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$\mathscr{A}$-Fixed Points of Multi-valued Contractions

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INTRODUCTION

A trend in today's literature on fixed point theory is the attempt at establishing the random versions of results which are well known in the deterministic case. To be more precise, we recall that, given a measurable space $(T, \mathscr{F})$ (so, $\mathscr{F}$ is a σ-algebra of subsets of T), a topological space X, and a multifunction F from $T \times X$ into X, a measurable function $\varphi: T \to X$ is said to be a random fixed point of F if $\varphi(t) \in F(t, \varphi(t))$ for all $t \in T$. In particular, S. Itoh proved in [5] the random version of a result by S. B. Nadler, Jr. [10, Theorem 5] which, in turn, was the first extension to multifunctions of the classical contraction mapping principle of Banach. Unlike the random case, it seems that very little is known about the following more general problem:

Given a family $\mathscr{A}$ of (single-valued) functions from T into X, find an $\mathscr{A}$-fixed point of F, i.e., a $\varphi \in \mathscr{A}$ such that $\varphi(t) \in F(t, \varphi(t))$ for all $t \in T$.

The aim of the present paper is to study the above problem, keeping always two basic assumptions: X is a complete metric space and, for every $t \in T$, $F(t, \cdot)$ is a multi-valued contraction, with closed values.

Our main abstract result is Theorem 2.1, whose formulation is based on the notion of $\mathscr{A}$-stability recently introduced by B. Ricceri in [12]. We then present a series of consequences of Theorem 2.1. In particular, besides a more refined version of the above-quoted result of Itoh, we establish two $\mathscr{A}$-fixed point theorems in the case where T is a topological space and $\mathscr{A}$ is either the family of all continuous functions or the family of all Baire functions of class α $(0 < \alpha < \omega_1)$. Finally, as further consequences, we obtain three results on fixed point stability. We will present some applications of them in forthcoming papers.

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1. Notation, Basic Definitions, and Preliminary Results

Let A, B be two non-empty sets. We will indicate by 2^B the family of all non-empty subsets of B. A multifunction from A into B is a function from A into 2^B. Let $\Phi: A \to 2^A$. A point $x \in A$ is said to be a fixed point of Φ if $x \in \Phi(x)$. We will denote by $\mathrm{Fix}(\Phi)$ the set of fixed points of Φ. Moreover, if $\Phi: A \to 2^B$, we will denote by $\mathrm{gr}(\Phi)$ (graph of Φ) the set $\{(x, y) \in A \times B: y \in \Phi(x)\}$. If Ω is a subset of B, we put $\Phi^-(\Omega) = \{x \in A: \Phi(x) \cap \Omega \neq \emptyset\}$. If A, B are topological spaces, a multifunction $\Phi: A \to 2^B$ is said to be lower semicontinuous at $x_0 \in A$ if for every open set $\Omega \subseteq B$ such that $x_0 \in \Phi^-(\Omega)$, one has $x_0 \in \mathrm{int}\, \Phi^-(\Omega)$. Φ is said to be lower semicontinuous in A if it is so at every point of A.

If A, B are two non-empty subsets of a generalized metric space (M, d), x a point of M, $r > 0$, we will put $B_d(x, r) = \{y \in M: d(x, y) < r\}$; $d(x, A) = \inf_{y \in A} d(x, y)$; $d^*(A, B) = \sup_{x \in A} d(x, B)$; $d_H(A, B) = \max(d^*(A, B), d^*(B, A))$. A multifunction $\Phi: A \to 2^M$ is said to be a multi-valued contraction if there exists a constant $L \in [0, 1[$ such that $d_H(\Phi(x), \Phi(y)) \leqslant L d(x, y)$ for every $x, y \in A$.

Now, let T be a non-empty set, $\mathscr{G} \subseteq 2^T \cup \{\emptyset\}$, X a topological space. We will say that $\Phi: T \to 2^X$ is $\mathscr{G}$-measurable if $\Phi^-(\Omega) \in \mathscr{G}$ for every open set $\Omega \subseteq X$. Thus, if $\mathscr{G}$ is a topology, $\mathscr{G}$-measurability means lower semicontinuity. If T is a topological space and $\mathscr{G}$ is the family of all Borel subsets of T of additive class α, with $0 < \alpha < \omega_1$ (ω_1 denotes the first uncountable ordinal), $\mathscr{G}$-measurability means being of lower class α (see [7, p. 401]). Finally, if $\mathscr{G}$ is a σ-algebra, we will say simply "measurability" instead of "$\mathscr{G}$-measurability."

From now on, T will indicate a non-empty set, (X, d) a metric space, and F a multifunction from $T \times X$ into X. Let us denote by $\mathscr{M}(T, X)$ the set of all (single-valued) functions from T into X. We will always consider $\mathscr{M}(T, X)$ endowed with the generalized metric ρ_d defined by

$$\rho_d(f, g) = \sup_{t \in T} d(f(t), g(t)) \qquad \text{for} \quad f, g \in \mathscr{M}(T, X).$$

Let $\mathscr{A}$ be a non-empty subset of $\mathscr{M}(T, X)$. We give the following

Definition 1.1. Let $f \in \mathscr{A}$. We say that f is an $\mathscr{A}$-fixed point of F if $f(t) \in F(t, f(t))$ for every $t \in T$.

Moreover, we recall the following definition (see [12, Definition 1.1]). Let $G: T \to 2^X$. We say that the multifunction G is $\mathscr{A}$-stable if the following two conditions are satisfied:

(I) there exists $f \in \mathscr{A}$ such that $f(t) \in G(t)$ for every $t \in T$;

(II) for every ε, $r \in \mathbb{R}^+$ and every $g \in \mathscr{A}$ such that $G(t) \cap B_d(g(t), r) \neq \varnothing$ for all $t \in T$, there exists $h \in \mathscr{A}$ such that $h(t) \in G(t) \cap B_d(g(t), r+\varepsilon)$ for all $t \in T$.

Finally, for every $t \in T$, we put $\Gamma_F(t) = \mathrm{Fix}(F(t, \cdot))$.

The following propositions will be useful in the sequel.

PROPOSITION 1.1. *Let $\mathscr{G} \subseteq 2^T \cup \{\varnothing\}$. Assume that either $\mathscr{G}$ is closed under arbitrary union or $\mathscr{G}$ is closed under countable union and X is separable. Let $\{\Phi_n\}_{n \in \mathbb{N}}$ be a sequence of $\mathscr{G}$-measurable multifunctions from T into X and let $\Phi\colon T \to 2^X$ be such that*

$$\lim_{n \to \infty} \sup_{t \in T} d_H(\Phi_n(t), \Phi(t)) = 0. \tag{1}$$

Then, Φ is $\mathscr{G}$-measurable.

Proof. Thanks to our first hypothesis, it suffices to prove that, for every $\varepsilon > 0$ and every $y \in X$, one has $\Phi^-(B_d(y, \varepsilon)) \in \mathscr{G}$. To this purpose, first of all we can construct, thanks to (1), a subsequence of $\{\Phi_n\}_{n \in \mathbb{N}}$, say $\{\Phi_{n_k}\}_{k \in \mathbb{N}}$, such that

$$\sup_{t \in T} d_H(\Phi_{n_k}(t), \Phi(t)) < 1/2k \qquad \text{for every } k \in \mathbb{N}. \tag{2}$$

Put $\mathbb{N}_\varepsilon = \{k \in \mathbb{N} : k > 1/2\varepsilon\}$. To prove our conclusion, it suffices to demonstrate the equality

$$\Phi^-(B_d(y, \varepsilon)) = \bigcup_{k \in \mathbb{N}_\varepsilon} \Phi^-_{n_k}\left(B_d\left(y, \varepsilon - \frac{1}{2k}\right)\right).$$

Then, let $t_0 \in \Phi^-(B_d(y, \varepsilon))$; hence, there exists $v_0 \in \Phi(t_0)$ such that $d(v_0, y) < \varepsilon$. Choose $k_0 \in \mathbb{N}$ such that $d(v_0, y) < \varepsilon - 1/k_0$. By (2), moreover, there exists $w_0 \in \Phi_{n_{k_0}}(t_0)$ such that $d(v_0, w_0) < 1/2k_0$. Hence, $d(w_0, y) < \varepsilon - 1/2k_0$ and so $t_0 \in \Phi^-_{n_{k_0}}(B_d(y, \varepsilon - 1/2k_0))$. Now, let $t_1 \in \Phi^-_{n_{k_1}}(B_d(y, \varepsilon - 1/2k_1))$ and let $w_1 \in \Phi_{n_{k_1}}(t_1)$ with $d(w_1, y) < \varepsilon - 1/2k_1$. By (2), there exists $v_1 \in \Phi(t_1)$ such that $d(v_1, w_1) < 1/2k_1$. Thus, $d(v_1, y) < \varepsilon$ and so $t_1 \in \Phi^-(B_d(y, \varepsilon))$. This completes the proof.

PROPOSITION 1.2. *Let $\mathscr{G} \subseteq 2^T \cup \{\varnothing\}$, (Y, d') be a metric space, $L \geqslant 0$, and $\Phi\colon T \times X \to 2^Y$ be such that one has $d'_H(\Phi(t, x), \Phi(t, y)) \leqslant Ld(x, y)$ for every $x, y \in X$, $t \in T$. Moreover, suppose that:*

(i) *there exists a dense subset D of X such that $F(\cdot, x)$ is $\mathscr{G}$-measurable for each $x \in D$;*

(ii) *$\mathscr{G}$ is a topology or $\mathscr{G}$ is closed under finite intersection and countable union and D is countable.*

Then, for every $\mathscr{G}$-measurable function $\varphi: T \to X$, the multifunction $\Phi(\cdot, \varphi(\cdot))$ is $\mathscr{G}$-measurable.

Proof. Fix φ. For every $k \in \mathbb{N}$, $t \in T$, $x \in X$, put

$$\Phi_k(t, x) = \Phi(t, B_d(x, 1/k) \cap D).$$

Fix $k \in \mathbb{N}$. It is easily seen that, for every open set $\Omega \subseteq X$, one has

$$\Phi_k^-(\Omega) = \bigcup_{y \in D} \left[\{t \in T: \Phi(t, y) \cap \Omega \neq \varnothing\} \times B_d\left(y, \frac{1}{k}\right)\right].$$

Hence we have

$$\begin{aligned}
&\{t \in T: \Phi_k(t, \varphi(t)) \cap \Omega \neq \varnothing\} \\
&\quad = \{t \in T: (t, \varphi(t)) \in \Phi_k^-(\Omega)\} \\
&\quad = \bigcup_{y \in D} \left[\{t \in T: \Phi(t, y) \cap \Omega \neq \varnothing\} \cap \varphi^{-1}\left(B_d\left(y, \frac{1}{k}\right)\right)\right].
\end{aligned}$$

Thanks to this equality and to hypotheses (i) and (ii) it follows that $\Phi_k(\cdot, \varphi(\cdot))$ is $\mathscr{G}$-measurable. Moreover, for every $(t, x) \in T \times X$, we have

$$d'_H(\Phi_k(t, x), \Phi(t, x)) \leqslant L/k.$$

Indeed, let $z \in \Phi_k(t, x)$, then there exists $y \in D$ such that $d(y, x) < 1/k$, $z \in \Phi(t, y)$. Hence, we have $d(z, \Phi(t, x)) \leqslant d'_H(\Phi(t, y), \Phi(t, x)) \leqslant Ld(x, y) < L/k$, and so $d'^*(\Phi_k(t, x), \Phi(t, x)) \leqslant L/k$. Now let $w \in \Phi(t, x)$, and let $y \in D \cap B_d(x, 1/k)$. We have $d'(w, \Phi_k(t, x)) \leqslant d'(w, \Phi(t, y)) \leqslant d'_H(\Phi(t, x), \Phi(t, y)) \leqslant L/k$. Thus, $d'^*(\Phi(t, x), \Phi_k(t, x)) \leqslant L/k$. Hence, we have

$$\lim_{k \to \infty} \sup_{t \in T} d'_H(\Phi_k(t, \varphi(t)), \Phi(t, \varphi(t))) = 0.$$

Our conclusion follows then from Proposition 1.1.

2. The Main Result

Our main result is the following

THEOREM 2.1. *Let (X, d) be a complete metric space, $\mathscr{A}$ a non-empty closed subset of $\mathscr{M}(T, X)$, and $F: T \times X \to 2^X$ a closed-valued multifunction.*

Suppose that $F(\cdot, \varphi(\cdot))$ is $\mathscr{A}$-stable for each $\varphi \in \mathscr{A}$, and that there exists $k \in [0, 1[$ such that $d_H(F(t, x), F(t, y)) \leqslant k\, d(x, y)$ for every $t \in T$, $x, y \in X$. Suppose moreover that there exists $\varphi_0 \in \mathscr{A}$ such that $\sup_{t \in T} d(\varphi_0(t), F(t, \varphi_0(t))) < +\infty$. Under such hypotheses, F admits an $\mathscr{A}$-fixed point φ^* such that $\rho_d(\varphi^*, \varphi_0) < +\infty$.

Proof. For every $\varphi \in \mathscr{A}$ put

$$G(\varphi) = \{\psi \in \mathscr{A}: \psi(t) \in F(t, \varphi(t)) \text{ for every } t \in T\}.$$

Thanks to the $\mathscr{A}$-stability of $F(\cdot, \varphi(\cdot))$, one has $G(\varphi) \neq \varnothing$. Let us prove that G is a contraction. Let $\varphi, \psi \in \mathscr{A}$, with $\rho_d(\varphi, \psi) < +\infty$. Fix $f \in G(\varphi)$. For every $t \in T$ we have

$$d(f(t), F(t, \psi(t))) \leqslant d_H(F(t, \varphi(t)), F(t, \psi(t))) \leqslant k\rho_d(\varphi, \psi).$$

Therefore, for every $\varepsilon > 0$ we have

$$F(t, \psi(t)) \cap B_d(f(t), k\rho_d(\varphi, \psi) + \varepsilon/2) \neq \varnothing.$$

Thanks to the $\mathscr{A}$-stability of $F(\cdot, \psi(\cdot))$ there exists $g \in \mathscr{A}$ such that $g(t) \in F(t, \psi(t)) \cap B_d(f(t), k\rho_d(\varphi, \psi) + \varepsilon)$ for every $t \in T$. Hence, $g \in G(\psi)$ and $\rho_d(f, g) \leqslant k\rho_d(\varphi, \psi) + \varepsilon$. Since ε is arbitrary, it follows that $\rho_d(f, G(\psi)) \leqslant k\rho_d(\varphi, \psi)$. This inequality holds for any $f \in G(\varphi)$, and so

$$\rho_d^*(G(\varphi), G(\psi)) \leqslant k\rho_d(\varphi, \psi).$$

Changing the roles of φ and ψ, likewise we obtain

$$\rho_d^*(G(\psi), G(\varphi)) \leqslant k\rho_d(\varphi, \psi).$$

Hence, G is a contraction. Moreover, since X is complete and $\mathscr{A}$ is closed in $\mathscr{M}(T, X)$, it follows that $\mathscr{A}$ is a complete generalized metric space. Now observe that, thanks to the $\mathscr{A}$-stability of the multifunction $F(\cdot, \varphi_0(\cdot))$, the condition $\sup_{t \in T} d(\varphi_0(t), F(t, \varphi_0(t))) < +\infty$ implies that $\rho_d(\varphi_0, G(\varphi_0)) < +\infty$. Therefore, by Corollary 1 of [1] we obtain a function $\varphi^* \in \mathscr{A}$ such that $\varphi^* \in G(\varphi^*)$. Moreover, by the proof of Theorem 1 of [1], we can derive that $\rho_d(\varphi^*, \varphi_0) < +\infty$. Thus, the proof is complete.

3. Some Consequences of Theorem 2.1

In this section we present some applications of Theorem 2.1. We begin with the following

THEOREM 3.1. *Let (X, d) be a separable and complete metric space, $(T, \mathscr{G})$ a measurable space, and $F\colon T \times X \to 2^X$ a closed-valued multifunction such that $F(\cdot, x)$ is measurable for every x in a countable dense subset D of X, and that $F(t, \cdot)$ is a multi-valued contraction for every $t \in T$. Then, for every $S \in \mathscr{G}$ and every measurable function $\psi\colon S \to X$ such that $\psi(t) \in F(t, \psi(t))$ for every $t \in S$, there exists a measurable function $\varphi\colon T \to X$ such that $\varphi(t) \in F(t, \varphi(t))$ for every $t \in T$, and that $\varphi_{|S} = \psi$.*

Proof. Let S and ψ be as in the statement. Define on T a real function L by putting

$$L(t) = \sup_{\substack{x, y \in D \\ x \neq y}} \frac{d_H(F(t, x), F(t, y))}{d(x, y)} \qquad (t \in T).$$

Observe that $L(t) < 1$ for every $t \in T$. Moreover, for fixed $x, y \in D$, thanks to Lemma 2.1 of [11], the real function $d_H(F(\cdot, x), F(\cdot, y))$ is measurable. Hence, the function L, as a supremum of a countable family of measurable functions, is measurable itself. Further, since D is dense in X, it is easily seen that the inequality

$$d_H(F(t, x), F(t, y)) \leqslant L(t)\, d(x, y)$$

holds for all $x, y \in X$, $t \in T$.

Now, choose a measurable function $\varphi_0\colon T \to X$ with $\varphi_{0|S} = \psi$.

Let $\{a_n\}_{n \in \mathbb{N}}$ be an increasing and unbounded sequence of positive real numbers such that, for every $n \in \mathbb{N}$, the set

$$T_n = \{t \in T\colon d(\varphi_0(t), F(t, \varphi_0(t))) < a_n\} \cap \{t \in T\colon L(t) < 1 - 1/a_n\}$$

is non-empty. Observe that T_n is measurable by Proposition 1.2 and by Lemma 2.1 of [11]. Now, for each $n \in \mathbb{N}$, denote by $\mathscr{A}_n$ the family of all measurable functions from T_n into X and, finally, put

$$\mathscr{B}_n = \{\varphi \in \mathscr{A}_n\colon \varphi_{|S \cap T_n} = \psi\}.$$

We want to prove that, for each $\varphi \in \mathscr{B}_n$, the multifunction $F(\cdot, \varphi(\cdot))$ is $\mathscr{B}_n$-stable. To this end, fix φ in $\mathscr{B}_n$ and introduce the multifunction $G\colon T_n \to 2^X$ defined by

$$G(t) = \begin{cases} F(t, \varphi(t)) & \text{if } t \in T_n \setminus S \\ \{\psi(t)\} & \text{if } t \in T_n \cap S. \end{cases}$$

G is measurable because, for each subset Ω of X, we have

$$G^-(\Omega) = \{t \in T_n \setminus S\colon F(t, \varphi(t)) \cap \Omega \neq \varnothing\} \cup (\psi^{-1}(\Omega) \cap T_n)$$

and, by Proposition 1.2, $F(\cdot, \varphi(\cdot))$ is measurable. Then, by Proposition 3.2 (part (β)) of [12], G is $\mathscr{A}_n$-stable. It is easy to see that the $\mathscr{A}_n$-stability of G and the $\mathscr{B}_n$-stability of $F(\cdot, \varphi(\cdot))$ are equivalent.

Finally, observe that $\mathscr{B}_n$ is closed in $(\mathscr{M}(T_n, X), \rho_d)$ and that $\sup_{t \in T_n} d(\varphi_0(t), F(t, \varphi_0(t))) \leqslant a_n$ and $d_H(F(t, x), F(t, y)) \leqslant (1 - 1/a_n)\, d(x, y)$ for all $t \in T_n$, $x, y \in X$. Therefore, by Theorem 2.1, there exists a function $\varphi_n \in \mathscr{B}_n$ such that $\varphi_n(t) \in F(t, \varphi_n(t))$ for each $t \in T_n$. For each $n \in \mathbb{N}$, put $T_n^* = T_n \setminus T_{n-1}$ $(T_0 = \varnothing)$. Observe that the sets T_n^* are measurable, pairwise disjoint, and $\bigcup_{n \in \mathbb{N}} T_n^* = T$.

Then, the function $\varphi\colon T \to X$ defined by putting

$$\varphi(t) = \varphi_n(t) \qquad \text{if} \quad t \in T_n^*,\ n \in \mathbb{N}$$

is the required function.

Remark 3.1. Theorem 3.1 is an improvement of the Theorem on p. 88 of [5].

Now, we state the following.

THEOREM 3.2. *Let T be a perfectly normal topological space, (X, d) a complete, separable, and bounded metric space, and $F\colon T \times X \to 2^X$ a closed-valued multifunction such that $F(\cdot, x)$ is of lower class α $(0 < \alpha < \omega_1)$ for every x in a countable dense subset of X.*

Suppose moreover that there exists $k \in [0, 1[$ such that $d_H(F(t, x), F(t, y)) \leqslant k d(x, y)$ for all $t \in T$, $x, y \in X$. Then, for every ambiguous set $S \subseteq T$ of class α and every function $\psi\colon S \to X$ of class α such that $\psi(t) \in F(t, \psi(t))$ for all $t \in S$, there exists a function $\varphi\colon T \to X$ of class α such that $\varphi(t) \in F(t, \varphi(t))$ for all $t \in T$ and $\varphi_{|S} = \psi$.

Proof. Let S an ψ be as in the statement. Denote by $\mathscr{A}$ the family of all functions of class α from T into X, and put $\mathscr{B} = \{\varphi \in \mathscr{A} : \varphi_{|S} = \psi\}$. Choose $\varphi_0 \in \mathscr{B}$. Now, let us prove that, for every $\varphi \in \mathscr{B}$, the multifunction $F(\cdot, \varphi(\cdot))$ is $\mathscr{B}$-stable. To this end, consider the multifunction $G\colon T \to 2^X$ defined by

$$G(t) = \begin{cases} F(t, \varphi(t)) & \text{if} \quad t \in T \setminus S \\ \{\psi(t)\} & \text{if} \quad t \in S. \end{cases}$$

By Proposition 1.2, the multifunction $F(\cdot, \varphi(\cdot))$ is of lower class α, then, since S is ambigous of class α, also G is of lower class α. Hence, by Proposition 3.2 (part (β)) of [12] (see also p. 401 of [7]), G is $\mathscr{A}$-stable, and so $F(\cdot, \varphi(\cdot))$ is $\mathscr{B}$-stable. Now, observe that, by Proposition 1.1, $\mathscr{B}$ is closed in $(\mathscr{M}(T, X), \rho_d)$. Moreover, we have $\sup_{t \in T} d(\varphi_0(t), F(t, \varphi_0(t))) < +\infty$. Therefore, by Theorem 2.1, there exists a $\mathscr{B}$-fixed point φ of F, and φ is the required function.

Another consequence of Theorem 2.1 is the following

THEOREM 3.3. *Let T be a paracompact topological space, X a closed convex subset of a Banach space $(E, \|\cdot\|)$, Z a subset of T, with* $\dim_T(Z) \leqslant 0$,[1] *and $F\colon T \times X \to 2^X$ a closed-valued multifunction such that $F(t, x)$ is convex for every $(t, x) \in (T \backslash Z) \times X$. Suppose moreover that $F(\cdot, x)$ is lower semicontinuous for every x in a dense subset of X and that there exists a continuous function $k\colon T \to [0, 1[$ such that $d_H(F(t, x), F(t, y)) \leqslant k(t)\|x - y\|$ for each $x, y \in X$, $t \in T$, where d is the metric on X induced by the norm $\|\cdot\|$. Then, for every closed subset S of T and every continuous function $\psi\colon S \to X$ such that $\psi(t) \in F(t, \psi(t))$ for every $t \in S$, there exists a continuous function $\varphi\colon T \to X$ such that $\varphi(t) \in F(t, \varphi(t))$ for every $t \in T$ and that $\varphi_{|S} = \psi$. If, in addition, $Z = T$, we can suppose that X is only closed.*

Proof. Let S and ψ be as in the statement. By Theorem 1.4 of [9], we can choose a continuous function $\tilde{\varphi}\colon T \to X$ such that $\tilde{\varphi}_{|S} = \psi$. Observe that the multifunction $F(\cdot, \tilde{\varphi}(\cdot))$ is closed-valued and, by Proposition 1.2, it is lower semicontinuous. Moreover, if $t \in T \backslash Z$, $F(t, \tilde{\varphi}(t))$ is convex. Then, by Theorem 7.1 of [9], there exists a continuous function $\varphi_0\colon T \to X$ such that $\varphi_0(t) \in F(t, \tilde{\varphi}(t))$ for every $t \in T$ and $\varphi_{0|S} = \psi$. Observe that, for every $t \in T$, one has

$$\begin{aligned} d(\varphi_0(t), F(t, \varphi_0(t))) &\leqslant d_H(F(t, \tilde{\varphi}(t)), F(t, \varphi_0(t))) \\ &\leqslant k(t)\|\tilde{\varphi}(t) - \varphi_0(t)\|. \end{aligned}$$

Let $\{a_n\}_{n \in \mathbb{N}}$ be an increasing and unbounded sequence of positive real numbers such that, for every $n \in \mathbb{N}$, the set

$$T_n = \{t \in T\colon \|\tilde{\varphi}(t)\| + \|\varphi_0(t)\| < a_n\} \cap \{t \in T\colon k(t) < 1 - 1/a_n\}$$

is non-empty. Then, $\{T_n\}_{n \in \mathbb{N}}$ is an increasing sequence of open sets and $\bigcup_{n \in \mathbb{N}} T_n = T$. We prove that there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ of continuous and bounded functions from T into X such that, for each $n \in \mathbb{N}$, one has

$$\varphi_n(t) \in F(t, \varphi_n(t)) \qquad \text{for every } t \in \bar{T}_n; \tag{3}$$

$$\varphi_{n|\bar{T}_{n-1}} = \varphi_{n-1} \qquad (\text{we put } T_0 = \varnothing); \tag{4}$$

$$\varphi_{n|S \cap \bar{T}_n} = \psi. \tag{5}$$

We construct such a sequence by induction. Let us construct φ_1. Denote by $\mathscr{A}_1$ the family of all continuous functions from $\bar{T}_1$ into X, and put

[1] $\dim_T(Z) \leqslant 0$ means that $\dim(U) \leqslant 0$ for every $U \subseteq Z$ which is closed in T, where $\dim(U)$ denotes the covering dimension of U.

$\mathscr{B}_1 = \{\varphi \in \mathscr{A}_1 : \varphi_{|S \cap \bar{T}_1} = \psi\}$. For each $\varphi \in \mathscr{B}_1$, the multifunction $F(\cdot, \varphi(\cdot))$ is $\mathscr{B}_1$-stable. Indeed, consider the multifunction $G_1 : \bar{T}_1 \to 2^X$ defined by putting

$$G_1(t) = \begin{cases} F(t, \varphi(t)) & \text{if } t \in \bar{T}_1 \backslash S \\ \{\varphi_0(t)\} & \text{if } t \in \bar{T}_1 \cap S. \end{cases}$$

By Example 1.3* of [8] and Proposition 1.2, G_1 is lower semicontinuous, then, by Proposition 3.2 (part (α)) of [12], it is $\mathscr{A}_1$-stable. It is easy to see that the $\mathscr{A}_1$-stability of G_1 and the $\mathscr{B}_1$-stability of $F(\cdot, \varphi(\cdot))$ are equivalent. Now, observe that $\mathscr{B}_1$ is closed in $(\mathscr{M}(\bar{T}_1, X), \rho_d)$, that $\varphi_{0|\bar{T}_1} \in \mathscr{B}_1$, that $d_H(F(t, x), F(t, y)) \leqslant (1 - 1/a_1)\|x - y\|$ for every $t \in \bar{T}_1$, $x, y \in X$, and that $\sup_{t \in \bar{T}_1} d(\varphi_0(t), F(t, \varphi_0(t))) \leqslant a_1$. Therefore, by Theorem 2.1, there exists a continuous function $\varphi_1^* : \bar{T}_1 \to X$ such that $\varphi^*_{1|S \cap \bar{T}_1} = \psi$, $\varphi_1^* \in F(t, \varphi_1^*(t))$ for each $t \in \bar{T}_1$, and $\sup_{t \in \bar{T}_1} \|\varphi_1^*(t) - \varphi_0(t)\| < +\infty$. Therefore, φ_1^* is bounded on $\bar{T}_1$. Then, by Theorem 1.4 of [9], it is possible to extend φ_1^* to a continuous and bounded function φ_1 on T, which satisfies (3), (4), (5) for $n = 1$. Suppose now that bounded and continuous functions $\varphi_1, \ldots, \varphi_h$ from T into X, satisfying (3), (4), (5) for $n = 1, 2, \ldots, h$, have been constructed. Let us construct φ_{h+1}. To this end, denote by $\mathscr{A}_{h+1}$ the family of all continuous functions from $\bar{T}_{h+1}$ into X and put $\mathscr{B}_{h+1} = \{\varphi \in \mathscr{A}_{h+1} : \varphi_{|\bar{T}_h} = \varphi_h\}$. Fix $\varphi \in \mathscr{A}_{h+1}$ and consider the multifunction $G_{h+1} : \bar{T}_{h+1} \to 2^X$ defined by putting

$$G_{h+1}(t) = \begin{cases} F(t, \varphi(t)) & \text{if } t \in \bar{T}_{h+1} \backslash \bar{T}_h \\ \{\varphi_h(t)\} & \text{if } t \in \bar{T}_h. \end{cases}$$

As in the case $n = 1$ it is seen that G_{h+1} is $\mathscr{A}_{h+1}$-stable and hence $F(\cdot, \varphi(\cdot))$ is $\mathscr{B}_{h+1}$-stable. Observe now that $\mathscr{B}_{h+1}$ is closed in $(\mathscr{M}(\bar{T}_{h+1}, X), \rho_d)$, and that $\varphi_{h|\bar{T}_{h+1}} \in \mathscr{B}_{h+1}$. Moreover, for all $t \in \bar{T}_{h+1}$, one has $d(\varphi_h(t), F(t, \varphi_h(t))) \leqslant \|\varphi_h(t) - \varphi_0(t)\| + d(\varphi_0(t), F(t, \varphi_0(t))) + d^*(F(t, \varphi_0(t)), F(t, \varphi_h(t))) \leqslant (2 - 1/a_{h+1})\|\varphi_h(t) - \varphi_0(t)\| + a_{h+1}$, hence $\sup_{t \in \bar{T}_{h+1}} d(\varphi_h(t), F(t, \varphi_h(t))) < +\infty$. Finally, one has $d_H(F(t, x), F(t, y)) \leqslant (1 - 1/a_{h+1})\|x - y\|$ for every $t \in \bar{T}_{h+1}$, $x, y \in X$. Therefore, by Theorem 2.1 there exists a continuous function $\varphi^*_{h+1} : \bar{T}_{h+1} \to X$ such that $\varphi^*_{h+1|\bar{T}_h} = \varphi_h$ and $\varphi^*_{h+1}(t) \in F(t, \varphi^*_{h+1}(t))$ for every $t \in \bar{T}_{h+1}$. Now, extend φ^*_{h+1} to a continuous and bounded function $\varphi_{h+1} : T \to X$. The function φ_{h+1} satisfies (3), (4), (5) for $n = h + 1$. So, the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ has been constructed. Now, define $\varphi : T \to X$ by putting

$$\varphi(t) = \varphi_n(t) \qquad \text{if } \; t \in T_n \backslash T_{n-1}, \quad n \in \mathbb{N}.$$

Of course, $\varphi(t) \in F(t, \varphi(t))$ for every $t \in T$ and $\varphi_{|S} = \psi$. Finally, to prove the continuity of φ, it suffices to observe that, thanks to (4), we have $\varphi_{|T_n} = \varphi_n$ for all $n \in \mathbb{N}$.

Remark 3.2. Since every complete metric space can be isometrically embedded in a Banach space as a closed set, when $T = Z$, in Theorem 3.3, we can suppose that (X, d) is an arbitrary complete metric space.

Remark 3.3. Theorem 3.3 improves in several directions the Proposition on p. 768 of [15]. It is also an extension of Theorem 1 of [14].

Before stating the next result, we recall the following definition (see [6, p. 335]).

Let (M, d) be a metric space and $\{A_n\}_{n \in \mathbb{N}}$ a sequence of non-empty subsets of M. The set

$$\operatorname*{Li}_{n \to \infty} A_n = \{x \in M \colon \lim_{n \to \infty} d(x, A_n) = 0\}$$

is called the topological lower limit of $\{A_n\}_{n \in \mathbb{N}}$.

The following result is a consequence of Theorem 3.3.

THEOREM 3.4. *Let (X, d) be a complete metric space and let Φ, $\Phi_1, \Phi_2, \ldots$ be a sequence of closed-valued multifunctions from X into itself, which are multi-valued contractions with the same constant k. Suppose that there exists a dense subset D of X, such that*

$$\Phi(x) \subseteq \operatorname*{Li}_{n \to \infty} \Phi_n(x) \qquad \textit{for every} \quad x \in D. \tag{6}$$

Then, $\operatorname{Fix}(\Phi) \subseteq \operatorname{Li}_{n \to \infty} \operatorname{Fix}(\Phi_n)$.

Proof. Let T be the one-point compactification of $\mathbb{N}$ with the usual topology. Define a multifunction $F \colon T \times X \to 2^X$ by putting

$$F(t, x) = \begin{cases} \Phi_n(x) & \text{if} \quad t = n,\, n \in \mathbb{N},\, x \in X \\ \Phi(x) & \text{if} \quad t = \infty,\, x \in X. \end{cases}$$

F satisfies the hypotheses of Theorem 3.3. In particular, relation (6) is equivalent to the lower semicontinuity of $F(\cdot, x)$ at the point $t = \infty$, for $x \in D$. Then, choose $x_0 \in \operatorname{Fix}(\Phi)$ and put $S = \{\infty\}$, $\psi(\infty) = x_0$. By Theorem 3.3 (see Remark 3.2 and take into account that T is zero-dimensional), there exists a continuous function $\varphi \colon T \to X$ such that $\varphi(n) \in F(n, \varphi(n))$ for all $n \in \mathbb{N}$ and $\varphi(\infty) = x_0$. Thus, if we put $x_n = \varphi(n)$, $n \in \mathbb{N}$, we have $x_n \in \operatorname{Fix}(\Phi_n)$. Moreover, by the continuity of φ, we have $\lim_{n \to \infty} x_n = x_0$, hence $x_0 \in \operatorname{Li}_{n \to \infty} \operatorname{Fix}(\Phi_n)$.

Remark 3.4. Observe that Theorem 3.4 extends to multifunctions Proposition 1 of [10].

The following result is the random version of Theorem 3.4.

THEOREM 3.5. *Let $(T, \mathscr{G}, \mu)$ be a complete σ-finite measure space, (X, d) a complete and separable metric space, and $F, F_1, F_2, \ldots$ a sequence of closed-valued multifunctions from $T \times X$ into X which are measurable with respect to t for every $x \in X$. Moreover, suppose that, for every $t \in T$, one has:*

(a) *$F(t, \cdot)$, $F_1(t, \cdot), \ldots$ are multi-valued contractions with the same Lipschitz constant k;*

(b) *there exists a dense subset D_t of X such that $F(t, x) \subseteq \mathrm{Li}_{n \to \infty} F_n(t, x)$ for every $x \in D_t$.*

Then, for every measurable function $\varphi: T \to X$ such that $\varphi(t) \in F(t, \varphi(t))$ for every $t \in T$, there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ of measurable functions from T into X, such that

$$\lim_{n \to \infty} \varphi_n(t) = \varphi(t) \qquad \text{for every} \quad t \in T;$$

$$\varphi_n(t) \in F_n(t, \varphi_n(t)) \qquad \text{for every } t \in T, n \in \mathbb{N}.$$

Proof. Consider the multifunctions $\Gamma_F, \Gamma_{F_1}, \Gamma_{F_2}, \ldots$, already defined in Section 1. Observe that they are closed-valued. Let us prove that Γ_F is measurable. To this end, introduce a real function h on $T \times X$ by putting

$$h(t, x) = d(x, F(t, x)), \qquad t \in T, \quad x \in X,$$

and observe that

$$\mathrm{gr}(\Gamma_F) = \{(t, x) \in T \times X : h(t, x) = 0\}.$$

By Theorem 3.3 of [3], $h(\cdot, x)$ is measurable for every $x \in X$ and by Proposition 1.1 of [13], $h(t, \cdot)$ is continuous for every $t \in T$. Then, taking into account Theorems 3.4 and 6.4 of [3], we realize that Γ_F is measurable. Similarly one proves the measurability of Γ_{F_n} for each $n \in \mathbb{N}$. Now, observe that hypothesis (b), jointly with Theorem 3.4, implies that $\lim_{n \to \infty} d(x, \Gamma_{F_n}(t)) = 0$ for every $t \in T$, $x \in \Gamma_F(t)$. Therefore, our conclusion follows from Theorem 1.1 of [11].

Now we state the following result, which derives from Theorem 3.4.

THEOREM 3.6. *Let T be a first-countable topological space, (X, d) a complete metric space, and F a closed-valued multifunction from $T \times X$ into X such that:*

(c) *$F(\cdot, x)$ is lower semicontinuous for every x in a dense subset D of X;*

(d) *there exists an upper semicontinuous function* $k: T \to [0, 1[$ *such that* $d_H(F(t, x), F(t, y)) \leqslant k(t)\, d(x, y)$ *for every* $t \in T$, $x, y \in X$.

Then, the multifunction Γ_F *is lower semicontinuous.*

Proof. Let $t_0 \in T$. By Proposition 2.1 of [4], which is valid also if T is first-countable, to prove the lower semicontinuity of Γ_F at t_0, it suffices to show that

$$\Gamma_F(t_0) \subseteq \operatorname*{Li}_{n \to \infty} \Gamma_F(t_n) \tag{7}$$

for every sequence $\{t_n\}_{n \in \mathbb{N}}$ of points of T convergent to t_0. To this purpose, consider the multifunctions from X into itself defined by putting

$$\Phi_n(x) = F(t_n, x) \qquad \text{for every } x \in X, n \in \mathbb{N};$$

$$\Phi(x) = F(t_0, x) \qquad \text{for every } x \in X.$$

Observe that, by the upper semicontinuity of the function k, all these multifunctions are contractions with the same Lipschitz constant $k' = \max(k(t_0), \sup_{n \in \mathbb{N}} k(t_n)\} < 1$. Moreover, by (c), one has $\Phi(x) \subseteq \operatorname{Li}_{n \to \infty} \Phi_n(x)$ for every $x \in D$. Then, by Theorem 3.4, one has $\operatorname{Fix}(\Phi) \subseteq \operatorname{Li}_{n \to \infty} \operatorname{Fix}(\Phi_n)$, that is, (7).

Taking into account Example 1.3* of [8], Theorem 2 of [2], and Theorem 3.6, we obtain the following result.

THEOREM 3.7. *Let the hypotheses of Theorem* 3.6 *be satisfied. In addition, let* T *be also paracompact and perfectly normal. Then, for every closed subset* S *of* T *and every continuous function* $\psi: S \to X$ *such that* $\psi(t) \in F(t, \psi(t))$ *for every* $t \in S$, *there exists a function* $\varphi: T \to X$, *of the first Baire class, such that* $\varphi(t) \in F(t, \varphi(t))$ *for every* $t \in T$ *and* $\varphi_{|S} = \psi$.

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