\sigma)$ and $\boldsymbol{K}$ contain exactly the same countable models, so therefore $\boldsymbol{K} = \mathrm{Mod}(\sigma)$. $\dashv$

The following example is indicative of the variety of closed classes that exist.

Example. Let $\boldsymbol{K}$ be

$$\{\langle A, <\rangle : \langle A, <\rangle \cong \langle \alpha + \alpha, \in\rangle \text{ for some ordinal } \alpha\}.$$

Then $\boldsymbol{K}$ is closed and co-closed, but not equal to Mod (σ) for any sentence σ of any $L_{\kappa\lambda}$.

The non-axiomatisability of $\boldsymbol{K}$ is due to Malitz [22].

We first show $\boldsymbol{K}$ is closed downward. Let $\mathfrak{A} \in \boldsymbol{K}$; we may suppose $\mathfrak{A}$ is $\langle \alpha + \alpha, \in\rangle$ for an ordinal α. Let f be defined on $\alpha + \alpha$ by $f(\xi) = \alpha + \xi$ and $f(\alpha + \xi) = \xi$ for all $\xi < \alpha$. Then any submodel of $\mathfrak{A}$ closed under f belongs to $\boldsymbol{K}$. Hence the set of all countable submodels of $\mathfrak{A}$ closed under f is a closed unbounded set contained in $\boldsymbol{K}$, which shows $\boldsymbol{K}$ is closed downward.

We next show $-\boldsymbol{K}$ is closed downward; this will complete the proof by 5.1(b). Let $\mathfrak{A} \notin \boldsymbol{K}$. If A is not well-ordered by $<^{\mathfrak{A}}$ then clearly $\mathfrak{A}' \notin \boldsymbol{K}$ a.e. and we are through. So we may assume that $\mathfrak{A}$ is $\langle \gamma, \in\rangle$ where γ cannot be written as $\alpha + \alpha$ for any α. If γ is closed under $+$ (that is, $\alpha_1, \alpha_2 < \gamma$ implies $\alpha_1 + \alpha_2 < \gamma$) then the set of countable submodels of $\mathfrak{A}$ closed under $+$ is closed unbounded and contains no element of $\boldsymbol{K}$, as required. If γ is not closed under $+$ then $\gamma = \beta_0 + \beta_1$ where $\beta_1 < \beta_0 < \gamma$ and $\alpha + \beta < \gamma$ whenever $\alpha < \beta_0$ and $\beta \leq \alpha$. Define f on γ by

$$f(\alpha, \beta) = \begin{cases} \alpha + \beta & \text{if} \quad \alpha < \beta_0 \quad \text{and} \quad \beta \leq \alpha \\ \alpha & \text{if} \quad \alpha < \beta_0 \quad \text{and} \quad \alpha < \beta \end{cases};$$

and

$$f(\alpha, \beta) = \alpha_1 \quad \text{if} \quad \alpha = \beta_0 + \alpha_1 \quad \text{for} \quad 0 \leq \alpha_1 < \beta_1.$$

Now let $\mathfrak{A}_0 \subseteq \mathfrak{A}$ be closed under f and contain the element β_1. If $\mathfrak{A}_0 \in \boldsymbol{K}$ then $A_0 = B_0 \cup B_1$ where

$$B_0 = \{\delta \in A_0 : \delta < \alpha_0\} \quad \text{and} \quad B_1 = \{\delta \in A_0 : \alpha_0 \leq \delta\}$$

for some $\alpha_0 \in A_0$ and $\langle B_0, \in\rangle \cong \langle B_1, \in\rangle$. We show this is impossible. First, assume that $\alpha_0 < \beta_0$. Then $g(y) = f(\alpha_0, y)$ for $y \in B_0$ is a 1–1 order-preserving map of B_0 into B_1, hence must be cofinal, which is impossible since $f(\alpha_0, \alpha_0) \in B_1$ but is greater than every value of g. So we must have $\beta_0 \leq \alpha_0$. But then $g(x) = f(x, \alpha_0)$ for $x \in B_1$ is a 1–1 order-preserving map of B_1 into B_0, hence must be cofinal, which is impossible since $\beta_1 \in B_0$ is greater than every value of g. Therefore the set of all countable submodels of $\mathfrak{A}$ closed under f and containing β_1 is closed unbounded and disjoint from $\boldsymbol{K}$, which completes the example. $\dashv$

We have two main interests. One is developing a natural logic adequate to axiomatise every closed class and investigating it. The other is developing some useful techniques for showing that classes are closed, or closed downward, and finding applications; this is done in the next section.

As one would expect from the game characterisation of "almost all," a logic which can axiomatise every closed class involves some game quantification. The logic we introduce here and call $L^P(\omega)$ contains the formulas of $L(\omega)$ which we can put into prenex form. A more explicit definition follows.

Definition. The formulas of $L^P(\omega)$ are inductively defined as follows
(a) every open formula of $L_{\infty\omega}$ with only countable many free variables belongs to $L^P(\omega)$;
(b) if $\{\varphi_k : k \in \omega\}$ is a countable set of formulas of $L^P(\omega)$, each having only finitely many free variables, then $\bigwedge_{k\in\omega}\varphi_k$ and $\bigvee_{k\in\omega}\varphi_k$ are formulas of $L^P(\omega)$;
(c) if φ is a formula of $L^P(\omega)$ with only finitely many free variables then so are $\exists x\varphi$ and $\forall x\varphi$;
(d) if φ is a formula of $L^P(\omega)$ then so is $(Q_nx_n)_{n<\omega}\varphi$ provided it has only finitely many free variables.

Notice that every formula of $L^P(\omega)$ is a subformula of a sentence of $L^P(\omega)$. Every formula of $L_{\omega_1\omega}$ with only finitely many free variables belongs to $L^P(\omega)$, as well as every prenex existential or universal sentence of $L_{\omega_1\omega_1}$. By Theorem 1.3 we know that $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ implies $\mathfrak{A} \equiv \mathfrak{B}(L^P(\omega))$. The following theorem gives a simple normal form for $L^P(\omega)$ and justifies our calling it the logic of the prenex formulas of $L(\omega)$.

Theorem 5.3. *Every formula of $L^P(\omega)$ with just finitely many free variables is equivalent to a formula of $L^P(\omega)$ with the same free variables of the form $(Q_nx_n)_{n<\omega}\varphi$ where φ is quantifier-free.*

Proof. The proof is by induction. Let ψ be a formula of $L^P(\omega)$ with finitely many free variables and assume the conclusion for all subformulas of ψ with finitely many free variables. If ψ is open or the result of quantifying on a single variable the conclusion is clear. Thus, the only cases needing consideration are those in which ψ is a conjunction or disjunction or obtained by game quantification.

Case 1. ψ is $\bigwedge_{k\in\omega}\psi_k$. By inductive hypothesis each ψ_k is equivalent to some $(Q_nx_n)_{n<\omega}\varphi_k$ where φ_k is open. By changing the names of bound variables and adding superfluous quantifiers on variables not occurring in φ_k, we may assume that they all begin with the same infinite quantifier-string but no x_n occurs in more than one φ_k. Then ψ is easily verified to be equivalent to

$$(Q_nx_n)_{n<\omega}\bigwedge_{k<\omega}\varphi_k$$

which has the right form.

Case 2. ψ is $\bigvee_{k<\omega}\psi_k$. We cannot proceed as in Case 1 since the equivalence we used there is not valid with $\wedge$ replaced by $\vee$ (a counterexample could be given using the result of Theorem 5.12). Instead, we may assume that each ψ_k is equivalent to $(Q_n x_n)_{k \le n < \omega} \varphi_k$ where φ_k is open and contains no x_n for $n < k$. Let ψ' be the following formula (where z_i, y_j are variables not occurring in any φ_k):

$$\forall z_1 z_2 \exists y_0 (\exists y_{n+1} Q_n x_n)_{n<\omega} \left(\left[z_1 = z_2 \wedge \bigvee_{k<\omega} \varphi_k \right] \vee \bigvee_{k<\omega} \left[\bigwedge_{i \le k} (y_i = y_k \wedge y_k \ne y_{k+1}) \wedge \varphi_k \right] \right).$$

We claim that $\models \psi \leftrightarrow \psi'$. Briefly, a justification is as follows, assuming ψ and ψ' have no free variables. $\mathfrak{A}$ is a model of ψ' if and only if (i) A has only one element ($\forall z_1 z_2 (z_1 = z_2)$ is true) and some φ_k is satisfied by the infinite sequence consisting of just that element, or (ii) there is some k and a sequence $y_0, \ldots, y_{k+1}$ such that $y_i = y_k$ for all $i \le k$ and $y_k \ne y_{k+1}$ and $(Q_n x_n)_{k \le n < \omega} \varphi_k$ is true. Clearly this is the same as saying that $\mathfrak{A}$ is a model of $\bigvee_{k \in \omega} (Q_n x_n)_{k \le n < \omega} \varphi_k$, and so ψ' is equivalent to ψ. Since ψ' has the desired form that concludes this case.

Case 3. ψ is $(Q_n x_n)_{n<\omega} \theta$. This is only non-trivial if θ has infinitely many free variables and is not open, in which case θ is either the conjunction or disjunction of formulas φ_k, $k \in \omega$, each having only finitely many free variables. Let us say that φ_k is $\varphi_k(x_0, \ldots, x_k, z)$ for each k, and take first the case in which θ is $\bigwedge_{k \in \omega} \varphi_k$. By our inductive hypothesis we may assume that each φ_k is equivalent to $(Q^1_m y_m)_{k \le m < \omega} \alpha_k$ where α_k is open, and each y_m occurs in at most one α_k. Let ψ' be

$$(Q_n x_n Q^1_n y_n)_{n<\omega} \bigwedge_{k \in \omega} \alpha_k.$$

We claim that $\models \psi \leftrightarrow \psi'$. It is easy to see that ψ logically implies ψ'. The implication in the other direction could be established directly, but it is easier to see that $\neg \psi \wedge \psi'$ cannot have a model. By Proposition 1.4, $\neg \psi$ is equivalent to $(Q'_n x_n)_{n<\omega} \bigvee_{k \in \omega} \neg \varphi_k$ where Q'_n is $\forall$ if Q_n is $\exists$ and Q'_n is $\exists$ if Q_n is $\forall$. One may readily verify that this contradicts ψ'. The case in which θ is a disjunction is treated in the manner of Case 2, and is left to the curious reader. $\dashv$

For the purposes of this section, we only need to know that the sentences of $L^P(\omega)$ are precisely those of the form given in Theorem 5.3 and are closed under countable conjunction and disjunction (up to logical equivalence). One could therefore take the prenex characterisation of Theorem 5.3 as the definition of $L^P(\omega)$, and simply use Cases 1 and 2 of its proof as a verification that they are closed under countable conjunctions and disjunctions.

The following important fact is an immediate consequence of Theorem 5.3.

Corollary 5.4. *If σ is a sentence of* $L^P(\omega)$ *then* Mod(σ) *is closed downward.*

Our main interest in $L^P(\omega)$ stems from the following result.

Theorem 5.5. *If K is a closed class then $K = \mathrm{Mod}(\sigma)$ for some sentence σ of* $L^P(\omega)$.

Proof. Let K be closed and let S be the class of countable models in K. If $\mathfrak{A} \in S$ and $a = \langle a_n : n \in \omega \rangle$ is any listing of all the elements of A, perhaps with repetitions, we let $\theta_{\mathfrak{A},a}$ be

$$\wedge \{\alpha(x_0, \ldots, x_n) : \alpha \text{ is atomic or negated atomic, } \mathfrak{A} \models \alpha[a_0, \ldots, a_n]\}.$$

Let $\varphi(x)$ be the disjunction over all $\mathfrak{A} \in S$ and all listings a of A of the formulas $\theta_{\mathfrak{A},a}$. Then φ is an open formula of $L_{\infty\omega}$ since there are only 2^ω different possible formulas $\theta_{\mathfrak{A},a}$. Finally, let σ be $(\forall x_{2n} \exists x_{2n+1})_{n<\omega} \varphi$. Then σ is a sentence of $L^P(\omega)$, and we will show that it axiomatises K. Given $\mathfrak{A}$, let

$$X = \{s \in \mathcal{P}_{\omega_1}(A) : \mathfrak{A}^s \in S \quad \text{and} \quad A^s = s\}.$$

Then $\{a_n : n \in \omega\} \in X$ iff $\mathfrak{A} \models \varphi[a_0, a_1, \ldots]$. We claim that the following statements are equivalent:

(i) $\mathfrak{A} \in K$,
(ii) $X \in D(A)$,
(iii) $(\forall a_{2n} \in A \exists a_{2n+1} \in A)_{n<\omega} [\{a_n : n \in \omega\} \in X]$,
(iv) $(\forall a_{2n} \in A \exists a_{2n+1} \in A)_{n<\omega} (\mathfrak{A} \models \varphi[a_0, a_1, \ldots])$,
(v) $\mathfrak{A} \models (\forall x_{2n} \exists x_{2n+1})_{n<\omega} \varphi$.

(i) and (ii) are equivalent since K is closed; (ii) and (iii) are equivalent by Proposition 2.1(c); (iii) and (iv) are equivalent by the characterisation of X given above; and (iv) is obviously equivalent to (v). Therefore $K = \mathrm{Mod}(\sigma)$ as claimed. ⊣

This theorem has several immediate consequences connecting sentences of $L^P(\omega)$ with classes which are closed downward.

Corollary 5.6. *If K is closed downward then there is some sentence σ of $L^P(\omega)$ such that $K \subseteq \mathrm{Mod}(\sigma)$ and every countable model of σ belongs to K.*

Proof. By Proposition 5.1(c) there is a closed class K_0 which has exactly the same countable models as K. If $\mathfrak{A} \in K$ then $\mathfrak{A}^s \in K_0$ a.e., hence $\mathfrak{A} \in K_0$ since K_0 is closed. Therefore $K \subseteq K_0$. By Theorem 5.5 $K_0 = \mathrm{Mod}(\sigma)$ for some sentence σ of $L^P(\omega)$, which is then the desired sentence. ⊣

Corollary 5.7. (Separation). *Let K_1 and K_2 be closed downward and assume that $K_1 \cap K_2 = 0$. Then there are sentences σ_1 and σ_2 of $L^P(\omega)$ such that $K_i \subseteq \mathrm{Mod}(\sigma_i)$ for $i = 1, 2$ and $\models \neg(\sigma_1 \wedge \sigma_2)$.*

Proof. Let σ_i be related to K_i as in the statement of 5.6, for $i = 1, 2$. If $\mathfrak{A} \models \sigma_1 \wedge \sigma_2$

then $\mathfrak{A}^s \models \sigma_1 \wedge \sigma_2$ a.e. by 5.4, in particular some countable model is a model of both σ_1 and σ_2 and hence belongs to $\boldsymbol{K}_1 \cap \boldsymbol{K}_2$, contradicting the hypothesis. $\dashv$

Combining 5.4, 5.1(d), and 5.7 we obtain the following interpolation theorem for $\mathrm{L}^P(\omega)$.

Theorem 5.8. *Let* L_1 *and* L_2 *be countable languages such that* $\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{L}$. *Let* θ_i *be a sentence of* $(\mathrm{L}_i)^P(\omega)$, *for* $i = 1, 2$, *and assume that* $\models \neg(\theta_1 \wedge \theta_2)$. *Then there are sentences* σ_1 *and* σ_2 *of* $\mathrm{L}^P(\omega)$ *such that* $\theta_i \models \sigma_i$, *for* $i = 1, 2$, *and* $\models \neg(\sigma_1 \wedge \sigma_2)$.

Interpolation holds also for formulas θ_i with free variables, as is seen by the standard procedure of replacing the free variables with new individual constants, interpolating, and then replacing the new constants in the interpolants by the original variables. Using this we obtain the following:

Corollary 5.9. (Definability). *Let* σ *be a sentence of* $(\mathrm{L} \cup \{R\})^P(\omega)$ *such that every* L*-model can be expanded in at most one way to a model of* σ. *Then there is a formula* $\varphi(\boldsymbol{x})$ *of* $\mathrm{L}^P(\omega)$ *such that*

$$\sigma \models \forall \boldsymbol{x}[R(\boldsymbol{x}) \leftrightarrow \varphi(\boldsymbol{x})].$$

Proof. Let σ' be the sentence of $(\mathrm{L} \cup \{R'\})^P(\omega)$ obtained by replacing R everywhere in σ by a new predicate symbol R'. Then the hypothesis on σ implies:

$$\models \neg([\sigma \wedge R(\boldsymbol{x})] \wedge [\sigma' \wedge \neg R'(\boldsymbol{x})]).$$

Let $\varphi(\boldsymbol{x})$ of $\mathrm{L}^P(\omega)$ be one of the interpolating formulas given by 5.8. Then

$$\sigma \models R(\boldsymbol{x}) \rightarrow \varphi(\boldsymbol{x}), \qquad \sigma' \models \neg R'(\boldsymbol{x}) \rightarrow \neg \varphi(\boldsymbol{x}).$$

Replacing R' by R in the second expression we see that φ is just as desired. $\dashv$

If in fact every L-model can be expanded in exactly one way to a model of σ, then $\mathrm{Mod}(\sigma)$ is closed, not just closed downward.

The special case of Corollary 5.6 in which $\boldsymbol{K}$ is $\mathrm{Mod}(\Sigma') \upharpoonright \mathrm{L}$ for some set Σ' of finitary sentences is due to Svenonius [28]. A similar result for $PC_{\omega_1\omega}$ classes is found in [32]. Malitz's counterexample [22] shows that we cannot require in 5.8 that the interpolants σ_i are in some $\mathrm{L}_{\kappa\lambda}$, even if we require the θ_i to also be sentences of $(\mathrm{L}_i)_{\omega_1\omega_1}$.

Takeuti [30] (see also [25], [29]) proved an interpolation theorem which is related to Theorem 5.8 and to its extension in Section 7. His theorem states that any valid implication in $\mathrm{L}_{\infty\infty}$ has an interpolant involving game quantifiers (of arbitrary lengths), which can be taken to occur only in a prefix. His theorem can be applied to sentences of $\mathrm{L}^P(\omega)$, since if the existentially quantified variables in a sentence of $\mathrm{L}^P(\omega)$ are replaced by new Skolem functions one obtains a universal sentence of $\mathrm{L}_{\infty\omega_1}$. One thus can obtain 5.8 except that the interpolants σ_1 and σ_2 might involve quantifier prefixes of length greater than ω. Thus his result does not appear to imply our Theorem 5.8.

It is also worth emphasising that 5.7 is stronger than its consequence 5.8. For example, consider sentences with partially-ordered quantifiers (introduced in [6]). If σ is a sentence consisting of a partially-ordered quantifier on only countably many variables followed by an open formula, and if each variable in the quantifier has only finitely many predecessors, then $\mathrm{Mod}(\sigma)$ is closed downward. Therefore we obtain 5.8 where θ_1 and θ_2 are allowed to be such sentences but the interpolants are still in $L^p(\omega)$. Interpolation with partially-ordered quantification is treated in [29].

The logic $L^p(\omega)$ has several other pleasant properties, in particular the following preservation results.

Definition. Let β be any non-zero cardinal. $\mathfrak{A}$ is the *β-union* of the set S of submodels of $\mathfrak{A}$ if every $B \subseteq A$ of cardinality less than β is contained in the universe of some model in S.

Definition. $(\forall^n \exists)^p(\omega)$ is the set of all sentences of $L^p(\omega)$ of the form $\forall x_0 \cdots \forall x_{n-1}(\exists y_k)_{k<\omega}\alpha$, with α open.

Theorem 5.10. (a) *K is closed downward and closed under $(n+1)$-unions iff $K = \mathrm{Mod}(\sigma)$ for some $\sigma \in (\forall^n \exists)^p(\omega)$.*

(b) *K is closed downward and closed under ω-unions iff $K = \mathrm{Mod}(\bigwedge_{n\in\omega}\sigma_n)$ where $\sigma_n \in (\forall^n \exists)^p(\omega)$ for each n.*

Proof. (a) It is easily verified that the class of models of any sentence of $(\forall^n \exists)^p(\omega)$ is closed under $(n+1)$-unions. For the other direction, let K be closed downward, and let $\varphi(x)$ be the formula defined during the proof of Theorem 5.5. Let σ be $\forall x_0 \cdots \forall x_{n-1}(\exists x_k)_{n\leq k<\omega}\varphi$. Then $\sigma \in (\forall^n \exists)^p(\omega)$ and $\mathfrak{A} \models \sigma$ if and only if $\mathfrak{A}$ is an $(n+1)$-union of countable models in K. So $K = \mathrm{Mod}(\sigma)$ if K is closed under $(n+1)$-unions.

(b) Let K be closed downward and for each $n \in \omega$ let σ_n be the sentence defined in the proof of (a). Then $\mathfrak{A} \models \bigwedge_{n\in\omega}\sigma_n$ if and only if $\mathfrak{A}$ is, for each n, the $(n+1)$-union of countable models in K, that is, if and only if $\mathfrak{A}$ is an ω-union of countable models in K. So $K = \mathrm{Mod}(\bigwedge_{n\in\omega}\sigma_n)$ if K is closed under ω-unions. ⊣

A sentence of $L^p(\omega)$ of the form $(\forall x_n)_{n<\omega}\alpha$ with α open is called *universal*. Notice that any conjunction (not just countable ones) of universal $L^p(\omega)$ sentences is equivalent to a universal $L^p(\omega)$ sentence. Since a universal $L^p(\omega)$ sentence is equivalent to the negation of a sentence of $(\forall^0\exists)^p(\omega)$, Theorem 5.10 implies the following, which is essentially a theorem of Tarski [31].

Corollary 5.11. *K is closed and closed under submodels iff $K = \text{Mod}(\sigma)$ for some universal sentence σ of $L^P(\omega)$.*

There are a number of obvious questions about closed classes and $L^P(\omega)$ — for example, does the converse of Theorem 5.5 hold? We now answer this, and several other questions, negatively using the example given in the following theorem.

Theorem 5.12. *The class of all countable well-orders is the union of two disjoint closed classes.*

Proof. By 2.1(d) $D(\omega_1)$ is not an ultrafilter, so there is some $X \subseteq \mathcal{P}_{\omega_1}(\omega_1)$ such that $X \notin D(\omega_1)$ and $\mathcal{P}_{\omega_1}(\omega_1) - X \notin D(\omega_1)$. Let S_1 be

$$\{\langle A, < \rangle : \langle A, < \rangle \cong \langle \alpha, < \rangle \quad \text{for some} \quad \alpha \in \omega_1 \cap X\},$$

and let S_2 be

$$\{\langle A, < \rangle : \langle A, < \rangle \cong \langle \alpha, < \rangle \quad \text{for some} \quad \alpha \in \omega_1 - X\}.$$

Then every countable well-order belongs to exactly one of S_1 and S_2. We will show that S_1 and S_2 are both closed. If S_1 is not closed, then there is some $\mathfrak{A}$ such that $\mathfrak{A}^s \in S_1$ a.e. and $|A| = \omega_1$. Let Y be a closed unbounded chain contained in $\{s : \mathfrak{A}^s \in S_1\}$, and let X' be

$$\{\alpha \in \omega_1 : \mathfrak{A}^s \cong \langle \alpha, < \rangle \quad \text{for some} \quad s \in Y\}.$$

Then $X' \subseteq X$. But X' contains arbitrarily large countable ordinals and is closed under unions of countable chains, since $\mathfrak{A}$ is uncountable and Y is a closed unbounded chain. Therefore X' is actually a closed unbounded subset of $\mathcal{P}_{\omega_1}(\omega_1)$, hence $X' \in D(\omega_1)$ and therefore $X \in D(\omega_1)$, contradicting the choice of X. Hence S_1 is closed. We similarly know that S_2 is closed, since $\omega_1 - X = \omega_1 \cap (\mathcal{P}_{\omega_1}(\omega_1) - X)$. $\dashv$

Corollary 5.13. (a) *There is a closed class whose complement is not closed downward.*

(b) *There are two closed classes whose union is not closed.*

(c) *There is a sentence of $L^P(\omega)$ whose negation is not equivalent to a sentence of $L^P(\omega)$.*

(d) *There are sentences σ of $L^P(\omega)$ such that $\text{Mod}(\sigma)$ is not closed; such a σ can be taken to be true of all countable models, or to be preserved under submodels (and therefore not equivalent to any universal $L^P(\omega)$ sentence).*

Proof. (a) S_1 is closed but $-S_1$ is not closed downward since $\langle \omega_1, < \rangle \in -S_1$ but not almost all of its countable submodels belong to $-S_1$.

(b) S_1 and S_2 are closed but $S_1 \cup S_2$ is not.

Let σ_1 and σ_2 be the sentences of $L^P(\omega)$ axiomatising S_1 and S_2.

(c) $\neg\sigma_1$ is not equivalent to a sentence of $L^P(\omega)$ since by (a) Mod $(\neg\sigma_1)$ is not closed downward.

(d) $\sigma_1 \vee \sigma_2$ is a sentence of $L^P(\omega)$, but Mod $(\sigma_1 \vee \sigma_2)$ is the class of all countable well-orders, which is closed under submodels but not closed. Let σ_3 be the sentence of $L^P(\omega)$ saying that $<$ is not a well-order. Then $\sigma_1 \vee \sigma_2 \vee \sigma_3$ is true of every countable model but does not define a closed class since no uncountable well-order is a model of it. ⊣

These counterexamples raise several questions. The first two are suggested by the fact that not every $L^P(\omega)$ sentence axiomatises a closed class.

(1) Is there some natural syntactical description of a logic whose sentences axiomatise precisely the closed classes?

(2) If $\mathfrak{A} \equiv \mathfrak{B}(L^P(\omega))$ then $\mathfrak{A}$ and $\mathfrak{B}$ belong to the same closed classes. Does the converse hold? If so, the sentences of $L^P(\omega)$ which do not define closed classes would be "harmless" as far as equivalence is concerned.

A more serious objection to $L^P(\omega)$ is that it is not closed under negations. Is this in any way essential in finding "small" extensions of $L_{\omega_1\omega}$ satisfying interpolation?

(3) Is there a natural sublogic of $L^P(\omega)$, properly extending $L_{\omega_1\omega}$, which is closed under negation and satisfies interpolation?

A natural logic to look at is the maximal sublogic of $L^P(\omega)$ closed under negation.

Definition. $L^\delta(\omega)$ is the class of all formulas of $L^P(\omega)$ whose negations are equivalent to formulas of $L^P(\omega)$.

Then $L^\delta(\omega)$ is closed under negation, countable conjunction and disjunction, and finite quantification of formulas with only finitely many free variables. The sentences of $L^\delta(\omega)$ are precisely those which axiomatise classes which are closed and co-closed.

$L^\delta(\omega)$ satisfies a weaker version of interpolation, called the *Δ-interpolation* or *Suslin–Kleene* property — that is, if $\boldsymbol{K} = \text{Mod}(\sigma_1)\restriction L$ and $-\boldsymbol{K} = \text{Mod}(\sigma_2)\restriction L$ for some sentences σ_1, σ_2 of the logic in a language containing L, then $\boldsymbol{K} = \text{Mod}(\sigma_0)$ for some σ_0 of $L^\delta(\omega)$ (for Δ-interpolation, see [3], [19]). The reason this property holds is that the hypotheses imply that both $\boldsymbol{K}$ and $-\boldsymbol{K}$ are closed downward, hence both are closed, and so $\boldsymbol{K}$ is $L^\delta(\omega)$ axiomatisable.

However, $L^\delta(\omega)$ does not satisfy full interpolation. The reason for this is that every closed class can be expressed as $\text{Mod}(\sigma_1)\restriction L$ for a sentence σ_1 of some $(L')^\delta(\omega)$ — in fact, σ_1 can be taken to be universal. So if this logic satisfied interpolation we would know that whenever $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are disjoint closed classes there would be a sentence σ_0 of $L^\delta(\omega)$ such that $\boldsymbol{K}_1 \subseteq \text{Mod}(\sigma_0)$ and $\boldsymbol{K}_2 \cap \text{Mod}(\sigma_0) = 0$. If $\boldsymbol{K}_2$ is the maximal closed class disjoint from $\boldsymbol{K}_1$ this would imply that $\boldsymbol{K}_1 = \text{Mod}(\sigma_0)$, which is impossible if $\boldsymbol{K}_1$ is taken not to be co-closed. (Note that this shows that no logic as in (3) could contain all universal $L^P(\omega)$ sentences.)

The interested reader should convince himself that a logic equivalent to $L^\delta(\omega)$ is obtained by taking all the formulas of $L^P(\omega)$ whose negations are equivalent to their duals — that is, a formula of $L^P(\omega)$ is in $L^\delta(\omega)$ iff it can be written as $(Q_n x_n)_{n<\omega}\alpha$ where α is open and $\models \neg(Q_n x_n)_{n<\omega}\alpha \leftrightarrow (Q'_n x_n)_{n<\omega}\neg\alpha$, where Q'_n is $\forall$ if Q_n is $\exists$ and Q'_n is $\exists$ if Q_n is $\forall$. (Hint: the sentence written down in the proof of Theorem 5.5 has this property if $\boldsymbol{K}$ is closed and co-closed.)

A syntactically very natural sublogic of $L^P(\omega)$ is the class $L^P_{\omega_1}(\omega)$ built using only countable conjunctions and disjunctions, that is the class of $L^P(\omega)$ formulas which can be written as $(Q_n x_n)_{n<\omega}\alpha$ where α is an open formula of $L_{\omega_1\omega}$. Sentences in this class correspond to Borel games, and Martin's recent proof of Borel determinacy [23] shows this logic is closed under negations.

(4) Does $L^P_{\omega_1}(\omega)$ satisfy interpolation, or Δ-interpolation?

Finally, let us note that extensions of $L^P(\omega)$ within $L(\omega)$ which are closed under negations do not even satisfy Δ-interpolation. Let σ be the sentence obtained above which is true of precisely the countable well-orders. Let σ^* be $\neg\sigma \wedge \theta$, where θ says "$<$ is a well-order of the universe." Then σ^* is the negation of a sentence of $L^P(\omega)$ and σ^* is true of precisely the uncountable well-orders. Let L_0 be the empty language. Then $\boldsymbol{K} = \mathrm{Mod}(\sigma)\restriction L_0$ contains exactly the countable L_0-models, and $-\boldsymbol{K} = \mathrm{Mod}(\sigma^*)\restriction L_0$. So any logic containing σ and σ^* and satisfying Δ-interpolation must contain an L_0-sentence axiomatising $\boldsymbol{K}$ and so not preserved by $L_{\infty\omega}$-elementary equivalence. By Keisler's Theorem 1.3 such a sentence cannot belong to $L(\omega)$.

If $\boldsymbol{K}_0$ is closed downward then, by 5.1(c), there is a unique closed class $\boldsymbol{K}$ containing $\boldsymbol{K}_0$ and having the same countable models as $\boldsymbol{K}_0$. We call this $\boldsymbol{K}$ the closed class *generated by* $\boldsymbol{K}_0$. The next theorem shows how the closed class generated by $\boldsymbol{K}_0$ is axiomatised by a class of $L_{\infty\omega}$ sentences. Any class closed under $L_{\infty\omega}$ elementary equivalence is axiomatisable by a class of $L_{\infty\omega}$ sentences, so the point of the theorem is exactly how these sentences are determined by $\boldsymbol{K}_0$.

Definition. Let $\boldsymbol{K}_0$ be closed downward and let σ be a sentence of $L_{\infty\omega}$. We say σ is *approximated* in $\boldsymbol{K}_0$ if, for almost all countable s, σ^s is true on some model in $\boldsymbol{K}_0$.

Theorem 5.14. *Let $\boldsymbol{K}_0$ be closed downward and let $\boldsymbol{K}$ be the closed class generated by $\boldsymbol{K}_0$.*

(a) $\mathfrak{A} \in \boldsymbol{K}$ *iff every $L_{\infty\omega}$-sentence true on $\mathfrak{A}$ is approximated in $\boldsymbol{K}_0$.*

(b) $\boldsymbol{K} = \mathrm{Mod}(\Sigma)$ *where Σ is*

$$\{\sigma \text{ of } L_{\infty\omega} : \neg\sigma \text{ is not approximated in } \boldsymbol{K}_0\}.$$

One could regard 5.14(a) as a very weak sort of compactness for closed classes. The main point in the proof of 5.14 is the following lemma, which could also be used to give an alternate proof of the implication from left to right in 3.5(a).

Lemma 5.15. *For any $\mathfrak{A}$ there is some sentence σ of $L_{\infty\omega}$ such that for almost all countable s we have for every $\mathfrak{B}$*

$$\mathfrak{B} \models \sigma^s \quad \textit{iff} \quad \mathfrak{B} \equiv_{\infty\omega} \mathfrak{A}^s.$$

Proof. We will take σ to be a standard Scott sentence for $\mathfrak{A}$. For any $a_0, \ldots, a_n \in A$ let $\varphi_{a_0 \cdots a_n}(x_0, \ldots, x_n)$ be a formula of $L_{\infty\omega}$ such that for any $a'_0, \ldots, a'_n \in A$,

$$\mathfrak{A} \models \varphi_a[a'] \quad \text{iff} \quad (\mathfrak{A}, a) \equiv_{\infty\omega} (\mathfrak{A}, a').$$

For each n, let $\{\alpha_k(x_0, \ldots, x_n) : k \in \omega\}$ list all the atomic and negated atomic formulas of L in the variables $x_0, \ldots, x_n$. Given any a from A, let I_a be

$$\{k \in \omega : \mathfrak{A} \models \alpha_k[a]\}.$$

Let σ be the conjunction of the following five sentences:

$$\forall x_0 \bigvee_{a \in A} \varphi_a, \qquad \bigwedge_{a \in A} \exists x_0 \varphi_a,$$

$$\bigwedge_{n \in \omega} \bigwedge_{a \in A^{n+1}} \forall x_0 \cdots \forall x_n \Big[\varphi_a \to \bigwedge_{a_{n+1} \in A} \exists x_{n+1} \varphi_{a, a_{n+1}} \Big],$$

$$\bigwedge_{n \in \omega} \bigwedge_{a \in A^{n+1}} \forall x_0 \cdots \forall x_n \Big[\varphi_a \to \forall x_{n+1} \bigvee_{a_{n+1} \in A} \varphi_{a, a_{n+1}} \Big],$$

$$\bigwedge_{n \in \omega} \bigwedge_{a \in A^{n+1}} \forall x_0 \cdots \forall x_n \Big[\varphi_a \to \bigwedge_{k \in I_a} \alpha_k \Big].$$

Then, as is well-known, $\mathfrak{B} \models \sigma$ iff $\mathfrak{B} \equiv_{\infty\omega} \mathfrak{A}$. In particular, $\mathfrak{A} \models \sigma$ and hence $\mathfrak{A}^s \models \sigma^s$ a.e. Now, if $\omega \subseteq s$ then σ^s is also the conjunction of five sentences which look like the preceding except that φ_a is replaced everywhere by φ_a^s, "$a \in A$" is replaced in conjunctions and disjunctions by "$a \in s \cap A$", and "$a \in A^{n+1}$" is replaced by "$a \in s \cap A^{n+1}$". But for almost all countable s, $s \cap A$ is A^s, the universe of $\mathfrak{A}^s$, and $s \cap A^{n+1}$ is $(s \cap A)^{n+1}$. Therefore, for almost all countable s, σ^s is a Scott sentence of $\mathfrak{A}^s$, and hence

$$\mathfrak{B} \models \sigma^s \quad \text{iff} \quad \mathfrak{B} \equiv_{\infty\omega} \mathfrak{A}^s. \quad \dashv$$

With this lemma we can easily prove the theorem.

Proof of Theorem 5.14. Assume the hypotheses on $\boldsymbol{K}_0$ and $\boldsymbol{K}$.

(a) If $\mathfrak{A} \in \boldsymbol{K}$ and $\mathfrak{A} \models \sigma$ then $\mathfrak{A}^s \models \sigma^s$ a.e. and $\mathfrak{A}^s \in \boldsymbol{K}_0$ a.e., so σ is approximated in $\boldsymbol{K}_0$. Conversely, assume that every $L_{\infty\omega}$ sentence true on $\mathfrak{A}$ is approximated in $\boldsymbol{K}_0$. Let σ be the sentence given by Lemma 5.15. Then for almost all countable s σ^s has a model in $\boldsymbol{K}_0$; since $\boldsymbol{K}_0$ is closed downward these models may be taken to be countable, and hence isomorphic to $\mathfrak{A}^s$ by 1.2(a). Therefore $\mathfrak{A}^s \in \boldsymbol{K}_0$ a.e., and hence $\mathfrak{A} \in \boldsymbol{K}$.

(b) If $\mathfrak{A} \in \boldsymbol{K}$ then every sentence true on $\mathfrak{A}$ is approximated in $\boldsymbol{K}_0$, hence $\mathfrak{A}$ is a model of every sentence whose negation is not approximated in $\boldsymbol{K}_0$, and thus $\mathfrak{A} \models \Sigma$. On the other hand, if $\mathfrak{A} \notin \boldsymbol{K}$ then by (a) there is a sentence σ of $L_{\infty\omega}$ such that $\mathfrak{A} \models \sigma$ but σ is not approximated in $\boldsymbol{K}_0$. Then $\neg\sigma \in \Sigma$, and therefore $\mathfrak{A}$ is not a model of Σ. $\dashv$

We introduce a third sort of Löwenheim–Skolem property, intermediate between being closed downward and being closed.

Definition. $\boldsymbol{K}$ is *locally closed* iff $\boldsymbol{K}$ is closed under isomorphism and whenever $\mathfrak{A} \in \boldsymbol{K}$ there is a set S of countable submodels of $\mathfrak{A}$ such that $\mathfrak{A}^s \in S$ a.e. and $\mathfrak{B} \in \boldsymbol{K}$ whenever $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B}^s \in S$ a.e.

Notice that the definition implies that $S \subseteq \boldsymbol{K}$, since if $\mathfrak{B} \in S$ then $\mathfrak{B} = \mathfrak{B}^s$ a.e. Therefore locally closed classes are closed downward. Further, if $\boldsymbol{K}$ is locally closed and $\mathfrak{A} \in \boldsymbol{K}$ then $\boldsymbol{K}$ also contains "many" submodels of $\mathfrak{A}$ of every infinite cardinality less than that of $\mathfrak{A}$. If σ is a sentence of $L^P(\omega)$ then $\mathrm{Mod}(\sigma)$ is locally closed, and Propositions 5.1(d)(e)(f) remain true for locally closed classes. We will refer to locally closed classes again in the next section, and for now give only the following interesting characterisation.

Proposition 5.16. *$\boldsymbol{K}$ is locally closed iff $\boldsymbol{K} = \boldsymbol{K}' \restriction \mathrm{L}$ where L' is some countable language extending L and $\boldsymbol{K}'$ is some class of L'-models closed under submodels and isomorphism.*

Proof. The implication from right to left is easily verified. For the other direction, assume $\boldsymbol{K}$ is locally closed and let L' be the extension of L which adds countably many n-place functions for each $n \in \omega$. If $\mathfrak{A} \in \boldsymbol{K}$ and if S is a set of countable submodels of $\mathfrak{A}$ as in the definition of locally closed, then we can interpret the functions in L'–L so that $\mathfrak{A}_0 \in S$ whenever A_0 is closed under all those functions and $|A_0| \leqslant \omega$ (cf. 2.2), and so $\mathfrak{B} \in \boldsymbol{K}$ whenever $\mathfrak{B} \subseteq \mathfrak{A}$ and B is closed under the functions. That is, we can expand $\mathfrak{A}$ to an L'-model $\mathfrak{A}'$ such that whenever $\mathfrak{B}' \subseteq \mathfrak{A}'$ then $\mathfrak{B}' \restriction \mathrm{L} \in \boldsymbol{K}$. Thus, if we let $\boldsymbol{K}'$ be the closure under submodels of the class of all such $\mathfrak{A}'$, then $\boldsymbol{K}$ is as desired. $\dashv$

If $\boldsymbol{K}$ is locally closed and closed under ultraproducts is $\boldsymbol{K}$ actually PC_Δ (in finitary logic)?

We could also, following Barwise [2], speak of relations between models as being closed downward, or closed, as in the following definition:

Definition. The n-ary relation $\mathcal{R}$ between models is *closed downward* if:

(i) $\mathcal{R}(\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ and $\mathfrak{A}_i \cong \mathfrak{A}'_i$ for $i = 1, \ldots, n$ imply $\mathcal{R}(\mathfrak{A}'_1, \ldots, \mathfrak{A}'_n)$, and

(ii) $\mathcal{R}(\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ implies $\mathcal{R}(\mathfrak{A}_1^s, \ldots, \mathfrak{A}_n^s)$ a.e.

For example, the relation $\mathfrak{A}_1 \prec_{\infty\omega} \mathfrak{A}_2$ is closed downward, and in fact closed (see the next section). We could handle relations as special sorts of classes of models by putting an n-tuple of L-models all into one model for a language L' which contains n distinct copies of each non-logical symbol of L and n new unary predicates to be interpreted as the universes of the old models. We will not carry this further since we will not require a theory of such relations.

To close this section we wish to consider what happens to our results if the filter $D(C)$ is changed, that is, if the notion of "almost all" is altered.

First of all, let D' be some filter on $\mathcal{P}_{\omega_1}(C)$ such that $D(C) \not\subseteq D'$. Then it is easy to find a model $\mathfrak{A}$ with $A \subseteq C$ such that

$$\{s \in \mathcal{P}_{\omega_1}(C) : \mathfrak{A}^s \equiv \mathfrak{A}\} \notin D',$$

and so Theorem 3.1 fails badly for such a filter.

If D' is some filter on $\mathcal{P}_{\omega_1}(C)$ which properly contains $D(C)$, such as an ultrafilter, then the results of Section 3 remain true, since they follow from the theorems as given for $D(C)$. But some results of Section 5 then fail. Let closed* and closed downward* be the new notions defined by a weaker notion of almost all*. Then clearly closed downward implies closed downward*. Let $\boldsymbol{K}$ be closed* and closed downward. Then $\boldsymbol{K}$ is in fact closed, since if $\mathfrak{A}^s \in \boldsymbol{K}$ a.e. then $\mathfrak{A}^s \in \boldsymbol{K}$ a.e.* hence $\mathfrak{A} \in \boldsymbol{K}$. Therefore if $\boldsymbol{K}$ is closed* but not closed then $\boldsymbol{K}$ is not closed downward and in particular is not axiomatised by a sentence of $L^P(\omega)$. Hence Theorem 5.5 fails for any weaker concept almost all*. This seems a serious drawback to any attempt to alter our definition of almost all.

6. Some Löwenheim–Skolem applications

This section is concerned with some applications of the concepts of the previous section. They depend mainly on the following two remarks.

(1) Let $\boldsymbol{K}_0$ be closed downward. If $\boldsymbol{K}$ is any class closed downward and $\boldsymbol{K} \cap \boldsymbol{K}_0 \neq 0$, then some countable model in $\boldsymbol{K}$ belongs to $\boldsymbol{K}_0$.

(2) Let $\boldsymbol{K}_0$ be closed. If $\boldsymbol{K}$ is any class closed downward and if every countable model in $\boldsymbol{K}$ belongs to $\boldsymbol{K}_0$, then $\boldsymbol{K} \subseteq \boldsymbol{K}_0$.

What we do is show that certain interesting classes $\boldsymbol{K}_0$ are closed or closed downward (Theorem 6.1). Remarks (1) and (2) then imply the results we are after (Theorems 6.3 and 6.4). The statements of some of our final results are improved by taking into consideration the interesting model-theoretic characterisations of the countable models in the classes we consider (Proposition 6.2). The results in 6.3 and 6.4 were announced in [16]

Those consequences using remark (1) are "transfer" results, that is, they state that if $\boldsymbol{K}$ contains some model with a certain property then it also contains a model of a certain cardinality with the same property. The applications using remark (2) are upward Löwenheim–Skolem results, but we could also call them "global definability" results (as in [18]), since they imply that every model in $\boldsymbol{K}$ has a semantic property if and only if every model in $\boldsymbol{K}$ satisfies a syntactical definability condition. The corresponding local theorems are the relevant results of 6.2, and the Löwenheim–Skolem argument is the "globalising" procedure.

The classes we will consider first are the following:

$\boldsymbol{K}^1 = \{\mathfrak{A} :$ every element of A is $L_{\infty\omega}$-definable in $\mathfrak{A}\}$.

$K^2 = \{\mathfrak{A}$: there is some finite $S \subseteq A$ such that every element of A is $L_{\infty\omega}$-definable in $\mathfrak{A}$ from the elements of $S\}$.

$K^3 = \{(\mathfrak{A}, P) : P$ is definable in $\mathfrak{A}$ by a formula of $L_{\infty\omega}\}$.

$K^4 = \{(\mathfrak{A}, P)$: there is some finite $S \subseteq A$ such that P is definable in $\mathfrak{A}$ from the elements of S by a formula of $L_{\infty\omega}\}$.

$K^5 = \{\mathfrak{A} : \mathfrak{A}$ has a proper $L_{\infty\omega}$-elementary submodel$\}$.

$K^6 = \{\mathfrak{A} : \mathfrak{A}$ has a proper $L_{\infty\omega}$-elementary submodel $\mathfrak{B}$ such that $U_{\mathfrak{A}} = U_{\mathfrak{B}}\}$.

In K^3 and K^4 it is understood that $(\mathfrak{A}, P)$ is a model for a language $L \cup \{P\}$ formed by adding a new predicate symbol to L. In K^6 U is a predicate of L, and $U_{\mathfrak{A}}$ and $U_{\mathfrak{B}}$ its interpretations in $\mathfrak{A}$ and $\mathfrak{B}$.

Theorem 6.1. (a) K^1, K^2, K^3 *and* K^4 *are closed and co-closed.*
(b) K^5 *and* K^6 *are closed downward.*

We could give direct proofs of 6.1 using the results in Section 3. We prefer, however, to proceed by writing down sentences of $L^P(\omega)$ which axiomatise K^1 through K^4 and by showing that K^5 and K^6 are reducts of classes which are so axiomatised. The details are left until after the statements of our end results.

The following proposition is a list of known results (see [14] concerning most of them). We first recall the following notation:

$$M(\mathfrak{A}, P) = \{Q : (\mathfrak{A}, P) \cong (\mathfrak{A}, Q)\}.$$

Proposition 6.2. *If* $\mathfrak{A}$ *is countable then*:
(a) $\mathfrak{A} \in K^1$ *iff* $\mathfrak{A}$ *has no non-trivial automorphism,*
(b) $\mathfrak{A} \in K^2$ *iff* $\mathfrak{A}$ *has* $\leq \omega$ (*or,* $< 2^\omega$) *automorphisms,*
(c) $(\mathfrak{A}, P) \in K^3$ *iff* $|M(\mathfrak{A}, P)| = 1$,
(d) $(\mathfrak{A}, P) \in K^4$ *iff* $|M(\mathfrak{A}, P)| \leq \omega$ (*or,* $< 2^\omega$),
(e) $\mathfrak{A} \in K^5$ *iff* $\mathfrak{A}$ *has an uncountable* $L_{\infty\omega}$*-elementary extension,*
(f) $\mathfrak{A} \in K^6$ *iff* $\mathfrak{A}$ *has an uncountable* $L_{\infty\omega}$*-elementary extension* $\mathfrak{B}$ *such that* $U_{\mathfrak{A}} = U_{\mathfrak{B}}$.

The following theorem contains the "transfer" consequences of the preceding. They are all immediate from 6.1, 6.2 and remark (1).

Theorem 6.3. *Let* K *be closed downward.*
(a) *If* $K \cap K^i \neq 0$ *then some countable model in* K *belongs to* K^i, *for* $i = 1, \ldots, 4$.
(b) *If* $K \cap K^5 \neq 0$ *then some countable model in* K *has an uncountable* $L_{\infty\omega}$*-elementary extension.*
(c) *If* $K \cap K^6 \neq 0$ *then there is some countable* $\mathfrak{A}$ *in* K *which has an uncountable* $L_{\infty\omega}$*-elementary extension with* $U_{\mathfrak{A}} = U_{\mathfrak{B}}$.

The following are the "upward" or "global definability" results.

Theorem 6.4. *Let* K *be closed downward. Then* (i) *is equivalent to* (ii) *in each of* (a), (b), (c), *and* (d).

(a) (i) *Every countable model in* $\boldsymbol{K}$ *has no proper automorphisms.*
(ii) $\boldsymbol{K} \subseteq \boldsymbol{K}^1$.
(b) (i) *Every countable model in* $\boldsymbol{K}$ *has less than* 2^ω *automorphisms.*
(ii) $\boldsymbol{K} \subseteq \boldsymbol{K}^2$.
(c) (i) *For every countable* $(\mathfrak{A}, P)$ *in* $\boldsymbol{K}$, $|M(\mathfrak{A}, P)| = 1$.
(ii) $\boldsymbol{K} \subseteq \boldsymbol{K}^3$.
(d) (i) *For every countable* $(\mathfrak{A}, P)$ *in* $\boldsymbol{K}$, $|M(\mathfrak{A}, P)| < 2^\omega$.
(ii) $\boldsymbol{K} \subseteq \boldsymbol{K}^4$.

Also note the negative consequences of Theorem 6.4, that is, that $\neg$(ii) implies $\neg$(i) in (a)–(d). For (b) this is an improvement of a result first proved by Hickin [7]; see below.

The results of 6.3 and 6.4 are interesting and largely new even for the special case in which $\boldsymbol{K}$ is defined by a sentence of $L_{\omega_1\omega}$. Theorem 6.4 could, for such $\boldsymbol{K}$, also be derived from the Motohashi-style global definability results (Proposition D and Theorems 2 and 3 in Makkai [18]), as has also been noticed by Makkai. Keisler [12] gives a two-cardinal theorem for $L_{\omega_1\omega}$ which neither implies nor is implied by 6.3(c) for $\boldsymbol{K}$ definable in $L_{\omega_1\omega}$.

The sentences we use in proving 6.1 are written using the formulas E_n and E_n^P which we now define.

Definition. (a) $E_n(u_0, \ldots, u_n, v_0, \ldots, v_n)$ is the following:

$$(\forall x_k \exists y_k)_{k<\omega} \bigwedge_{\text{atomic } \alpha} [\alpha(\boldsymbol{u}, x_0, y_1, x_2, y_3, \ldots) \leftrightarrow \alpha(\boldsymbol{v}, y_0, x_1, y_2, x_3, \ldots)]$$

(b) If P is a unary predicate not in L then $E_n^P(u_0, \ldots, u_n, v_0, \ldots, v_n)$ is the following:

$$(\forall x_k \exists y_k)_{k<\omega}\left[\bigwedge_{m\in\omega}\left[\bigwedge_{i\leq m} P(x_{2i})\right] \rightarrow (P(y_{2m+1}) \wedge \right.$$
$$\left.\bigwedge_{\alpha\in I_m} [\alpha(\boldsymbol{u}, x_0, y_1, x_2, y_3, \ldots, x_{2m}, y_{2m+1}) \leftrightarrow \alpha(\boldsymbol{v}, y_0, x_1, \ldots, y_{2m}, x_{2m+1})])\right]$$

where I_m is the set of all atomic formulas $\alpha(\boldsymbol{u}, x_0, y_1, \ldots, x_{2m}, y_{2m+1})$.

We drop the subscript n when the number of free variables is clear or irrelevant. E is a formula of $L^P(\omega)$, and $\neg E$ is (equivalent to) a formula of $L^P(\omega)$ by Theorem 1.4. Similarly, E^P and $\neg E^P$ both belong to $(L \cup \{P\})^P(\omega)$. What these formulas say is contained in the following lemma.

Lemma 6.5. (a) $\mathfrak{A} \models E[\boldsymbol{a}, \boldsymbol{b}]$ *iff* $(\mathfrak{A}, \boldsymbol{a}) \equiv_{\infty\omega} (\mathfrak{A}, \boldsymbol{b})$.
(b) *If* $a_0, \ldots, a_n \in P$ *then* $(\mathfrak{A}, P) \models E^P[\boldsymbol{a}, \boldsymbol{b}]$ *iff* $\mathfrak{A} \upharpoonright P$ *has universe* P *and* $(\mathfrak{A} \upharpoonright P, \boldsymbol{a}) \equiv_{\infty\omega} (\mathfrak{A}, \boldsymbol{b})$.

Proof. We just prove (a) and leave (b) to the reader. Assume first that $\mathfrak{A} \models E[\boldsymbol{a}, \boldsymbol{b}]$

and let $g_k(x_0, \ldots, x_k)$ be winning functions for E picking the y_k's in $\mathfrak{A}$. Then for every atomic formula α,

$$\mathfrak{A} \models \alpha[\boldsymbol{a}, x_0, g_1(x_0, x_1), x_2, \ldots] \leftrightarrow \alpha[\boldsymbol{b}, g_0(x_0), x_1, g_2(x_0, x_1, x_2), \ldots]$$

for all $x_0, x_1, \ldots$ in A. By induction we easily show that this equivalence holds for every formula α of $L_{\infty\omega}$ with only finitely many free variables, and so $(\mathfrak{A}, \boldsymbol{a}) \equiv_{\infty\omega} (\mathfrak{A}, \boldsymbol{b})$.

Next, assume that $(\mathfrak{A}, \boldsymbol{a}) \equiv_{\infty\omega} (\mathfrak{A}, \boldsymbol{b})$. Using Theorem 1.1 we know there are functions $g_k(x_0, \ldots, x_k)$ such that for every $n \in \omega$ and any $x_0, \ldots, x_n \in A$

$$(\mathfrak{A}, \boldsymbol{a}, x_0, g_1(x_0, x_1), \ldots) \equiv_{\infty\omega} (\mathfrak{A}, \boldsymbol{b}, g_0(x_0), x_1, \ldots).$$

Using these functions to pick the y_k's clearly wins the game of E, hence $\mathfrak{A} \models E[\boldsymbol{a}, \boldsymbol{b}]$. ⊣

Proof of Theorem 6.1. (a) Let σ_1 be

$$\forall u \forall v [u = v \vee \neg E(u, v)].$$

Then $\boldsymbol{K}^1 = \mathrm{Mod}(\sigma_1)$ and both σ_1 and $\neg\sigma_1$ are sentences of $L^P(\omega)$, hence both $\boldsymbol{K}^1$ and its complement are closed downward and therefore closed.

Let σ_2 be

$$\bigvee_{n \in \omega} \exists u_0 \cdots \exists u_n \forall z_0 \forall z_1 [z_0 = z_1 \vee \neg \exists (u_0, \ldots, u_n, z_0, u_0, \ldots, u_n, z_1].$$

Then $\boldsymbol{K}^2 = \mathrm{Mod}(\sigma_2)$ and σ_2 and $\neg\sigma_2$ are sentences of $L^P(\omega)$, hence $\boldsymbol{K}^2$ is closed and co-closed.

We similarly axiomatise $\boldsymbol{K}^3$ and $\boldsymbol{K}^4$; details are left to the reader.

(b) Let P be a new unary predicate symbol, and let σ_5 be the following sentence of $(L \cup \{P\})^P(\omega)$:

$$\bigwedge_{n \in \omega} \forall u_0 \cdots u_n [P(u_0) \wedge \cdots \wedge P(u_n) \rightarrow E^P(\boldsymbol{u}, \boldsymbol{u})] \wedge \exists x \neg P(x).$$

Then $(\mathfrak{A}, P) \models \sigma_5$ if and only if P is the universe of a proper $L_{\infty\omega}$-elementary submodel of $\mathfrak{A}$. Hence $\boldsymbol{K}^5 = \mathrm{Mod}(\sigma_5) \restriction L$ is closed downward, in fact locally closed.

Let σ_6 be

$$\sigma_5 \wedge \forall z_1 \cdots z_k [U(z_1, \ldots, z_k) \rightarrow P(z_1) \wedge \cdots \wedge P(z_k)].$$

Then $\boldsymbol{K}^6 = \mathrm{Mod}(\sigma_6) \restriction L$ is also closed downward. ⊣

Another approach to the proof of Theorem 6.1 uses just the approximation results of Section 3. For example, let $\mathfrak{A}$ belong to $\boldsymbol{K}^1$. Then there are formulas $\varphi_a(x)$ of $L_{\infty\omega}$ for $a \in A$, such that

$$\mathfrak{A} \models \forall x \bigvee_{a \in A} (\varphi_a(x) \wedge \neg \exists y [y \neq x \wedge \varphi_a(y)]).$$

Therefore, by Theorem 3.1,

$$\mathfrak{A}^s \models \forall x \bigvee_{a \in s \cap A} (\varphi_a^s(x) \wedge \neg \exists y[y \neq x \wedge \varphi_a^s(y)]) \text{ a.e.}$$

This means that every element of $\mathfrak{A}^s$ is definable by a formula of $L_{\omega_1\omega}$ a.e., and hence $\mathfrak{A}^s \in \boldsymbol{K}^1$ a.e. Therefore $\boldsymbol{K}^1$ is closed downward. Let $\mathfrak{A}$ not belong to $\boldsymbol{K}^1$. Then there are a, b in A such that $a \neq b$ but $(\mathfrak{A}, a) \equiv_{\infty\omega} (\mathfrak{A}, b)$. Therefore by Theorem 3.5,

$$(\mathfrak{A}^s, a) \cong (\mathfrak{A}^s, b) \text{ a.e.,}$$

in particular $\mathfrak{A}^s \notin \boldsymbol{K}^1$ a.e. Therefore $-\boldsymbol{K}^1$ is also closed downward. Similar proofs could be given of the other parts of 6.1.

We could, of course, immediately define many more classes which could easily be shown to be closed downward. It seems to be harder, however, to determine whether a class is closed. For example, is $\boldsymbol{K}^5$ closed? If true this would be interesting since it would imply, for example, that if σ is a sentence of $L^P(\omega)$ and every countable model of σ is $L_{\infty\omega}$-elementarily equivalent to an uncountable model, then every model of σ has a proper $L_{\infty\omega}$-elementary submodel. If L contains a binary predicate $<$ then we can easily show that

$$\{\mathfrak{A} : \text{some proper } <\text{-initial segment of } \mathfrak{A} \text{ is an } L_{\infty\omega}\text{-elementary submodel of } \mathfrak{A}\}$$

is closed. But if the $L_{\infty\omega}$-elementary submodels are not definable in some way, we do not know the answer.

We can ask the same question for $\boldsymbol{K}^6$ and

$$\boldsymbol{K}^7 = \{\mathfrak{A} : \mathfrak{A} \equiv_{\infty\omega} \mathfrak{A}_0 \text{ for some proper submodel } \mathfrak{A}_0 \text{ of } \mathfrak{A}\},$$

which is easily seen to be closed downward. The class

$$\boldsymbol{K}^{7*} = \{\mathfrak{A} : \mathfrak{A} \mathrel{\exists_{\infty\omega}} \mathfrak{A}_0 \text{ for some proper submodel } \mathfrak{A}_0 \text{ of } \mathfrak{A}\}$$

is easily shown to be closed and contain precisely the same countable models as $\boldsymbol{K}^7$. Therefore $\boldsymbol{K}^7$ is closed if and only if it equals $\boldsymbol{K}^{7*}$.

We now wish to look at the uncountable models in our classes. Using the concept of a "locally closed" class introduced at the end of Section 5, we have the following immediate consequence of Theorem 6.1.

Corollary 6.6. *Let $\boldsymbol{K}$ be locally closed. Then, for each i from 1 to 7, if $\boldsymbol{K} \cap \boldsymbol{K}^i$ contains a model of cardinality κ then $\boldsymbol{K} \cap \boldsymbol{K}^i$ contains models of all infinite cardinalities less than κ. The same is true of $\boldsymbol{K} \cap (-\boldsymbol{K}^i)$ for $1 \leq i \leq 4$.*

Consider the following class:

$$\boldsymbol{K}^8 = \{\mathfrak{A} : \text{for every finite } S \subseteq A, \mathfrak{A} \text{ has some non-trivial automorphism fixing the elements of } S\}.$$

Then $\boldsymbol{K}^8 \subseteq -\boldsymbol{K}^2$ and they contain the same countable models; hence $\boldsymbol{K}^8$ is closed downward, a fact first proved by Hickin [7]. $\boldsymbol{K}^8$ is locally closed, in fact PC_Δ, so we can add $i = 8$ to 6.6.

Of probably more interest are the uncountable models in the classes corresponding to $-\boldsymbol{K}^2$ and $\boldsymbol{K}^8$ but in which "S finite" is replaced by "S of cardinality less than $|A|$". Corresponding to $\boldsymbol{K}^8$ we have

$$\boldsymbol{K}^9 = \{\mathfrak{A} : \text{for every } S \subsetneq A \text{ with } |S| < |A| \text{ there is some non-trivial automorphism of } \mathfrak{A} \text{ fixing each element of } S\},$$

and corresponding to $-\boldsymbol{K}^2$ there is

$$\boldsymbol{K}^{10} = \{\mathfrak{A} : \text{there is no } S \subseteq A \text{ with } |S| < |A| \text{ such that every element of } \mathfrak{A} \text{ is } \mathrm{L}_{\infty\omega}\text{-definable from elements of } S\}.$$

There are at least two significant ways of understanding the phrase "every element of $\mathfrak{A}$ is $\mathrm{L}_{\infty\omega}$-definable from elements of S".

(a) For every $b_1, b_2 \in A$ if $b_1 \neq b_2$ then $(\mathfrak{A}, b_1, a)_{a\in S} \not\equiv_{\infty\omega} (\mathfrak{A}, b_2, a)_{a\in S}$.

(b) For every $b_1, b_2 \in A$ if $b_1 \neq b_2$ then there are $a_1, \ldots, a_n \in S$ such that $(\mathfrak{A}, b_1, \boldsymbol{a}) \equiv_{\infty\omega} (\mathfrak{A}, b_2, \boldsymbol{a})$.

Everything we say about $\boldsymbol{K}^{10}$ will be true for both senses (a) and (b).

$\boldsymbol{K}^9 \subseteq \boldsymbol{K}^{10} \subsetneq -\boldsymbol{K}^2$ and all three contain precisely the same countable models. Hence $\boldsymbol{K}^9$ and $\boldsymbol{K}^{10}$ are closed downward, but not closed. Although they are not locally closed, the following is true.

Theorem 6.7. *If $\boldsymbol{K}$ is locally closed and $\boldsymbol{K} \cap \boldsymbol{K}^9$ (or $\boldsymbol{K} \cap \boldsymbol{K}^{10}$) contains a model of cardinality κ, then this intersection contains models of each regular cardinality less than or equal to the cofinality of κ.*

Proof. The proof is essentially the same for $\boldsymbol{K}^9$ and both senses of $\boldsymbol{K}^{10}$. We phrase it in terms of $\boldsymbol{K}^9$. Let $<$ be a new binary predicate symbol, and let $\boldsymbol{K}^\#$ be the class of all models $(\mathfrak{A}, <)$ where $<$ well-orders A and for every $a \in A$ there is some non-trivial automorphism of $\mathfrak{A}$ fixing every element less than a. Then $\boldsymbol{K}^\#$ is locally closed and $\boldsymbol{K}^9 \subseteq \boldsymbol{K}^\# \restriction \mathrm{L}$.

Let $\mathfrak{A} \in \boldsymbol{K} \cap \boldsymbol{K}^9$ have cardinality κ and let $<$ well-order A in type κ. Then $(\mathfrak{A}, <) \in \boldsymbol{K}^\#$. By Theorem 5.16 there are finitary functions f_n for $n \in \omega$ such that $\mathfrak{A}_0 \in \boldsymbol{K}$ and $(\mathfrak{A}_0, <) \in \boldsymbol{K}^\#$ for every $\mathfrak{A}_0 \subseteq \mathfrak{A}$ closed under every f_n. Let λ be regular, $\lambda \leq \mathrm{cf}(\kappa)$. Take $\mathfrak{A}_0 \subseteq \mathfrak{A}$ of cardinality λ, closed under all f_n's. We might not have $\mathfrak{A}_0 \in \boldsymbol{K}^9$ since some $S \subsetneq A_0$ with $|S| < \lambda$ might be $<$-cofinal in A_0. We define a chain $\{\mathfrak{A}_\xi\}_{\xi\leq\lambda}$ of submodels of $\mathfrak{A}$, each of cardinality λ and closed under all f_n's as follows. $\mathfrak{A}_0$ is the given submodel, and $\mathfrak{A}_\alpha = \bigcup_{\xi<\alpha} \mathfrak{A}_\xi$ if $\alpha \leq \lambda$ is a limit ordinal. $\mathfrak{A}_{\xi+1}$ is some extension of $\mathfrak{A}_\xi$ of cardinality λ, contained in $\mathfrak{A}$ and closed under all f_n's, such that $A_{\xi+1}$ contains some element which is greater than every element of A_ξ unless A_ξ is cofinal in A, in which case we let $\mathfrak{A}_{\xi+1} = \mathfrak{A}_\xi$. Then $\mathfrak{A}_\lambda \in \boldsymbol{K}$, $(\mathfrak{A}_\lambda, <) \in \boldsymbol{K}^\#$, and every cofinal subset of A_λ has cardinality λ, hence $\mathfrak{A}_\lambda \in \boldsymbol{K}^9$ as desired. $\dashv$

The complement of K^{10} is not closed downward, but we do not know if it has the following property: if K is closed and $K \cap (-K^{10})$ contains an uncountable model then this intersection contains a model of cardinality ω_1. If true, we would have the following interesting transfer property: if a closed class contains a model of cardinality $\kappa > \omega$ every element of which is definable from some subset of cardinality $< \kappa$, then the class contains an uncountable model every element of which is definable from some countable subset. This would be worth establishing for either sense of definability, and seems open even if the closed class is defined by a set of sentences of finitary logic.

An interesting question on uncountable models concerns what could be called the "Hanf number" for particular closed properties. Let K_0 be some closed class. Then for any sentence σ of $L_{\omega_1\omega}$, $\mathrm{Mod}(\sigma) \cap K_0$ either contains models of all infinite cardinalities or there is some κ such that $\mathrm{Mod}(\sigma) \cap K_0$ contains infinite models of precisely the infinite cardinalities less than κ. The *Hanf number* (w.r.t. $L_{\omega_1\omega}$) for K_0 is the least κ such that, for every sentence σ of $L_{\omega_1\omega}$ if $\mathrm{Mod}(\sigma) \cap K_0$ contains a model of cardinality κ then it contains models of all infinite cardinalities. What is the Hanf number for certain interesting K_0? In general, of course, this number may be enormous, simply because the Hanf number of the logic of closed classes is enormous. But it seems that for some K_0 this number could be small. In particular we raise this question for the class K^1 of all models in which every element is $L_{\infty\omega}$ definable. The Hanf number w.r.t. finitary theories (where sentences of $L_{\omega_1\omega}$ are replaced by finitary theories) is equally as interesting. Determining these numbers of K^1 takes an additional interest due to Shelah's destruction of Ehrenfeucht's rigid spectrum problem.

It seems to us that the methods of this section should prove useful in establishing further transfer results.

The following easy lemma gives two more useful properties of classes closed downward.

Lemma 6.8. *Let P be a unary predicate not in* L.
(a) *If K' is a class of $L \cup \{P\}$-models which is closed downward, then*

$$K = \{\mathfrak{A} \restriction P : (\mathfrak{A}, P) \in K'\}$$

is closed downward.
(b) *If K is a class of* L*-models which is closed downward, then*

$$K' = \{(\mathfrak{A}, P) : \mathfrak{A} \restriction P \in K\}$$

is closed downward.

Part (a) implies, for example, that the class of relativised reducts of a class axiomatised by a sentence of $L^P(\omega)$ is closed downward. Since it can be shown that every Σ_1-definable class of models closed under isomorphism is such a class of relativised reducts, this therefore implies that Σ_1 classes of models are closed downward, a version of Barwise's result, Theorem 2.6.

As an example of an application of part (b) of 6.8 we prove a theorem about groups. We first recall some group-theoretic terminology.

Definition. (a) Let $\mathfrak{B}$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be groups. $\mathfrak{B}$ is an *extension of* $\mathfrak{A}_1$ *by* $\mathfrak{A}_2$ if $\mathfrak{A}_1 \subseteq \mathfrak{B}$ and there is a homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}_2$ whose kernel is A_1.

(b) If $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are classes of groups, then $\boldsymbol{K}_1\boldsymbol{K}_2$ is the class of all extensions of groups in $\boldsymbol{K}_1$ by groups in $\boldsymbol{K}_2$.

Proposition 6.9. *If $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are closed downward then so is $\boldsymbol{K}_1\boldsymbol{K}_2$.*

Proof. Expand the language L of group theory by adding new unary predicates P_1 and P_2, a new unary function symbol H and a new binary function symbol F. Let $\boldsymbol{K}^*$ be the class of all models $(\mathfrak{B}, P_1, P_2, F, H)$ where $\mathfrak{B}$ is a group, $\mathfrak{B} \upharpoonright P_1 \in \boldsymbol{K}_1$, $(P_2, F \upharpoonright P_2)$ is a group belonging to $\boldsymbol{K}_2$ and H is a homomorphism of $\mathfrak{B}$ onto $(P_2, F \upharpoonright P_2)$ whose kernel is P_1. Then, using 6.8(b), we see that $\boldsymbol{K}^*$ is closed downward. Therefore $\boldsymbol{K}^* \upharpoonright \mathrm{L}$ is closed downward, and this class is precisely $\boldsymbol{K}_1\boldsymbol{K}_2$. ⊣

It is convenient to have the following definition for the subsequent discussion.

Definition. A class $\boldsymbol{K}$ closed under isomorphism is *closed upward* if, for any model $\mathfrak{A}$, $\mathfrak{A} \in \boldsymbol{K}$ whenever $\mathfrak{A}^s \in \boldsymbol{K}$ a.e.

$\boldsymbol{K}$ is closed iff it is closed downward and closed upward. $\boldsymbol{K}$ is closed upward if (but not only if, by 5.13(a)) its complement is closed downward. $\boldsymbol{K}$ is closed upward iff it includes some closed class with exactly the same countable models.

Determining whether classes of groups are closed upward is related to group-theoretical questions concerning *local-type* theorems.

Recall that a class of groups is *local* if a group belongs to the class whenever all its finitely generated subgroups do. Various people (see [7], [8] for an account) have considered the following countable generalisation: a class is *countably local*, or of *countable character*, if a group belongs to the class whenever all its countable subgroups do. A class which is closed upward is countably local; the converse is also true if the class is closed under subgroups.

Quite recently Hickin and Phillips [8] have considered classes of groups closed upward, and show that various classes of groups are closed upward or of countable character. For example, let $\boldsymbol{A}$ be the class of abelian groups and let $\boldsymbol{S}$ be the class of simple groups. They establish [8, Theorem 5] that $\boldsymbol{AS}$ is closed upward. Since $\boldsymbol{A}$ and $\boldsymbol{S}$ are both closed ($\boldsymbol{S}$ is axiomatisable by a sentence of $\mathrm{L}_{\omega_1\omega}$ by Kopperman and Mathias [13]) we obtain the following from 6.9.

Corollary 6.10. *$\boldsymbol{AS}$ is closed.*

Notice that by Proposition 5.1(a), $\boldsymbol{AS}$ is therefore closed under $L_{\infty\omega}$-elementary equivalence.

We refer the reader to [7], [8] for other results and techniques concerning upward and local theorems for classes of groups. We conclude this section by mentioning a problem adapted from [13]. Let $\boldsymbol{K}_0$ be the class of all groups with some automorphism which is not inner. $\boldsymbol{K}_0$ is easily seen to be closed downward, but it seems to be open whether or not it is closed. We can write down the sentence σ of $L^P(\omega)$ axiomatising the closed class generated by $\boldsymbol{K}_0$, and show that the complement of this closed class is also closed, but we do not know whether or not σ axiomatises $\boldsymbol{K}_0$.

7. Uncountable approximations

Our basic framework of countable approximations generalises to approximations of cardinality at most κ. In this section we give these generalisations and indicate, without proof, the major results which carry over.

For any set s we define $\mathfrak{A}^s$ to be $\mathfrak{A} \restriction s \cap A$, as before. If φ is a formula of $L_{\infty\lambda}$ then φ^s is defined by the obvious extension of the definition in Section 2; that is, conjunctions and disjunctions in φ indexed by I are replaced by the subconjunctions and subdisjunctions indexed by $I \cap s$. If $\kappa \geqslant \lambda$ and $\kappa \geqslant |s|$, then φ^s is a formula of $L_{\kappa^+\lambda}$.

In dealing with approximations by sets s of cardinality at most κ we tacitly assume that the language L has at most κ non-logical symbols.

The generalisation of the filter $D(C)$ is defined by a game of length κ.

Definition. The filter $D_{\kappa^+}(C)$ on $\mathcal{P}_{\kappa^+}(C)$ is defined as follows: $X \in D_{\kappa^+}(C)$ iff

$$(\forall x_{2\xi} \in C \exists x_{2\xi+1} \in C)_{\xi<\kappa} [\{x_\xi : \xi < \kappa\} \in X].$$

The following proposition lists the relevant properties of these filters.

Proposition 7.1. (a) *$D_{\kappa^+}(C)$ is a κ^+-complete filter.*
(b) *Assume $\kappa^{|\alpha|} = \kappa$, and that $X_a \in D_{\kappa^+}(C)$ for every α-termed sequence a from C. Then $\bar{X} \in D_{\kappa^+}(C)$ where*

$$\bar{X} = \{s \in \mathcal{P}_{\kappa^+}(C) : s \in X_a \text{ for every } a \text{ from } s\}.$$

(c) *If $\kappa^{\underline{\kappa}} = \kappa$, then $X \in D_{\kappa^+}(C)$ iff there is some $X' \subseteq X$ such that:*
  (i) *for every $s \in \mathcal{P}_{\kappa^+}(C)$ there is some $s' \in X'$ with $s \subseteq s'$,*
  (ii) *X' is closed under unions of chains of length equal to κ.*

As in Section 2 we say that C is large enough to approximate $\mathfrak{A}$ if $A \subseteq C$, and C is large enough to approximate φ if every conjunction and disjunction in φ is indexed by elements of C. Using $\mathbf{a}_1, \ldots, \mathbf{a}_n$ to stand for either models or formulas of some $L_{\infty\lambda}$, we make the following definition.

Definition. $P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s)$ holds for *almost all s of cardinality* $\leq \kappa$ iff:

$$\{s \in \mathcal{P}_{\kappa^+}(C) : P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s) \text{ holds}\} \in D_{\kappa^+}(C) \qquad (*)$$

for some C large enough to approximate $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

As before, the property in the definition is independent of the choice of C. We use κ-*a.e.* as an abbreviation for "for almost all s of cardinality $\leq \kappa$".

Proposition 7.2. *If* $P(\mathbf{a}_1^s, \ldots, \mathbf{a}_n^s)$ *holds* κ-a.e. *then* (*) *holds for every C large enough to approximate* $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Our general downward Löwenheim–Skolem theorem is now proved just like Theorem 3.1.

Theorem 7.3. *If* σ *is a sentence of* $L_{\infty\lambda}$ *and* $\kappa^{\underset{\smile}{\lambda}} = \kappa$, *then* $\mathfrak{A} \models \sigma$ *iff* $\mathfrak{A}^s \models \sigma^s$ κ-*a.e.*

As a particular consequence we have the following biconditional strengthening of the usual downward Löwenheim–Skolem theorem for $L_{\mu^+\lambda}$.

Corollary 7.4. *Let* σ *be a sentence of* $L_{\mu^+\lambda}$ *and let* κ *be such that* $\kappa \geq \mu$ *and* $\kappa^{\underset{\smile}{\lambda}} = \kappa$. *Then* $\mathfrak{A} \models \sigma$ *iff* $\mathfrak{A}^s \models \sigma$ κ-*a.e.* (*that is, iff almost all submodels of* $\mathfrak{A}$ *of power* κ *are models of* σ).

Note that the obvious analogues of Corollaries 3.2(b)–3.4 also hold.

Definition. Let κ^* be the least infinite cardinal such that $(\kappa^*)^{\underset{\smile}{\kappa}} = \kappa^*$.

Using Theorem 7.3 and a back-and-forth argument the following is proved just like 3.5.

Theorem 7.5. (a) $\mathfrak{A} \equiv_{\infty\kappa} \mathfrak{B}$ *iff* $\mathfrak{A}^s \equiv_{\infty\kappa} \mathfrak{B}^s$ κ^*-*a.e.*
(b) $\mathfrak{A} \not\equiv_{\infty\kappa} \mathfrak{B}$ *iff* $\mathfrak{A}^s \not\equiv_{\infty\kappa} \mathfrak{B}^s$ κ^*-*a.e.*

Unfortunately, this theorem for $\kappa > \omega$ does not have the interest of Theorem 3.5, even if $\kappa = \kappa^*$, since $L_{\infty\kappa}$-elementarily equivalent models of power κ need not be isomorphic. Thus, for example, it says nothing about the relationship between two $L_{\infty\omega_1}$-elementarily equivalent models of power ω_1.

Also for this reason, the results in Section 4 do not generalise (except for those on Horn sentences, which we omit here).

On the other hand, the results in Section 5 on closed classes and sentences with game quantifiers generalise rather well.

Definition. Let $\boldsymbol{K}$ be a class of models closed under isomorphism.
(a) $\boldsymbol{K}$ is κ-*closed downward* if: $\mathfrak{A} \in \boldsymbol{K}$ implies $\mathfrak{A}^s \in \boldsymbol{K}$ κ-a.e.
(b) $\boldsymbol{K}$ is κ-*closed* if: $\mathfrak{A} \in \boldsymbol{K}$ iff $\mathfrak{A}^s \in \boldsymbol{K}$ κ-a.e.

If σ is a sentence of $L_{\kappa^+\lambda}$ then $\text{Mod}(\sigma)$ is κ'-closed for all $\kappa' \geqslant \kappa$ such that $(\kappa')^{\underline{\lambda}} = \kappa'$, by Corollary 7.4. However a κ-closed class need not be closed under $L_{\infty\kappa}$-elementary equivalence (even assuming $\kappa = \kappa^*$). The obvious generalisations of parts (b)–(f) of Proposition 5.1 do hold, but we omit the statements here.

A logic adequate to axiomatise every κ-closed class is obtained, as one would expect, by allowing sentences with game quantifiers of length κ. Instead of giving an inductive definition paralleling that of $L^p(\omega)$ we simply use the prenex game characterisation as a definition.

Definition. The sentences of $L^p(\kappa)$ are the sentences of the form $(Q_\xi x_\xi)_{\xi<\kappa}\varphi$, where φ is open.

One can show that every sentence of $L_{\kappa^+\kappa}$ is equivalent to a sentence of $L^p(\kappa)$. If σ is a sentence of $L^p(\kappa)$ then $\text{Mod}(\sigma)$ is λ-closed downward for every λ such that $\lambda^{\underline{\kappa}} = \lambda$ (in particular for $\lambda = \kappa^*$).

Our main result is the following, whose proof exactly follows that of Theorem 5.5.

Theorem 7.6. *If $\boldsymbol{K}$ is κ-closed then $\boldsymbol{K} = \text{Mod}(\sigma)$ for some sentence σ of $L^p(\kappa)$.*

As an immediate consequence we have a separation theorem.

Corollary 7.7. *Let $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ be κ-closed downward and assume that $\boldsymbol{K}_1 \cap \boldsymbol{K}_2 = 0$. There are sentences σ_1 and σ_2 of $L^p(\kappa)$ such that $\boldsymbol{K}_i \subseteq \text{Mod}(\sigma_i)$ for $i = 1, 2$ and $\models \neg(\sigma_1 \wedge \sigma_2)$.*

The immediate interpolation consequence is less satisfactory.

Corollary 7.8. *Let L_1 and L_2 be languages with at most κ non-logical symbols such that $L_1 \cap L_2 = L$. Let θ_i be a sentence of $L_i^p(\kappa)$, $i = 1, 2$ and assume that $\models \neg(\theta_1 \wedge \theta_2)$. Then there are sentences σ_1 and σ_2 of $L^p(\kappa^*)$ such that $\theta_i \models \sigma_i$ for $i = 1, 2$, and $\models \neg(\sigma_1 \wedge \sigma_2)$.*

Due to the necessity of passing from $L^p(\kappa)$ to $L^p(\kappa^*)$ in 7.8, this result is perhaps better formulated concerning a larger logic containing every $L^p(\kappa)$.

Definition. $L^+(\infty)$ is the logic containing all atomic and negated atomic formulas and closed under arbitrary conjunctions, disjunctions, and homogeneous and game quantification.

Then any sentence of $L^+(\infty)$ is equivalent to a sentence of some $L^p(\kappa)$, hence 7.8 implies:

$L^+(\infty)$ *satisfies interpolation.*

Takeuti's interpolation theorem [30], already mentioned in Section 5, can be phrased as saying that any valid implication in $L_{\infty\infty}$ has an interpolant in $L^{+}(\infty)$. At least as proved by Nebres [25], it is true allowing functions with infinitely many arguments in the language. And in this form it implies that $L^{+}(\infty)$ satisfies interpolation, since replacing existential quantifiers in an $L^{+}(\infty)$ sentence by (infinitary) Skolem functions results in an $L_{\infty\infty}$-sentence. That is, a class which is *PC* in $L^{+}(\infty)$ is already *PC* in $L_{\infty\infty}$ with infinitary functions. Once again, however, we wish to point out that this does not appear to imply 7.8, and certainly not 7.7.

$L^{P}(\kappa)$ is also easily seen to satisfy preservation theorems analogous to Theorem 5.10.

There are many questions and problems concerned with uncountable approximations.

There is a sentence σ of $L_{\omega_1\omega_1}$ which has a model of power κ iff $\kappa^{\omega} = \kappa$, and so $\mathrm{Mod}(\sigma)$ is κ-closed downward iff $\kappa^{\omega} = \kappa$. In particular, κ_1-closed and $\kappa_1 < \kappa_2$ need not imply κ_2-closed. Does the example indicate the only difficulties that can arise? That is, assuming G.C.H. for simplicity, if κ_1 and κ_2 are regular, $\kappa_1 < \kappa_2$, and $\boldsymbol{K}$ is κ_1-closed, must $\boldsymbol{K}$ be κ_2-closed? We know it must at least be κ_2-closed downward.

The situation as far as applications like those in Section 6 is not satisfactory. For example, let $\boldsymbol{K}_0$ be the class of all models with a non-trivial automorphism, and let $\boldsymbol{K}_1$ be the class of all models in which some element is not $L_{\infty\kappa}$-definable. Then $\boldsymbol{K}_0$ is κ-closed downward, $\boldsymbol{K}_0 \subsetneqq \boldsymbol{K}_1$ and $\boldsymbol{K}_1$ is κ-closed assuming $\kappa = \kappa^{*}$. But $\boldsymbol{K}_1$ is not the smallest κ-closed class containing $\boldsymbol{K}_0$, if $\kappa > \omega$, since there are models of cardinality κ in $\boldsymbol{K}_1$ but not in $\boldsymbol{K}_0$. What is the smallest κ-closed class containing $\boldsymbol{K}_0$? Is it the class of all models in which some element is not definable in some natural logic (like $L(\kappa)$, see [11])?

Related to this is the problem of finding a logic $L^{\#}$ such that we obtain a full generalisation of Theorem 3.5, that is, such that $\mathfrak{A} \equiv \mathfrak{B}(L^{\#})$ iff $\mathfrak{A}^{s} \cong \mathfrak{B}^{s}$ κ-a.e.

Just as with $L^{P}(\omega)$ one can find sentences of $L^{P}(\kappa)$ whose negations are not equivalent to sentences of $L^{P}(\kappa)$. Let $L^{\delta}(\kappa)$ be the maximal sublogic of $L^{P}(\kappa)$ closed under negation. Then, assuming $\kappa = \kappa^{*}$, $L^{\delta}(\kappa)$ satisfies Δ-interpolation but not full interpolation. Are there "large" natural sublogics of $L^{P}(\kappa)$ closed under negation and satisfying full interpolation?

The sentences of $L^{P}(\omega)$ we constructed whose negations are not equivalent to sentences of $L^{P}(\omega)$ all have the property that their negations are equivalent to sentences of $L^{P}(\kappa)$ for some $\kappa > \omega$. Is this in fact always true? More generally, if σ is a sentence of $L^{P}(\kappa_1)$ is $\neg\sigma$ equivalent to a sentence of $L^{P}(\kappa_2)$ for some $\kappa_2 \geqslant \kappa_1$? If true, this would mean that $L^{+}(\infty)$ itself is closed under negations, and is therefore equivalent to $L(\infty)$ (defined in [11]; we use Shelah's result [27] that non-well-ordered quantifiers are superfluous).

Let $L^{\delta}(\infty)$ be the maximal sublogic of $L^{+}(\infty)$ closed under negations. Then $L^{\delta}(\infty)$ satisfies Δ-interpolation and, as just pointed out, may be equivalent to $L^{+}(\infty)$. In any case, does $L^{\delta}(\infty)$ satisfy interpolation?

We are interested in these questions because it seems to us that the concept of

κ-closed is one abstract criterion to help distinguish second-order properties from those which are, in some weak sense, first-order. Even the simplest, most natural second-order sentences (for example, "P has cardinality strictly less than the cardinality of the universe") define classes which are not κ-closed, or even κ-closed downward, for any κ. It seems to us that any logic all of whose sentences define classes which are κ-closed for some κ has some right to be considered a generalised first-order logic, rather than second-order. Thus, if $L^*(\infty)$ does turn out to be closed under negation, this would say that even negations of game-quantified sentences are not too outrageously second-order.

Finally, returning to $L_{\infty\kappa}$, it seems worthwhile to remedy the defect in Theorem 7.5 by finding a criterion for $\mathfrak{A} \equiv_{\infty\kappa} \mathfrak{B}$ which is interesting when $|A| = |B| = \kappa$. Results on this line for free algebras, or models having similar characteristics, have been announced by the author [17].

References

[1] K.J. Barwise, Back and forth through infinitary logic, in: Studies in Model Theory, MAA Studies 8 (1973) 5–34.

[2] K.J. Barwise, Mostowski's collapsing function and the closed unbounded filter, Fund. Math. 82 (1974) 95–103.

[3] K.J. Barwise, Axioms for abstract model theory, Ann. Math. Logic 7 (1974) 221–265.

[4] C.C. Chang, Some remarks on the model theory of infinitary languages, in: The Syntax and Semantics of Infinitary Languages (Springer-Verlag, 1968) 36–63.

[5] G.V. Čudnovskii, Some results in the theory of infinitely long expressions, Soviet Math. Doklady 9 (1968) 556–559.

[6] L. Henkin, Some remarks on infinitely long formulas, in: Infinitistic Methods (PWN, 1961) 167–183.

[7] K. Hickin, Countable type local theorems in algebra, J. of Algebra 27 (1973) 523–537.

[8] K. Hickin and R. Phillips, Local theorems and group extensions, Proc. Camb. Phil. Soc. 73 (1973) 7–20.

[9] T. Jech, Some combinatorial problems concerning uncountable cardinals, Ann. of Math. Logic 5 (1973) 165–198.

[10] C. Karp, Finite quantifier equivalence, in: The Theory of Models (North-Holland, 1965) 407–412.

[11] H.J. Keisler, Formulas with linearly ordered quantifiers, in: The Syntax and Semantics of Infinitary Languages (Springer-Verlag, 1968) 96–130.

[12] H.J. Keisler, Model Theory for Infinitary Logic (North-Holland, 1971).

[13] R.D. Kopperman and A.R.D. Mathias, Some problems in group theory, in: The Syntax and Semantics of Infinitary Languages (Springer-Verlag, 1968) 131–138.

[14] D.W. Kueker, Definability, automorphisms, and infinitary languages, in: The Syntax and Semantics of Infinitary Languages (Springer-Verlag, 1968) 152–165.

[15] D.W. Kueker, Löwenheim–Skolem and interpolation theorems in infinitary languages, Bull. Amer. Math. Soc. 78 (1972) 211–215.

[16] D.W. Kueker, Some Löwenheim-Skolem theorems, abstract 72T–E72, Notices Amer. Math. Soc. 19 (1972) A–600.

[17] D.W. Kueker, Free and almost free algebras (to appear).

[18] M. Makkai, Global definability theory in $L_{\omega_1\omega}$, Bull. Amer. Math. Soc. 79 (1973) 916–921.

[19] J.A. Makowsky, S. Shelah, and J. Stavi, Δ-logics and generalized quantifiers, to appear.

[20] J.I. Malitz, The Hanf number for complete $L_{\omega_1\omega}$ sentences, in: The Syntax and Semantics of Infinitary Languages (Springer-Verlag, 1968) 166–181.

[21] J.I. Malitz, Universal classes in infinitary languages, Duke Math. Journal 36 (1969) 621–630.

[22] J.I. Malitz, Infinitary analogs of theorems from first order model theory, J. of Symb. Logic 36 (1971) 216–228.
[23] D.A. Martin, Borel determinacy, to appear.
[24] Y.N. Moschovakis, Elementary Induction on Abstract Structures (North-Holland, 1974).
[25] B.F. Nebres, Herbrand uniformity theorems for infinitary languages, J. of Math. Soc. Japan 24 (1972) 1–19.
[26] D. Scott, Logic with denumerably long formulas, in: The Theory of Models (North-Holland, 1965) 329–341.
[27] S. Shelah, On languages with non-homogeneous strings of quantifiers, Israel J. of Math. 8 (1970) 75–79.
[28] L. Svenonius, On the denumerable models of theories with extra predicates, in: The Theory of Models (North-Holland, 1965) 376–389.
[29] A.K. Swett, Herbrand's theorem in infinitary logic, Ph.D. dissertation, University of Toronto (1973).
[30] G. Takeuti, A determinate logic, in: The Syntax and Semantics of Infinitary Languages (Springer-Verlag, 1968) 237–264.
[31] A. Tarski, Remarks on predicate logic with infinitely long expressions, Colloq. Math. 6 (1958) 171–176.
[32] R.L. Vaught, Descriptive set theory in $L_{\omega_1\omega}$, in: Cambridge Summer School in Mathematical Logic (Springer-Verlag, 1973) 574–598.