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Categorical Hopf kernels and representations of semisimple Hopf algebras

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ABSTRACT

It is shown that in the category of semisimple Hopf algebras the categorical Hopf kernels introduced by Andruskiewitsch and Devoto (1995) in [1] coincide with the kernels of representations introduced by the present author in 2009 [2]. New results concerning the normality of these kernels are also presented. It is proven that the Hopf algebra property to have all representation kernels as normal Hopf subalgebras is a self-dual property.

1. Introduction

Semisimple Hopf algebras were intensively studied in the last twenty years. During this study many concepts and properties from finite groups were extended to the more general settings of semisimple Hopf algebras. An example of such a concept is that of a kernel of a semisimple Hopf algebra representation recently introduced in [2]. This extends the notion of kernel of a finite group representation.

Another example of such a concept is that of Hopf kernel of a Hopf algebra map. In order to work inside the category of Hopf algebras Andruskiewitsch and Devoto introduced in [1] the notion of a categorical Hopf kernel of a morphism between Hopf algebras. In this paper we prove that as in the group situation, the two notions of kernels coincide. It will be proven in Proposition 3.2 that the Hopf kernel of a morphism between semisimple Hopf algebras is the kernel of a certain associated repre-

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sentation. Theorem 3.4 shows that the converse of this fact is also true, any kernel of a representation is also the categorical Hopf kernel of an associated Hopf algebra projection.

Although many properties of kernels were transferred from groups to this more general setting of semisimple Hopf algebras there are still some unanswered questions in this direction. For example the normality of kernels of representations was proven in [2] with an additional assumption, that of centrality of its character in the dual Hopf algebra. It is not yet known if this additional assumption is really necessary. In this paper we study Hopf algebras where all these kernels are normal Hopf subalgebras. We say that a Hopf algebra has property (N) if any kernel H_χ is a normal Hopf subalgebra of H for any $\chi \in \mathrm{Irr}(H)$. It will be shown in Theorem 4.15 that property (N) is a self-dual notion, H has property (N) if and only if H^* has property (N). We also give new information on these kernels in terms of the central characters described in [15]. Besides group algebras, it follows from Theorem 5.7 of [3] that Drinfeld doubles of finite groups also have property (N).

The paper is organized as follows. The second section recalls the basic properties of kernels Hopf algebra representations. Section 3 shows the coincidence between Hopf kernels of morphisms and kernels of Hopf algebra representations. Section 4 studies the kernels of Hopf algebras with property (N). Theorem 4.15 shows that (N) is a self-dual Hopf algebra property. Proposition 4.5 gives a description of kernels in terms of some minimal normal Hopf subalgebras of H.

We work over the base field $\mathbb{C}$ and all Hopf algebra notations are those from [7]. We drop the sigma symbol in Sweedler's notations for comultiplication.

2. Preliminaries

Through all this paper H is a finite dimensional semisimple Hopf algebra over $\mathbb{C}$. It follows that H is also cosemisimple and $S^2 = \mathrm{id}_H$. The set of irreducible characters of H is denoted by $\mathrm{Irr}(H)$ and this is a $\mathbb{C}$-basis of the character algebra $C(H) \subset H^*$. Moreover $C(H)$ is a semisimple algebra [16]. On $C(H)$ there is also an involution "$*$" defined as follows. For $\chi \in \mathrm{Irr}(H)$, an irreducible character of H, χ^* denotes the character of the dual module M^*.

To any irreducible character $d \in \mathrm{Irr}(H^*)$ is associated a simple comatrix coalgebra $C = \mathbb{C}\langle x_{ij}\rangle_{1 \leqslant i,j \leqslant q} \subset H$ as in [5].

If Λ is the idempotent integral of H (with $\epsilon(\Lambda) = 1$) it follows that $\dim_{\mathbb{C}}(H)\Lambda$ is the regular character of H^*, that is:

$$\dim_{\mathbb{C}}(H)\Lambda = \sum_{d \in \mathrm{Irr}(H^*)} \epsilon(d)d. \tag{2.1}$$

Let H be a semisimple Hopf algebra over $\mathbb{C}$ and M be an H-module affording the character χ. Proposition 1.2 from [2] shows that $|\chi(d)| \leqslant \epsilon(d)\chi(1)$ for all $d \in \mathrm{Irr}(H^*)$. In fact the equivalence of the assertions of the next proposition also follows from the same Proposition 1.2 of [2].

Proposition 2.2. *(See also [2, Remark 1.3].) Let H be a semisimple Hopf algebra over $\mathbb{C}$ and M be an H-module affording the character χ. Then the following are equivalent for $d \in \mathrm{Irr}(H^*)$:*

(1) $\chi(d) = \epsilon(d)\chi(1)$.
(2) $\chi(x_{ij}) = \delta_{ij}\chi(1)$ *for all i, j.*
(3) $dm = \epsilon(d)m$ *for all $m \in M$.*
(4) $x_{ij}m = \delta_{ij}m$ *for all i, j and $m \in M$.*

Recall that a subset $X \subset \mathrm{Irr}(H^*)$ is closed under multiplication if for every $\chi, \mu \in X$ in the decomposition of $\chi\mu = \sum_{\gamma \in \mathrm{Irr}(H^*)} m_\gamma \gamma$ one has $\gamma \in X$ if $m_\gamma \neq 0$. Also, a subset $X \subset \mathrm{Irr}(H^*)$ is closed under "$*$" if $x^* \in X$ for all $x \in X$.

If $X \subset \mathrm{Irr}(H^*)$ is closed under multiplication and "$*$" then it generates a Hopf subalgebra of H denoted by H_X (see Theorem 6 of [9] or page 23 of [8]). One has $H_X = \bigoplus_{d \in X} C_d$. Alternatively, H_X is the smallest Hopf subalgebra of H containing the set X.

The kernel H_M (or H_χ) is defined as follows (see item 2 of Remark 1.5 from [2]). Let $\ker_H(\chi)$ be the set of all irreducible characters $d \in \mathrm{Irr}(H^*)$ which satisfy the equivalent conditions above. It can be proven that this set is closed under multiplication and "$*$" and therefore it generates a Hopf subalgebra H_M (or H_χ) of H which is called the kernel of the representation M.

Remark 2.3.

(1) For later use let us notice that $\ker_H(\chi) \subset \ker_H(\chi^n)$ for all $n \geqslant 0$. This is item 1 of Remark 1.5 from [2]. It follows that $\bigcap_{n\geqslant 0} \ker_H(\chi^n) = \ker_H(\chi)$ which can also be written as $\bigcap_{n\geqslant 0} H_{M^{\otimes n}} = H_M$.
(2) We also need the following result proven in [10]. Suppose that d is a character of H^* and χ a character of H. Then $\chi(d^*) = \overline{\chi(d)}$, the complex conjugate of $\chi(d)$.
(3) Suppose that $\chi_i \in \mathrm{Irr}(H)$ and $\chi = \sum_{i=1}^r m_i \chi_i$ where $m_i > 0$. Then it is easy to see from the definition that

$$\ker_H(\chi) = \bigcap_{i=1}^{r} \ker_H(\chi_i).$$

The above equation can also be written as $H_\chi = \bigcap_{i=1}^r H_{\chi_i}$.

Proposition 2.4. *Let H be a semisimple Hopf algebra over $\mathbb{C}$ and M be a representation of H with character χ. Consider the subalgebra of H given by*

$$\mathcal{S}_M = \{h \in H \mid hm = \epsilon(h)m \text{ for all } m \in M\}.$$

Then H_M is the largest Hopf subalgebra of H contained in $\mathcal{S}_M$.

Proof. By the definition of the kernel it is clear that $H_M \subset \mathcal{S}_M$. Item (2) of Remark 2.3 implies that if a simple subcoalgebra C of H satisfies $C \subset \mathcal{S}_M$ then $S(C) \subset \mathcal{S}_M$. Previous proposition also shows that if a subcoalgebra C is included in $\mathcal{S}_M$ then C is also included in H_M. It is easy to see that if C and D are two subcoalgebras included in $\mathcal{S}_M$ then the product coalgebra CD is also included in H_M. It follows that the largest Hopf subalgebra included in $\mathcal{S}_M$ is the sum of all subcoalgebras of H that are included in $\mathcal{S}_M$. Therefore this Hopf subalgebra is included in H_M. Thus this largest Hopf subalgebra coincides with H_M. □

Remark 2.5. Instead of using the second item of Remark 2.3 in the proof of above proposition one can use the fact that any bialgebra of a finite dimensional Hopf algebra is a Hopf subalgebra (see Lemma 6 of [11] or Lemma 1 of [12]).

Corollary 2.6. *Let H be a semisimple Hopf algebra over $\mathbb{C}$ and M be a representation of H. Then H_M is the largest Hopf subalgebra K of H such that $HK^+H \subset \mathrm{Ann}_H(M)$.*

Proof. It is easy to see that for any Hopf subalgebra K of H one has $K^+ \subset \mathrm{Ann}_H(M)$ if and only if $K \subset \mathcal{S}_M$. Then one can apply the previous proposition. □

Remark 2.7. Let K be a normal Hopf subalgebra of H and $L := H//K$ be the quotient Hopf subalgebra. Recall that if K is a normal Hopf subalgebra then HK^+ is a Hopf ideal and by the definition of the quotient Hopf algebra one has $L = H/HK^+$. From the previous corollary it follows that $\mathrm{Irr}(L) = \{\chi \in \mathrm{Irr}(H) \mid H_\chi \supset K\}$.

3. Kernels of Hopf algebra representations and Hopf algebra maps

The coincidence between kernels of Hopf algebra representations and categorical Hopf kernels of Hopf algebra maps is proven in this section.

3.1. The Hopf kernel of a Hopf algebra map

In this subsection it will be shown that any Hopf kernel is the kernel of a representation. The converse of this fact will be proven in Theorem 3.4.

Recall the Hopf kernel of a Hopf algebra map defined [13]. It also appeared in [1]. If $f : A \to B$ is a Hopf algebra map then the Hopf kernel of f is defined as follows:

$$\mathrm{HKer}(f) = \left\{a \in A \;\middle|\; a_1 \otimes f(a_2) \otimes a_3 = a_1 \otimes f(1) \otimes a_2\right\}. \tag{3.1}$$

It is easy to see that the Hopf kernel is a Hopf subalgebra of A [1].

Proposition 3.2. *Let I be a Hopf ideal of H and $\pi : H \to H/I$ be the canonical Hopf projection. Regarding $M := H/I$ as an H-module via π it follows that*

$$\mathrm{HKer}(\pi) = H_M.$$

Proof. It is straightforward to verify that

$$\mathcal{S}_M = \left\{h \in H \;\middle|\; \pi(h) = \epsilon(h)1\right\}.$$

Then it is easy to see that $H_M \subset \mathrm{HKer}(\pi)$. Indeed, if $h \in H_M$ then $\sum h_1 \otimes \pi(h_2) \otimes h_3 = \sum h_1 \otimes \epsilon(h_2)1 \otimes h_3 = \sum h_1 \otimes 1 \otimes h_2$ since $\pi(h) = \epsilon(h)1$ for all $h \in H_M$.

One can also see that $\mathrm{HKer}(\pi) \subset H_M$. Indeed if $h \in \mathrm{HKer}(\pi)$ then $\sum h_1 \otimes \pi(h_2) \otimes h_3 = \sum h_1 \otimes 1 \otimes h_2$. Applying $\epsilon \otimes \mathrm{id} \otimes \epsilon$ to this identity it follows that $\pi(h) = \epsilon(h)1$ and therefore $\mathrm{HKer}(\pi) \subset \mathcal{S}_M$. Since $\mathrm{HKer}(\pi)$ is a Hopf subalgebra it follows from Proposition 2.4 that $\mathrm{HKer}(\pi) \subset H_M$. Thus $\mathrm{HKer}(\pi) = H_M$. □

3.2. Description of the kernel of a representation as the kernel of a Hopf algebra map

In this subsection it will be shown that the kernel of a representation M of H coincides with the categorical kernel of a certain Hopf algebra map $\pi : H \to B$.

Let M be an H-module with character χ. Consider

$$I_M := \bigcap_{n \geqslant 0} \mathrm{Ann}_H\left(M^{\otimes n}\right).$$

Then by Proposition 1 of [11] and Lemma 5.1 of [4] it follows that I_M is a Hopf ideal and it is the largest Hopf ideal contained in the annihilator $\mathrm{Ann}_H(M)$ of M. Therefore $B := H/I_M$ is a Hopf algebra and one has a canonical projection of Hopf algebras

$$\pi : H \to B.$$

Lemma 3.3. *Let H be a semisimple Hopf algebra and M be an H-module. Using the above notations it follows that $H_M = H_B$ where B is regarded as H-module via the Hopf algebra map π.*

Proof. The representations of B are exactly those representations of H that are constituents of a tensor power $M^{\otimes n}$ of M. Thus one has $H_B = \bigcap_{n \geqslant 0} H_{M^{\otimes n}}$ by item (3) of Remark 2.3. Also item (1) of the same Remark 2.3 implies that $H_M = H_B$. □

The next theorem gives the characterization of the kernel of a representation as a categorical Hopf kernel.

Theorem 3.4. *Let H be a semisimple Hopf algebra and M be a representation of H. Let $\pi : H \to B$ be as above the canonical Hopf projection. Then*

$$H_M = \mathrm{HKer}(\pi).$$

Proof. By previous lemma one has that $H_M = H_B$. The equality $\mathrm{HKer}(\pi) = H_B$ follows from Proposition 3.2. □

4. Hopf algebras with all kernels normal

In this section we describe some kernel properties of Hopf subalgebras with all kernels H_χ normal Hopf subalgebras. It will also be shown that this property of all normal kernels is a self-dual property.

4.1. Central characters in the dual Hopf algebra

Let H be a finite dimensional semisimple Hopf algebra over $\mathbb{C}$ and Λ_H be its idempotent integral. Define a central subalgebra of H by $\hat{\mathrm{Z}}(H) := \mathrm{Z}(H) \cap C(H^*)$. It is the subalgebra of H^*-characters which are central in H. Let $\hat{\mathrm{Z}}(H^*) := \mathrm{Z}(H^*) \cap C(H)$ be the dual concept, the subalgebra of H-characters which are central in H^*.

Let $\phi : H^* \to H$ given by $f \mapsto f \rightharpoonup \Lambda_H$ where $f \rightharpoonup \Lambda_H = f(S(\Lambda_{H1}))\Lambda_{H2}$ and Λ_H is the idempotent integral of H. Then it is well known that ϕ is an isomorphism of vector spaces, see for example Theorem 4.1.1 of [14].

Remark 4.1. It can be checked that $\phi(\xi_d) = \frac{\epsilon(d)}{\dim H} d^*$ and $\phi^{-1}(\xi_\chi) = \chi(1)\chi$ for all $d \in \mathrm{Irr}(H^*)$ and $\chi \in \mathrm{Irr}(H)$ (see for example Lemma 3 of [15]). Here $\xi_\chi \in H$ is the central primitive idempotent of H associated to χ. Dually, $\xi_d \in H^*$ is the central primitive idempotent of H^* associated to $d \in \mathrm{Irr}(H^*)$.

The following description of $\hat{\mathrm{Z}}(H^*)$ and $\hat{\mathrm{Z}}(H)$ was given in Proposition 5 of [15]. Since $\phi(C(H)) = \mathrm{Z}(H)$ and $\phi(\mathrm{Z}(H^*)) = C(H^*)$ it follows that the restriction

$$\phi|_{\hat{\mathrm{Z}}(H^*)} : \hat{\mathrm{Z}}\big(H^*\big) \to \hat{\mathrm{Z}}(H)$$

is an isomorphism of vector spaces.

Since $\hat{\mathrm{Z}}(H^*)$ is a commutative semisimple algebra it has a vector space basis given by its primitive idempotents. Since $\hat{\mathrm{Z}}(H^*)$ is a subalgebra of $\mathrm{Z}(H^*)$ each primitive idempotent of $\hat{\mathrm{Z}}(H^*)$ is a sum of primitive idempotents of $\mathrm{Z}(H^*)$. But the primitive idempotents of $\mathrm{Z}(H^*)$ are of the form ξ_d where $d \in \mathrm{Irr}(H^*)$. Thus, there is a partition $\{\mathcal{Y}_j\}_{j\in J}$ of the set of irreducible characters of H^* such that the elements $(e_j)_{j\in J}$ given by

$$e_j = \sum_{d\in\mathcal{Y}_j} \xi_d$$

form a basis for $\hat{\mathrm{Z}}(H^*)$. Since $\phi(\hat{\mathrm{Z}}(H^*)) = \hat{\mathrm{Z}}(H)$ it follows that $\hat{e}_j := (\dim H)\phi(e_j)$ is a basis for $\hat{\mathrm{Z}}(H)$. Using the first formula from Remark 4.1 one has

$$\hat{e}_j = \sum_{d\in\mathcal{Y}_j} \epsilon(d) d^*. \tag{4.2}$$

Proposition 3.3 of [2] shows that kernels of central characters are normal Hopf subalgebras. Thus with the above notations $H^*_{\hat{e}_j}$ is a normal Hopf subalgebra of H^*.

Remark 4.3. By duality, the set of irreducible characters of H can be partitioned into a finite collection of subsets $\{\mathcal{X}_i\}_{i\in I}$ such that the elements $(f_i)_{i\in I}$ given by

$$f_i = \sum_{\chi\in\mathcal{X}_i} \chi(1)\chi \tag{4.4}$$

form a $\mathbb{C}$-basis for $\hat{Z}(H^*)$. Then the elements $\phi(f_i) = \sum_{\chi\in\mathcal{X}_i} \xi_\chi$ are the central orthogonal primitive idempotents of $\hat{Z}(H)$ and therefore they form a basis for this space. Clearly $|I| = |J|$.

For two irreducible H^*-characters write $d \sim d'$ if both characters appear as constituents in the same central character $\hat{e}_j$ of H. Clearly $\sim$ is an equivalence relation with equivalence classes $\mathcal{Y}_j$.

For an irreducible character $d \in \mathrm{Irr}(H^*)$ let $N(d)$ be the smallest normal Hopf subalgebra of H containing d. This always exists since intersection of normal Hopf subalgebras is always a normal Hopf subalgebra.

Proposition 4.5. *Suppose that $d, d' \in \mathrm{Irr}(H^*)$ with $d \sim d'$. Then $N(d) = N(d')$.*

Proof. Since $N(d)$ is normal it follows from [6] that the idempotent integral Λ of $N(d)$ is central in H. Since $N(d)$ is a semisimple Hopf algebra Eq. (2.1) implies that Λ is a scalar multiple of the sum $\sum_{e\in\mathrm{Irr}(N(d)^*)} \epsilon(e)e$. But $\mathrm{Irr}(N(d)^*) \subset \mathrm{Irr}(H^*)$ and decomposition (4.2) of central characters of H^* shows that $d' \in \mathrm{Irr}(N(d)^*)$. Therefore $N(d') \subset N(d)$. Symmetry implies that $N(d') = N(d)$. □

Remark 4.6. (See Remark 2.3 of [2].)

(1) Suppose that K is a normal Hopf subalgebra of H and let $L = H//K$ be the quotient Hopf algebra of H via $\pi : H \to L$. Then $\pi^* : L^* \to H^*$ is an injective Hopf algebra map. It follows that $\pi^*(L^*)$ is a normal Hopf subalgebra of H^*. Moreover $(H^*//L^*)^* \cong K$ as Hopf algebras.
(2) There is a bijection between normal Hopf subalgebras of H and H^*. To any normal Hopf subalgebra K of H one associates $(H//K)^*$ as normal Hopf subalgebra of H^*. Conversely to any L, a normal Hopf subalgebra in H^*, one associates $(H^*//L)^*$ as normal Hopf subalgebra of H. The fact that these maps are inverse one to the other follows from the previous item of this remark.

4.2. *Hopf algebras with all kernels normal*

Definition 4.7. We say that a semisimple Hopf algebra H has property (N) if and only if H_χ is a normal Hopf subalgebra of H for all irreducible characters $\chi \in \mathrm{Irr}(H)$.

Dually H^* has property (N) if and only if H^*_d is normal for any irreducible character $d \in \mathrm{Irr}(H^*)$.

Remark 4.8. Clearly group algebras of finite groups and their duals have property (N). It follows from Theorem 5.7 of [3] that Drinfeld doubles of finite groups have property (N). In a future preprint the author will also prove that Kac algebras of the type $k^G\#kF$ with F and G finite groups have property (N).

For a semisimple Hopf algebra H denote by $t_H \in H^*$ the idempotent integral of H^*. Then as in Eq. (2.1) that $\dim_{\mathbb{C}}(H)t_H$ is the regular character of H. The following theorem from [2] will be used in the sequel.

Theorem 4.9. *Let H be a finite dimensional semisimple Hopf algebra. Any normal Hopf subalgebra K of H is the kernel of a character which is central in H^*. More precisely, with the above notations one has*:

$$K = H_{\dim_{\mathbb{C}}(L)\pi^*(t_L)},$$

where $L = H//K$, $t_L \in L^*$ is the idempotent integral on L and $\pi : H \to L$ is the canonical Hopf projection.

Here $\pi^* : L^* \to H^*$ is the dual map of π. Clearly π^* is an injection since π is surjective.

From the proof of this theorem also follows that the regular character $\dim_{\mathbb{C}}(L)t_L$ of L equals $\epsilon_K \uparrow_K^H$ where ϵ_K is the character of the trivial representation of K. This means that

$$\epsilon_K \uparrow_K^H = \sum_{\chi \in \mathrm{Irr}(H//K)} \chi(1)\chi. \tag{4.10}$$

Theorem 4.11. *Let H be a semisimple Hopf algebra. If H^* has property (N) then $\ker_{H^*}(d) = \ker_{H^*}(d')$ for all $d \sim d'$.*

Proof. Suppose H_d^* is a normal Hopf subalgebra of H^*. Then by Remark 4.6 there is some normal Hopf subalgebra K of H such that $H_d^* = (H//K)^*$. From the definition of the kernel and Remark 2.7 one has that

$$\ker_{H^*}(d) = \{\chi \in \mathrm{Irr}(H) \mid K \subset H_\chi\}. \tag{4.12}$$

We claim that $d \in K$. Corollary 2.5 of [2] implies that $H_{\epsilon_K \uparrow_K^H} = K$. Thus it is enough to show that $d \in \ker_H(\epsilon_K \uparrow_K^H)$. But as above $\epsilon_K \uparrow_K^H$ is the regular character of $H//K$. Then formula (4.10) and item (3) of Remark 2.3 imply that

$$\ker_H(\epsilon_K \uparrow_K^H) = \ker_H\left(\sum_{\chi \in \ker_{H^*}(d)} \chi(1)\chi\right) = \bigcap_{\chi \in \ker_{H^*}(d)} \ker_H(\chi).$$

Note that from Proposition 2.2 one has $d \in \ker_H(\chi)$ if and only if $\chi \in \ker_{H^*}(d)$. This implies that $d \in \ker_H(\epsilon_K \uparrow_K^H)$. Since $N(d)$ is the minimal normal Hopf subalgebra containing d one also has that $N(d) \subset K$.

On the other hand since K is normal the integral Λ_K of K is central in H. Recall that $\dim_{\mathbb{C}}(K)\Lambda_K$ is the regular character of K^* and formula (2.1) implies that

$$\dim_{\mathbb{C}}(K)\Lambda_K = \sum_{e \in \mathrm{Irr}(K^*)} \epsilon(e)e. \tag{4.13}$$

Since $d \in \mathrm{Irr}(K^*)$ by decomposition (4.2) of central characters of H^* one obtains as above that $d' \in K$ for all $d' \sim d$. This implies that $\ker_{H^*}(d') \supset \ker_{H^*}(d)$. Indeed, if $\chi \in \ker_{H^*}(d)$ then from formula (4.12) it follows that $d' \in K \subset H_\chi$, i.e. $d' \in \ker_H(\chi)$. Thus as above $\chi \in \ker_{H^*}(d')$.

By symmetry one obtains $\ker_{H^*}(d') \subset \ker_{H^*}(d)$ and thus the equality $\ker_{H^*}(d') = \ker_{H^*}(d)$. □

Corollary 4.14. *Let H be a semisimple Hopf algebra. Then H^* has property (N) if and only if $\ker_{H^*}(d) = \ker_{H^*}(d')$ for all $d \sim d'$.*

Proof. We have already shown that if H^* has (N) then $\ker_{H^*}(d) = \ker_{H^*}(d')$ for all $d \sim d'$.

For the converse suppose that $d \in \mathcal{Y}_j$ for some j as above. Then by the hypothesis one has $\ker_{H^*}(d) = \ker_{H^*}(d')$ for all other $d' \in \mathcal{Y}_j$. Thus $\ker_{H^*}(d) = \bigcap_{d' \in \mathcal{Y}_j} \ker_{H^*}(d') = \ker_{H^*}(\hat{e}_j)$ by item (3) of Remark 2.3. But it is known that $H_{\hat{e}_j}^*$ is a normal Hopf subalgebra of H by Proposition 3.3 of [2]. □

Now we can prove the main result of this subsection.

Theorem 4.15. *Let H be a semisimple Hopf algebra. Then H^* has property (N) if and only if H has property (N).*

Proof. We show that if H^* has (N) then H has also (N). For two irreducible H-characters write as above $\chi \sim \chi'$ if both characters appear in a decomposition of a central character f_i of H defined in Remark 4.3. In other words $\chi, \chi' \in \mathcal{X}_i$ for some set $\mathcal{X}_i$. Suppose $\chi \sim \chi'$ and let $d \in \ker_H(\chi)$. Since H^* has (N) as before we have that $H^*_d = (H//K)^*$ for some K, a normal Hopf subalgebra of H. Then Theorem 4.9 implies that χ is a constituent of $\epsilon_K \uparrow^H_K$. By the same theorem $\epsilon_K \uparrow^H_K$ is central in H^* and it follows from decomposition (4.4) of central characters of H that χ' is also a constituent of $\epsilon_K \uparrow^H_K$. Therefore also $d \in \ker_H(\chi')$. This shows $\ker_H(\chi) \subset \ker_H(\chi')$. By symmetry one has the equality $\ker_H(\chi) = \ker_H(\chi')$. Previous corollary shows that H has also property (N). □

Proposition 4.16. *Suppose that H is a semisimple Hopf algebra with property (N) and let $d \in \mathrm{Irr}(H^*)$. In this situation $H^*_d = (H//N(d))^*$.*

Proof. It is enough to show that $\ker_{H^*}(d) = \mathrm{Irr}(H//N(d))$. Clearly $\mathrm{Irr}(H//N(d)) \subset \ker_{H^*}(d)$ by Remark 2.7. Suppose now that $\chi \in \ker_{H^*}(d)$. Since H has (N) it follows that H_χ is a normal Hopf subalgebra of H. Since $d \in \ker_H(\chi)$ definition of $N(d)$ shows that $N(d) \subset H_\chi$. Thus $\chi \in \mathrm{Irr}(H//N(d))$. □

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