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Computation of Milnor numbers and critical values at infinity

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Abstract

We describe how to compute topological objects associated to a polynomial map of several complex variables with isolated singularities. These objects are the affine critical values, the affine Milnor numbers for all irregular fibers, the critical values at infinity, and the Milnor numbers at infinity for all irregular fibers. Then for a family of polynomials we detect parameters where the topology of the polynomials can change. Implementation and examples are given with the computer algebra system SINGULAR.

MSC (1991): 32S25; 14Q10; 32S30

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1. Introduction

1.1. Review on the local case

Let $g : \mathbb{C}^n, 0 \longrightarrow \mathbb{C}, 0$ be a germ of a polynomial map with isolated singularities. One of the most important topological objects attached to g is its *local Milnor number* (Milnor, 1968):

$$\mu_0 = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\}/\mathrm{Jac}(g)$$

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where $\mathrm{Jac}(g) = (\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n})$ is the Jacobian ideal of g. It is possible to compute μ_0 with the help of a Gröbner basis. For example such a computation can be done with the computer algebra system SINGULAR (Greuel et al., 2001).

Let us consider a family $(g_s)_{s\in[0,1]}$, with $g_s : \mathbb{C}^n, 0 \longrightarrow \mathbb{C}, 0$ germs of isolated singularities, such that g_s is a smooth function of s. To each $s \in [0, 1]$ we associate the local Milnor number $\mu_0(g_s)$. The main topological result for families is the μ-constant theorem of Lê and Ramanujam (1976) and Timourian (1977).

Theorem 1. *If $n \neq 3$ and $\mu_0(g_s)$ is constant $(s \in [0, 1])$ then the family $(g_s)_{s\in[0,1]}$ is a topologically trivial family.*

1.2. *Motivation and aims for the global case*

Let us consider now a polynomial function $f : \mathbb{C}^n \longrightarrow \mathbb{C}$. The study of the topology of f is not just the glueing of local studies because of the behaviour of f at infinity; see Broughton (1988). For example the polynomial $f(x, y) = x(xy - 1)$ has no affine singularities but the fiber $f^{-1}(0)$ has two connected components while the other fibers $f^{-1}(c)$, $c \neq 0$, have only one.

We attach to the polynomial f "Milnor numbers" μ, λ and finite sets of critical values $\mathcal{B}_{\mathrm{aff}}$, $\mathcal{B}_\infty$, $\mathcal{B} = \mathcal{B}_{\mathrm{aff}} \cup \mathcal{B}_\infty$ (see definitions below). The first aim of this work is to compute these objects and to give the topology of the fibers $f^{-1}(c)$ for all $c \in \mathbb{C}$.

There is a global version of the local μ-constant theorem (see Theorem 2) where the Milnor number μ_0 is replaced by a *Milnor multi-integer* $\mathfrak{m} = (\mu, \#\mathcal{B}_{\mathrm{aff}}, \lambda, \#\mathcal{B}_\infty, \#\mathcal{B})$. In order to verify if $\mathfrak{m}(f_s)$ remains constant in a family $(f_s)_{s\in[0,1]}$ we have to compute $\mathfrak{m}(f_s)$ for infinitely many values. The second aim of the work is to give (and compute) a finite set $\mathcal{S}$ such that $\mathfrak{m}(f_s)$ is constant for $s \in [0, 1]\setminus\mathcal{S}$. Hence we finally just have to compute $\mathfrak{m}(f_s)$ for finitely many values.

The rest of this section is devoted to definitions and results.

1.3. *Critical values*

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map, $n \geq 2$. By a result of Thom (1969) there is a finite minimal *set of critical values* $\mathcal{B}$ of points of $\mathbb{C}$ such that $f : f^{-1}(\mathbb{C}\setminus\mathcal{B}) \longrightarrow \mathbb{C}\setminus\mathcal{B}$ is a locally trivial fibration. In the next two paragraphs we give a description of $\mathcal{B}$.

1.4. *Affine singularities*

We suppose that *affine singularities are isolated*, i.e. that the set $\{x \in \mathbb{C}^n \mid \mathrm{grad}_f\, x = 0\}$ is a finite set. Let μ_c be the sum of the local Milnor numbers at the points of $f^{-1}(c)$. Let

$$\mathcal{B}_{\mathrm{aff}} = \{c \mid \mu_c > 0\} \qquad \text{and} \qquad \mu = \sum_{c\in\mathbb{C}} \mu_c$$

be the *affine critical values* and the *affine Milnor number*. Morally μ is the number of affine singularities counted with multiplicities.

1.5. Singularities at infinity

To explain the behaviour of the polynomials at infinity we need some definitions; see Broughton (1988). Let d be the degree of $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, and let $f = f^d + f^{d-1} + \cdots + f^0$, where f^j is homogeneous of degree j. Let $\bar{f}(x, z)$ (with $x = (x_1, \ldots, x_n)$) be the homogenisation of f with the new variable z: $\bar{f}(x, z) = f^d(x) + f^{d-1}(x)z + \cdots + f^0(x)z^d$. Let

$$X = \{((x : z), t) \in \mathbb{P}^n \times \mathbb{C} \mid \bar{f}(x, z) - tz^d = 0\}.$$

X is a compactification of $\mathbb{C}^n$ associated to the polynomial f. Let $\mathcal{H}_\infty$ be the hyperplane at infinity of $\mathbb{P}^n$ defined by $(z = 0)$. The singular locus of X at infinity has the form $\Sigma \times \mathbb{C}$, where

$$\Sigma = \left\{ (x : 0) \;\middle|\; \frac{\partial f^d}{\partial x_1} = \cdots = \frac{\partial f^d}{\partial x_n} = f^{d-1} = 0 \right\} \subset \mathcal{H}_\infty.$$

We suppose that f has *isolated singularities at infinity*; that is to say that Σ is finite. This is always true for $n = 2$. We say that f has *strong isolated singularities at infinity* if

$$\Sigma' = \left\{ (x : 0) \;\middle|\; \frac{\partial f^d}{\partial x_1} = \cdots = \frac{\partial f^d}{\partial x_n} = 0 \right\}$$

is finite.

For a point $(x : 0) \in \mathcal{H}_\infty$ assume for example that $x = (x_1, \ldots, x_{n-1}, 1)$ and set $\check{x} = (x_1, \ldots, x_{n-1})$ and

$$F_c(\check{x}, z) = \bar{f}(x_1, \ldots, x_{n-1}, 1) - cz^d.$$

This is the localisation of f at the point at infinity $(x : 0)$. Let $\mu_{\check{x}}(F_c)$ be the local Milnor number of F_c at the point $(\check{x}, 0)$. If $(x : 0) \in \Sigma$ then $\mu_{\check{x}}(F_c) > 0$. For a generic s, $\mu_{\check{x}}(F_s) = \nu_{\check{x}}$, and for finitely many c, $\mu_{\check{x}}(F_c) > \nu_{\check{x}}$. We set $\lambda_{c,\check{x}} = \mu_{\check{x}}(F_c) - \nu_{\check{x}}$, $\lambda_c = \sum_{(x:0)\in\Sigma} \lambda_{c,\check{x}}$. Then $\lambda_c > 0$ if and only if at some point at infinity the compactification of the fiber $f^{-1}(c)$ is more singular than a generic fiber. Let

$$\mathcal{B}_\infty = \{c \in \mathbb{C} \mid \lambda_c > 0\} \qquad \text{and} \qquad \lambda = \sum_{c \in \mathbb{C}} \lambda_c$$

be the *critical values at infinity* and the *Milnor number at infinity*.

We can now describe the set of critical values $\mathcal{B}$ as follows (see Hà and Lê (1984) and Parusiński (1995)):

$$\mathcal{B} = \mathcal{B}_{\text{aff}} \cup \mathcal{B}_\infty.$$

Moreover, by Hà and Lê (1984) and Siersma and Tibăr (1995) for all $c \in \mathbb{C} \backslash \mathcal{B}$, $f^{-1}(c)$ has the homotopy type of a wedge of $\mu + \lambda$ spheres of real dimension $n - 1$. And for all $c \in \mathbb{C}$ the Euler characteristic of a fiber is $\chi(f^{-1}(c)) = 1 - (-1)^{n-1}(\mu + \lambda - \mu_c - \lambda_c)$.

For our example $f(x, y) = x(xy - 1)$ there are no affine singularities; hence $\mathcal{B}_{\text{aff}} = \varnothing$ and $\mu = 0$. But at the point $(0 : 1 : 0)$ in $\mathbb{P}^2$ we have a singularity at infinity such that $\mathcal{B}_\infty = \{0\}$ and $\lambda = 1$. Then the fiber $f^{-1}(c)$, $c \neq 0$, is homotopic to a circle, and $f^{-1}(0)$ is homotopic to the union of a point and a circle.

1.6. Families of polynomials

We associate to a polynomial its *Milnor multi-integer* $\mathfrak{m} = (\mu, \#\mathcal{B}_{\text{aff}}, \lambda, \#\mathcal{B}_\infty, \#\mathcal{B})$. Two polynomial maps $f, g : \mathbb{C}^n \longrightarrow \mathbb{C}$ are *topologically equivalent* if there exist homeomorphisms $\Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and $\Psi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $f \circ \Phi = \Psi \circ g$. The topological equivalence preserves the topology of the singularities and in particular the Milnor multi-integer is a topological invariant; that is to say, if f and g are topologically equivalent then $\mathfrak{m}(f) = \mathfrak{m}(g)$. We recall a result that is a kind of converse of this property.

Let $(f_s)_{s\in[0,1]}$ be a family of polynomials, such that f_s has *strong* isolated singularities at infinity and isolated affine singularities for all $s \in [0, 1]$. For each $s \in [0, 1]$ we consider the Milnor multi-integer of f_s, $\mathfrak{m}(f_s) = (\mu(s), \#\mathcal{B}_{\text{aff}}(s), \lambda(s), \#\mathcal{B}_\infty(s), \#\mathcal{B}(s))$. We suppose that the coefficients of the family are polynomials in s, and that the degree $\deg f_s$ is constant. The result of Bodin (2003) and Bodin and Tibăr (2003) is:

Theorem 2. *Let $n \neq 3$. If $\mathfrak{m}(f_s)$ is constant ($s \in [0, 1]$), then f_0 is topologically equivalent to f_1.*

How is it possible to verify the hypotheses from a computable point of view? We compute $\mathfrak{m}(f_s)$ for infinitely many $s \in [0, 1]$ by using the fact that $\mathfrak{m}(f_s)$ is constant except for finitely many s; we denote by $\mathcal{S}$ the set of these *critical parameters*. The aim of Section 4 is to give a computation of these critical parameters.

1.7. Implementation

The results of this paper have been implemented in two libraries, `critic` and `defpol`. The first one enables the calculation of all the objects defined above: $\mathcal{B}_{\text{aff}}$, μ, μ_c for $c \in \mathcal{B}_{\text{aff}}$, $\mathcal{B}_\infty$, λ, and λ_c for $c \in \mathcal{B}_\infty$. These programs are written for SINGULAR (Greuel et al., 2001). They are based on polar curves and on the article of Siersma and Tibăr (1995). For polynomials in two variables ($n = 2$) a program in MAPLE has been written by Bailly-Maître (2000) based on a discriminant formula of Hà (1989). For families of polynomials the second library computes a finite set $\mathcal{S}'$ that contains the critical parameters.

2. Milnor numbers and critical values in affine space

2.1. Milnor number

The computation of the affine Milnor number μ is easy and well-known. For details see Greuel and Pfister (2002, Chapter 3). Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Let J be the Jacobian ideal of the partial derivatives $(\partial f/\partial x_i)_i$. Then by definition μ is the vector space dimension (over $\mathbb{C}$) of the quotient $\mathbb{C}[x_1, \ldots, x_n]/J$. But if $L(J)$ denotes the leading ideal of J and if G is a Gröbner basis of J then

$$\mu := \dim \mathbb{C}[x_1, \ldots, x_n]/J = \dim \mathbb{C}[x_1, \ldots, x_n]/L(J) = \dim \mathbb{C}[x_1, \ldots, x_n]/L(G).$$

And the last dimension can be recursively computed.

2.2. Critical values

We add a new variable t. We consider the variety

$$C = \{(x,t) \in \mathbb{C}^n \times \mathbb{C} \mid f(x) - t = 0 \text{ and } \operatorname{grad}_f x = 0\}.$$

The critical values are the projection of C on the t-coordinate: $\mathcal{B}_{\text{aff}} = \operatorname{pr}_t(C)$.

2.3. Milnor number of a fiber

Let $c \in \mathbb{C}$. We would like to compute μ_c, the sum of the Milnor numbers of the points of $f^{-1}(c)$. Let J be the Jacobian ideal of f and let x be a critical point. We denote by J_x the localisation of J at x. Let $I_x = (t-c, J_x)$; the dimension of I_x is equal to the Milnor number of f at x. For $k \geq 1$ we consider $K_x^k = ((f-t)^k, I_x)$. Then $f(x) = c$ if and only if K_x^k has non-zero dimension (as a vector space). Moreover, if $f(x) = c$ then by the Nullstellensatz $(f-t)^k$ is in I_x for a sufficiently large k. For such a k, the dimension of K_x^k is the Milnor number at x if $f(x) = c$, and it is 0 otherwise. Such a k is less than or equal to the Milnor number at x, but k can often be chosen much less. The minimal k is the first integer such that the vector space dimension of K_x^k is equal to the one of K_x^{k+1}.

3. Milnor numbers and critical values at infinity

We give the computation of the objects at infinity and its implementation in SINGULAR. We will suppose that f has isolated singularities at infinity; in fact computations are valid for a larger class of polynomials but it is not possible to compute if f belongs to this class. The algorithm is based on the article of Siersma and Tibăr (1995) that gives critical values at infinity and Milnor numbers at infinity with the help of polar curves.

3.1. Working space

We will work in $\mathbb{P}^n \times \mathbb{C}$, with the homogeneous coordinates of $\mathbb{P}^n$: $(x_1 : \ldots : x_n : z)$; we still need t, which is a parameter or a variable depending on the context.

We recall that

$$X = \{((x:z),t) \in \mathbb{P}^n \times \mathbb{C} \mid \bar{f}(x,z) - tz^d = 0\}.$$

The part at infinity of X is $X_\infty = X \cap (\mathcal{H}_\infty \times \mathbb{C})$:

$$X_\infty = \{((x:0),t) \in \mathbb{P}^n \times \mathbb{C} \mid f^d(x) = 0\},$$

where $f = f^d + f^{d-1} + \cdots$ is the decomposition in homogeneous polynomials.

In SINGULAR we write:

```
ring r = 0, (x(1..n),z,t), dp;
poly f = ...;
poly fH = homog(f,z)-t*z^deg(f);
ideal X = fH;
ideal Xinf = z, fH;
```

3.2. Polar curve

Let k be in $\{1, \ldots, n\}$. The polar curve $\mathcal{P}$ is the critical locus of the map $\phi : \mathbb{C}^n \longrightarrow \mathbb{C}^2$ defined for $x = (x_1, \ldots, x_n)$ by $\phi(x) = (f(x), x_k)$:

$$\mathcal{P} = \left\{ x \in \mathbb{C}^n \;\middle|\; \frac{\partial f}{\partial x_i}(x) = 0, \forall i \neq k \right\}.$$

We have that $\mathcal{P}$ is a curve or is void. We call $\mathcal{P}_H$ the projective closure of $\mathcal{P}$. This curve intersects the hyperplane at infinity $\mathcal{H}_\infty$ in finitely many points.

```
ideal P = diff(f,x(1)),..., diff(f,x(k-1)), diff(f,x(k+1)),...;
ideal PH = homog(P,z);
```

The former objects can be viewed in X; we will also denote by $\mathcal{P}_H$ the set $(\mathcal{P}_H \times \mathbb{C}) \cap X$. In the chart $x_k = 1$ we denote the curve $\mathcal{P}_H$ by $\bar{\mathcal{C}}$. The “real” polar curve $\mathcal{C}$ in this chart is the closure of $\bar{\mathcal{C}} \setminus X_\infty$:

```
ideal Cbar = x(k)-1, PH, X;
ideal C = sat(Cbar,Xinf)[1];
```

3.3. Critical values at infinity

We need the following result of Siersma and Tibăr (1995). A value c is a critical values at infinity if and only if there is a coordinate x_k and a point $(x : 0, t)$ in X_∞ (with $x_k \neq 0$) such that $(x : 0, t) \in \mathcal{C}$. That is to say, $\mathcal{B}_\infty$ is the projection of $\mathcal{C}_\infty = X_\infty \cap \mathcal{C}$ on the space of parameters $t \in \mathbb{C}$.

Then the critical values are computed with:

```
ideal Cinf = z, C;
poly Binf = eliminate(Cinf,x(1)x(2)..x(n)z)[1];
```

The set of critical values at infinity are the roots of the polynomial `Binf`, which belongs to $\mathbb{C}[t]$.

3.4. Milnor numbers at infinity

Actually the results of Siersma and Tibăr (1995) are more precise. For a fixed t let $X_t = \{(x : z, t) \in X\}$; this is a projective model for the fiber $f^{-1}(t)$.

Theorem 3. *The Milnor number at infinity at a point $(x : 0, t) \in \mathcal{C}_\infty$ is given by the intersection number (in X) of $\mathcal{C}$ with X_t at $(x : 0, t)$.*

Roughly speaking, the polar curve arrives at infinity exactly on the critical fiber at infinity, and the order of contact with this fiber is the Milnor number at infinity. So, for $c \in \mathcal{B}_\infty$, the Milnor number at infinity λ_c (for the chart $x_k \neq 0$) is equal to the sum of all intersection numbers of X_c and $\mathcal{C}$ in X_∞.

We compute an ideal I which corresponds to $X_c \cap \mathcal{C}$; then we only deal with points at infinity by intersecting this set with $z^q = 0$ for a sufficiently large q.

```
number c = ...;
ideal Xc = t-c, X;
```

```
ideal I = Xc, C;
ideal K = z^q, I;     // q >> 1
lambdac = vdim(std(K));
```

Once we have computed λ_c for all $c \in \mathcal{B}_\infty$ we have $\lambda = \sum_{c\in\mathcal{B}_\infty} \lambda_c$.

4. Families of polynomials

Let $(f_s)_{s\in[0,1]}$ be a family of complex polynomials in n variables. We suppose that the coefficients are polynomial functions of s, and that for all $s \in [0, 1]$, f_s has affine isolated singularities and strong isolated singularities at infinity. The implementation is similar to the one of Section 3 and will be omitted. The *set of critical parameters* is the finite minimal set $\mathcal{S}$ such that for $s \in [0, 1]\backslash\mathcal{S}$ the Milnor multi-integer $\mathfrak{m}(f_s) = (\mu(s), \#\mathcal{B}_{\text{aff}}(s), \lambda(s), \#\mathcal{B}_\infty(s), \#\mathcal{B}(s))$ is constant.

4.1. Change in affine space

The Milnor number $\mu(s)$ is constant excepted at finitely many values; we do not need to compute it for all values but we detect a change of $\mu(s)$. The Milnor number $\mu(s)$ changes if and only if some critical points escape at infinity. Then we can detect critical parameters for μ as follows. Let $J = \{(x_1, \dots, x_n, s) \in \mathbb{C}^n \times \mathbb{C} \mid \frac{\partial f_s}{\partial x_1} = \cdots, \frac{\partial f_s}{\partial x_n} = 0\}$ be the set of critical points (that corresponds to the Jacobian ideal in $\mathbb{C}[x_1, \dots, x_n, s]$). Let $\bar{J}$ be the homogenisation of J with the new variable z, while s is considered as a parameter. The part at infinity of J corresponds to the $J_\infty = \bar{J} \cap (z = 0)$, and the affine part of J is $J_{\text{aff}} = \overline{\bar{J}\backslash J_\infty}$. Now the critical parameters for μ are $\text{pr}_s(J_{\text{aff}}) \subset \mathbb{C}$, where pr_s is the projection to the s-coordinate.

It is possible to compute $\mathcal{B}_{\text{aff}}(s)$ for all $s \in [0, 1]$ by a direct extension of the work of Section 2. Then we can compute the parameters where the cardinality of this set changes.

4.2. Change at infinity

Again we look for the parameters where $\lambda(s)$ changes. We extend the definition of Section 3 by adding a parameter s. We set $d = \deg f_s$ and

$$X = \{((x : z), t, s) \in \mathbb{P}^n \times \mathbb{C} \times \mathbb{C} \mid \bar{f}_s(x, z) - tz^d = 0\}.$$

The part at infinity of X is $X_\infty = X \cap (\mathcal{H}_\infty \times \mathbb{C} \times \mathbb{C})$:

$$X_\infty = \{((x : 0), t, s) \in \mathbb{P}^n \times \mathbb{C} \mid f_s^d(x) = 0\}.$$

The polar "curve" is

$$\mathcal{P} = \left\{(x, s) \in \mathbb{C}^n \times \mathbb{C} \;\middle|\; \frac{\partial f_s}{\partial x_i}(x) = 0, \forall i \neq k\right\}.$$

In the chart $x_k = 1$ we denote the homogenisation of $\mathcal{P}$ (with s a parameter) by $\bar{\mathcal{C}}$, and the "real" polar curve $\mathcal{C}$ in this chart is the closure of $\bar{\mathcal{C}}\backslash X_\infty$. The part at infinity of $\mathcal{C}$ is $\mathcal{C}_\infty = \mathcal{C} \cap X_\infty$.

Let $B_\infty(s) = \mathrm{pr}_t\{(x : 0, t, s) \in \mathcal{C}_\infty\}$. For a generic s', $\mathcal{B}_\infty(s') = B_\infty(s')$. Then the critical parameters for $\mathcal{B}_\infty(s)$ are included in the set of parameters where $\#B_\infty(s)$ fails to be equal to $\#\mathcal{B}_\infty(s')$ (in fact $B_\infty(s)$ may be infinite).

We set $X_* = \{(x : z, c, s) \in X \mid (x : 0, c, s) \in \mathcal{C}_\infty\}$; for non-critical parameters this corresponds to union of the irregular fibers at infinity. Now a change of λ corresponds to a change in the value of the intersection multiplicity of the polar curve $\mathcal{C}$ with X_*. The critical parameters for λ are given as the projection to the s-coordinate of

$$\overline{(\mathcal{C} \cap X_*)\backslash\mathcal{C}_\infty} \cap (z = 0).$$

At last we compute parameters where the cardinal of $\mathcal{B}(s) = \mathcal{B}_{\mathrm{aff}}(s) \cup \mathcal{B}_\infty(s)$ changes.

As a conclusion, if $\mathcal{S}$ is the set of critical parameters of (f_s), then we have constructed a finite and computable set $\mathcal{S}'$ such that

$$\mathcal{S} \subset \mathcal{S}'.$$

Now to check if a value $s \in \mathcal{S}'$ is in $\mathcal{S}$, we compute $\mathfrak{m}(f_s)$ and we compare it with $\mathfrak{m}(f_{s'})$, where s' is any value of $[0, 1]\backslash\mathcal{S}'$; now $s \in \mathcal{S}$ if and only if $\mathfrak{m}(f_s) \neq \mathfrak{m}(f_{s'})$.

5. Examples

5.1. *Briançon polynomial*

The following example shows how to use the program once SINGULAR is started. We have to load the library `critic.lib`, then we set the ring, with $n + 1$ variables; the last variable will enable us to have the critical values (as the zeros of a polynomial) in return. The following code gives the critical values at infinity of the Briançon polynomial. Let $q = xy + 1$, $p = xq + 1$ and $f(x, y) = 3yp^3 + 3p^2q - 5pq - q$.

```
LIB "critic.lib";
ring r = 0, (x,y,t), dp;
poly q = xy+1;
poly p = x*q+1;
poly f = 3*y*p^3+3*p^2*q-5*p*q-q;
crit(f);
```

The result is:

```
> Affine critical values are the roots of 1
> Affine Milnor number : 0
> Critical values at infinity are the roots of 3t2+16t
> Milnor number at infinity : 4
> Details of critical values at infinity :
>   t       1
>   3t+16   3
```

This shows that there is no affine critical value (as the root of the polynomial 1) and that $\mathcal{B}_\infty = \{0, -16/3\}$ (as the root of the polynomial t and $3t + 16$) are the critical values at infinity with Milnor number at infinity respectively equal to 1 and 3.

5.2. More variables

Let $f(a, b, c, d) = a + a^4b + b^2c^3 + d^5$ be the example of Choudary and Dimca (1994) and Artal et al. (1998). This polynomial has isolated singularities at infinity. The only singularity is a singularity at infinity for the critical value 0. Let us check it.

```
ring r = 0, (a,b,c,d,t), dp;
poly f = a+a^4*b+b^2*c^3+d^5;
crit(f);
> Affine critical values are the roots of 1
> Affine Milnor number : 0
> Critical values at infinity are the roots of t
> Milnor number at infinity : 8
```

The computation shows that actually $\mathcal{B}_\infty = \{0\}$, and moreover $\lambda = 8$.

5.3. A family

We give an example of deformation. We first need to load the library `defpoly.lib`; then we introduce a ring in $n + 1$ variables, where the last variable is the parameter of the deformation. For instance, we consider the deformation $f_s(x, y) = y(1-sx)(y-(s-1)x)$.

```
LIB "defpol.lib";
ring r = 0, (x,y,s), dp;
poly f = y*(1-sx)*(y-(s-1)*x);
parCrit(f);
> Critical parameters are included in the roots of s2-s
```

Then the critical parameters are $s = 0$ and $s = 1$.

5.4. A trivial family

Another deformation is $f_s(x, y) = x(x^3y + sx^2 + s^2x + 1)$.

```
LIB "defpol.lib";
ring r = 0, (x,y,s), dp;
poly f = x*(x^3*y+s*x^2+s^2*x+1);
parCrit(f);
> Critical parameters are included in the roots of 1
```

Then $\mathfrak{m}(f_s)$ and the degree are constant; Theorem 2 implies that for all $s, s' \in \mathbb{C}$, f_s and $f_{s'}$ are topologically equivalent.

5.5. Combination

We consider the family $f_s(x, y) = (x - s^2 - 1)(x^2y + 1)$.

```
LIB "defpol.lib";
ring r = 0, (x,y,s), dp;
poly f = (x-s^2-1)*(x^2*y+1);
```

```
parCrit(f);
> Critical parameters are included in the roots of s2+1
```

The critical parameters are $+i$ and $-i$ (with $i^2 = -1$).

For a generic value $s \neq \pm i$ we have

```
LIB "critic.lib";
ring r = (0,s), (x,y,t), dp;
poly f = (x-s^2-1)*(x^2*y+1);
crit(f);
> Affine critical values are the roots of t
> Affine Milnor number : 1
> Critical values at infinity are the roots of t+(s2+1)
> Milnor number at infinity : 1
```

So for $s \neq \pm i$, $\mathcal{B}_{\text{aff}}(s) = \{0\}$, $\mu(s) = 1$ and $\mathcal{B}_\infty(s) = \{-s^2 - 1\}$, with $\lambda(s) = 1$.

And for a critical parameter ($s = i$ or $s = -i$) we have $\mathcal{B}_{\text{aff}}(s) = \varnothing$ and $\mathcal{B}_\infty(s) = \{0\}$, with $\lambda(s) = 1$:

```
ring r = (0,s), (x,y,t), dp;
minpoly = s^2+1;
poly f = (x-s^2-1)*(x^2*y+1);
crit(f);
> Affine critical values are the roots of 1
> Affine Milnor number : 0
> Critical values at infinity are the roots of t
> Milnor number at infinity : 1
```

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