Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 228 (2006) 440–485
Journal of Differential Equations
www.elsevier.com/locate/jde

Besov estimates in the high-frequency Helmholtz equation, for a non-trapping and C^2 potential

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Received 4 June 2005

Available online 21 July 2006

Abstract

We study the high-frequency Helmholtz equation, for a potential having C^2 smoothness, and satisfying the non-trapping condition. We prove optimal Morrey–Campanato estimates that are both homogeneous in space and uniform in the frequency parameter. The homogeneity of the obtained bounds, together with the weak assumptions we require on the potential, constitute the main new result in the present text. Our result extends previous bounds obtained by Perthame and Vega, in that we do not assume the potential satisfies the virial condition, a strong form of non-trapping.

MSC: primary 35Q40; secondary 35J10, 81Q20

Keywords: Helmholtz equation; Non-trapping; Limiting absorption principle; Morrey–Campanato spaces; Besov space; Wigner transform

Contents

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.06.006

1. Introduction

This article is devoted to establishing *homogeneous* and *uniform* bounds on the high-frequency Helmholtz equation, with *minimal* geometric and analytic assumptions on the potential.

Technically, the question is the following.

We take a (small) absorption parameter $\alpha > 0$, and a (small) wavelength parameter $\varepsilon > 0$. We also pick up a potential $V(x) \in \mathbb{R}$ ($x \in \mathbb{R}^d$ with $d \geqslant 3$), whose regularity and geometric properties is made precise below, together with an energy parameter $\lambda \in \mathbb{R}$. Last, we choose a given function $f(x)$, whose regularity is made precise below. Associated with all these data, we consider the solution $u(x)$ to the following high-frequency Helmholtz equation with source term,[1]

$$+i\varepsilon\alpha u(x) - \varepsilon^2 \Delta_x u(x) + \big[V(x) - \lambda\big]u(x) = f(x), \quad x \in \mathbb{R}^d,\ d \geqslant 3. \tag{1.1}$$

In the terminology of the Helmholtz equation, the function $\lambda - V(x)$ in (1.1) plays the role of a refraction index, $f(x)$ is a source of radiation, and $u(x)$ is the signal created by f in the whole space.

The first main goal of this text is to prove the existence of a constant $C > 0$, independent of $\alpha > 0$ and $\varepsilon > 0$, such that

$$\|u\|_{B^*} \leqslant \frac{C}{\varepsilon}\|f\|_B, \tag{1.2}$$

where the *homogeneous* Morrey–Campanato spaces B and B^* are defined as the completion of smooth function under the following norms,[2]

$$\|u\|_{B^*} := \sup_{j\in\mathbb{Z}} 2^{-j/2}\|u\|_{L^2(C_j)}, \qquad \|f\|_B := \sum_{j\in\mathbb{Z}} 2^{j/2}\|f\|_{L^2(C_j)}, \tag{1.3}$$

$$\text{and the annulus } C_j \text{ is } C_j := \big\{x \in \mathbb{R}^d \text{ s.t. } 2^j \leqslant |x| \leqslant 2^{j+1}\big\}. \tag{1.4}$$

These norms were introduced by Agmon and Hörmander in [2], and have been used recently in two articles by Perthame and Vega [28,29]. As we explain below, the particular homogeneity of these norms, and the weights $2^{j/2}$, respectively $2^{-j/2}$ in (1.3), make estimate (1.2) the *optimal* bound in this context.

Our second aim is to establish (1.2) under *weak* assumptions on V (assumptions (H1)–(H5) below). From the analytic standpoint, we shall mainly assume that V has the (low) C^2 regularity. More importantly, from the geometric point of view, we shall mainly assume that the potential satisfies a *non-trapping* property. In particular, we shall *not* make any *virial assumption* on

[1] Admitting that $u(x)$ is well defined for a while.
[2] In particular, we assert that u is well defined and belongs to B^* provided $f \in B$.

the potential, a reinforced version of the non-trapping condition. This is a key point. We refer to (1.10) and (1.18) for the precise definition of these two notions.

Both the homogeneity of the uniform bound (1.2) under space rescaling, and the weak assumptions we require on the potential V, constitute the main new points in the present text.

To illustrate the importance of dealing with homogeneous norms, we readily mention the following scaling property, previously underlined in [28], as it is an important motivation for the present work. Define $w(x)$ as the solution to the *low-frequency* Helmholtz equation, with *almost-constant* refraction index

$$+i\varepsilon\alpha w(x) - \Delta_x w(x) + \big[V(\varepsilon x) - \lambda\big] w(x) = f(x), \tag{1.5}$$

where the potential $V(x)$ satisfies assumptions (H1)–(H5) below. Using the change of unknown $u(x) = \varepsilon^{-d/2} w(x/\varepsilon)$, the reader may easily check that estimate (1.2) *implies* w satisfies the uniform bound

$$\|w\|_{B^*} \leqslant C\|f\|_B, \tag{1.6}$$

i.e., w is bounded as the parameters α and ε go to zero, and one may try to pass to the limit in (1.5) (see [6,32]).

The uniform estimate (1.6) lies at the core of the high-frequency analysis of the Helmholtz equation, as performed in [4,7]. It is important to stress that, dealing with inhomogeneous norms, i.e., replacing the norms $\|\cdot\|_B$, respectively $\|\cdot\|_{B^*}$ appearing in (1.2) and (1.6), by $\|\cdot\|_{B_{\mathrm{inh}}}$, respectively $\|\cdot\|_{B^*_{\mathrm{inh}}}$, or even by $\|\cdot\|_{L^2(\langle x\rangle^s\,dx)}$, respectively $\|\cdot\|_{L^2(\langle x\rangle^{-s}\,dx)}$ for $s > 1$ (these norms are defined in Remark 1 below), would give a diverging factor C/ε instead of C in (1.6), as the reader may readily check. Hence the point in using homogeneous B–B^* norms lies in obtaining a w which is correctly estimated by an $O(1)$ quantity, rather than by an incorrect $O(1/\varepsilon)$.

Uniform bounds on w, of the form (1.6), have first been obtained in [28], and later generalized in [32]. Their analysis strongly uses a *virial assumption* on the potential V. For a potential which is merely *non-trapping*, an $O(1)$ bound on w in the distribution sense has first and recently been obtained in [6]. In this text, we go to the $O(1)$ bound on w in the natural B^* norm.

Needless to say, the nice scaling property of $\|\cdot\|_B$ and $\|\cdot\|_{B^*}$, which estimate (1.6) illustrates, has a counterpart. Namely, the homogeneity of the space B prevents it to be a subspace of L^2_{loc}, due to a divergence close to the origin $x = 0$. As the analysis provided in the present text shows, this fact turns out to create considerable difficulties while establishing (1.2).

Let us now come to more detailed statements.

Throughout this text, our assumptions on the potential V are the following:

(H1) $\lambda - V(x)$ is positive at infinity, i.e., there exists $R_0 > 0$ and $c_0 > 0$ such that

$$\forall |x| \geqslant R_0, \quad \lambda - V(x) \geqslant (c_0)^2 > 0. \tag{1.7}$$

(H2) $\lambda - V(x)$ is positive at the origin:

$$\lambda - V(0) > 0. \tag{1.8}$$

(H3) The zero energy is non-trapping for the Hamiltonian flow of $H(x,\xi) := \xi^2 + V(x) - \lambda$. In other words, for any (x,ξ) such that $\xi^2 + V(x) - \lambda = 0$, the trajectory $(X(t), \Xi(t))$ issued from (x,ξ), which satisfies Hamilton's equations

$$\begin{cases} dX(t)/dt = 2\Xi(t), & X(0) = x, \\ d\Xi(t)/dt = -\nabla_x V(X(t)), & \Xi(0) = \xi, \end{cases} \tag{1.9}$$

is required to go to infinity as time goes to infinity:

$$\left|X(t)\right| \underset{t\to\pm\infty}{\longrightarrow} \infty. \tag{1.10}$$

(H4) The potential V is bounded as well as its first two derivatives, i.e., $V \in C_b^2(\mathbb{R}^d)$. For later convenience, we define $c_1 > 0$ as

$$\left\|\lambda - V(x)\right\|_{L^\infty(\mathbb{R}^d)} =: (c_1)^2 < \infty. \tag{1.11}$$

(H5) The first and second derivatives of V decay faster than $1/|x|$ at infinity. More precisely, there exists a $\rho > 0$,[3] and a $C_\rho > 0$, such that:[4]

$$\forall x \in \mathbb{R}^d, \quad \left|\partial_x V(x)\right| \leqslant C_\rho \langle x\rangle^{-1-\rho}, \qquad \left|\partial_x^2 V(x)\right| \leqslant C_\rho \langle x\rangle^{-1-\rho}. \tag{1.12}$$

Under all the above assumptions (commented below), we prove the following theorem.

Main Theorem. *Assume $f \in B$. Then, for any $\alpha > 0$ and $\varepsilon > 0$, there exists a unique solution $u \in B^*$ to* (1.1). *Under assumptions* (H1)–(H5)*, and provided the space dimension satisfies $d \geqslant 3$, there exists a constant C, and an $\varepsilon_0 > 0$, such that for any $\alpha > 0$, and any $0 < \varepsilon \leqslant \varepsilon_0$, we have*

$$\|u\|_{B^*} \leqslant \frac{C}{\varepsilon}\|f\|_B. \tag{1.13}$$

Remark 1. The uniform estimate (1.13) is enough to establish the limiting absorption principle, i.e., to pass to the limit $\alpha \to 0^+$ in (1.1), whenever $f \in B$. The limiting value of u obtained in this way still belongs to the space B^*.

More importantly, and as the reader may easily check, our Main Theorem *implies* the following *inhomogeneous* B–B^* estimates as well:

$$\|u\|_{B^*_{\text{inh}}} \leqslant \frac{C}{\varepsilon}\|f\|_{B_{\text{inh}}}, \tag{1.14}$$

where the inhomogeneous B–B^* norms are

$$\|u\|_{B^*_{\text{inh}}} := \|u\|_{L^2(B(0,1))} + \sup_{j\geqslant 0} 2^{-j/2}\|u\|_{L^2(C_j)},$$

$$\|f\|_{B_{\text{inh}}} := \|f\|_{L^2(B(0,1))} + \sum_{j\geqslant 0} 2^{j/2}\|f\|_{L^2(C_j)}.$$

[3] Without loss of generality, we assume throughout the paper that $\rho \leqslant 1$.

[4] Here and below, we use the standard notation $\langle x\rangle = (1+x^2)^{1/2}$.

It also implies the usual *inhomogeneous* weighted L^2 estimates, valid for any given $s > 1$,

$$\|u\|_{L^2(\langle x\rangle^{-s}\,dx)} \leqslant \frac{C}{\varepsilon}\|f\|_{L^2(\langle x\rangle^{s}\,dx)}. \tag{1.15}$$

In this sense, estimate (1.13) is optimal in the context. The B and B^* spaces correspond to the limiting case $s = 1$ in (1.15).

Note in passing that the spaces B and B^* are in duality:

$$\left|\langle f, u\rangle\right| = \left|\sum_{j\in\mathbb{Z}}\langle f, u\rangle_{L^2(C_j)}\right| \leqslant \|f\|_B\,\|u\|_{B^*}. \tag{1.16}$$

Similarly, the inhomogeneous B–B^* spaces, together with the weighted L^2 spaces $L^2(\langle x\rangle^{-s}\,dx)$ and $L^2(\langle x\rangle^{s}\,dx)$, are in duality.

Remark 2. The constraint $d \geqslant 3$ in the Main Theorem is natural, since the Helmholtz equation somehow degenerates in dimensions 2 and 1. It might be, however, that our Main Theorem holds in dimension $d = 2$ as well. Technically, the need for a dimension larger than 3 comes up in Section 9 only, where we make use of the Perthame and Vega multiplier method [28].

Let us comment on our assumptions.

Assumptions (H1) through (H3) are the main geometric hypotheses needed to establish (1.13). As is well known, the energy $|u(x)|^2$ in the Helmholtz equation is "propagated" along the rays of geometric optics, i.e., along the solutions $(X(t), \Xi(t))$ to Hamilton's equations (1.9) with zero energy. In particular, these rays propagate with a "speed" ξ given by $\xi^2 = \lambda - V(x)$. In this picture, all three assumptions (H1) through (H3) tend to ensure that energy is "well propagated" from bounded regions of space up to unbounded regions, at a non-zero speed.

More precisely, (H1) means the speed of propagation of waves is uniformly non-zero at infinity in x, so that the energy $|u(x)|^2$ cannot "accumulate" at infinity in x. This is a minimal assumption, and the estimates we provide here become false, in general, when (H1) is not fulfilled.

Similarly, (H2) means the energy brought by the source term f is shot from the origin $x = 0$ with a non-zero initial speed, hence it immediately leaves $x = 0$. The special role played by the origin is dictated by the very norms B and B^* we use in the sequel, for which $x = 0$ is a distinguished point. We do not know whether (H2) is an optimal assumption or not.

Finally, the non-trapping hypothesis (H3) is a standard assumption. It is in the same spirit, though much weaker, than the virial condition, see (1.18) below. The non-trapping condition ensures the rays of geometric optics leave any compact set after some finite time, preventing again accumulation of energy in bounded regions of space. When the potential is C^∞ and has the long-range behaviour, i.e., when

$$\forall\alpha \in \mathbb{N}^d, \quad \left|\partial_x^\alpha V(x)\right| \leqslant C_\alpha\langle x\rangle^{-|\alpha|-\rho'}, \tag{1.17}$$

for some $\rho' > 0$, the non-trapping assumption is known as a necessary and sufficient condition to have weighted L^2 estimates (1.15) on $u(x)$, as proved by Wang in [31]. Since weighted L^2 estimates are *implied* by the B–B^* estimates we provide here (see Remark 1), non-trapping is clearly a minimal assumption, and the estimates we provide here become false in general when (H3) is not fulfilled.

The last two assumptions (H4) and (H5) are of more technical nature.

Assumption (H4) requires $V \in C_b^2$. This is a natural condition in order to have a well-defined and smooth, C^1, Hamiltonian flow for $H(x,\xi) = \xi^2 + V(x) - \lambda$. Hence it is a natural assumption in order to even *state* the non-trapping condition. One aspect of the present result precisely lies in this low regularity on V. In many places, this lack of smoothness prevents us from using standard pseudo-differential calculus, making the proof much more involved than in the C^∞ case. In any circumstance, our method can probably *not* go below the C^2 regularity, though the $C^{1,1}$ regularity may be attainable. However, it might be that our Main Theorem still holds true for less smooth, "non-trapping," potentials.

The last condition (H5) is a weak version of the long-range condition (1.17) quoted before. Assumption (H5) actually implies the potential goes to a constant (zero) at infinity. Note that an even weaker assumption on the potential is required in Ref. [29]: in this text the authors merely assume the potential V possesses *radial limits* as $|x| \to \infty$, i.e.,

$$V_\infty(\hat{x}) = \lim_{r \to +\infty} V(r\hat{x}) \quad \text{exists}, \quad \hat{x} = x/|x|,$$

where $V_\infty(\hat{x})$ is not assumed to be a constant. We do not know whether our analysis may include potentials that are non-constant at infinity.

We end this introduction with bibliographical comments.

First, for fixed values of ε (say $\varepsilon = 1$), we recall that the Mourre theory has proved to be a powerful tool in order to show the limiting absorption principle in a very general setting. We refer to [27], as well as [9,20], and, more recently, to [3]. The Mourre theory typically allows to recover the *inhomogeneous* bounds (1.14) and (1.15) when $\varepsilon = 1$. We also refer to [1] for similar inhomogeneous bounds.

Next, when $\varepsilon > 0$ goes to zero, many results in the literature establish the *inhomogeneous* bounds (1.14) and (1.15) for C^∞ potentials satisfying the non-trapping condition, and having the long range behaviour, see (1.17). We refer to [16,30,31]. These works use a parameter dependent version of the Mourre theory, and the key tool is the construction of a so-called "global escape function." We stress the fact that, when the potential is C^∞ and long-range, Wang's result [31] establishes that the weighted L^2 bound (1.15) actually is *equivalent* with the non-trapping condition.

In this perspective, the present paper aims at dealing with a potential satisfying the minimal smoothness and geometric requirements, and at going to the optimal, *homogeneous*, B–B^* estimates.

The case of B and B^* spaces has been treated recently by Perthame and Vega in [28], for a potential with limited, C^1 regularity. They use a new and elegant multiplier method in the spirit of the Morawetz estimate [26], and the work by Lions and Perthame for kinetic equations [24]. The inequality $\|u\|_{B^*} \leqslant C\varepsilon^{-1}\|f\|_B$, is established in [28], for potentials that are non-positive, i.e., $V(x) - \lambda \leqslant 0$. Their work also requires potentials V that satisfy the following *virial condition*:

$$2\sum_{j\in\mathbb{Z}} \sup_{x\in C_j} \frac{(x\cdot\nabla_x[\lambda - V(x)])_-}{\lambda - V(x)} < 1, \tag{1.18}$$

where the subscript "−" means "negative part of." Here, the assumption that $V(x) - \lambda \leqslant 0$ is a reinforced version of our requirements (1.7) and (1.8): it gives a uniformly non-zero speed of propagation at any point x in space. More importantly, the virial condition requires, in essence,

$2[\lambda - V(x)] + x \cdot \nabla_x[\lambda - V(x)] \geqslant c > 0$. A short computation shows that the virial condition *implies* the non-trapping assumption (1.10). It implies even more, namely all the trajectories $X(t)$ in (1.10) satisfy $|X(t)| \geqslant c|t|$. Hence the virial condition clearly is a reinforced version of the non-trapping condition (see the book [11]).

Under similar virial-like conditions, Fouassier has recently established in [13] a B–B^* estimate for a potential having discontinuities. Again, for discontinuous potentials, and in the easier case when $\varepsilon = 1$, we wish to quote De Bièvre and Pravica's result [10].

In a very different spirit, Burq has introduced in [5] an original method to derive estimates in Helmholtz-like equations posed in bounded domains: his method is based on a contradiction argument, and uses semiclassical measures as a key ingredient of the proof. His approach has been recently adapted in [18] to establish the weighted L^2 estimate (1.15), for a C^∞ and long-range potential. The argument in [18] uses, as an additional ingredient, an "escape function at infinity" that gives the key tool to deduce the relevant estimates. The idea of using escape functions in [18] actually comes from the so-called Mourre estimates in scattering theory.

The present paper may be seen as an extension of the work by Perthame and Vega [28], in order to replace the virial condition, by the weaker non-trapping assumption. Our method of proof combines some estimates derived in [28], together with the idea of using an escape function at infinity as in [18], and the contradiction argument introduced by Burq [5]. In this perspective, the present approach has the advantage of decoupling the phenomena that occur at infinity in x, from those occurring locally in x. This is a new point in comparison with the global approach developed by Perthame and Vega.

Another important motivation for the present work stems from the analysis of high-frequency Helmholtz equations with source terms performed in [4,7] (see also [8]), and more recently [6]. In particular, the difficult question of analyzing the *radiation condition at infinity* for such equations is analyzed in detail in [6]. This work uses the non-trapping condition, and gives a quite detailed geometric analysis of the propagation phenomena in this context. We also wish to quote the recent results by Wang and Zhang [32] on these questions. Here, various original and elegant bounds are proved in the context, using however virial-like assumptions.

To end this introduction, we mention that Mourre's theory and Burq's method have also been used to derive weighted L^2 estimates of the form (1.15) for u's and f's satisfying a *system* of Helmholtz equations, or, in other words, to estimate the resolvent of *matrix-valued* Schrödinger operators. We refer to [17,19,21,22].

2. Outline of the proof

We prove (1.13) by contradiction, using an idea originally due to Burq [5]. In the core of the proof, the propagation phenomena at infinity are treated using an escape function at infinity, in the spirit of [18]. On the other hand, the propagation phenomena at the origin are dealt with using the Perthame and Vega multipliers [28]. The non-trapping assumption makes the link between both regions of space.

Let us come to a more detailed description.

First, for any given $f \in B$, $\alpha > 0$ and $\varepsilon > 0$, an immediate energy estimate gives, using the fact that the spaces B and B^* are in duality (see (1.16)), the a priori bound $\|u\|_{B^*} \leqslant \alpha^{-1}\|f\|_B$. Hence u clearly is uniquely defined, and belongs to the space B^*. This trivial remark being made, the point is to prove the uniform estimate (1.13).

We argue by contradiction and assume that inequality (1.13) is false. Hence we may build up sequences $\alpha_n > 0$, $\varepsilon_n > 0$, as well as f_n and u_n, such that

$$\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0, \qquad \alpha_n \underset{n\to\infty}{\longrightarrow} \alpha \in \mathbb{R}^+ \cup \{+\infty\},$$
$$\varepsilon_n^{-1} \|f_n\|_B =: \eta_n \underset{n\to\infty}{\longrightarrow} 0, \qquad \|u_n\|_{B^*} = 1, \tag{2.1}$$

while the high-frequency Helmholtz equation is satisfied

$$+i\varepsilon_n\alpha_n u_n(x) - \varepsilon_n^2 \Delta_x u_n(x) + V(x)u_n(x) - \lambda u_n(x) = f_n(x). \tag{2.2}$$

Under these circumstances, we claim we may assume

$$\alpha = \lim \alpha_n = 0. \tag{2.3}$$

Indeed, the obvious energy estimate yields

$$\alpha_n \|u_n\|^2_{L^2(\mathbb{R}^d)} = \varepsilon_n^{-1} \operatorname{Im}\langle f_n, u_n\rangle \leqslant \varepsilon_n^{-1} \|f_n\|_{B^*} \|u_n\|_{B^*} = \eta_n \underset{n\to\infty}{\longrightarrow} 0, \tag{2.4}$$

which implies $\alpha_n \to 0$ *or* $\|u_n\|_{L^2} \to 0$. In the latter case, using the information $\|u_n\|_{B^*} = 1$, we recover $\int_{|x|\geqslant r} |u_n(x)|^2\,dx \to 0$, for any $r > 0$. From this, it turns out the argument we give in Section 9 allows then to obtain a contradiction quite trivially (see Remark 15 after Proposition 10). Hence we are left with the case $\alpha_n \to 0$.

Now, to analyze the asymptotic behaviour of the sequence $\{u_n\}$, we readily observe that $B^* \subset L^2_{\mathrm{loc}}$. For that reason, the sequence $\{u_n\}$ is bounded in L^2_{loc} at least. It then becomes natural to define $\mu(x,\xi)$ as the (unique, up to subsequences) semi-classical measure associated to u_n. It satisfies,

$$\left\langle \big(a(x,\xi)\big)^w_{\varepsilon_n} u_n, u_n \right\rangle \underset{n\to\infty}{\longrightarrow} (2\pi)^{-d} \int_{T^*\mathbb{R}^d} a(x,\xi)\mu(dx,d\xi),$$

whenever a is a symbol which belongs to $C_c^\infty(T^*\mathbb{R}^d)$. Here and throughout the paper, we use the notation $a^w_{\varepsilon_n}$ or, equivalently, $[a(x,\xi)]^w_{\varepsilon_n}$, for the ε_n-Weyl quantization of $a(x,\xi)$. In other words, for any $v(x)$, we set

$$\begin{aligned}\big[a^w_{\varepsilon_n} v\big](x) &= \big[\big(a(x,\xi)\big)^w_{\varepsilon_n} v\big](x) \\ &= \int_{\mathbb{R}^{2d}} \frac{dy\,d\xi}{(2\pi\varepsilon_n)^d} \exp\left(i\frac{(x-y)\cdot\xi}{\varepsilon_n}\right) a\left(\frac{x+y}{2},\xi\right) v(y).\end{aligned} \tag{2.5}$$

It is a well-known fact that μ is a non-negative Radon measure over $T^*\mathbb{R}^d$ (see [14,15]). Equivalently, the measure μ may be defined as the limit, in the distribution sense, of the Wigner transform $\mu_n(x,\xi)$ associated with the sequence $\{u_n\}$, i.e.,

$$\begin{aligned}\mu_n(x,\xi) &:= \int_{\mathbb{R}^d} \exp(-iy\cdot\xi) u_n\left(x + \varepsilon_n\frac{y}{2}\right) u_n^*\left(x - \varepsilon_n\frac{y}{2}\right) dy \\ &= \mathcal{F}_{y\to\xi}\left(u_n\left(x+\varepsilon_n\frac{y}{2}\right)u_n^*\left(x-\varepsilon_n\frac{y}{2}\right)\right).\end{aligned} \tag{2.6}$$

With these notations, we have

$$\mu_n(x,\xi) \underset{n\to\infty}{\longrightarrow} \mu(x,\xi) \quad \text{in } \mathcal{D}'\big(\mathbb{R}^{2d}\big). \tag{2.7}$$

Also, μ_n satisfies the useful algebraic property, whenever the symbol a is smooth enough,

$$\big\langle \big(a(x,\xi)\big)^w_{\varepsilon_n} u_n, u_n\big\rangle = \big\langle a(x,\xi), \mu_n\big\rangle. \tag{2.8}$$

Adopting this "semiclassical measures" point of view, the contradiction then comes as follows. On the one hand, the information $\|u_n\|^2_{B^*} = 1$ roughly gives that the "mass" of the limiting measure $\mu = \lim \mu_n$ is one. On the other hand, the several steps given below allow to establish that the measure μ is zero everywhere. This is, in essence, the contradiction.

Let us now describe more precisely the way we prove that $\mu = 0$ everywhere.

First, in Section 6, we prove the measure μ carries "no mass" away from resonant frequencies, i.e., away from ξ's such that $\xi^2 = \lambda - V(x)$. This is the meaning of Propositions 4–6, which actually give a much stronger result. Naturally, this fact is fairly standard when dealing with L^2 solutions of PDEs with C^∞ coefficients. It is generally obtained through symbolic calculus. Here, the difficulty is twofolds. On the one hand, the potential V has limited smoothness (it is only C^2). On the other hand, we are here dealing with a sequence $\{u_n\}$ which is merely bounded in B^* (and not in L^2), a space of the type $l^\infty(L^2)$ which encodes in the optimal way the integrability properties of the sequence $\{u_n\}$. These two facts prevent from using standard symbolic calculus in any direct fashion.

For that reason, in Section 3, we preliminarily develop alternative tools that "replace" symbolic calculus in our specific context. This is not only a technical question, because the limited smoothness, and the optimality of the B^* estimates, are a central aspect of our effort. In that direction, Proposition 1 identifies in a close to optimal way those symbols $a(x,\xi)$ such that $a^w_{\varepsilon_n}$ acts continuously on B and/or on B^*, thus preserving the sharp decay encoded in the B and/or B^* norms (this is the "$l^\infty(L^2)$" aspect). Such symbols are only required to depend continuously on the variable x (this is the "low regularity" aspect). The proof of Proposition 1 is the most difficult task in order to get an appropriate "symbolic calculus" in the present situation. Then, as a direct application of Proposition 1, it is established in Proposition 3 that μ satisfies the usual transport equation along the rays of geometric optics, while Proposition 2 establishes a version of the Garding inequality that is suited to our purposes.

At this level, we have only proved that μ vanishes away from resonant frequencies ξ. There remains to prove that μ vanishes close to resonant frequencies. This task is achieved upon distinguishing large, moderate, and small values of the space variable x.

Section 7 is a key step of the present paper. There, we prove the measure μ has no mass at infinity in x. This is Proposition 7, which actually gives a much stronger result. This step uses assumption (1.7) on the potential, according to which $\lambda - V(x)$ is positive at infinity. In the spirit of [18], our proof uses an "escape function at infinity," denoted by $a_J(x,\xi)$ in the text (see (7.8)). This function a_J satisfies, very roughly, the following inequality, valid for sufficiently large values of $x \geqslant 2^J$:

$$\big\{\xi^2 + V(x) - \lambda, a_J(x,\xi)\big\} \geqslant \sum_{j\geqslant J}\bigg(\frac{\mathbf{1}[x\in C_j]}{|x|}\beta_j\bigg), \tag{2.9}$$

where $\beta_j \geqslant 0$ is an l^1 sequence and $\{\cdot,\cdot\}$ denotes the Poisson bracket. Relation (2.9) is then combined with the appropriate symbolic calculus developed in Section 3, to deduce

$$\left\| u_n(x)\mathbf{1}\left[|x| \geqslant 2^J\right]\right\|_{B^*} \underset{n\to\infty}{\longrightarrow} 0,$$

for J large enough. The basic estimate (2.9) is very much in the spirit of the Mourre estimate. It means that the function a_J grows *at least* like $\mathbf{1}[x \in C_j]\beta_j/|x|$ along the Hamiltonian flow of $\xi^2 + V(x) - \lambda$ (at infinity in x), a quantity that decays slightly faster than $1/|x|$ at infinity (thanks to the l^1 sequence β_j). Hence (2.9) somehow prescribes a minimal rate at which energy escapes to infinity in the Helmholtz equation. This rate is directly related with the weighted norms for which the solution to the Helmholtz equation remains bounded. The construction of the escape function a_J should also be compared with the multiplier method of Perthame and Vega. In our case, a_J is essentially $\frac{x}{|x|} \cdot \frac{\xi}{|\xi|}$, plus a crucial corrective factor, and we stress the similar multiplier $\frac{x}{|x|} \cdot \nabla_x +$ corrector is used in [28].

In Section 8, we prove the measure μ vanishes for moderate values of x, i.e., for values that are bounded away from infinity and from zero. This is Proposition 9, which actually gives a much stronger result. The proof goes as follows. The previous step establishes $\mu = 0$ at infinity in x. On the other hand, μ is invariant along the Hamiltonian flow of $H(x,\xi) = \xi^2 + V(x) - \lambda$. Besides, the non-trapping assumption states that the relevant flow sends bounded value of x outside any compact set. Hence μ necessarily vanishes for bounded values of x as well. This is the point where we use non-trapping in our proof.

Last, in Section 9, we prove μ vanishes close to the origin as well. This is Proposition 10, which actually gives a much stronger result. In the present paper, the origin needs a special treatment, because the homogeneous norms B and B^* are somehow singular at $x = 0$. To show the measure μ also vanishes close to the origin, we go back to an estimate established in [28]. This estimate relates (amongst others) the mass close to the origin with the mass at infinity. Then, using the already established fact that the mass vanishes away from any neighbourhood of the origin, we deduce μ vanishes at $x = 0$ as well. This step uses assumption (1.8), according to which $\lambda - V(0) > 0$.

The remainder part of the paper is devoted to the proof of all the above mentioned steps. We present the proof in the case where V has limited smoothness. We mention without giving further details that the proof might be shortened in various places when V is smooth ($V \in C^\infty$) and V satisfies the long-range condition (1.17).

Notation. Throughout this text, we will use the notation $a^w_{\varepsilon_n}$ for the ε_n Weyl quantized operator associated with the symbol $a(x,\xi)$, see (2.5) (see also [12]). Also, we sometimes will need symbols $a(x,\xi)$ that are C^∞ and satisfy

$$\forall\alpha \in \mathbb{N}^d,\ \forall\beta \in \mathbb{N}^d, \quad \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leqslant C_{\alpha,\beta}\langle x\rangle^{m-|\alpha|}, \tag{2.10}$$

for some real number m and some constants $C_{\alpha,\beta} > 0$. Such symbols will be said to belong to the class $S(\langle x\rangle^m)$. Last, we normalize the Fourier transform as

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-ix\cdot\xi} u(x)\,dx, \qquad u(x) = \mathcal{F}^{-1}(\hat{u})(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{+ix\cdot\xi}\hat{u}(\xi)\,d\xi. \tag{2.11}$$

3. Symbolic calculus on B and B^*, for symbols having limited smoothness

In this section, we give (close to) optimal conditions on the function $\varphi(x,\xi)$, so that the Weyl operator $\varphi^w_{\varepsilon_n}$ is bounded from B^* to B^*. We also give conditions that make it bounded[5] from B^* to B. It gives us a symbolic calculus on the B and B^* spaces. These bounds will be of constant use in the sequel.

Here, the word "optimal" refers to two aspects. First, we deal here with symbols $\varphi(x,\xi)$ that are merely *continuous* in x. This will be very important in the sequel since our potential $V(x)$ only has C^2 regularity. Second, for $\varphi^w_{\varepsilon_n}$ to send B^* to B^*, we only require φ to be *bounded* in the x variable. This is a natural assumption. Third, for $\varphi^w_{\varepsilon_n}$ to send B^* to B, we only require φ to *decay slightly faster than* $1/|x|$ *at infinity*, in that φ should behave like $\sum_j \alpha_j \mathbf{1}[x\in C_j]/|x|$ for some sequence α_j that is l^1 in the j variable. This is also a natural requirement.

As a counterpart though, our symbols need to be very smooth and fastly decaying in the ξ variable.

The difficulty is twofolds. First, the very definition of $\langle \varphi^w_{\varepsilon_n} u_n, u_n\rangle = \langle \varphi, \mu_n\rangle$ (see (2.6)) involves a delocalisation in the two arguments of u_n, which has to be combined with the dyadic decomposition that is used to define the B^* norms. This is a technically delicate task in itself, and we refer to the splitting (3.10) and the associated analysis on this point. Second, while it is easily checked that B^* is a subset of L^2_{loc}, yet the space B *is not*: in the B norm, the small annuli, i.e., the C_j's for which j is large and negative, are allowed to carry a mass that may be infinite in the L^2_{loc} norm. This leads to a special treatment of the small annuli in the analysis below.

The main result of the present section is the following proposition.

Proposition 1. *Let $\varphi(x,\xi)\in C^0_b(\mathbb{R}^{2d})$ be a bounded, continuous function.*[6] *Define its Fourier transform, in the sense of distributions*

$$\hat\varphi(x,y) := \mathcal{F}_{\xi\to y}\big(\varphi(x,\xi)\big) = \int_{\mathbb{R}^d} \mathrm{e}^{-iy\cdot\xi}\varphi(x,\xi)\,d\xi.$$

Then, the following holds:

(1) (A rough bound on $\langle \varphi^w_{\varepsilon_n} u_n, u_n\rangle$.) *Assume $\hat\varphi(x,y)$ has the decay property*

$$\|\varphi\|_{W_s} := \int_{\mathbb{R}^d} \langle y\rangle^s \sup_{x\in\mathbb{R}^d} \langle x\rangle^s \big|\hat\varphi(x,y)\big|\,dy < \infty, \quad \textit{for some } s>1. \tag{3.1}$$

Then, there is a constant $C_s>0$, that depends on s only, such that

$$\big|\langle \varphi^w_{\varepsilon_n} u_n, u_n\rangle\big| = \big|\langle \mu_n, \varphi\rangle\big| \leqslant C_s \|\varphi\|_{W_s}\|u_n\|^2_{B^*}. \tag{3.2}$$

[5] Strictly speaking, bounded from B^* to the second dual B^{**}. We shall not dwell on the difference between B^{**} and B, since it anyhow plays absolutely no role in our analysis.

[6] Strictly speaking, the assertions in this proposition should first be stated for very smooth φ's, satisfying $\hat\varphi(x,y)\in C^\infty_c(\mathbb{R}^{2d})$, and the natural density argument should be performed next. We do not dwell on that harmless point.

(2) (An "optimal" bound on $\langle \varphi^w_{\varepsilon_n} u_n, u_n\rangle$.) *Assume $\hat\varphi(x,y)$ has the decay property*

$$\|\varphi\|_{X_N} := \sup_{|x|\leqslant 1}\sup_{y\in\mathbb{R}^d}\langle y\rangle^N\big|\hat\varphi(x,y)\big| + \sum_{j\in\mathbb{N}} 2^j \sup_{x\in C_j}\sup_{y\in\mathbb{R}^d}\langle y\rangle^N\big|\hat\varphi(x,y)\big| < \infty,$$
$$\textit{for some } N > d+1. \tag{3.3}$$

Then, there is a constant $C_N > 0$, that depends on N only, such that

$$\big|\langle \varphi^w_{\varepsilon_n} u_n, u_n\rangle\big| = \big|\langle \mu_n, \varphi\rangle\big| \leqslant C_N\|\varphi\|_{X_N}\|u_n\|^2_{B^*}. \tag{3.4}$$

(3) (An "optimal" bound on $\langle \varphi^w_{\varepsilon_n} u_n, f_n\rangle$.) *Assume $\hat\varphi(x,y)$ has the decay property*

$$\|\varphi\|_{Y_N} := \sup_{x\in\mathbb{R}^d}\sup_{y\in\mathbb{R}^d}\langle y\rangle^N\big|\hat\varphi(x,y)\big| < \infty, \quad \textit{for some } N > d+1. \tag{3.5}$$

Then, there is a constant $C_N > 0$, that depends on N only, such that

$$\big|\langle \varphi^w_{\varepsilon_n} u_n, f_n\rangle\big| = \left|\int_{\mathbb{R}^{2d}} u_n\left(x+\varepsilon_n\frac{y}{2}\right) f_n\left(x-\varepsilon_n\frac{y}{2}\right)\hat\varphi(x,y)\,dx\,dy\right| \leqslant C_N\|\varphi\|_{Y_N}\|u\|_{B^*}\|f_n\|_B. \tag{3.6}$$

Remark 3. The bound in point (1) will *not* be used in the sequel, being too weak for our purposes. At variance, points (2) and (3) will be exploited repeatedly. We feel the (easy) proof of point (1) is very illustrative, and allows to smoothly introduce the technically more delicate proof of points (2) and (3). This is the reason why we still give statement (1) here, and its proof.

The bounds of points (1)–(3) naturally hold as well when u_n and f_n are replaced by *any* functions $u \in B^*$ and $f \in B$. In that perspective, the above proposition roughly asserts that the operator $\varphi^w_{\varepsilon_n}$ sends B^* in B continuously, provided $\varphi \in W_s$ $(s > 1)$, or $\varphi \in X_N$ $(N > d+1)$. Similarly, it asserts that $\varphi^w_{\varepsilon_n}$ sends B^* in B^* continuously, provided $\varphi \in Y_N$.

Remark 4. Points (1) and (2) of the above proposition immediately allow to improve the weak convergence $\langle \mu_n, \varphi\rangle \to \langle \mu, \varphi\rangle$, valid for test functions $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ (see (2.7)), into the stronger convergence $\langle \mu_n, \varphi\rangle \to \langle \mu, \varphi\rangle$ for test functions in the larger space $\varphi \in W_s$ $(s > 1)$, or even for $\varphi \in X_N$ $(N > d+1)$. In particular, μ acts continuously on the space of test functions $\varphi \in W_s$ $(s > 1)$ or $\varphi \in X_N$ $(N > d+1)$. This implies the measure μ grows at most polynomially at infinity in the x variable.

Remark 5. Both norms W_s $(s > 1)$ and X_N $(N > d+1)$ require, in essence, that $\varphi(x,\xi)$ should decay slightly faster than $\langle x\rangle^{-1}$ at infinity in x (and it should be smooth enough in the ξ variable). Similarly, the norm Y_N requires that $\varphi(x,\xi)$ is merely bounded in the x variable (and it should be smooth as well in ξ). Note in passing the clear continuous embedding $X_N \subset Y_N$.

From the point of view of decay at infinity in x, the above bounds are thus very natural, since the space B^* (respectively B) is essentially a weighted L^2 space, with a weight that grows slightly slower than $\langle x\rangle$ at infinity (respectively decays slightly faster than $\langle x\rangle^{-1}$). Note that the X_N norm gives an "optimal" version of the W_s norm, corresponding to the limiting decay $\langle x\rangle^{-s}$ with $s = 1$.

We stress the X_N norm is *inhomogeneous* in x: small annuli C_j ($j \leqslant 0$) are put apart in this norm. The homogeneous version of the X_N norm would be

$$\sum_{j\in\mathbb{Z}} 2^j \sup_{x\in C_j} \sup_{y\in\mathbb{R}^d} \langle y\rangle^N \left|\hat{\varphi}(x,y)\right|.$$

Remark 6. As already mentioned, the norms W_s, X_N, and Y_N do not require regularity in the x variable, and $\varphi(x,\xi)$ only needs to be *continuous* in x. As a counterpart, strong smoothness of $\varphi(x,\xi)$ in the ξ variable is required: roughly, $\partial_\xi^\alpha \varphi(x,\xi)$ should be integrable in ξ, up to the order $|\alpha| = d+1$.

Proof of Proposition 1. *Proof of part* (1). Assuming $\varphi(x,\xi)$ is smooth enough ($\varphi \in \mathcal{S}(\mathbb{R}^d)$ will do), one may write,

$$\left|\langle \mu_n, \varphi\rangle\right| = \left| \int_{\mathbb{R}^{2d}} u_n\left(x + \varepsilon_n \frac{y}{2}\right) u_n^*\left(x - \varepsilon_n \frac{y}{2}\right) \hat{\varphi}(x,y)\, dx\, dy \right|. \tag{3.7}$$

On the other hand, we clearly have the rough estimate

$$\left\| \langle x\rangle^{-s} u_n \right\|_{L^2} \leqslant C_s \|u_n\|_{B^*}, \quad \text{whenever } s > 1/2. \tag{3.8}$$

Hence we may upper-bound (3.7) in the following way

$$\begin{aligned}\left|\langle \mu_n, \varphi\rangle\right| &\leqslant \int_{\mathbb{R}^{2d}} \frac{|u_n|(x+\varepsilon_n y/2)}{\langle x+\varepsilon_n y/2\rangle^s} \frac{|u_n|(x-\varepsilon_n y/2)}{\langle x-\varepsilon_n y/2\rangle^s} \left\langle x + \varepsilon_n \frac{y}{2}\right\rangle^s \left\langle x - \varepsilon_n \frac{y}{2}\right\rangle^s \left|\hat{\varphi}(x,y)\right| dx\, dy \\ &\leqslant C_s \|u_n\|_{B^*}^2 \int_{\mathbb{R}^d} \langle y\rangle^{2s} \sup_{x\in\mathbb{R}^d} \langle x\rangle^{2s} \left|\hat{\varphi}(x,y)\right| dy.\end{aligned}$$

This establishes (3.2).

Proof of part (2). We take a smooth test function $\varphi(x,\xi)$, and observe again the standard identity

$$\langle \mu_n, \varphi\rangle = \left\langle \left(\varphi(x,\xi)\right)_{\varepsilon_n}^w u_n, u_n \right\rangle = \int_{\mathbb{R}^{2d}} \hat{\varphi}(x,y) u_n\left(x + \varepsilon_n \frac{y}{2}\right) u_n^*\left(x - \varepsilon_n \frac{y}{2}\right) dx\, dy.$$

Hence we upper bound

$$\left|\langle \mu_n, \varphi\rangle\right| \leqslant \int_{\mathbb{R}^{2d}} \left|u\left(x + \varepsilon_n \frac{y}{2}\right)\right| \left|u\left(x - \varepsilon_n \frac{y}{2}\right)\right| \left|\hat{\varphi}(x,y)\right| dx\, dy =: \int_{\mathbb{R}^{2d}} B_n(x,y)\, dx\, dy. \tag{3.9}$$

This serves as a definition of $B_n(x,y)$. In order to estimate $\int B_n(x,y)\, dx\, dy$ in an optimal way, we decompose u_n into contributions due to the various annuli C_j. This is naturally imposed by the dyadic decomposition that defines the B^* norm of u_n. In doing so, it turns out that small annuli C_j (i.e., large negative values of j) need a specific analysis (see definition (3.3) of the X_N-norm).

Our first step is to decompose $\int B_n(x,y)\,dx\,dy$ upon distinguishing the relative size of x and $\varepsilon_n y$, as follows:

$$\int_{\mathbb{R}^{2d}} B_n(x,y)\,dx\,dy = \int_{|x|\geqslant\varepsilon_n|y|} B_n + \int_{|x|\leqslant\varepsilon_n|y|/4} B_n + \int_{\varepsilon_n|y|/4\leqslant|x|\leqslant\varepsilon_n|y|} B_n. \tag{3.10}$$

Each of the above terms is estimated separately. The first term in (3.10) is estimated upon using the Cauchy–Schwarz inequality in x:

$$\begin{aligned}
\int_{|x|\geqslant\varepsilon_n|y|} B_n &= \int_{|x|\geqslant\varepsilon_n|y|} \left|u_n\left(x+\varepsilon_n\frac{y}{2}\right)\right|\left|u_n\left(x-\varepsilon_n\frac{y}{2}\right)\right|\left|\hat\varphi(x,y)\right|dx\,dy \\
&= \sum_{j\in\mathbb{Z}} \int_{x\in C_j,\ |x|\geqslant\varepsilon_n|y|} \left|u_n\left(x+\varepsilon_n\frac{y}{2}\right)\right|\left|u_n\left(x-\varepsilon_n\frac{y}{2}\right)\right|\left|\hat\varphi(x,y)\right|dx\,dy \\
&\leqslant \sup_{j\in\mathbb{Z}}\sup_{y\in\mathbb{R}^d}\left\{2^{-j}\int_{x\in C_j,\ |x|\geqslant\varepsilon_n|y|}\left|u_n\left(x+\varepsilon_n\frac{y}{2}\right)\right|\left|u_n\left(x-\varepsilon_n\frac{y}{2}\right)\right|dx\right\} \\
&\quad\times\sum_{j\in\mathbb{Z}}2^j\int_{\mathbb{R}^d}\sup_{x\in C_j}\left\{\left|\hat\varphi(x,y)\right|\right\}dy \\
&\leqslant C_N\sup_{j\in\mathbb{Z}}\left(2^{-j}\|u_n\|_{L^2(\widetilde{C}_j)}\|u_n\|_{L^2(\widetilde{C}_j)}\right)\sum_{j\in\mathbb{Z}}2^j\sup_{x\in C_j}\sup_{y\in\mathbb{R}^d}\langle y\rangle^N\left|\hat\varphi(x,y)\right| \\
&\leqslant C_N\|\varphi\|_{X_N}\|u_n\|_{B^*}^2,
\end{aligned} \tag{3.11}$$

whenever $N>d$. Here we used the notation $\widetilde{C}_j := C_{j-1}\cup C_j\cup C_{j+1}$.

The second term in (3.10) is bounded in the same spirit, using this time the Cauchy–Schwarz inequality in y:

$$\begin{aligned}
\int_{|x|\leqslant\varepsilon_n|y|/4} B_n &= \int_{|x|\leqslant\varepsilon_n|y|/4} \left|u_n\left(x+\varepsilon_n\frac{y}{2}\right)\right|\left|u_n\left(x-\varepsilon_n\frac{y}{2}\right)\right|\left|\hat\varphi(x,y)\right|dx\,dy \\
&= \sum_{j\in\mathbb{Z}} \int_{y\in C_j,\ |y|\geqslant 4|x|/\varepsilon_n} \left|u_n\left(x+\varepsilon_n\frac{y}{2}\right)\right|\left|u_n\left(x-\varepsilon_n\frac{y}{2}\right)\right|\left|\hat\varphi(x,y)\right|dx\,dy \\
&\leqslant \sup_{j\in\mathbb{Z}}\sup_{x\in\mathbb{R}^d}\left\{2^{-j}\int_{y\in C_j,\ |y|\geqslant 4|x|/\varepsilon_n}\left|u_n\left(x+\varepsilon_n\frac{y}{2}\right)\right|\left|u_n\left(x-\varepsilon_n\frac{y}{2}\right)\right|dy\right\} \\
&\quad\times\sum_{j\in\mathbb{Z}}2^j\int_{\mathbb{R}^d}\sup_{y\in C_j,\ |y|\geqslant 4|x|/\varepsilon_n}\left\{\left|\hat\varphi(x,y)\right|\right\}dx,
\end{aligned}$$

and, rescaling x and y by ε_n in both integrals, we obtain

$$\int\limits_{|x|\leqslant\varepsilon_n|y|/4} B_n \leqslant \sup_{j\in\mathbb{Z}}\sup_{x\in\mathbb{R}^d}\left\{2^{-j}\varepsilon_n^{-1}\int\limits_{y\in\varepsilon_n C_j,\ |y|\geqslant 4|x|}\left|u_n\left(x+\frac{y}{2}\right)\right|\left|u_n\left(x-\frac{y}{2}\right)\right|dy\right\}$$
$$\times\sum_{j\in\mathbb{Z}}2^j\varepsilon_n\int\limits_{\mathbb{R}^d}\sup_{y\in C_j,\ |y|\geqslant 4|x|}\left\{\left|\hat\varphi(\varepsilon_n x,y)\right|\right\}dx.$$

Hence, making use of the X_N norm of φ to upper-bound the second factor, we recover the estimate

$$\int\limits_{|x|\leqslant\varepsilon_n|y|/4} B_n \leqslant C\|u_n\|_{B^*}^2\left[\int\limits_{\mathbb{R}^d}\langle x\rangle^{-N}dx\right]\left(\sum_{2^j\varepsilon_n\leqslant 1}2^j\varepsilon_n\sup_{|\varepsilon_n x|\leqslant 1}\sup_{y\in\mathbb{R}^d}\left\{\langle y\rangle^N\left|\hat\varphi(\varepsilon_n x,y)\right|\right\}\right.$$
$$\left.+\sum_{2^j\varepsilon_n\geqslant 1}2^j\varepsilon_n 2^{-Pj}\sup_{|\varepsilon_n x|\leqslant 2^j}\sup_{y\in\mathbb{R}^d}\left\{\langle y\rangle^{N+P}\left|\hat\varphi(\varepsilon_n x,y)\right|\right\}\right)$$
$$\leqslant C_N\|u_n\|_{B^*}^2\left(\|\varphi\|_{X_N}+\|\varphi\|_{X_{N+P}}\sum_{2^j\varepsilon_n\geqslant 1}2^j\varepsilon_n 2^{-Pj}\right)\leqslant C_{N,P}\|\varphi\|_{X_{N+P}}\|u_n\|_{B^*}^2, \tag{3.12}$$

whenever $N>d$ and $P>1$.

Finally, the third term in (3.10) is estimated upon using the Cauchy–Schwarz inequality in *both* x and y. Indeed, this term has the value,

$$\int\limits_{\varepsilon_n|y|/4\leqslant|x|\leqslant\varepsilon_n|y|} B_n=\varepsilon_n^{-d}\int\limits_{\mathbb{R}^{2d}}\left|u_n(X)\right|\left|u_n(Y)\right|\left|\hat\varphi\left(\frac{X+Y}{2},\frac{X-Y}{\varepsilon_n}\right)\right|$$
$$\times\mathbf{1}\left[\frac14|X-Y|\leqslant\frac12|X+Y|\leqslant|X-Y|\right]dX\,dY.$$

The reader may easily check, for any (large) integer N, the inequality

$$\mathbf{1}\left[\frac14|X-Y|\leqslant\frac12|X+Y|\leqslant|X-Y|\right]\left|\hat\varphi\left(\frac{X+Y}{2},\frac{X-Y}{\varepsilon_n}\right)\right|$$
$$\leqslant C_N\langle X/\varepsilon_n\rangle^{-N/2}\langle Y/\varepsilon_n\rangle^{-N/2}\sup_{x,y}\left\{\langle y\rangle^N|\hat\varphi|(x,y)\right\}$$
$$\leqslant C_N\langle X/\varepsilon_n\rangle^{-N/2}\langle Y/\varepsilon_n\rangle^{-N/2}\|\varphi\|_{X_N}. \tag{3.13}$$

Hence, the third term in (3.10) is estimated by

$$\int\limits_{\varepsilon_n|y|/4\leqslant|x|\leqslant\varepsilon_n|y|} B_n\leqslant C_N\varepsilon_n^{-d}\left(\int\limits_{\mathbb{R}^d}\left|u_n(X)\right|\left\langle\frac{X}{\varepsilon_n}\right\rangle^{-N/2}dX\right)\left(\int\limits_{\mathbb{R}^d}|u_n|(Y)\left\langle\frac{Y}{\varepsilon_n}\right\rangle^{-N/2}dY\right)\|\varphi\|_{X_N}$$
$$\leqslant C_N\|u_n\|_{B^*}^2\|\varphi\|_{X_N}\varepsilon_n^{-d}\left(\sum_{j\in\mathbb{N}}2^{j/2}\left\|\left\langle\frac{X}{\varepsilon_n}\right\rangle^{-N/2}\right\|_{L^2(X\in C_j)}\right)^2\leqslant C_N\|\varphi\|_{X_N}\|u\|_{B^*}^2, \tag{3.14}$$

whenever $N>d+1$. Putting together (3.11), (3.12) and (3.14), establishes (3.4).

Proof of part (3). This part of the analysis is similar to the proof given for point (2). We start with the following observation, similar to (3.9):

$$\begin{aligned}\left|\left\langle\left(\varphi(x,\xi)\right)^w_{\varepsilon_n}u_n, f_n\right\rangle\right| &\leqslant \int\limits_{\mathbb{R}^{2d}}\left|u_n\left(x+\varepsilon_n\frac{y}{2}\right)\right|\left|f_n\left(x-\varepsilon_n\frac{y}{2}\right)\right|\left|\hat{\varphi}(x,y)\right|dx\,dy\\ &=: \int\limits_{\mathbb{R}^{2d}}\widetilde{B}_n(x,y)\,dx\,dy. \qquad (3.15)\end{aligned}$$

The technique to estimate $\int\widetilde{B}_n(x,y)$ is the same as for $\int B_n(x,y)$. The only difference lies in the fact that we do not need to treat the small annuli C_j apart.

We write

$$\int\limits_{\mathbb{R}^{2d}}\widetilde{B}_n(x,y)=\int\limits_{|x|\geqslant\varepsilon_n|y|}\widetilde{B}_n+\int\limits_{|x|\leqslant\varepsilon_n|y|/4}\widetilde{B}_n+\int\limits_{\varepsilon_n|y|/4\leqslant|x|\leqslant\varepsilon_n|y|}\widetilde{B}_n. \qquad (3.16)$$

As before, the first term in (3.16) is estimated upon using the Cauchy–Schwarz inequality in x:

$$\begin{aligned}\int\limits_{|x|\geqslant\varepsilon_n|y|}\widetilde{B}_n &\leqslant \sum_{j\in\mathbb{Z}}\sup_{y\in\mathbb{R}^d}\left\{\int\limits_{x\in C_j,\ |x|\geqslant\varepsilon_n|y|}\left|u_n\left(x+\varepsilon_n\frac{y}{2}\right)\right|\left|f_n\left(x-\varepsilon_n\frac{y}{2}\right)\right|dx\right\}\\ &\quad\times\int\limits_{\mathbb{R}^d}\sup_x\left\{\left|\hat{\varphi}(x,y)\right|\right\}dy\\ &\leqslant C_N\sum_{j\in\mathbb{Z}}\left(\|u_n\|_{L^2(\widetilde{C}_j)}\|f_n\|_{L^2(\widetilde{C}_j)}\right)\sup_{x,y}\langle y\rangle^N\left|\hat{\varphi}(x,y)\right|\\ &\leqslant C_N\|\varphi\|_{Y_N}\|u_n\|_{B^*}\|f_n\|_B, \qquad (3.17)\end{aligned}$$

whenever $N>d$, and we used the notation $\widetilde{C}_j:=C_{j-1}\cup C_j\cup C_{j+1}$ as before.

The second term in (3.16) is bounded using the Cauchy–Schwarz inequality in y:

$$\begin{aligned}\int\limits_{|x|\leqslant\varepsilon_n|y|/4}\widetilde{B}_n &\leqslant \sum_{j\in\mathbb{Z}}\sup_{x\in\mathbb{R}^d}\left\{\int\limits_{y\in C_j,\ |y|\geqslant 4|x|/\varepsilon_n}\left|u_n\left(x+\varepsilon_n\frac{y}{2}\right)\right|\left|f_n\left(x-\varepsilon_n\frac{y}{2}\right)\right|dy\right\}\\ &\quad\times\int\limits_{\mathbb{R}^d}\sup_{|y|\geqslant 4|x|/\varepsilon_n}\left\{\left|\hat{\varphi}(x,y)\right|\right\}dx\\ &=\sum_{j\in\mathbb{Z}}\sup_{x\in\mathbb{R}^d}\left\{\int\limits_{y\in\varepsilon_n C_j,\ |y|\geqslant 4|x|}\left|u_n\left(x+\frac{y}{2}\right)\right|\left|f_n\left(x-\frac{y}{2}\right)\right|dy\right\}\\ &\quad\times\int\limits_{\mathbb{R}^d}\sup_{|y|\geqslant 4|x|}\left\{\left|\hat{\varphi}(\varepsilon_n x,y)\right|\right\}dx\leqslant C_N\|\varphi\|_{Y_N}\|u_n\|_{B^*}\|f_n\|_B, \qquad (3.18)\end{aligned}$$

whenever $N>d$.

Finally, the third term in (3.10) is estimated upon using the Cauchy–Schwarz inequality in *both* x and y. Indeed, this term has the value,

$$\int_{\varepsilon_n|y|/4\leqslant|x|\leqslant\varepsilon_n|y|} \widetilde{B}_n = \varepsilon_n^{-d} \int_{\mathbb{R}^{2d}} \big|u_n(X)\big|\big|f_n(Y)\big| \times \mathbf{1}\left[\frac{1}{4}|X-Y| \leqslant \frac{1}{2}|X+Y| \leqslant |X-Y|\right]\left|\hat{\varphi}\left(\frac{X+Y}{2}, \frac{X-Y}{\varepsilon_n}\right)\right| dX\,dY.$$

Estimating as before (see (3.13))

$$\mathbf{1}\left[\frac{1}{4}|X-Y| \leqslant \frac{1}{2}|X+Y| \leqslant |X-Y|\right]\left|\hat{\varphi}\left(\frac{X+Y}{2}, \frac{X-Y}{\varepsilon_n}\right)\right| \leqslant C_N\langle X/\varepsilon_n\rangle^{-N/2}\langle Y/\varepsilon_n\rangle^{-N/2}\sup\langle y\rangle^N\big|\hat{\varphi}(x,y)\big|,$$

gives the upper-bound

$$\begin{aligned}\int_{\varepsilon_n|y|/4\leqslant|x|\leqslant\varepsilon_n|y|} \widetilde{B}_n &\leqslant C_N\|\varphi\|_{Y_N}\varepsilon_n^{-d}\left(\int_{\mathbb{R}^d}\big|u_n(X)\big|\left\langle\frac{X}{\varepsilon_n}\right\rangle^{-N/2} dX\right) \\ &\quad\times\left(\int_{\mathbb{R}^d}\big|f_n(Y)\big|\left\langle\frac{Y}{\varepsilon_n}\right\rangle^{-N/2} dY\right) \\ &\leqslant C_N\|\varphi\|_{Y_N}\|u_n\|_{B^*}\|f_n\|_B\varepsilon_n^{-d} \\ &\quad\times\left(\sum_{j\in\mathbb{N}}2^{j/2}\left\|\left\langle\frac{X}{\varepsilon_n}\right\rangle^{-N/2}\right\|_{L^2(X\in C_j)}\right)\left(\sup_{j\in\mathbb{N}}2^{j/2}\left\|\left\langle\frac{X}{\varepsilon_n}\right\rangle^{-N/2}\right\|_{L^2(X\in C_j)}\right) \\ &\leqslant C_N\|\varphi\|_{Y_N}\|u_n\|_{B^*}\|f_n\|_B, \end{aligned}\tag{3.19}$$

whenever $N > d+1$.

Putting together (3.17)–(3.19) gives estimate (3.6) of point (3). □

4. First consequence: a version of the Garding inequality

This section is devoted to the proof of a consequence of Proposition 1, namely, a version of the Garding inequality, valid for symbols with limited smoothness. This result plays a key role in our analysis (see Section 7).

Needless to say, the result we give here is completely standard for smooth symbols, acting on L^2 functions.

The main result of the present section is the following proposition.

Proposition 2 *(Garding's inequality). Let $\varphi(x,\xi) \in C_b^0(\mathbb{R}^{2d})$ be a bounded and continuous*[7] *function. Assume φ has the regularity $\varphi \in X_N$ (see* (3.3)*), together with $D_x\varphi \in X_N$, for some*

[7] Again, this assertion should first be stated for very smooth φ's, and the natural density argument should be performed in a second step.

$N > d+2$. Assume, finally, that φ is a non-negative function

$$\varphi(x,\xi) \geqslant 0.$$

Then, there exists a universal constant $C_N > 0$, that depends on N only, such that the following Garding inequality holds true:

$$\big\langle \big(\varphi(x,\xi)\big)^w_{\varepsilon_n} u_n, u_n\big\rangle = \langle \mu_n, \varphi\rangle \geqslant -C_N\sqrt{\varepsilon_n}\big(\|\varphi\|_{X_N} + \|D_x\varphi\|_{X_N}\big)\|u_n\|^2_{B^*}. \tag{4.1}$$

Remark 7. In other words, the quantity $\langle \mu_n, \phi\rangle$ is non-negative, up to a corrective term of size $\sqrt{\varepsilon_n}$. Note the usual Garding inequality gives a smaller corrective term, of size ε_n. This gap is due to our assumptions, namely, to the fact that φ is here assumed to be only *once* differentiable in x. The reader may easily check that our proof gives the more standard $O(\varepsilon_n)$ corrective term if φ is assumed *twice* differentiable in x, with $D_x^2\varphi \in X_N$.

Remark 8. It is a well-known fact that $\mu = \lim \mu_n$ is a non-negative measure, while μ_n is *not* a non-negative measure for *finite* values of n. The above proposition quantifies the defect of positivity of μ_n when n is finite. Note the test functions we use here are in the class X_N, i.e., they should decay slightly faster than $1/|x|$ at infinity. This is due to the fact that u_n is uniformly bounded in B^* only (and not in L^2).

Proof of Proposition 2. We start with the standard observation (see [23], or [25]),

$$\tilde{\mu}_n(x,\xi) \geqslant 0,$$

where $\tilde{\mu}_n$, known as the Husimi transform of the sequence $\{u_n\}$, is defined as

$$\begin{aligned}\tilde{\mu}_n(x,\xi) &:= \mu_n(x,\xi) *_{x,\xi} G_{\varepsilon_n}(x,\xi)\\ &= \int_{\mathbb{R}^{2d}} \mu_n(x-z,\xi-\eta)\underbrace{(\pi\varepsilon_n)^{-d}\exp\left(-\frac{z^2+\eta^2}{\varepsilon_n}\right)}_{=:G_{\varepsilon_n}(y,\eta)} dz\,d\eta.\end{aligned}$$

Hence, taking a non-negative test function φ as in Proposition 2, we have

$$0 \leqslant \langle \tilde{\mu}_n, \varphi\rangle = \big\langle \mu_n, \varphi(x,\xi) *_{x,\xi} G_{\varepsilon_n}(x,\xi)\big\rangle.$$

As a consequence,

$$\langle \mu_n, \varphi\rangle \geqslant -\big|\langle \mu_n, \varphi * G_{\varepsilon_n} - \varphi\rangle\big| \geqslant -C_N\|\varphi * G_{\varepsilon_n} - \varphi\|_{X_N}\|u_n\|_{B^*}, \tag{4.2}$$

whenever $N > d+1$. Here we made use of Proposition 1(2).

We perform a Taylor expansion on the term $\varphi * G_{\varepsilon_n}$ to estimate the right-hand side of (4.2) and to establish that $\varphi * G_{\varepsilon_n} - \varphi = O(\sqrt{\varepsilon_n})$ in X_N norm. For obvious convenience, the Taylor expansion is performed directly on the Fourier transformed term $\mathcal{F}_{\xi\to y}(\varphi * G_{\varepsilon_n} - \varphi)$. We write

$$\begin{aligned}
&\mathcal{F}_{\xi\to y}(\varphi * G_{\varepsilon_n} - \varphi)(x,y) \\
&= (\pi\varepsilon_n)^{-d/2} \int_{\mathbb{R}^d} \hat\varphi(x-z,y)\exp\left(-\frac{z^2}{\varepsilon_n}\right)\exp\left(-\varepsilon_n\frac{y^2}{4}\right)dz - \hat\varphi(x,y) \\
&= \left[\pi^{-d/2}\int_{\mathbb{R}^d}\hat\varphi\big(x-\sqrt{\varepsilon_n}z,y\big)\exp\big(-z^2\big)\,dz - \hat\varphi(x,y)\right]\exp\big(-\varepsilon_n y^2/4\big) \\
&\quad + \hat\varphi(x,y)\big[\exp\big(-\varepsilon_n y^2/4\big)-1\big] \\
&=: \widehat{A}(x,y) + \widehat{B}(x,y).
\end{aligned}$$

This serves as a definition for the functions $A(x,\xi) = \mathcal{F}^{-1}_{y\to\xi}(\widehat{A}(x,y))$, and $B(x,\xi)$. Up to this point, we have,

$$\|\varphi * G_{\varepsilon_n} - \varphi\|_{X_N} \leqslant \|A\|_{X_N} + \|B\|_{X_N}.$$

We estimate separately each term $\|A\|_{X_N}$ and $\|B\|_{X_N}$.

On the one hand, we have

$$\big|\widehat{B}(x,y)\big| \leqslant C\sqrt{\varepsilon_n}|y|\big|\hat\varphi(x,y)\big|.$$

This implies

$$\|B\|_{X_N} \leqslant C_N\sqrt{\varepsilon_n}\|\varphi\|_{X_{N+1}}.$$

On the other hand, we turn to the more difficult analysis of A. First order Taylor expansion gives

$$\big|\widehat{A}(x,y)\big| \leqslant C\sqrt{\varepsilon_n}\int_{\mathbb{R}^d}dz\int_0^1 d\theta\,|z|\big|D_x\hat\varphi\big(x-\theta\sqrt{\varepsilon_n}z,y\big)\big|\exp\big(-z^2\big).$$

When $x \in C_j$ with $j \geqslant 0$, we may thus estimate

$$\begin{aligned}
&\sup_{x\in C_j}\sup_{y\in\mathbb{R}^d}\langle y\rangle^N\big|\widehat{A}(x,y)\big| \\
&\leqslant C\sqrt{\varepsilon_n}\sup_{x\in C_j}\int_{\mathbb{R}^d}dz\int_0^1 d\theta\,\sup_y\langle y\rangle^N|z|\big|D_x\hat\varphi\big(x-\theta\sqrt{\varepsilon_n}z,y\big)\big|\exp\big(-z^2\big) \\
&= C\sqrt{\varepsilon_n}\sup_{x\in C_j}\int_{|z|\leqslant 2^{j-1}/\sqrt{\varepsilon_n}}\int_0^1 d\theta\,\sup_y\langle y\rangle^N|z|\big|D_x\hat\varphi\big(x-\theta\sqrt{\varepsilon_n}z,y\big)\big|\exp\big(-z^2\big)
\end{aligned}$$

$$
\begin{aligned}
&+ C\sqrt{\varepsilon_n} \sup_{x\in C_j} \int_{|z|\geqslant 2^{j-1}/\sqrt{\varepsilon_n}} \int_0^1 d\theta \sup_y \langle y\rangle^N |z| \big| D_x\hat{\varphi}\big(x-\theta\sqrt{\varepsilon_n}z, y\big)\big| \exp\big(-z^2\big) \\
&\leqslant C\sqrt{\varepsilon_n} \sup_{x\in\widetilde{C}_j} \sup_y \langle y\rangle^N \big| D_x\hat{\varphi}(x,y)\big| \\
&\quad + C\sqrt{\varepsilon_n} \exp\left(-\frac{2^{2j}}{C\varepsilon_n}\right) \sup_{x\in\mathbb{R}^d} \sup_y \langle y\rangle^N \big| D_x\hat{\varphi}(x,y)\big|.
\end{aligned}
$$

Here, we made use of the notation $\widetilde{C}_j := C_{j-1} \cup C_j \cup C_{j+1}$. As a consequence, multiplying by 2^j and summing over all values of $j \geqslant 0$, we deduce

$$
\sum_{j\geqslant 0} 2^j \sup_{x\in C_j} \sup_{y\in\mathbb{R}^d} \langle y\rangle^N \big|\widehat{A}(x,y)\big| \leqslant C\sqrt{\varepsilon_n} \|D_x\varphi\|_{X_N}.
$$

Similarly, when $|x| \leqslant 1$, we may estimate

$$
\begin{aligned}
\sup_{|x|\leqslant 1} \sup_{y\in\mathbb{R}^d} \langle y\rangle^N \big|\widehat{A}(x,y)\big| &\leqslant C\sqrt{\varepsilon_n} \int_{\mathbb{R}^d} dz \int_0^1 d\theta \sup_{x\in\mathbb{R}^d} \sup_{y\in\mathbb{R}^d} \langle y\rangle^N |z| \big| D_x\hat{\varphi}\big(x-\theta\sqrt{\varepsilon_n}z, y\big)\big| \exp\big(-z^2\big) \\
&\leqslant C\sqrt{\varepsilon_n} \|D_x\varphi\|_{X_N}.
\end{aligned}
$$

All these estimates finally result in the following

$$
\|\varphi * G_{\varepsilon_n} - \varphi\|_{X_N} \leqslant C\sqrt{\varepsilon_n}\big(\|\varphi\|_{X_{N+1}} + \|D_x\varphi\|_{X_N}\big). \tag{4.3}
$$

Plugging (4.3) into (4.2), ends the proof of Proposition 2. □

5. Second consequence: a version of the transport equation

This section is devoted to the proof of another important application of Proposition 1, namely, the fact that μ_n almost satisfies a transport equation, or, in other words, μ_n is almost invariant along the Hamiltonian flow of $H(x,\xi) = \xi^2 + V(x)$. Again, the point is that we can prove the relevant transport equation is satisfied in a weak sense, when tested along symbols with *limited* smoothness. The corresponding statement when the symbols are smooth is completely standard.

The main result of the present section is the following proposition.

Proposition 3 *(Transport equation). Let $\varphi(x,\xi) \in C_b^0(\mathbb{R}^{2d})$ be a bounded and continuous function.*[8] *Assume $\varphi \in Y_N$ (see (3.3)), for some $N > d+5$.*

Then, the distribution $\mu_n(x,\xi)$ satisfies

$$
\big|\big\langle -\alpha_n\mu_n + 2\xi\cdot\nabla_x\mu_n - \nabla_x V(x)\cdot\nabla_\xi\mu_n, \varphi\big\rangle\big| \leqslant C_N(\varepsilon_n + \eta_n)\|\varphi\|_{Y_N}. \tag{5.1}
$$

[8] Again, the present statement should first be stated for very smooth φ's, and the natural density argument should be performed next.

In particular, the limiting measure $\mu = \lim \mu_n$ satisfies the transport equation

$$2\xi \cdot \nabla_x \mu(x,\xi) - \nabla_x V(x) \cdot \nabla_\xi \mu(x,\xi) = 0 \tag{5.2}$$

in the sense of distributions. Equivalently, if Φ_t denotes the Hamiltonian flow associated with the Hamiltonian $\xi^2 + V(x) - \lambda$, we have the invariance

$$\forall t \in \mathbb{R}, \quad \mu \circ \Phi_t = \mu, \tag{5.3}$$

where the equality holds between measures.

Remark 9. As already mentioned, it is a standard fact that $\mu = \lim \mu_n$ satisfies a transport equation, in the sense of distributions. Here, the point is twofolds. First, estimate (5.1) keeps track of the fact that, for *finite* values of n, μ_n satisfies a transport equation as well, up to an error term that we are able to upper-bound. Second, the transport equation is not only established in the sense of distributions: we work here with test functions φ in the class Y_N. We recall that such functions are merely *continuous* and *bounded* in the x variable. This turns out to be a useful aspect in the subsequent sections.

Proof of Proposition 3. We Wigner transform the Helmholtz equation (2.2) itself. It gives the usual transport equation with remainder:

$$-\alpha_n \mu_n + \xi \cdot \nabla_x \mu_n - \nabla_x V(x) \cdot \nabla_\xi \mu_n = R_n(x,\xi), \tag{5.4}$$

up to defining $R_n(x,\xi) := \mathcal{F}^{-1}_{\xi \to y}(\widehat{R}_n(x,y))$ through the formula

$$\begin{aligned}\widehat{R}_n(x,y) := {} & -\varepsilon_n^{-1} \operatorname{Im}\left(f_n\left(x + \varepsilon_n \frac{y}{2}\right) u_n^*\left(x - \varepsilon_n \frac{y}{2}\right)\right) \\ & + \frac{V(x+\varepsilon_n \frac{y}{2}) - V(x - \varepsilon_n \frac{y}{2}) - \varepsilon_n y \cdot \nabla_x V(x)}{2i\varepsilon_n} u_n\left(x + \varepsilon_n \frac{y}{2}\right) u_n^*\left(x - \varepsilon_n \frac{y}{2}\right). \end{aligned} \tag{5.5}$$

Taking φ as in Proposition 3 and testing R_n against φ, we estimate the duality product $\langle R_n, \varphi\rangle$ as follows.

The first term appearing in $\langle R_n, \varphi\rangle$ (see (5.5)) is upper-bounded by

$$\begin{aligned} & \leqslant \varepsilon_n^{-1} \int_{\mathbb{R}^{2d}} \left| u_n\left(x - \varepsilon_n \frac{y}{2}\right)\right| \left| f_n\left(x + \varepsilon_n \frac{y}{2}\right)\right| \left|\hat{\varphi}(x,y)\right| dx\, dy \\ & \leqslant C_N \varepsilon_n^{-1} \|\varphi\|_{Y_N} \|f_n\|_B \|u_n\|_{B^*} \\ & \leqslant C_N \eta_n \|\varphi\|_{Y_N} \underset{n\to\infty}{\longrightarrow} 0, \end{aligned} \tag{5.6}$$

whenever $N > d+1$. Here we used Proposition 1(3), together with the assumed bounds (2.1) on u_n and f_n.

As for the second term, it is estimated by

$$\leqslant C\varepsilon_n \int_{\mathbb{R}^{2d}} dx\,dy \int_{-1}^{1} dt \left| u_n\left(x+\varepsilon_n \frac{y}{2}\right)\right| \left| u_n\left(x-\varepsilon_n \frac{y}{2}\right)\right| \left|y^2 \hat{\varphi}(x,y)\right| \left|D_x^2 V(x+t\varepsilon_n y/2)\right| dt$$

$$\leqslant C_N \varepsilon_n \|u_n\|_{B^*}^2 \int_{-1}^{1} dt \Big(\sup_{|x|\leqslant 1} \sup_y \langle y\rangle^N \left|y^2 \hat{\varphi}(x,y)\right| \left|D_x^2 V(x+t\varepsilon_n y/2)\right|$$

$$+ \sum_{j\geqslant 0} 2^j \sup_{x\in C_j} \sup_y \langle y\rangle^N \left|y^2 \hat{\varphi}(x,y)\right| \left|D_x^2 V(x+t\varepsilon_n y/2)\right| \Big), \tag{5.7}$$

whenever $N > d+1$. Here we have used Proposition 1(2). There remains to estimate the two suprema on the right-hand side of (5.7). On the one hand, we clearly have

$$\sup_{|x|\leqslant 1} \sup_y \langle y\rangle^N \left|y^2 \hat{\varphi}(x,y)\right| \left|D_x^2 V(x+t\varepsilon_n y/2)\right|$$
$$\leqslant C \left\| D_x^2 V \right\|_{L^\infty} \sup_{|x|\leqslant 1} \sup_y \langle y\rangle^{N+2} \left|\hat{\varphi}(x,y)\right| \leqslant C\|\varphi\|_{Y_{N+2}}.$$

Also, using the assumed decay of $D_x^2 V$ at infinity in x (see (1.12)), we may estimate, on the other hand,

$$2^j \sup_{x\in C_j} \sup_y \langle y\rangle^N \left|y^2 \hat{\varphi}(x,y)\right| \left|D_x^2 V(x+t\varepsilon_n y/2)\right|$$
$$\leqslant C \sup_{x\in C_j} \sup_y \langle y\rangle^{N+2} \frac{\langle x\rangle}{\langle x+t\varepsilon_n y/2\rangle^{1+\rho}} \left|\hat{\varphi}(x,y)\right|$$
$$\leqslant C \sup_{x\in C_j} \sup_y \langle y\rangle^{N+4} \left|\hat{\varphi}(x,y)\right| \sup_{x\in C_j} \sup_y \langle y\rangle^{-2} \frac{\langle x\rangle}{\langle x+t\varepsilon_n y/2\rangle^{1+\rho}}$$
$$\leqslant C \sup_{x\in C_j} \sup_y \langle y\rangle^{N+4} \left|\hat{\varphi}(x,y)\right|$$
$$\leqslant C\|\varphi\|_{Y_{N+4}}, \tag{5.8}$$

as a direct inspection shows. Hence the second term appearing in $\langle R_n, \varphi\rangle$ is eventually estimated by

$$\leqslant C_N \varepsilon_n \|\varphi\|_{Y_{N+4}} \underset{n\to\infty}{\longrightarrow} 0.$$

This ends the proof of the proposition. □

6. Energy localization: the sequence $\{u_n\}$ carries no mass outside resonant frequencies

This section is devoted to the proof of various relations which assert the sequence $\{u_n\}$ carries "no mass" for frequencies that do not satisfy the zero energy condition $\xi^2 = \lambda - V(x)$, i.e., away from resonant frequencies.

It is a very important statement. Indeed, it allows us in the next sections to reduce our argument to the asymptotic analysis of the sequence $\{u_n\}$, conveniently restricted to frequencies that do satisfy the zero energy condition. For those frequencies, the non-trapping condition (1.10) gives an additional, and crucial, information.

As in Sections 5 and 6, the statement of interest is fairly standard when dealing with L^2 solutions to PDEs with C^∞ coefficients. The difficulty lies again in the fact that $V \in C^2$ only, and u_n belongs to B^*. Also, we prove here statements that hold for *finite* values of n, and not only when $n = \infty$.

The main results of this section are the following three propositions. All three results assert, in one way or another, the localization on resonant frequencies. Proposition 4 gives a rough statement, while the next Propositions 5 and 6 give refined, and technically more involved, estimates.

Proposition 4. *Let $\varphi(x,\xi) \in C_b^0(\mathbb{R}^{2d})$ be a bounded and continuous function. Assume φ has the regularity $\varphi \in Y_N$, (see* (3.3)*), together with $\Delta_x\varphi \in X_N$, for some $N > d+4$.*

Then, the distribution $\mu_n(x,\xi)$ satisfies

$$\left|\left\langle\left(\xi^2 + V(x) - \lambda\right)\mu_n(x,\xi), \varphi\right\rangle\right| \leqslant C_N\varepsilon_n\left(\|\varphi\|_{Y_N} + \|\Delta_x\varphi\|_{X_N}\right). \tag{6.1}$$

In particular, the limit $\mu = \lim \mu_n$ satisfies $(\xi^2 + V(x) - \lambda)\mu(x,\xi) = 0$, in the sense of distributions, so that the measure μ only carries resonant frequencies:

$$\operatorname{supp}\mu \subset \left\{(x,\xi) \in \mathbb{R}^{2d} \text{ s.t. } \xi^2 + V(x) - \lambda = 0\right\}. \tag{6.2}$$

Remark 10. Note that estimate (6.1) needs $\Delta_x\phi \in X_N$, but only $\varphi \in Y_N$. In others words, φ may be merely bounded at infinity in x, while $\Delta_x\varphi$ needs to decay slightly faster than $1/|x|$.

Proposition 5. *Introduce two C^∞ cutoff functions in x and ξ, satisfying respectively*

$$\Theta(\xi) \equiv 0 \quad \text{whenever } |\xi| \leqslant 2c_1, \qquad \Theta(\xi) \equiv 1 \quad \text{whenever } |\xi| \geqslant 3c_1,$$
$$\chi(x) \equiv 0 \quad \text{whenever } |x| \leqslant 1, \qquad \chi(x) \equiv 1 \quad \text{whenever } |x| \geqslant 2.$$

Here we recall that $c_1 = \|\lambda - V(x)\|_{L^\infty}^{1/2}$, see (1.11).

Then, the following L^2 bound holds true:

$$\left\|\Theta_{\varepsilon_n}^w \chi(x)u_n(x)\right\|_{L^2} \leqslant C\left(\sqrt{\varepsilon_n} + \eta_n\right) \underset{n\to\infty}{\longrightarrow} 0,$$

for some C that does not depend on n.

Remark 11. In other words, up to putting apart small annuli in x, the sequence $\{u_n\}$ carries no mass, *in L^2 norm*, for frequencies above $\|V(x)-\lambda\|_{L^\infty}^{1/2} = c_1$. This is indeed a quantitative version of (6.2), valid for bounded values on n.

Proposition 6. *Introduce two C^∞ cutoff functions in x and ξ, still denoted by $\Theta(\xi)$ an $\chi(x)$ not to overweight notations, satisfying respectively*

$$\Theta(\xi) \equiv 0 \quad \text{whenever } |\xi| \geqslant c_0/2, \qquad \Theta(\xi) \equiv 1 \quad \text{whenever } |\xi| \leqslant c_0/3,$$
$$\chi(x) \equiv 0 \quad \text{whenever } |x| \leqslant 2R_0, \qquad \chi(x) \equiv 1 \quad \text{whenever } |x| \geqslant 3R_0.$$

Here, we recall that the constants c_0 and R_0 are defined in (1.7).

Then, the following L^2 bound holds true:

$$\left\| \Theta_{\varepsilon_n}^w \chi(x) u_n(x) \right\|_{L^2} \leqslant C(\varepsilon_n + \eta_n) \underset{n\to\infty}{\longrightarrow} 0,$$

for some C that does not depend on n.

Remark 12. In other words, up to restricting to x's larger than R_0, for which $\lambda - V(x)$ is away from zero, i.e., $\lambda - V(x) \geqslant (c_0)^2$, the sequence $\{u_n\}$ carries no mass, *in L^2 norm*, for frequencies below c_0. This is again a quantitative version of (6.2), valid for bounded values of n.

Proof of Proposition 4. An easy computation gives

$$\begin{aligned}
&2\big(\xi^2 + V(x) - \lambda\big)\mu_n(x,\xi) \\
&\quad = \frac{\varepsilon_n^2}{2}\Delta_x \mu_n(x,\xi) + \mathcal{F}_{y\to\xi}\left(\left[-\varepsilon_n^2 \Delta u_n + V u_n - \lambda u_n\right]\left(x + \varepsilon_n \frac{y}{2}\right) u_n^*\left(x - \varepsilon_n \frac{y}{2}\right)\right) \\
&\qquad + \mathcal{F}_{y\to\xi}\left(u_n\left(x + \varepsilon_n \frac{y}{2}\right)\left[-\varepsilon_n^2 \Delta u_n^* + V u_n^* - \lambda u_n^*\right]\left(x - \varepsilon_n \frac{y}{2}\right)\right) \\
&\qquad + \mathcal{F}_{y\to\xi}\left(\left[V(x) - \frac{1}{2}V\left(x + \varepsilon_n \frac{y}{2}\right) - \frac{1}{2}V\left(x - \varepsilon_n \frac{y}{2}\right)\right] u_n\left(x + \varepsilon_n \frac{y}{2}\right) u_n^*\left(x - \varepsilon_n \frac{y}{2}\right)\right).
\end{aligned}$$

Hence, using the Helmholtz equation (2.2) satisfied by u_n, we recover

$$\big(\xi^2 + V(x) - \lambda\big)\mu_n(x,\xi) = R_n(x,\xi), \tag{6.3}$$

up to introducing the remainder term $R_n(x,\xi) = \mathcal{F}_{\xi\to y}^{-1}(\widehat{R}_n(x,y))$ through

$$\begin{aligned}
\widehat{R}_n(x,y) &:= \frac{\varepsilon_n^2}{4}\Delta_x \hat{\mu}_n(x,y) + \mathrm{Re}\left(f_n\left(x + \varepsilon_n \frac{y}{2}\right) u_n^*\left(x - \varepsilon_n \frac{y}{2}\right)\right) \\
&\qquad + \frac{1}{2}\left[V(x) - \frac{1}{2}V\left(x + \varepsilon_n \frac{y}{2}\right) - \frac{1}{2}V\left(x - \varepsilon_n \frac{y}{2}\right)\right] u_n\left(x + \varepsilon_n \frac{y}{2}\right) u_n^*\left(x - \varepsilon_n \frac{y}{2}\right).
\end{aligned} \tag{6.4}$$

Testing R_n against a test function φ gives three terms, which we estimate as follows.

The first term, when tested against φ, is estimated using Proposition 1(2), by

$$\begin{aligned}&\leqslant C_N\varepsilon_n^2\|\Delta_x\varphi\|_{X_N}\|u_n\|_{B^*}^2\\&\leqslant C_N\varepsilon_n^2\|\Delta_x\varphi\|_{X_N}\underset{n\to\infty}{\longrightarrow}0,\end{aligned}$$

whenever $N>d+1$. The second line uses the bound (2.1) on u_n.

In the same spirit, the second term in (6.4) is estimated thanks to Proposition 1(3), by

$$\begin{aligned}&\leqslant C_N\|\varphi\|_{Y_N}\|u_n\|_{B^*}\|f_n\|_B\\&\leqslant C_N\varepsilon_n\eta_n\|\varphi\|_{Y_N}\underset{n\to\infty}{\longrightarrow}0,\end{aligned}$$

whenever $N>d+1$. The second line uses the bounds at hand on u_n and f_n.

Finally, the third term is upper-bounded by

$$\begin{aligned}&\leqslant C\varepsilon_n\int_{\mathbb{R}^{2d}}dx\,dy\int_{-1}^{1}dt\left|u_n\left(x+\varepsilon_n\frac{y}{2}\right)\right|\left|u_n\left(x-\varepsilon_n\frac{y}{2}\right)\right|\left|y\hat{\varphi}(x,y)\right|\left|D_xV\left(x+t\varepsilon_n\frac{y}{2}\right)\right|\\&\leqslant C_N\varepsilon_n\|u_n\|_{B^*}^2\int_{-1}^{1}dt\biggl(\sup_{|x|\leqslant1}\sup_y\langle y\rangle^N\bigl|y\hat{\varphi}(x,y)\bigr|\bigl|D_xV(x+t\varepsilon_ny/2)\bigr|\\&\quad+\sum_{j\geqslant0}2^j\sup_{x\in C_j}\sup_y\langle y\rangle^N\bigl|y\hat{\varphi}(x,y)\bigr|\bigl|D_xV(x+t\varepsilon_ny/2)\bigr|\biggr),\end{aligned}\tag{6.5}$$

whenever $N>d+1$. Here we have used Proposition 1(2). There remains to estimate the two suprema on the right-hand side of (6.5). On the one hand, we clearly have

$$\begin{aligned}\sup_{|x|\leqslant1}\sup_y\langle y\rangle^N\bigl|y\hat{\varphi}(x,y)\bigr|\bigl|D_xV(x+t\varepsilon_ny/2)\bigr|&\leqslant C\|D_xV\|_{L^\infty}\sup_{|x|\leqslant1}\sup_y\langle y\rangle^{N+1}\bigl|\hat{\varphi}(x,y)\bigr|\\&\leqslant C\|\varphi\|_{Y_{N+1}}.\end{aligned}$$

Also, using the assumed decay of D_xV at infinity in x (see (1.12)), we may estimate as in (5.8)

$$\begin{aligned}&2^j\sup_{x\in C_j}\sup_y\langle y\rangle^N\bigl|y\hat{\varphi}(x,y)\bigr|\bigl|D_xV(x+t\varepsilon_ny/2)\bigr|\\&\leqslant C\sup_{x\in C_j}\sup_y\langle y\rangle^{N+1}\frac{\langle x\rangle}{\langle x+t\varepsilon_ny/2\rangle^{1+\rho}}\bigl|\hat{\varphi}(x,y)\bigr|\\&\leqslant C\sup_{x\in C_j}\sup_y\langle y\rangle^{N+3}\bigl|\hat{\varphi}(x,y)\bigr|\sup_{x\in C_j}\sup_y\langle y\rangle^{-2}\frac{\langle x\rangle}{\langle x+t\varepsilon_ny/2\rangle^{1+\rho}}\\&\leqslant C\sup_{x\in C_j}\sup_y\langle y\rangle^{N+3}\bigl|\hat{\varphi}(x,y)\bigr|\\&\leqslant C\|\varphi\|_{Y_{N+3}},\end{aligned}$$

as a direct inspection shows. Hence the third term appearing in $\langle R_n, \varphi\rangle$ is eventually estimated by

$$\leqslant C_N \varepsilon_n \|\varphi\|_{Y_{N+3}} \underset{n\to\infty}{\longrightarrow} 0.$$

This ends the proof of Proposition 4. □

Proof of Proposition 5. We take two cutoff functions $\Theta(\xi)$ and $\chi(x)$ as in Proposition 5. For later convenience, we take some large parameter $\Lambda > 0$, and we decompose accordingly the cutoff in frequency $\Theta(\xi)$ into large and moderate frequencies, namely,

$$\Theta(\xi) \equiv \Theta^1_\Lambda(\xi) + \Theta^2_\Lambda(\xi),$$

where both functions $\Theta^i_\Lambda(\xi)$ $(i = 1, 2)$ are smooth, non-negative, and they satisfy:

$$\operatorname{supp} \Theta^1_\Lambda \subset \big\{2c_1 \leqslant |\xi| \leqslant 2\Lambda\big\}, \qquad \operatorname{supp} \Theta^2_\Lambda \subset \big\{|\xi| \geqslant \Lambda\big\}.$$

Associated with this decomposition, we write

$$\big\|\Theta^w_{\varepsilon_n} \chi u_n\big\|_{L^2} \leqslant \big\|\big(\Theta^1_\Lambda\big)^w_{\varepsilon_n} \chi u_n\big\|_{L^2} + \big\|\big(\Theta^2_\Lambda\big)^w_{\varepsilon_n} \chi u_n\big\|_{L^2}, \tag{6.6}$$

and we start estimating separately each of the above terms.

First step: *estimating* $\|(\Theta^1_\Lambda)^w_{\varepsilon_n} \chi u_n\|_{L^2}$. Roughly, this contribution is estimated thanks to the energy localization already stated in Proposition 4, as we show below.

To begin with, we express the norm $\|(\Theta^1_\Lambda)^w_{\varepsilon_n} \chi u_n\|^2_{L^2}$ in terms of symbols. We write, using standard symbolic calculus

$$\chi(x)\big(\Theta^1_\Lambda\big)^w_{\varepsilon_n}\big(\Theta^1_\Lambda\big)^w_{\varepsilon_n}\chi(x) = \big(\chi(x)^2\big(\Theta^1_\Lambda\big)^2\big)^w_{\varepsilon_n} + O\big(\varepsilon_n\langle x\rangle^{-\infty}\big),$$

as operators on $L^2(\mathbb{R}^d)$. Here the $O(\varepsilon_n\langle x\rangle^{-\infty})$ term denotes an operator whose symbol belongs to the class $S(\langle x\rangle^{-N})$ for any N, and whose size is $O(\varepsilon_n)$ in each of these classes. As a consequence, we may write:

$$\begin{aligned}
\big\|\big(\Theta^1_\Lambda\big)^w_{\varepsilon_n} \chi u_n\big\|^2_{L^2} &= \big\langle \chi\big(\Theta^1_\Lambda\big)^w_{\varepsilon_n}\big(\Theta^1_\Lambda\big)^w_{\varepsilon_n}\chi u_n, u_n\big\rangle \\
&= \big\langle\big(\chi(x)^2\big(\Theta^1_\Lambda\big)^2\big)^w_{\varepsilon_n} u_n, u_n\big\rangle + O(\varepsilon_n) \\
&= \big\langle \chi(x)^2\big(\Theta^1_\Lambda(\xi)\big)^2, \mu_n\big\rangle + O(\varepsilon_n).
\end{aligned} \tag{6.7}$$

Naturally, the second line uses the fact that

$$\int\limits_{\mathbb{R}^d} \frac{|u_n(x)|^2}{\langle x\rangle^N}\, dx \leqslant C_N \|u_n\|^2_{B^*} \leqslant C_N,$$

for any $N > 1$.

We now apply Proposition 4 to the right-hand side of (6.7). To do so, we first observe that the symbol $\chi(x)^2(\Theta^1_\Lambda(\xi))^2$ clearly belongs to the class Y_N for any N, being essentially constant at infinity in x, and C_c^∞ in ξ. This is actually the reason why we have put apart bounded frequencies, through the decomposition $\Theta = \Theta_\Lambda^1 + \Theta_\Lambda^2$ in (6.6). On top of that, the Laplacian of this symbol, having the value $(\Delta_x \chi(x)^2)(\Theta_\Lambda^1(\xi))^2$, clearly belongs to the class X_N for any N: it is actually C_c^∞ in both x and ξ. Last, we observe

$$|\xi| \geqslant 2c_1 \quad \text{on } \operatorname{supp} \Theta_\Lambda^1,$$

from which it follows:

$$\left|\xi^2 + V(x) - \lambda\right| \geqslant 4c_1^2 - c_1^2 = 3c_1^2 > 0 \quad \text{on } \operatorname{supp} \Theta_\Lambda^1. \tag{6.8}$$

For this reason, we do have the following regularity/decay properties in x and ξ for the symbol $\chi(x)^2(\Theta_\Lambda^1(\xi))^2/(\xi^2 + V(x) - \lambda)$:

$$\frac{\chi(x)^2(\Theta_\Lambda^1(\xi))^2}{\xi^2 + V(x) - \lambda} \in Y_N, \quad \text{for any } N, \quad \text{and}$$
$$\Delta_x\left(\frac{\chi(x)^2(\Theta_\Lambda^1(\xi))^2}{\xi^2 + V(x) - \lambda}\right) \in X_N, \quad \text{for any } N.$$

Note the important distinction between the symbol (which does not decay in x), and its second derivative (which has the good decay). Note also that the above statement only requires V to be twice differentiable. This being observed, we recover in this way, using Proposition 4:

$$\begin{aligned}\left\langle \chi(x)^2\left(\Theta_\Lambda^1(\xi)\right)^2, \mu_n\right\rangle &= \left\langle \frac{\chi(x)^2(\Theta_\Lambda^1(\xi))^2}{\xi^2 + V(x) - \lambda}, \left(\xi^2 + V(x) - \lambda\right)\mu_n\right\rangle \\ &= O(\varepsilon_n + \eta_n).\end{aligned}$$

Eventually, we have established

$$\left\|\left(\Theta_\Lambda^1\right)_{\varepsilon_n}^w \chi u_n\right\|_{L^2}^2 \leqslant C(\varepsilon_n + \eta_n), \tag{6.9}$$

where $C > 0$ is independent of n (but it does depend on Λ).

Second step: *estimating* $\|(\Theta_\Lambda^2)_{\varepsilon_n}^w \chi u_n\|_{L^2}^2$. Let us now come to the analysis of the term involving Θ_Λ^2. We are here dealing with unbounded frequencies, so that Proposition 4 is of no use. The alternative idea is, essentially, to perform a standard energy estimate.

We test the Helmholtz equation (2.2) against

$$\chi(x)\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n,$$

and start estimating. It gives at once

$$\begin{aligned}&\left|\left\langle \left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x)\left(-\varepsilon_n^2 \Delta_x u_n(x)\right), \left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\rangle\right| \\ &\quad\leqslant \left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) f_n(x)\right\|_{L^2} \left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\|_{L^2} + \varepsilon_n \alpha_n \left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\|_{L^2}^2 \\ &\qquad + \left|\left\langle \left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x)\left[V(x) - \lambda\right] u_n(x), \left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\rangle\right|.\end{aligned}$$

As a consequence, standard symbolic calculus and the bounds at hand on f_n and u_n give

$$\begin{aligned}&\left|\left\langle\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x)\left(-\varepsilon_n^2 \Delta_x u_n(x)\right),\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\rangle\right| \\ &\quad\leqslant C\left(\varepsilon_n \eta_n+\varepsilon_n \alpha_n\right)\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\|_{L^2} \\ &\qquad+\left|\left\langle\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x)\left[V(x)-\lambda\right] u_n(x),\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\rangle\right|.\end{aligned} \tag{6.10}$$

We now come to estimate, in (6.10), the term

$$\left|\left\langle\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x)\left(-\varepsilon_n^2 \Delta_x u_n(x)\right),\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\rangle\right| \tag{6.11}$$

from below, and the term

$$\left|\left\langle\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x)\left[V(x)-\lambda\right] u_n(x),\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\rangle\right| \tag{6.12}$$

from above.

First, we estimate the term involving $V(x)-\lambda$. This necessitates to commute the truncation in frequency, $(\Theta_\Lambda^2)_{\varepsilon_n}^w$, with the potential V, a function of limited regularity. This cannot be done using standard symbolic calculus. For that reason, we claim here that for any functions v and w in B^*, the following commutator estimate holds:

$$\left|\left\langle\left[\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w V(x)-\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w V(x)\right] v(x), w(x)\right\rangle\right| \leqslant C \varepsilon_n\|v\|_{B^*}\|w\|_{B^*}. \tag{6.13}$$

Assuming for a while that (6.13) has been proved, we immediately deduce

$$\begin{aligned}&\left|\left\langle\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x)\left[V(x)-\lambda\right] u_n(x),\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\rangle\right| \\ &\quad=\left|\left\langle\left[V(x)-\lambda\right]\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x),\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\rangle\right|+O\left(\varepsilon_n\right) \\ &\quad\leqslant c_1^2\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n(x)\right\|_{L^2}^2+O\left(\varepsilon_n\right),\end{aligned} \tag{6.14}$$

where the constant c_1^2 is simply $\|V(x)-\lambda\|_{L^\infty}$ (in particular, it does not depend upon Λ). To complete the proof of (6.14), there remains to prove (6.13). We proceed as in the proof of Proposition 4:

$$\begin{aligned}&\left|\left\langle\left[\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w V(x)-\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w V(x)\right] v(x), w(x)\right\rangle\right| \\ &\quad=\left|\left\langle\left[\left(1-\Theta_\Lambda^2\right)_{\varepsilon_n}^w V(x)-\left(1-\Theta_\Lambda^2\right)_{\varepsilon_n}^w V(x)\right] v(x), w(x)\right\rangle\right| \\ &\quad=\left|\int_{\mathbb{R}^d} dx\, dy\, v\left(x+\varepsilon_n \frac{y}{2}\right) w^*\left(x-\varepsilon_n \frac{y}{2}\right)\left(1-\widehat{\Theta}_\Lambda^2(y)\right)\left[V\left(x+\varepsilon_n \frac{y}{2}\right)-V\left(x-\varepsilon_n \frac{y}{2}\right)\right]\right| \\ &\quad\leqslant C \varepsilon_n \int_{\mathbb{R}^d} dx\, dy \int_{-1}^1 dt\left|v\left(x+\varepsilon_n \frac{y}{2}\right)\right|\left|w^*\left(x-\varepsilon_n \frac{y}{2}\right)\right|\left|\left(1-\widehat{\Theta}_\Lambda^2(y)\right)\right||y|\left|D_x V\left(x+t \varepsilon_n \frac{y}{2}\right)\right| \\ &\quad\leqslant C \varepsilon_n\|v\|_{B^*}\|w\|_{B^*},\end{aligned}$$

where the last line uses exactly the same argument as in the proof of Proposition 4 (see Eq. (6.5) and the estimates that follow). Note that the exchange between Θ_Λ^2 and $1-\Theta_\Lambda^2$ in the commutator, is needed in order to have $(1-\widehat{\Theta}_\Lambda^2(y))$ fastly decaying in y ($\widehat{\Theta}_\Lambda^2$ is the Fourier transform of Θ_Λ^2). Estimate (6.13) hence (6.14) are proved.

Second, we turn to estimating (6.11) from below. Here, we need to commute the Laplacian with the truncation χ:

$$\begin{aligned}
&\left|\left\langle\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)\left(-\varepsilon_n^2\Delta_x u_n(x)\right),\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n(x)\right\rangle\right|\\
&\quad\geqslant\left|\left\langle\left(-\varepsilon_n^2\Delta_x\right)\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n(x),\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n(x)\right\rangle\right|-|\text{commutator}|\\
&\quad\geqslant\Lambda^2\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n\right\|_{L^2}^2-|\text{commutator}|.
\end{aligned}\tag{6.15}$$

We are left with the task of estimating the commutator. An explicit computation gives:

$$\begin{aligned}
&|\text{commutator}|\\
&\quad\leqslant C\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n\right\|_{L^2}\left(\varepsilon_n\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\nabla_x\chi(x)\varepsilon_n\nabla_x u_n\right\|_{L^2}+\varepsilon_n^2\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\Delta_x\chi(x)u_n\right\|_{L^2}\right)\\
&\quad\leqslant C\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n\right\|_{L^2}\left(\varepsilon_n\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\nabla_x\chi(x)\varepsilon_n\nabla_x u_n\right\|_{L^2}+\varepsilon_n^2\right).
\end{aligned}$$

Here we used the bounds at hand on u_n, together with the fact that $\Delta_x\chi$ has support in an annulus whose big radius is bounded from above, and whose small radius is bounded from below. Now the term involving $\varepsilon_n\nabla_x u_n$ is easily bounded. Indeed, we know that $\|\chi(x)u_n\|_{B^*}\leqslant C$, and the Helmholtz equation (2.2) clearly implies

$$\begin{aligned}
\left\|\chi(x)\varepsilon_n^2\Delta_x u_n\right\|_{B^*}&\leqslant C\left(\left\|\chi(x)f_n\right\|_{B^*}+\left\|\left(V(x)-\lambda\right)u_n\right\|_{B^*}\right)\\
&\leqslant C\left(\|f_n\|_B+\|u_n\|_{B^*}\right)\leqslant C.
\end{aligned}$$

Note that the function χ here cuts off a small ball around the origin, so that χf is bounded in B^*. We stress at variance that the function f_n itself is *not* bounded in this space, due to the diverging contribution of small annuli. Then, a straightforward interpolation gives

$$\|\chi\varepsilon_n\nabla_x u_n\|_{B^*}\leqslant C.$$

We deduce

$$|\text{commutator}|\leqslant C\varepsilon_n\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n\right\|_{L^2}.\tag{6.16}$$

Eventually we have obtained

$$\begin{aligned}
&\left|\left\langle\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)\left(-\varepsilon_n^2\Delta_x u_n(x)\right),\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n(x)\right\rangle\right|\\
&\quad\geqslant\Lambda^2\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n\right\|_{L^2}^2-C\varepsilon_n\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n\right\|_{L^2}.
\end{aligned}\tag{6.17}$$

Putting (6.10), (6.14), and (6.17) together, we infer

$$\left(\Lambda^2-c_1^2\right)\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n\right\|_{L^2}^2\leqslant C\varepsilon_n\left\|\left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w\chi(x)u_n\right\|_{L^2}+C\varepsilon_n.$$

Hence, taking Λ large enough, we eventually obtain:

$$\left\| \left(\Theta_\Lambda^2\right)_{\varepsilon_n}^w \chi(x) u_n \right\|_{L^2} \leqslant C \varepsilon_n^{1/2}. \tag{6.18}$$

Third step: *conclusion*. Estimates (6.9) and (6.18) give in (6.6):

$$\left\| \Theta_{\varepsilon_n}^w \chi u_n \right\|_{L^2} \leqslant C\left(\varepsilon_n^{1/2} + \eta_n\right).$$

Proposition 5 is proved. □

Proof of Proposition 6. The proof is essentially the same as that of Proposition 5, up to the following changes. First, dealing here with bounded frequencies, we do not need to split Θ into two terms (= bounded frequencies + unbounded ones). Second, the estimate from below (6.8) simply has to be replaced by

$$\left|\xi^2 + V(x) - \lambda\right| \geqslant c_0^2 - \frac{c_0^2}{4} = \frac{3c_0^2}{4} > 0, \quad \text{on } \operatorname{supp} \Theta \cap \operatorname{supp} \chi. \qquad \square$$

7. Building up an escape function at infinity: the sequence $\{u_n\}$ has no mass at infinity

In this step we prove the sequence $\{u_n\}$ is small at infinity in the x variable, when measured in the B^* norm.

More precisely, the main result of this section is the following proposition.

Proposition 7. *There exist a large integer J and a constant C such that*

$$\left\| u_n(x) \mathbf{1}\left[|x| \geqslant 2^J\right] \right\|_{B^*} \leqslant C\left(\alpha_n + \sqrt{\varepsilon_n} + \eta_n\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{7.1}$$

The proof of (7.1) is done upon introducing an appropriate "escape function at infinity," denoted a_J below. This part of our argument is the core of the proof. It gives an independent information on the mass carried by u_n at infinity.

Our function $a_J(x, \xi)$, which is precisely defined below, essentially is

$$a_J(x, \xi) \sim \frac{x}{|x|} \cdot \frac{\xi}{|\xi|} + \text{corrector}.$$

It measures the speed at which the solid angles $x/|x|$ and $\xi/|\xi|$ tend to become parallel for large times, along the Hamiltonian flow of $H(x, \xi) = \xi^2 + V(x) - \lambda$ (think of the simple case $V \equiv 0$). Our proof needs this speed to be as large as possible. The idea of introducing such a function is borrowed from [18]. The corrector term is crucial, for two reasons. First, the speed at which $x/|x|$ and $\xi/|\xi|$ align, tends to zero along the Hamiltonian flow of $H(x, \xi)$, i.e., as $x/|x| \cdot \xi/|\xi|$ tends to ± 1. Hence we really need to estimate the corrector accurately. Second, this additional term is entirely taylored to the (optimal) B^* estimate we aim at proving.

Needless to say, this escape function is very much related with the Morawetz-like multiplier

$$\frac{x}{|x|} \mathbf{1}\left[|x| \geqslant R\right] \cdot \nabla_x + \text{corrector}$$

used in [28]. Also, it is certainly related with the more standard multiplier $x \cdot \nabla_x + d/2$ used in the Mourre theory for Schrödinger operators.

Before coming to the proof of Proposition 7, we now define the function a_J that lies at the core of our proof. In order to precisely define our escape function, we first need to introduce some cutoffs, and a dyadic partition, as follows.

First, we take some non-negative, C_c^∞, and radial cutoff function ψ satisfying

$$\psi(x) \equiv 0 \quad \text{whenever } |x| \geqslant 1, \qquad \psi(x) \equiv 1 \quad \text{whenever } |x| \leqslant 1/2,$$
$$0 \leqslant \psi(x) \leqslant 1 \quad \text{for any } x. \tag{7.2}$$

Then, with the help of ψ we build up a dyadic partition in the usual way. We define the function $\phi(x) = \psi(x/2) - \psi(x)$. It obviously satisfies:

$$\operatorname{supp} \phi \subset \{1/2 \leqslant |x| \leqslant 1\}, \qquad \phi(x) \geqslant 0 \quad \text{for any } x.$$

Associated with ϕ, we define, for any $j \in \mathbb{Z}$, the ϕ_j's as

$$\phi_j(x) = \phi\left(\frac{x}{2^j}\right). \tag{7.3}$$

Clearly, the ϕ_j's build up a dyadic partition of unity: $1 \equiv \sum_{j\in\mathbb{Z}} \phi_j(x)$. Finally, associated with the ϕ_j's, which truncate around each *annulus* $\{2^{j-1} \leqslant |x| \leqslant 2^j\}$, we define functions χ_j that truncate away from the *ball* of radius 2^{j-1}, as follows. For any $t \geqslant 0$, we set

$$\chi(t) = \left(\int_0^t \phi(s)\,ds\right) \Big/ \left(\int_0^{+\infty} \phi(s)\,ds\right),$$

where we have identified the radial function $\phi(x)$ with a function of one real variable $t \geqslant 0$, still denoted $\phi(t)$ for convenience. Then, again identifying $\chi(t)$ and the radial function $\chi(x) = \chi(|x|)$, we also set, for any $j \in \mathbb{Z}$

$$\chi_j(x) = \chi\left(\frac{x}{2^j}\right). \tag{7.4}$$

The χ_j's clearly satisfy:

$$\chi_j(x) \equiv 1 \quad \text{whenever } |x| \geqslant 2^j, \qquad \chi_j(x) \equiv 0 \quad \text{whenever } |x| \leqslant 2^{j-1},$$
$$\nabla_x \chi_j(x) = 2^{-j} \frac{x}{|x|} \phi_j(x).$$

The functions ϕ_j and χ_j will be used below as cutoff functions in the space variable x.

Second, it turns out our argument requires a cutoff function in the *angular* variable $\hat{x} \cdot \hat{\xi} = (x/|x|) \cdot (\xi/|\xi|)$, together with a cutoff in *frequency* $|\xi|$ as well. To that aim, we take a non-negative, C_c^∞ function θ_δ satisfying (for some $\delta > 0$ small enough: see below):

$$\theta_\delta(t) \equiv 1 \quad \text{when } 1-\delta \leqslant t \leqslant 1+\delta, \qquad \theta_\delta(t) \equiv 0 \quad \text{when } |t-1| \geqslant 2\delta,$$
$$\theta'_\delta(t) \geqslant 0 \quad \text{when } t \in [0,1]. \tag{7.5}$$

The function θ_δ will serve as a cutoff in $\hat{x} \cdot \hat{\xi}$. Also, we take a non-negative, radial C_c^∞ function $\Theta_\Lambda(\xi)$ such that

$$\Theta_\Lambda(\xi) \equiv 1 \quad \text{when } 1/\Lambda \leqslant |\xi| \leqslant \Lambda,$$
$$\Theta_\Lambda(t) \equiv 0 \quad \text{when } |\xi| \leqslant 1/(2\Lambda) \text{ or } |\xi| \geqslant 2\Lambda. \tag{7.6}$$

Here $\Lambda > 0$ is a parameter to be chosen later.

We claim the proof of Proposition 7 is based on the following estimate.

Proposition 8. *We use the previously defined notations* (*Eqs.* (7.2)–(7.6)). *The small parameter $\delta > 0$ and the large parameter Λ are fixed. Similarly, we also pick up a small parameter $\gamma > 0$ and a large integer $J \geqslant 0$. The way all these parameters are chosen is made precise in the proof. Next, we pick up an arbitrary sequence $\{\beta_j\}_{j \geqslant J}$ in l^1, such that*

$$\forall j \geqslant J, \quad 0 < \beta_j \leqslant 1, \quad \textit{and} \quad \sum_{j \geqslant J} \beta_j = 1. \tag{7.7}$$

Accordingly, we define the "escape functions at infinity"

$$a_J(x,\xi) := \hat{x} \cdot \hat{\xi} + \gamma \sum_{j \geqslant J} \beta_j \chi_j(x) \big[\theta_\delta(\hat{x} \cdot \hat{\xi}) - \theta_\delta(-\hat{x} \cdot \hat{\xi})\big], \tag{7.8}$$
$$b_J(x,\xi) := a_J(x,\xi) \chi_J(x)^2 \Theta_\Lambda(\xi)^2. \tag{7.9}$$

Then, there is a constant $C > 0$, that depends on the choice of γ, δ, J, and Λ, but does not depend on the chosen sequence $\{\beta_j\}$, such that the following estimate holds true, for any $(x,\xi) \in \mathbb{R}^{2d}$:

$$\big\{\xi^2 + V(x), b_J\big\} \geqslant \frac{C}{|x|} \bigg(\sum_{j \geqslant J} \beta_j \phi_j(x) \bigg) \chi_J(x)^2 \Theta_\Lambda(\xi)^2. \tag{7.10}$$

Remark 13. As shown below, the escape function a_J has good properties when $|x| \geqslant 2^{J-1}$ and $|\xi| \geqslant 1/\Lambda$. In this picture, the function b_J simply is equal to a_J, conveniently extended outside the values $|x| \geqslant 2^{J-1}$ and $|\xi| \geqslant 1/\Lambda$. For this reason, both functions a_J and b_J essentially measure the same phenomena, up to the technically useful fact that b_J is defined globally.

Remark 14. The proof of Proposition 8 actually gives more information. Indeed, estimate (7.10) turns out to be true when $\xi^2 + V(x)$ is replaced by ξ^2. In these estimates indeed, the fact that $DV(x)$ behaves like $\langle x \rangle^{-1-\rho}$ at infinity makes the contribution of V in $\{V(x), a_J\}$ or $\{V(x), b_J\}$ negligible, of size $\langle x \rangle^{-1-\rho}$.

We begin with the easy proof of Proposition 8.

Proof of Proposition 8. *First step*. We first prove an estimate similar to (7.10), that involves the function a_J, namely:

$$\{\xi^2 + V(x), a_J\} \geqslant \frac{C}{|x|} \sum_{j \geqslant J} \beta_j \phi_j(x), \quad \text{whenever } |\xi| \geqslant 1/\Lambda,\ |x| \geqslant 2^{J-1}. \tag{7.11}$$

In order to prove (7.11), we compute the Poisson bracket:

$$\begin{aligned}\{\xi^2 + V(x), a_J\} &= 2\frac{|\xi|}{|x|}\big(1 - (\hat{x}\cdot\hat{\xi})^2\big)\Big[1 + \gamma \sum_{j\geqslant J} \beta_j \chi_j(x)\big[\theta'_\delta(\hat{x}\cdot\hat{\xi}) + \theta'_\delta(-\hat{x}\cdot\hat{\xi})\big]\Big] \\ &\quad + 2\gamma\frac{|\xi|}{|x|}\hat{\xi}\cdot\hat{x}\Big[\sum_{j\geqslant J}\beta_j \frac{|x|}{2^j}\phi_j(x)\big[\theta_\delta(\hat{x}\cdot\hat{\xi}) - \theta_\delta(-\hat{x}\cdot\hat{\xi})\big]\Big] \\ &\quad - \frac{\hat{x} - (\hat{x}\cdot\hat{\xi})\hat{\xi}}{|\xi|}\cdot\nabla_x V(x)\Big[1 + \gamma\sum_{j\geqslant J}\beta_j\chi_j(x)\big[\theta'_\delta(\hat{x}\cdot\hat{\xi}) + \theta'_\delta(-\hat{x}\cdot\hat{\xi})\big]\Big].\end{aligned} \tag{7.12}$$

Let us now estimate each term on the right-hand side of (7.12). The last term in (7.12) is negligible for $|x| \geqslant 2^{J-1}$, provided J is large enough. Indeed, its absolute value is estimated by

$$\leqslant \frac{C}{|\xi|}\langle x\rangle^{-1-\rho} \leqslant C\langle x\rangle^{-1-\rho},$$

whenever $|\xi| \geqslant 1/\Lambda > 0$. Here $C > 0$ does not depend on $\{\beta_j\}$. The interesting (dominant) terms are the two other contributions in (7.12). The first term in (7.12) is $\geqslant 0$ in any case. For $||\hat{\xi}\cdot\hat{x}| - 1| \geqslant \delta$ however, i.e., away from the critical values $\hat{\xi}\cdot\hat{x} = \pm 1$, one may improve this lower bound into

$$\geqslant C\delta\frac{|\xi|}{|x|}(1 + C\gamma) \geqslant \frac{C}{|x|},$$

for some constant $C > 0$ independent of $\{\beta_j\}$, whenever $|\xi| \geqslant 1/\Lambda > 0$. Similarly, the second term is $\geqslant 0$ in any case. For $||\hat{\xi}\cdot\hat{x}| - 1| \leqslant \delta$ however, and under the requirement $|x| \geqslant 2^{J-1}$, one may improve this lower bound into

$$\geqslant C\gamma\frac{|\xi|}{|x|}\sum_{j\geqslant J}\beta_j\phi_j(x) \geqslant \frac{C}{|x|}\sum_{j\geqslant J}\beta_j\phi_j(x),$$

for some $C > 0$ independent of $\{\beta_j\}$. Note this inequality requires δ small enough. Putting together the three estimates above allows to estimate the Poisson bracket in (7.12) as in the claimed lower-bound (7.11).

Second step. We now deduce (7.10) from (7.11). The basic observation is the following:

$$\begin{aligned}\{\xi^2 + V(x), a_J\chi_J^2\} &= \{\xi^2 + V(x), a_J\}\chi_J^2 + \{\xi^2 + V(x), \chi_J^2\}a_J \\ &\geqslant \frac{C}{|x|}\Big(\sum_{j\geqslant J}\beta_j\phi_j\Big)\chi_J^2 + \frac{4|\xi|}{2^J}\chi_J\phi_J\underbrace{(\hat{\xi}\cdot\hat{x})a_J}_{\geqslant 0},\end{aligned}$$

where the last line uses the very value of a_J, as well as inequality (7.11), valid for $|\xi| \geqslant 1/\Lambda$, $|x| \geqslant 2^J$. Next we deduce

$$\begin{aligned}\{\xi^2 + V(x), b_J\} &= \{\xi^2 + V(x), a_J\chi_J^2(x)\Theta_\Lambda(\xi)^2\} \\ &= \{\xi^2 + V(x), a_J\chi_J^2(x)\}\Theta_\Lambda(\xi)^2 + \{\xi^2 + V(x), \Theta_\Lambda(\xi)^2\}a_J\chi_J(x)^2 \\ &\geqslant \frac{C}{|x|}\Big(\sum_{j\geqslant J}\beta_j\phi_j\Big)\chi_J(x)^2\Theta_\Lambda(\xi)^2 + \Theta_\Lambda(\xi)\chi_J^2(x)\big(\nabla\Theta_\Lambda(\xi)\cdot\nabla_x V(x)\big) \\ &\geqslant \frac{C}{|x|}\Big(\sum_{j\geqslant J}\beta_j\phi_j\Big)\chi_J(x)^2\Theta_\Lambda(\xi)^2 - C\Theta_\Lambda(\xi)\chi_J^2(x)\langle x\rangle^{-1-\rho} \\ &\geqslant \frac{C}{|x|}\Big(\sum_{j\geqslant J}\beta_j\phi_j\Big)\chi_J(x)^2\Theta_\Lambda(\xi)^2,\end{aligned}$$

provided J is large enough, independently of the sequence $\{\beta_j\}$. This ends the proof of Proposition 8. □

We now come to the more delicate proof of Proposition 7.

Proof of Proposition 7. *First step*: *preliminary reduction*. Our strategy is the following. In order to establish the desired

$$\big\|u_n(x)\mathbf{1}\big[|x| \geqslant 2^J\big]\big\|_{B^*} = \sup_{j\geqslant J} 2^{-j/2}\|u_n\|_{L^2(C_j)} \underset{n\to\infty}{\longrightarrow} 0, \tag{7.13}$$

we prove

$$\sum_{j\geqslant J}\beta_j\int_{\mathbb{R}^d}\phi_j(x)\chi_J^2(x)\frac{|u_n(x)|^2}{|x|}\,dx \underset{n\to\infty}{\longrightarrow} 0, \tag{7.14}$$

independently of the l^1 sequence $\{\beta_j\}$. Clearly, (7.14) implies (7.13).

In the sequel, we actually reduce the statement a bit. Indeed, Propositions 5 and 6 already assert that the sequence $\chi_J u_n$ carries no mass, in L^2 norm, for frequencies ξ above $2c_1 > 0$ or below $c_0/2 > 0$, provided $|x| \geqslant 2^{J-1} \geqslant R_0$. Hence, an easy estimate based on symbolic calculus and Propositions 5 and 6, proves that (7.14) is implied by the weaker

$$\sum_{j\geqslant J}\beta_j\int_{\mathbb{R}^d}\phi_j(x)\chi_J(x)\frac{|(\Theta_\Lambda)^w_{\varepsilon_n}u_n(x)|^2}{|x|}\,dx \underset{n\to\infty}{\longrightarrow} 0, \tag{7.15}$$

independently of the l^1 sequence $\{\beta_j\}$, whenever Λ is so large that $1/\Lambda \leqslant c_0/2$ and $\Lambda \geqslant 2c_1$, and J satisfies $2^{J-1} \geqslant R_0$.

With these reductions in mind, the remainder part of our proof is actually devoted to establishing the estimate

$$\sum_{j\geqslant J} \beta_j \int_{\mathbb{R}^d} \phi_j(x)\chi_{J+1}(x)^2 \frac{|(\Theta_{\Lambda-1})^w_{\varepsilon_n} u_n(x)|^2}{|x|}\, dx \leqslant C\big(\alpha_n + \sqrt{\varepsilon_n} + \eta_n\big), \tag{7.16}$$

for some constant C independent of n and the β_j's. Here, $\Theta_{\Lambda-1}$ is as the function Θ_Λ, with Λ replaced by $\Lambda - 1$, see (7.6). This piece of information is clearly enough to establish (7.15), hence to conclude the proof of the proposition.

Second step: *combining the estimate in Proposition* 8*, with the Garding inequality of Proposition* 2. In essence, estimate (7.16) comes from applying the Garding inequality in Proposition 2, to inequality (7.10). This procedure indeed allows to control, roughly,

$$\sum_{j\geqslant J} \beta_j \int_{\mathbb{R}^d} \phi_j(x)\chi_J(x)^2 \frac{|(\Theta_\Lambda)^w_{\varepsilon_n} u_n(x)|^2}{|x|}\, dx$$

by

$$\big\langle\big(\big\{\xi^2 + V(x), b_J\big\}\big)^w_{\varepsilon_n} u_n, u_n\big\rangle = \big\langle\big\{\xi^2 + V(x), b_J\big\}, \mu_n\big\rangle.$$

One then concludes using the fact that μ_n almost satisfies a transport equation (Proposition 3), so that $\langle\{\xi^2 + V(x), b_J\}, \mu_n\rangle$ is a small term. Let us detail these rough statements.

In this step, we perform the above mentioned combination of the Garding inequality, with Proposition 8. A basic difficulty is the following. Clearly, inequality (7.10) involves (amongst others) symbols that belong to the class $S(\langle x\rangle^{-1})$. This class is critical for our purposes. Recall indeed that the quantity $\int dx\, |u_n(x)|^2/\langle x\rangle$ *is not* controlled by the B^* norm of u_n: only weights that decay slightly faster than $1/|x|$ at infinity are allowed. For this reason, one cannot directly apply the Garding inequality, and an indirect path is needed.

Let us come to the technical details.

First, we start from estimate (7.10) with $\xi^2 + V(x)$ replaced by ξ^2 (see Remark 14), namely,

$$\big\{\xi^2, b_J\big\} \geqslant \frac{C}{|x|}\bigg(\sum_{j\geqslant J} \beta_j\phi_j(x)\bigg)\chi_J^2\Theta_\Lambda^2, \tag{7.17}$$

and the symbols appearing in (7.17) belong to the class $S(\langle x\rangle^{-1})$. As a consequence, the difference between both sides of the above inequality may be written as a *squared*, up to reducing the value of the constant C. Quantitatively, there exists a symbol $b(x,\xi)$ belonging to the class $S(\langle x\rangle^{-1/2})$ such that

$$\bigg(\big\{\xi^2, b_J\big\} - \frac{C}{|x|}\bigg(\sum_{j\geqslant J} \beta_j\phi_j(x)\bigg)\bigg)\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2 = b(x,\xi)^2\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2. \tag{7.18}$$

Note the slight reduction consisting in the replacement of χ_J by χ_{J+1}, and Θ_Λ by $\Theta_{\Lambda-1}$, in order to avoid taking the square root of symbols close to the set where they vanish. Note also that

b certainly depends upon the β_j's, yet its norm in the class $S(\langle x\rangle^{-1/2})$ is bounded, uniformly with respect to the β_j's.

Second, Weyl-quantizing relation (7.18) and testing against the function u_n, we recover

$$\big\langle \big(\{\xi^2, b_J\}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\big)^w_{\varepsilon_n} u_n, u_n\big\rangle = C\Bigg\langle \Bigg(\sum_{j\geqslant J} \beta_j \frac{\phi_j(x)}{|x|}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\Bigg)^w_{\varepsilon_n} u_n, u_n\Bigg\rangle + \big\langle \big(b(x,\xi)^2\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\big)^w_{\varepsilon_n} u_n, u_n\big\rangle. \tag{7.19}$$

On top of that, the term involving $b(x,\xi)$ in (7.19) is easily bounded from below. Indeed, using standard symbolic calculus, we observe

$$\begin{aligned}
&\big\langle \big[b(x,\xi)^2\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\big]^w_{\varepsilon_n} u_n, u_n\big\rangle \\
&\quad= \big\langle \big[b(x,\xi)\chi_{J+1}(x)\Theta_{\Lambda-1}(\xi)\big]^w_{\varepsilon_n} u_n, \big[b(x,\xi)\chi_{J+1}(x)\Theta_{\Lambda-1}(\xi)\big]^w_{\varepsilon_n} u_n\big\rangle \\
&\qquad+ \big\langle \big(O\big(\varepsilon_n\langle x\rangle^{-3/2}\big)\big)^w_{\varepsilon_n} u_n, u_n\big\rangle.
\end{aligned}$$

Here the $O(\varepsilon_n\langle x\rangle^{-3/2})$ denotes some symbol in the class $S(\langle x\rangle^{-3/2})$, and of size $O(\varepsilon_n)$ in this class, uniformly with respect to the β_j's. This implies

$$\begin{aligned}
&\big\langle \big[b(x,\xi)^2\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\big]^w_{\varepsilon_n} u_n, u_n\big\rangle \\
&\quad= \big\|\big[b(x,\xi)\chi_{J+1}(x)\Theta_{\Lambda-1}(\xi)\big]^w_{\varepsilon_n} u_n\big\|^2_{L^2} + \big\langle \big[O\big(\varepsilon_n\langle x\rangle^{-3/2}\big)\big]^w_{\varepsilon_n} u_n, u_n\big\rangle \\
&\quad\geqslant -C\varepsilon_n \int_{\mathbb{R}^d} \langle x\rangle^{-3/2}\big|u_n(x)\big|^2\,dx \\
&\quad\geqslant -C\varepsilon_n,
\end{aligned} \tag{7.20}$$

uniformly with respect to the β_j's. The last line uses the information $\|u_n\|_{B^*}=1$. Note that the squared term in (7.20) may be *unbounded*, because it only carries the critical weight $\langle x\rangle^{-1}$. We are able to discard it because it has the good sign. Similarly, standard symbolic calculus also allows to write

$$\begin{aligned}
\Bigg\langle \Bigg(\frac{\sum_j \beta_j\phi_j}{|x|}\chi^2_{J+1}\Theta^2_{\Lambda-1}\Bigg)^w_{\varepsilon_n} u_n, u_n\Bigg\rangle &= \Bigg\langle \frac{\sum_j \beta_j\phi_j}{|x|}\chi^2_{J+1}(\Theta_{\Lambda-1})^w_{\varepsilon_n} u_n, (\Theta_{\Lambda-1})^w_{\varepsilon_n} u_n\Bigg\rangle + O(\varepsilon_n) \\
&= \sum_j \beta_j \int_{\mathbb{R}^d} \phi_j\chi^2_{J+1}\frac{|(\Theta_{\Lambda-1})^w_{\varepsilon_n} u_n(x)|^2}{|x|} + O(\varepsilon_n).
\end{aligned} \tag{7.21}$$

Summarizing, we have established at this stage the estimate

$$\big\langle \big(\{\xi^2, b_J\}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\big)^w_{\varepsilon_n} u_n, u_n\big\rangle \geqslant C\sum_j \beta_j \int_{\mathbb{R}^d} \phi_j\chi^2_{J+1}\frac{|(\Theta_{\Lambda-1})^w_{\varepsilon_n} u_n(x)|^2}{|x|} - C\varepsilon_n. \tag{7.22}$$

Third, we may estimate from below the term involving $\{V, b_J\}$ in the similar way. Indeed, at the level of symbols, we have

$$\{V, b_J\}\chi_{J+1}^2\Theta_{\Lambda-1}^2 \geqslant -C\langle x\rangle^{-1-\rho}\chi_{J+1}^2\Theta_{\Lambda-1}^2. \tag{7.23}$$

Hence, upon Weyl-quantizing (7.23), and using the Garding inequality of Proposition 2, which is licit thanks to the assumed decay/smoothness of the potential V, we recover

$$\begin{aligned}\big\langle\big(\big\{V(x), b_J\big\}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\big)_{\varepsilon_n}^w u_n, u_n\big\rangle &\geqslant -C\big\langle\big(\langle x\rangle^{-1-\rho}\chi_{J+1}^2\Theta_{\Lambda-1}^2\big)_{\varepsilon_n}^w u_n(x), u_n\big\rangle - C\sqrt{\varepsilon_n} \\ &\geqslant -C\int_{\mathbb{R}^d}\chi_{J+1}^2\frac{|(\Theta_{\Lambda-1})_{\varepsilon_n}^w u_n|^2}{\langle x\rangle^{1+\rho}}\,dx - C\sqrt{\varepsilon_n}, \end{aligned} \tag{7.24}$$

where the last inequality uses standard symbolic calculus again.

Fourth, we put together (7.22) and (7.24). This gives eventually, possibly taking a larger value of J,

$$\begin{aligned}&\big\langle\big(\big\{\xi^2 + V(x), b_J\big\}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\big)_{\varepsilon_n}^w u_n, u_n\big\rangle \\ &\quad\geqslant C\sum_j\beta_j\int_{\mathbb{R}^d}\phi_j\chi_{J+1}^2\frac{|(\Theta_{\Lambda-1})_{\varepsilon_n}^w u_n(x)|^2}{|x|} - C\int_{\mathbb{R}^d}\chi_{J+1}^2\frac{|(\Theta_{\Lambda-1})_{\varepsilon_n}^w u_n|^2}{\langle x\rangle^{1+\rho}}\,dx - C\sqrt{\varepsilon_n}. \end{aligned}\tag{7.25}$$

This is the final estimate of the present step.

Third step: *estimating from above the left-hand side of* (7.25). In this step, we prove the left-hand side of (7.25) goes to zero as n goes to infinity.

First, we write the obvious

$$\begin{aligned}&\big\langle\big[\big\{\xi^2 + V(x), b_J\big\}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\big]_{\varepsilon_n}^w u_n, u_n\big\rangle \\ &\quad= \big\langle\big\{\xi^2 + V(x), b_J\big\}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2, \mu_n\big\rangle. \end{aligned}\tag{7.26}$$

We next start taking back the product $\chi_{J+1}^2\Theta_{\Lambda-1}^2$ inside the Poisson bracket.

Second, at the level of symbols, we readily have the commutator inequality

$$\big\{\xi^2 + V(x), b_J\big\}\chi_{J+1}(x)^2 - \big\{\xi^2 + V(x), b_J\chi_{J+1}(x)^2\big\} = -\frac{4|\xi|\chi_{J+1}, \phi_{J+1}}{2^{J+1}}\big([\hat{\xi}\cdot\hat{x}]b_J\big) \leqslant 0,$$

and we note that the symbols involved here are C_c^∞ in the ξ variable, and they behave like $O(\langle x\rangle^{-\infty})$ at infinity in x. As a consequence, these symbols certainly belong to the class X_N for any N, as well as all their derivatives. This allows to use the Garding inequality derived in Proposition 2, and to write in (7.26)

$$\begin{aligned}&\big\langle\big\{\xi^2 + V(x), b_J\big\}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2, \mu_n\big\rangle \\ &\quad\leqslant \big\langle\big\{\xi^2 + V(x), b_J\chi_{J+1}(x)^2\big\}\Theta_{\Lambda-1}(\xi)^2, \mu_n\big\rangle + O\big(\sqrt{\varepsilon_n}\big). \end{aligned}\tag{7.27}$$

Third, and similarly, we now take $\Theta_{\Lambda-1}$ back in the Poisson bracket. We write, at the level of symbols, the commutator estimate:

$$\begin{aligned}&\left|\left\{\xi^2+V(x), b_J\chi_{J+1}(x)^2\right\}\Theta_{\Lambda-1}(\xi)^2-\left\{\xi^2+V(x), b_J\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\right\}\right|\\&\quad\leqslant C\langle x\rangle^{-1-\rho}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi).\end{aligned}$$

Here, all symbols clearly belong to the class X_N for any N. Using again the version of the Garding inequality derived in Proposition 2, we recover in (7.27)

$$\begin{aligned}&\left\langle\left\{\xi^2+V(x), b_J\chi_{J+1}(x)^2\right\}\Theta_{\Lambda-1}(\xi)^2, \mu_n\right\rangle\\&\quad\leqslant\left\langle\left\{\xi^2+V(x), b_J\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\right\}, \mu_n\right\rangle\\&\qquad+C\left\langle\langle x\rangle^{-1-\rho}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi), \mu_n\right\rangle+O\left(\sqrt{\varepsilon_n}\right)\\&\quad=\left\langle\left\{\xi^2+V(x), b_J\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\right\}, \mu_n\right\rangle\\&\qquad+C\int_{\mathbb{R}^d}\frac{\chi_{J+1}(x)^2[(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n](x)u_n^*(x)}{\langle x\rangle^{1+\rho}}\,dx+O\left(\sqrt{\varepsilon_n}\right).\end{aligned}\tag{7.28}$$

The last line uses standard symbolic calculus again. Yet the transport equation (Proposition 3, estimate (5.1)) readily allows to estimate the first term on the right-hand side of (7.28) as

$$\left|\left\langle\left\{\xi^2+V(x), b_J\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\right\}, \mu_n\right\rangle\right|\leqslant C(\alpha_n+\varepsilon_n+\eta_n).\tag{7.29}$$

On top of that, using Propositions 5 and 6, we may estimate the second term on the right-hand side of (7.28) as

$$\begin{aligned}&\int_{\mathbb{R}^d}\frac{\chi_{J+1}(x)^2[(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n](x)u_n^*(x)}{\langle x\rangle^{1+\rho}}\,dx\\&\quad=\int_{\mathbb{R}^d}\frac{\chi_{J+1}(x)^2|(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n|^2(x)}{\langle x\rangle^{1+\rho}}\,dx\\&\qquad+\int_{\mathbb{R}^d}\frac{\chi_{J+1}(x)^2[(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n](x)[(1-\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n]^*(x)}{\langle x\rangle^{1+\rho}}\,dx\\&\quad=\int_{\mathbb{R}^d}\frac{\chi_{J+1}(x)^2|(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n|^2(x)}{\langle x\rangle^{1+\rho}}\,dx+O\left(\sqrt{\varepsilon_n}+\eta_n\right),\end{aligned}\tag{7.30}$$

and the last $O(\sqrt{\varepsilon_n}+\eta_n)$ comes from the fact that the function $1-\Theta_{\Lambda-1}$ has support away from resonant frequencies $\xi^2=\lambda-V(x)$, provided Λ is large enough.

Fourth, putting together (7.27)–(7.30), we recover

$$\begin{aligned}&\left|\left\langle\left[\left\{\xi^2+V(x),b_J\right\}\chi_{J+1}(x)^2\Theta_{\Lambda-1}(\xi)^2\right]^w_{\varepsilon_n}u_n,u_n\right\rangle\right|\\&\quad\leqslant C\int\limits_{\mathbb{R}^d}\frac{\chi_{J+1}(x)^2|(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n|^2(x)}{\langle x\rangle^{1+\rho}}\,dx+C\left(\alpha_n+\sqrt{\varepsilon_n}+\eta_n\right).\end{aligned}\tag{7.31}$$

Fourth step: *conclusion.* Estimate (7.31) now gives in (7.25):

$$\begin{aligned}&\sum_j\beta_j\int\limits_{\mathbb{R}^d}\phi_j\chi_{J+1}^2\frac{|(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n(x)|^2}{|x|}\\&\quad\leqslant C\int\limits_{\mathbb{R}^d}\frac{\chi_{J+1}(x)^2|(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n|^2(x)}{\langle x\rangle^{1+\rho}}\,dx+C\left(\alpha_n+\sqrt{\varepsilon_n}+\eta_n\right).\end{aligned}$$

Now, taking the supremum over all possible l^1 sequences $\{\beta_j\}$ as in (7.7), gives

$$\begin{aligned}&\sup_{j\geqslant J}2^{-j/2}\left\|\chi_{J+1}(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n\right\|_{L^2(C_j)}\\&\quad\leqslant C\left\|\chi_{J+1}(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n\right\|_{L^2(\langle x\rangle^{-1-\rho}\,dx)}+C\left(\alpha_n+\sqrt{\varepsilon_n}+\eta_n\right)\\&\quad\leqslant C2^{-\rho J}\sup_{j\geqslant J}2^{-j/2}\left\|\chi_{J+1}(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n\right\|_{L^2(C_j)}+C\left(\alpha_n+\sqrt{\varepsilon_n}+\eta_n\right).\end{aligned}$$

Hence, upon possibly taking an even larger value of J, we recover

$$\sup_{j\geqslant J}2^{-j/2}\left\|\chi_{J+1}(\Theta_{\Lambda-1})^w_{\varepsilon_n}u_n\right\|_{L^2(C_j)}\leqslant C\left(\alpha_n+\sqrt{\varepsilon_n}+\eta_n\right).\tag{7.32}$$

This ends the proof of (7.16). Proposition 7 is now proved. □

8. Using the transport equation: the sequence $\{u_n\}$ has no mass away from the origin

The previous section establishes u_n has no mass "at infinity" in x. In this section we use the transport equation satisfied by μ_n and the non-trapping assumption to deduce that u_n has no mass on any bounded set away from the origin $x=0$. Again, the special treatment of the origin is made necessary because of the very norms B and B^*. Needless to say, the idea is to use the non-trapping assumption, together with the invariance of μ along the flow Φ_t, to infer that μ vanishes locally, from the fact that μ vanishes at infinity.

Our main result in this section is:

Proposition 9. *Let $0<r_0<r_1<\infty$ be two arbitrary radii. Then the following holds*:

$$\left\|u_n(x)\mathbf{1}\left[r_0\leqslant|x|\leqslant r_1\right]\right\|_{B^*}\underset{n\to\infty}{\longrightarrow}0.$$

In particular, for any test function $\varphi(x)\in C_c^\infty(\mathbb{R}^d)$ whose support lies in $r_0\leqslant|x|\leqslant r_1$, one has $\langle\mu,\varphi\rangle=0$.

Proof. Take a test function $\varphi(x)$ as in Proposition 9. It is enough to prove

$$\big\|\varphi(x)u_n(x)\big\|_{L^2} \to 0.$$

To reduce the problem a bit, let $\Theta(\xi)$ be a C_c^∞ function that cuts off large frequencies ξ, i.e., such that $\Theta(\xi) \equiv 0$ when $|\xi| \geqslant 3c_1$, and $\Theta(\xi) \equiv 1$ when $|\xi| \leqslant 2c_1$, say. Energy localization (and more precisely Proposition 5) readily gives

$$\big\|(1-\Theta)^w_{\varepsilon_n}\varphi(x)u_n(x)\big\|_{L^2} \to 0.$$

On the other hand, standard symbolic calculus gives

$$\begin{aligned}\big\|\Theta^w_{\varepsilon_n}\varphi(x)u_n(x)\big\|^2_{L^2} &= \big\langle \mu_n(x,\xi), \Theta(\xi)^2\varphi(x)^2\big\rangle + O(\varepsilon_n)\\ &\underset{n\to\infty}{\sim} \big\langle \mu(x,\xi), \Theta(\xi)^2\varphi(x)^2\big\rangle.\end{aligned}$$

It is thus enough to prove $\langle \mu, \Theta^2\varphi^2\rangle = 0$.

To reduce the problem further, let $\delta > 0$ be a small parameter to be chosen later, and let $\chi(x,\xi)$ be a function that cuts-off (x,ξ)'s away from the zero energy, i.e., $\chi(x,\xi) \equiv 0$ when $|\xi^2 + V(x) - \lambda| \geqslant \delta$, $\chi(x,\xi) \equiv 1$ when $|\xi^2 + V(x) - \lambda| \leqslant \delta/2$. Again, energy localization (and more precisely Proposition 4) readily gives

$$\big\langle \mu(x,\xi), \Theta(\xi)^2\varphi(x)^2\big\rangle = \big\langle \mu(x,\xi), \Theta(\xi)^2\varphi(x)^2\chi(x,\xi)\big\rangle.$$

It is thus enough to prove $\langle \mu(x,\xi), \Theta(\xi)^2\varphi(x)^2\chi(x,\xi)\rangle = 0$.

To do so, we use the fact that, from Proposition 3, the measure μ is invariant under the Hamiltonian flow Φ_t of $H(x,\xi) = \xi^2 + V(x) - \lambda$. Hence, for any time t, we certainly have

$$\big\langle \mu(x,\xi), \Theta(\xi)^2\varphi(x)^2\chi(x,\xi)\big\rangle = \big\langle \mu(x,\xi), \big(\Theta(\xi)^2\varphi(x)^2\chi(x,\xi)\big)\circ\Phi_{-t}\big\rangle,$$

where the support of the function $(\Theta(\xi)\varphi(x))^2 \circ \Phi_{-t}$ is

$$\Phi_t\big(\big\{|x| \leqslant r_1,\ |\xi| \leqslant 2c_1,\ \big|\xi^2 + V(x) - \lambda\big| \leqslant \delta\big\}\big).$$

The idea is to use the non-trapping assumption to establish there is a (large) time t_*, such that

$$\Phi_{t_*}\big(\big\{|x| \leqslant r_1,\ |\xi| \leqslant 2c_1,\ \big|\xi^2 + V(x) - \lambda\big| \leqslant \delta\big\}\big) \subset \big\{|x| \geqslant 2^J\big\}. \tag{8.1}$$

Clearly, (8.1) is enough to conclude since Proposition 7 allows to write under these circumstances

$$\big\langle \mu(x,\xi), \Theta(\xi)^2\varphi(x)^2\big\rangle = \big\langle \mu(x,\xi), \big(\Theta(\xi)^2\varphi(x)^2\chi(x,\xi)\big)\circ\Phi_{-t_*}\big\rangle = 0.$$

Let us come to the proof of (8.1).

Non-trapping asserts that for any (x,ξ) satisfying $\xi^2 = \lambda - V(x)$, there is a time $T(x,\xi)$ such that for any $t \geqslant T(x,\xi)$ we have $|X(t,x,\xi)| \geqslant 2^{2J}$. Actually, and as is well known (see, e.g., [11]), non-trapping is an open property: there is a δ such that any (x,ξ) satisfying $|\xi^2 + V(x) - \lambda| \leqslant \delta$ verifies $|X(t,x,\xi)| \to \infty$ as $t \to \infty$. By compactness and continuous dependence of the flow upon the initial data, there is hence a time T_0 such that for any (x,ξ) satisfying

$|\xi^2 + V(x) - \lambda| \leqslant \delta$ together with $|x| \leqslant r_1$ and $|\xi| \leqslant c_1$, one has $|X(t,x,\xi)| \geqslant 2^{2J}$. This ends the proof of (8.1).

Proposition 9 is proved. □

9. Using the Perthame and Vega multiplier: the sequence $\{u_n\}$ has no mass at the origin

This section is devoted to proving the last piece of information that allows to glue together the vanishing of u_n at infinity, away from the origin, and close to the origin, and to conclude $\|u_n\|_{B^*} \to 0$. This gives the desired contradiction (see Section 2), and finishes the proof of our Main Theorem.

Our main result in this section is the following:

Proposition 10. *Let $r_0 > 0$ be any positive radius. Then, the following holds*:

$$\left\| u_n(x)\mathbf{1}\big[|x| \leqslant r_0\big]\right\|_{B^*} \underset{n\to\infty}{\longrightarrow} 0.$$

Remark 15. The proof we give of Proposition 10 actually establishes the following stronger fact. *If* we are able to prove that, for some reason, $\|u_n\mathbf{1}[|x| \geqslant r]\|_{B^*} \to 0$ as $n \to \infty$, whenever $r > 0$, *then*, necessarily, for any $r > 0$, $\|u_n\mathbf{1}[|x| \leqslant r]\|_{B^*} \to 0$ as well.

In the present case, we already know from Propositions 7 and 9 that $\|u_n\mathbf{1}[|x| \geqslant r]\|_{B^*} \to 0$ as $n \to \infty$, whenever $r > 0$.

As already mentioned, the idea of proof is to use an estimate established by Perthame and Vega, which relates the mass of u_n close to $x = 0$, with the mass of u_n away from $x = 0$. This together with the already established fact that u_n vanishes away from the origin, gives the result.

Proof of Proposition 10. The proof is in several steps.

First step: *an a priori estimate obtained from* [28]—*its consequences.* A straightforward rescaling in estimate (3.7) of [28, p. 346] gives, for any $R > 0$:

$$\begin{aligned}
&\frac{1}{R}\int_{|x|\leqslant R} \left|\varepsilon_n \nabla_x u_n(x)\right|^2 dx + \varepsilon_n^2 \frac{d-1}{2R^2}\int_{|x|=R} \left|u_n(x)\right|^2 d\sigma_R(x)\\
&\qquad + \frac{1}{R}\int_{|x|\leqslant R} \big[\lambda - V(x) + \big(x\cdot\nabla_x V(x)\big)_-\big]\left|u_n(x)\right|^2 dx\\
&\quad \leqslant C\int_{\mathbb{R}^d} \left|f_n(x)\right|\left|u_n(x)\right| dx + C\int_{\mathbb{R}^d} \frac{|f_n(x)||u_n(x)|}{|x|}\, dx\\
&\qquad + \int_{\mathbb{R}^d} \big(\hat{x}\cdot\nabla_x V(x)\big)_+ \left|u_n(x)\right|^2 dx + \alpha_n \int_{\mathbb{R}^d} \left|u_n(x)\right|\left|\varepsilon_n \nabla_x u_n(x)\right| dx,
\end{aligned} \tag{9.1}$$

for some constant C independent of n and R. We now start estimating all terms on the right-hand side of (9.1), except the one including the contribution of $(\hat{x}\cdot\nabla_x V(x))_+$, which we exploit later.

First, using the bounds (2.1) on f_n and u_n, we clearly have

$$\int_{\mathbb{R}^d} \left| f_n(x) \right| \left| u_n(x) \right| dx \leqslant C \varepsilon_n \eta_n,$$

which gives an estimate for the first term on the right-hand side of (9.1). Second, a straightforward energy estimate in the Helmholtz equation (2.2) gives

$$\alpha_n \int_{\mathbb{R}^d} \left| u_n(x) \right|^2 dx \leqslant \varepsilon_n^{-1} \int_{\mathbb{R}^d} \left| f_n(x) \right| \left| u_n(x) \right| dx \leqslant C \eta_n,$$

together with

$$\begin{aligned} \int_{\mathbb{R}^d} \left| \varepsilon_n \nabla_x u_n(x) \right|^2 dx &\leqslant \int_{\mathbb{R}^d} \left| \lambda - V(x) \right| \left| u_n(x) \right|^2 dx + \int_{\mathbb{R}^d} \left| f_n(x) \right| \left| u_n(x) \right| dx \\ &\leqslant C \int_{\mathbb{R}^d} \left| u_n(x) \right|^2 dx + C \varepsilon_n \eta_n \leqslant C \alpha_n^{-1} \eta_n. \end{aligned}$$

This allows to estimate the last term on the right-hand side of (9.1) as

$$\alpha_n \int_{\mathbb{R}^d} \left| u_n(x) \right| \left| \varepsilon_n \nabla_x u_n(x) \right| dx \leqslant C \eta_n.$$

Last, we may estimate the third term as in [28]:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|f_n(x)||u_n(x)|}{|x|} dx &= \sum_{j \in \mathbb{Z}} \left(2^{-j} \int_{C_j} \frac{|u_n(x)|^2}{|x|^2} dx \right)^{1/2} \left(2^j \int_{C_j} \left| f_n(x) \right|^2 dx \right)^{1/2} \\ &\leqslant C \varepsilon_n \eta_n \left(\sup_{r>0} \frac{1}{r^2} \int_{|x|=r} \left| u_n(x) \right|^2 d\sigma_r(x) \right)^{1/2}. \end{aligned}$$

These three estimates give in (9.1), upon discarding the first term on the left-hand side:

$$\begin{aligned} &\varepsilon_n^2 \frac{d-1}{R^2} \int_{|x|=R} \left| u_n(x) \right|^2 d\sigma_R(x) + \frac{1}{R} \int_{|x| \leqslant R} \left[\lambda - V(x) + \left(x \cdot \nabla_x V(x) \right)_- \right] \left| u_n(x) \right|^2 dx \\ &\quad \leqslant C \eta_n + C \eta_n \left(\sup_{r>0} \frac{\varepsilon_n^2}{r^2} \int_{|x|=r} \left| u_n(x) \right|^2 d\sigma_r(x) \right)^{1/2} + \int_{\mathbb{R}^d} \left(\hat{x} \cdot \nabla_x V(x) \right)_+ \left| u_n(x) \right|^2 dx. \quad (9.2) \end{aligned}$$

Next, for n large enough, estimate (9.2) allows to upper bound the term $\sup_{r>0}(\varepsilon_n^2/r^2 \ldots)$ upon estimating the right-hand side with the help of the left-hand side. Indeed, taking the supremum over R of the left-hand side, exploiting the information $\lim_{n\to\infty} \eta_n = 0$, and using the fact that

$$\frac{1}{R}\int\limits_{|x|\leqslant R}\big|u_n(x)\big|^2\,dx\leqslant C\|u_n\|_{B^*}^2\leqslant C,$$

$$\int\limits_{\mathbb{R}^d}\big(\hat{x}\cdot\nabla_x V(x)\big)_+\big|u_n(x)\big|^2\,dx\leqslant C\int\limits_{\mathbb{R}^d}\langle x\rangle^{-1-\rho}\big|u_n(x)\big|^2\,dx\leqslant C,$$

we easily deduce from (9.2) the estimate

$$\sup_{r>0}\frac{\varepsilon_n^2}{r^2}\int\limits_{|x|=r}\big|u_n(x)\big|^2\,d\sigma_r(x)\leqslant C.$$

As a consequence, we are now able to transform estimate (9.2) into the simpler:

$$\begin{aligned}&\frac{1}{R}\int\limits_{|x|\leqslant R}\big[\lambda-V(x)+\big(x\cdot\nabla_x V(x)\big)_-\big]\big|u_n(x)\big|^2\,dx\\&\quad\leqslant C\eta_n+\int\limits_{\mathbb{R}^d}\big(\hat{x}\cdot\nabla_x V(x)\big)_+\big|u_n(x)\big|^2\,dx.\end{aligned}\tag{9.3}$$

This is the key a priori estimate we needed.

Second step: *the right-hand side of* (9.3) *vanishes as $n\to\infty$*. To prove that the right-hand side of (9.3) vanishes asymptotically, we split the integral on the right-hand side of (9.3) according to the distinction $|x|\geqslant 2^J$, $2^{-j_0}\leqslant|x|\leqslant 2^J$ and $|x|\leqslant 2^{-j_0}$. Here, the (large positive) integer J is as in Proposition 7, (large positive) integer j_0 is to be chosen later. More precisely, we write:

$$\begin{aligned}\int\limits_{\mathbb{R}^d}\big(\hat{x}\cdot\nabla_x V(x)\big)_+\big|u_n(x)\big|^2&=\int\limits_{\mathbb{R}^d}\big[\hat{x}\cdot\nabla_x V(x)\big]_+\big|u_n(x)\big|^2\\&\quad\times\big(\chi_J(x)+\big[1-\chi_J(x)\big]\chi_{-j_0}(x)+\big[1-\chi_{-j_0}(x)\big]\big).\end{aligned}$$

First, Proposition 7 allows to estimate the contribution of large values of x:

$$\int\limits_{\mathbb{R}^d}\chi_J(x)\big[\hat{x}\cdot\nabla_x V(x)\big]_+\big|u_n(x)\big|^2\,dx\leqslant C\int\limits_{\mathbb{R}^d}\chi_J(x)\frac{|u_n(x)|^2}{\langle x\rangle^{1+\rho}}\,dx\underset{n\to\infty}{\longrightarrow}0.$$

Second, the contribution of moderate values of x is estimated using the fact that the measure μ vanishes everywhere (Proposition 9). Indeed, for any *given* value of j_0, we have

$$\begin{aligned}&\int\limits_{\mathbb{R}^d}\big(1-\chi_J^2\big)(x)\chi_{-j_0}^2(x)\big(\hat{x}\cdot\nabla_x V(x)\big)_+\big|u_n(x)\big|^2\,dx\\&\quad\underset{n\to\infty}{\longrightarrow}\int\limits_{\mathbb{R}^{2d}}\big(1-\chi_J^2\big)(x)\chi_{-j_0}^2(x)\big(\hat{x}\cdot\nabla_x V(x)\big)_+\mu(x,\xi)\,dx\,d\xi=0.\end{aligned}$$

To be precise, this step uses the fact that the measure μ has compact support in ξ. Last, using the mere boundedness of $\nabla_x V$ close to the origin, we may estimate the contribution of small values of x in the natural way:

$$\int_{\mathbb{R}^d} \big[1-\chi_{-j_0}(x)\big]\big[\hat{x}\cdot\nabla_x V(x)\big]_+ \big|u_n(x)\big|^2\,dx$$

$$\leqslant C \int_{|x|\leqslant 2^{-j_0+2}} \big|u_n(x)\big|^2\,dx = C\sum_{j\leqslant -j_0+2} \|u_n\|^2_{L^2(C_j)} \leqslant C\|u_n\|^2_{B^*}\sum_{j\leqslant j_0+2} 2^j \leqslant C2^{-j_0}.$$

These three estimates clearly imply

$$\int_{\mathbb{R}^d} \big(\hat{x}\cdot\nabla_x V(x)\big)_+ \big|u_n(x)\big|^2 \underset{n\to\infty}{\longrightarrow} 0.$$

As a consequence, the upper-bound (9.3), finally gives

$$\frac{1}{R}\int_{|x|\leqslant R} \big[\lambda - V(x) + \big(x\cdot\nabla_x V(x)\big)_-\big]\big|u_n(x)\big|^2\,dx \underset{n\to\infty}{\longrightarrow} 0. \tag{9.4}$$

Third step: *conclusion.* We now exploit (9.4) to obtain the proposition. To do so, we use the fact that

$$\lambda - V(0) > 0, \quad \text{and } V \text{ is continuous.}$$

Hence, for j_0 large enough, and some $C>0$ independent of n, we have the following lower bound, valid for any $R\leqslant 2^{-j_0}$:

$$\frac{1}{R}\int_{|x|\leqslant R} \big[\lambda - V(x) + \big(x\cdot\nabla_x V(x)\big)_-\big]\big|u_n(x)\big|^2\,dx \geqslant \frac{C}{R}\int_{|x|\leqslant R} \big|u_n(x)\big|^2\,dx.$$

This together with (9.4) finally gives

$$\sup_{R\leqslant 2^{j_0}} \frac{1}{R}\int_{|x|\leqslant R} \big|u_n(x)\big|^2\,dx \underset{n\to\infty}{\longrightarrow} 0,$$

or, in other words

$$\sup_{j\leqslant -j_0} 2^{-j/2}\|u_n\|_{L^2(C_j)} \underset{n\to\infty}{\longrightarrow} 0.$$

The proposition is proved. □

Acknowledgments

The first author thanks X.P. Wang, P. Zhang as well as B. Perthame and L. Vega for stimulating discussions on the present subject.

This work has been partially supported by the "ACI Jeunes Chercheurs—Méthodes haute fréquence pour les Equations différentielles ordinaires, et aux dérivées partielles. Applications," by the GDR "Amplitude Equations and Qualitative Properties" (GDR CNRS 2103: EAPQ) and the European Program 'Improving the Human Potential' in the framework of the 'HYKE' network HPRN-CT-2002-00282.

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