Physics Letters B 692 (2010) 43–46

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Qubits and chirotopes

J.A. Nieto [a,b,c,*]

[a] Mathematical, Computational & Modeling Science Center, Arizona State University, PO Box 871904, Tempe, AZ 85287, USA
[b] Facultad de Ciencias Físico-Matemáticas de la Universidad Autónoma de Sinaloa, 80010, Culiacán Sinaloa, Mexico
[c] Departamento de Investigación en Física de la Universidad de Sonora, 83000, Hermosillo Sonora, Mexico

ARTICLE INFO

Article history:
Received 6 May 2010
Received in revised form 23 June 2010
Accepted 28 June 2010
Available online 13 July 2010
Editor: M. Cvetič

Keywords:
Qubits
Oriented matroid theory

ABSTRACT

We show that qubit and chirotope concepts are closely related. In fact, we prove that the qubit concept leads to a generalization of the chirotope concept, which we call qubitope. Moreover, we argue that a possible qubitope theory may suggest interesting applications of oriented matroid theory in at least three physical contexts, in which qubits make their appearance, namely string theory, black holes and quantum information.

Recently, in a number of remarkable developments [1–8], a relation between, apparently two different scenarios, black holes and quantum information, has been established. The key concept for this link has been the so-called quantum bit notion, or qubit, which is the smallest unit of quantum information. In appropriate qubit basis, the components of a pure state $|\psi\rangle$ can be written in terms hypermatrix $a_{a_1a_2...a_N}$ which in turn leads to a density matrix ρ. It turns out that ρ can be defined in terms of the hyperdeterminant associated with $a_{a_1a_2...a_N}$, a quantity introduced for the first time by Cayley in 1845 [9]. Surprisingly, in some cases the quantity $a_{a_1a_2...a_N}$ can also be related to the entropy of STU black holes via also its hyperdeterminant (see Ref. [4] for details).

On the other hand, it is known that the chirotope concept plays a fundamental role in oriented matroid theory [10]. In fact, the emergence of this concept can be traced back to the origin of matroids [11] which can be understood as a generalization of matrices. From a modern perspective, however, one may introduce the mathematical notion of chirotope, or oriented matroid, by considering a generalization of the Grassmann–Plücker relations of ordinary determinants [12].

Thus, we have two generalizations of the matrix notion, namely hypermatrix and matroid. Since a qubit is related with hyperdeterminants of hypermatrices and a chirotope is connected with a generalization of ordinary determinants via the Grassmann–Plücker relations one may wonder whether these two qubit-chirotope concepts are related. If we achieve such a relation then one may be in a position to bring a variety of mathematical tools from oriented matroid theory to black-hole physics and vice versa.

Our starting point is to consider a possible scenario in which the qubit concept makes its appearance [1], namely the $(2+2)$-signature flat target "spacetime" of the Nambu–Goto action. Let us first observe that the line element,

$$ds^2 = dx^\mu \, dx^\nu \, \eta_{\mu\nu}, \tag{1}$$

of flat space with $(2+2)$-signature, with $\eta_{\mu\nu} = \mathrm{diag}(-1,-1,1,1)$, may also be written as

$$ds^2 = \frac{1}{2} dx^{ab} \, dx^{cd} \, \varepsilon_{ac}\varepsilon_{bd}, \tag{2}$$

where the matrix coordinates x^{ab} are given by

$$x^{ab} = \begin{pmatrix} x^1 + x^3 & x^2 + x^4 \\ x^2 - x^4 & -x^1 + x^3 \end{pmatrix}, \tag{3}$$

and ε_{ab} is the completely antisymmetric symbol with $\varepsilon_{12} = 1$.

Similarly, it is not difficult to show [1] (see also Ref. [2]) that the world sheet metric

$$\gamma_{ab} = \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} \tag{4}$$

can also be written as

$$\gamma_{ab} = \frac{1}{2} \partial_a x^{cd} \partial_b x^{ef} \varepsilon_{ce}\varepsilon_{df}. \tag{5}$$

This expression motivates to write the determinant of γ_{ab},

* Address for correspondence: Facultad de Ciencias Físico-Matemáticas de la Universidad Autónoma de Sinaloa, 80010, Culiacán Sinaloa, Mexico.
E-mail addresses: nieto@uas.uasnet.mx, janieto1@asu.edu.

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doi:10.1016/j.physletb.2010.07.010

$$\det\gamma=\frac{1}{2}\varepsilon^{ab}\varepsilon^{cd}\gamma_{ac}\gamma_{bd}, \tag{6}$$

in the form

$$\det\gamma=\frac{1}{2}\varepsilon^{ab}\varepsilon^{cd}\varepsilon_{eg}\varepsilon_{fh}\varepsilon_{ru}\varepsilon_{sv}a_a^{ef}a_c^{gh}a_b^{rs}a_d^{uv}=Deta, \tag{7}$$

with

$$a_a^{cd}\equiv\partial_a x^{cd}. \tag{8}$$

One recognizes in (7) the hyperdeterminant of the hypermatrix a_a^{cd}. Thus, this proves that the Nambu–Goto action [13,14]

$$S=\frac{1}{2}\int d^2\xi\,\sqrt{\det\gamma}, \tag{9}$$

for a flat target "spacetime" with $(2+2)$-signature can also be written as [1]

$$S=\frac{1}{2}\int d^2\xi\,\sqrt{Deta}. \tag{10}$$

We shall now show that the hyperdeterminant (7) can be linked to the chirotope concept. For this purpose by using (4) we first write (6) in the alternative Schild-type [15] form

$$\det\gamma=\frac{1}{2}\sigma^{\mu\nu}\sigma^{\alpha\beta}\eta_{\mu\alpha}\eta_{\nu\beta}, \tag{11}$$

where

$$\sigma^{\mu\nu}=\varepsilon^{ab}a_a^{\mu}a_b^{\nu}. \tag{12}$$

Here, we have used the definition

$$a_a^{\mu}=\partial_a x^{\mu}. \tag{13}$$

It turns out that the quantity $\chi^{\mu\nu}=\operatorname{sign}\sigma^{\mu\nu}$ is a chirotope of an oriented matroid (see Refs. [16–18]). In fact, since $\sigma^{\mu\nu}$ satisfies the identity

$$\sigma^{\mu[\nu}\sigma^{\alpha\beta]}\equiv 0, \tag{14}$$

one can verify that $\chi^{\mu\nu}$ satisfies the Grassmann–Plücker relation

$$\chi^{\mu[\nu}\chi^{\alpha\beta]}=0, \tag{15}$$

and therefore $\chi^{\mu\nu}$ is a realizable chirotope (see Ref. [10] and references therein). Here, the bracket $[\nu\alpha\beta]$ means completely antisymmetric.

Since the Grassmann–Plücker relation (15) holds, the ground set

$$E=\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\} \tag{16}$$

and the alternating map

$$\chi^{\mu\nu}\to\{-1,0,1\} \tag{17}$$

determine a 2-rank realizable oriented matroid $M=(E,\chi^{\mu\nu})$. The collection of bases for this oriented matroid is

$$\mathcal{B}=\big\{\{\mathbf{1},\mathbf{2}\},\{\mathbf{1},\mathbf{3}\},\{\mathbf{1},\mathbf{4}\},\{\mathbf{2},\mathbf{3}\},\{\mathbf{2},\mathbf{4}\},\{\mathbf{3},\mathbf{4}\}\big\}, \tag{18}$$

which can be obtained by just given values to the indices μ and ν in $\chi^{\mu\nu}$. Actually, the pair $(E,\mathcal{B})$ determines a 2-rank uniform nonoriented ordinary matroid.

Using the definition

$$\sigma^{efrs}\equiv\varepsilon^{ab}a_a^{ef}a_b^{rs}, \tag{19}$$

one can show that the hyperdeterminant (7) can also be written as

$$\det\gamma=\frac{1}{2}\sigma^{efrs}\sigma^{ghuv}\varepsilon_{eg}\varepsilon_{fh}\varepsilon_{ru}\varepsilon_{sv}=Deta. \tag{20}$$

So, we have achieved our goal of writing the hyperdeterminant (7) in terms of a "chirotope" structure (19). Our strategy was to translate the "chirotope" given in (12) to the form (19). However, by comparing (12) and (19) one finds that there are important differences between these two expressions which suggest a possible generalization of the chirotope concept. From (12) we obtain the property

$$\sigma^{\mu\nu}=-\sigma^{\nu\mu}, \tag{21}$$

that is, $\sigma^{\mu\nu}$ is completely antisymmetric (alternative) quantity, while in (19) we have the weaker condition

$$\sigma^{efrs}=-\sigma^{rsef}. \tag{22}$$

This means that the quantity σ^{efrs}, which we shall call qubitope (qubit-chirotope), is not completely antisymmetric but only alternative in pair of indices. Further, while in the case of (12) the ground set E is given by (16), the expressions (19) and (20) suggest to introduce the underlying ground bitset (from bit and set)

$$\mathcal{E}=\{1,2\} \tag{23}$$

and the pre-ground set

$$E_0=\big\{(1,1),(1,2),(2,1),(2,2)\big\}. \tag{24}$$

So, our task is to find the relation between E_0 and E. By comparing (16) and (24) one sees that by establishing the labels

$$(1,1)\leftrightarrow\mathbf{1},\qquad(1,2)\leftrightarrow\mathbf{2},$$
$$(2,1)\leftrightarrow\mathbf{3},\qquad(2,2)\leftrightarrow\mathbf{4}, \tag{25}$$

such a relation is achieved. This can be understood considering that (25) is equivalent to make the identification of indices $\{a,b\}\leftrightarrow\mu,\ldots$, etc. Observe that considering this identification the family of bases (18) becomes

$$\mathcal{B}_0=\big\{\{(1,1),(1,2)\},\{(1,1),(2,1)\},\{(1,1),(2,2)\},$$
$$\{(1,2),(2,1)\},\{(1,2),(2,2)\},\{(2,1),(2,2)\}\big\}. \tag{26}$$

Thus, from the qubitope σ^{efrs}, we have discovered the underlying structure $Q=(\mathcal{E},E_0,B_0)$. By convenience we call this new structure Q a qubitoid. The word "qubitoid" is short for qubit-matroid.

Let us try to generalize the above scenario to higher dimensions. First, we would like to extend the steps in the expressions (1) and (2). If we consider the coordinates x^{abc} instead of x^{ab} one finds that the null line element

$$ds^2=\frac{1}{2}dx^{abc}dx^{def}\varepsilon_{ad}\varepsilon_{be}\varepsilon_{cf} \tag{27}$$

vanishes identically. This follows because $dx^{abc}dx^{def}\varepsilon_{ad}\varepsilon_{be}=s^{cf}$ is a symmetric quantity, while ε_{cf} is antisymmetric. Similarly one can verify that the hyperdeterminant of the hypermatrix $a_a^{bcd}\equiv\partial_a x^{bcd}$ presents some difficulties due to the fact that the analogue of (7) cannot be obtained. In fact, the quantity $\lambda_{ab}=\partial_a x^{efg}\partial_b x^{hrs}\varepsilon_{eh}\varepsilon_{fr}\varepsilon_{gs}$ is antisymmetric rather than symmetric as the metric γ_{ab} and therefore in this case the steps (4) and (5) cannot follow. Hence, from the Nambu–Goto action point of view this case, which corresponds to $(4+4)$-signature, is not very interesting, although in the Polyakov action context may still be interesting. So, we jump to the next possibility, namely the line element

$$ds^2=\frac{1}{2}dx^{abcr}dx^{defs}\varepsilon_{ad}\varepsilon_{be}\varepsilon_{cf}\varepsilon_{rs}, \tag{28}$$

which, one can show, implies a line element of the type (1), but now associated with a flat target (8+8)-signature "spacetime". Explicitly, we have the relations

$$\begin{aligned}
&x^{1111} \leftrightarrow x^1 + x^9, \quad x^{2222} \leftrightarrow -x^1 + x^9, \quad x^{1112} \leftrightarrow x^2 + x^{10},\\
&x^{2221} \leftrightarrow x^2 - x^{10}, \quad x^{1122} \leftrightarrow x^3 + x^{11}, \quad x^{2211} \leftrightarrow -x^3 + x^{11},\\
&x^{1121} \leftrightarrow x^4 + x^{12}, \quad x^{2212} \leftrightarrow x^4 - x^{12}, \quad x^{1212} \leftrightarrow x^5 + x^{13},\\
&x^{2121} \leftrightarrow -x^5 + x^{13}, \quad x^{1211} \leftrightarrow x^6 + x^{14}, \quad x^{2122} \leftrightarrow x^6 - x^{14},\\
&x^{1221} \leftrightarrow x^7 + x^{15}, \quad x^{2112} \leftrightarrow -x^7 + x^{15}, \quad x^{1222} \leftrightarrow x^8 + x^{16},\\
&x^{2111} \leftrightarrow x^8 - x^{16}.
\end{aligned} \tag{29}$$

In this case, the hyperdeterminant of the hypermatrix

$$a_a^{bcde} = \partial_a x^{bcde} \tag{30}$$

is given by (see Eq. (32) of Ref. [19])

$$\begin{aligned}
\det\gamma &= \frac{1}{2}\varepsilon^{ab}\varepsilon^{cd}\varepsilon_{e_1f_1}\varepsilon_{e_2f_2}\varepsilon_{e_3f_3}\varepsilon_{e_4f_4}\varepsilon_{g_1h_1}\varepsilon_{g_2h_2}\varepsilon_{g_3h_3}\varepsilon_{g_4h_4}\\
&\quad\times a_a^{e_1e_2e_3e_4} a_c^{f_1f_2f_5f_4} a_b^{g_1g_2g_3g_4} a_d^{h_1h_2h_3h_4}\\
&= Deta.
\end{aligned} \tag{31}$$

Thus, by substituting (31) into (10) we find a Nambu–Goto action for a flat target "spacetime" with (8+8)-signature written in terms of the hyperdeterminant *Deta*.

The qubitoid now is determined by the underlying set

$$\mathcal{E} = \{1, 2\}, \tag{32}$$

and the pre-ground set

$$\begin{aligned}
E_0 = \big\{&(1,1,1,1), (1,1,1,2), (1,1,2,1), (1,1,2,2),\\
&(1,2,1,1), (1,2,1,2), (1,2,2,1), (1,2,2,2),\\
&(2,1,1,1), (2,1,1,2), (2,1,2,1), (2,1,2,2),\\
&(2,2,1,1), (2,2,1,2), (2,2,2,1), (2,2,2,2)\big\}.
\end{aligned} \tag{33}$$

It is not difficult to see that by making the identifications

$$\begin{aligned}
&(1,1,1,1) \leftrightarrow \mathbf{1}, \quad (1,1,1,2) \leftrightarrow \mathbf{2}, \quad (1,1,2,1) \leftrightarrow \mathbf{3},\\
&(1,1,2,2) \leftrightarrow \mathbf{4}, \quad (1,2,1,1) \leftrightarrow \mathbf{5}, \quad (1,2,1,2) \leftrightarrow \mathbf{6},\\
&(1,2,2,1) \leftrightarrow \mathbf{7}, \quad (1,2,2,2) \leftrightarrow \mathbf{8}, \quad (2,1,1,1) \leftrightarrow \mathbf{9},\\
&(2,1,1,2) \leftrightarrow \mathbf{10}, \quad (2,1,2,1) \leftrightarrow \mathbf{11}, \quad (2,1,2,2) \leftrightarrow \mathbf{12},\\
&(2,2,1,1) \leftrightarrow \mathbf{13}, \quad (2,2,1,2) \leftrightarrow \mathbf{14}, \quad (2,2,2,1) \leftrightarrow \mathbf{15},\\
&(2,2,2,2) \leftrightarrow \mathbf{16},
\end{aligned} \tag{34}$$

one obtains a relation between the pre-ground set E_0 given in (33) and the ground set

$$E = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{15}, \mathbf{16}\}. \tag{35}$$

This can be understood by considering that (34) is equivalent to make the identification of indices $(a, b, c, d) \leftrightarrow \mu, \ldots$, etc. It turns out that considering these relations one finds that the collection of bases $\mathcal{B}$ contains $\binom{16}{2} = 120$ two-element subsets of the 16-element set E, given in (35). This 2-element subset can be obtained by considering a lexicographic order of all 120 two-subsets of $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{15}, \mathbf{16}\}$. For instance, the first 35 two-subsets of $\mathcal{B}$ are

$$\begin{aligned}
&\{\mathbf{1},\mathbf{2}\}, \quad \{\mathbf{1},\mathbf{3}\}, \quad \{\mathbf{1},\mathbf{4}\}, \quad \{\mathbf{1},\mathbf{5}\}, \quad \{\mathbf{1},\mathbf{6}\}, \quad \{\mathbf{1},\mathbf{7}\},\\
&\{\mathbf{1},\mathbf{8}\}, \quad \{\mathbf{1},\mathbf{9}\}, \quad \{\mathbf{1},\mathbf{10}\}, \quad \{\mathbf{1},\mathbf{11}\}, \quad \{\mathbf{1},\mathbf{12}\},\\
&\{\mathbf{1},\mathbf{13}\}, \quad \{\mathbf{1},\mathbf{14}\}, \quad \{\mathbf{1},\mathbf{15}\}, \quad \{\mathbf{1},\mathbf{16}\}, \quad \{\mathbf{2},\mathbf{3}\},\\
&\{\mathbf{2},\mathbf{4}\}, \quad \{\mathbf{2},\mathbf{5}\}, \quad \{\mathbf{2},\mathbf{6}\}, \quad \{\mathbf{2},\mathbf{7}\}, \quad \{\mathbf{2},\mathbf{8}\}, \quad \{\mathbf{2},\mathbf{9}\},\\
&\{\mathbf{2},\mathbf{10}\}, \quad \{\mathbf{2},\mathbf{11}\}, \quad \{\mathbf{2},\mathbf{12}\}, \quad \{\mathbf{2},\mathbf{13}\}, \quad \{\mathbf{2},\mathbf{14}\},\\
&\{\mathbf{2},\mathbf{15}\}, \quad \{\mathbf{2},\mathbf{16}\}, \quad \{\mathbf{3},\mathbf{4}\}, \quad \{\mathbf{3},\mathbf{5}\}, \quad \{\mathbf{3},\mathbf{6}\},\\
&\{\mathbf{3},\mathbf{7}\}, \quad \{\mathbf{3},\mathbf{8}\}, \quad \{\mathbf{3},\mathbf{9}\}, \quad \ldots.
\end{aligned} \tag{36}$$

The sequence follows until the last term $\{\mathbf{15}, \mathbf{16}\}$. By using (34) one finds that the first terms of $\mathcal{B}_0$ look like

$$\begin{aligned}
\mathcal{B}_0 = \Big\{&\big\{(1,1,1,1), (1,1,1,2)\big\}, \big\{(1,1,1,1), (1,1,2,1)\big\},\\
&\big\{(1,1,1,1), (1,1,2,2)\big\}, \big\{(1,1,1,1), (1,2,1,1)\big\},\\
&\big\{(1,1,1,1), (1,2,1,2)\big\}, \ldots\Big\}.
\end{aligned} \tag{37}$$

Thus, associated with the quantity a_a^{bcde} we have again a qubitoid structure of the form $Q = (\mathcal{E}, E_0, \mathcal{B}_0)$ which corresponds to a flat target "spacetime" of (8+8)-dimensions. The corresponding qubitope is given by

$$\sigma^{a_1a_2a_3a_4a_5a_6a_7a_8} = \frac{1}{2}\varepsilon^{bc} a_b^{a_1a_2a_3a_4} a_c^{a_5a_6a_7a_8}. \tag{38}$$

It is worth mentioning that while in (2+2)-dimensions the quantity *Deta* is invariant under $SL(2,R)^3$ in the case of (8+8)-dimensions *Deta* must be invariant under $SL(2,R)^5$.

The method, of course, can be extended to $(\frac{2^{2n+2}}{2} + \frac{2^{2n+2}}{2})$-signature, $n = 0, 1, 2, \ldots$, etc. For the cases of $(\frac{2^{2n+1}}{2} + \frac{2^{2n+1}}{2})$-signature the corresponding line element vanishes identically.

It remains to explore whether the present qubitoid and qubitope formalism will allow us a deeper understanding of other two scenarios, namely black holes and quantum information. In the first case, as in (2+2)-dimensions, one may think in a black hole with $\frac{2^{2n+2}}{2}$-electric and $\frac{2^{2n+2}}{2}$-magnetic charges with entropy

$$S = \pi\sqrt{Deta}. \tag{39}$$

While in the second case, one may introduce pure states $|\psi\rangle$ associated with the $\frac{2^{2n+2}}{2}$-qubitoid system. For instance in the case of (8+8)-dimensions the pure states $|\psi\rangle$ must be given by (see Refs. [19] and [20])

$$|\psi\rangle = \sum_{a_1a_2a_3a_4a_5} a_{a_1}^{a_2a_3a_4a_5} |a_1a_2a_3a_4a_5\rangle. \tag{40}$$

It is worth mentioning that the complete classification of N-qubit systems is a difficult, or perhaps an impossible, task. In Ref. [19] an interesting development for characterizing a subclass of N-qubit entanglement has been considered. An attractive aspect of this construction is that the N-qubit entanglement can be understood in geometric terms. The idea is based on the bipartite partitions of the Hilbert space in the form $C^{2^N} = C^L \otimes C^l$, with $L = 2^{N-n}$ and $l = 2^n$. Such a partition allows a geometric interpretation in terms of the complex Grassmannian variety $Gr(L,l)$ of l-planes in C^L via the Plücker embedding. In this case, the Plücker coordinates of the Grassmannians are natural invariants of the theory. In this scenario the 5-qubit given in (40) admits a geometric interpretation in terms of the complex Grassmannian $Gr(8,4)$. Considering such an interpretation it has been proved that the expression (31) is a hyperdeterminant associated with the Plücker coordinates of the Grassmannian $Gr(8,4)$ (see Eq. (32) of Ref. [19] and Ref. [20] for 5-qubit discussion).

Furthermore, it is interesting that the line element (28) also appears on several physical contexts. In particular, extremal black hole solutions in the STU model of $D=4$, $\mathcal{N}=2$ supergravity admit a description in terms of 4-qubit systems [21,22] (for a 4-qubit entanglement see [23] and references therein). In this case, the line element corresponds to the moduli space $\mathcal{M}_4=[U(1)\backslash SL(2;R)]^3$ rather than to the "spacetime". Upon dimensional reduction $\mathcal{M}_4$ becomes $\mathcal{M}_3=[SO(4)]^2\backslash SO(4,4)$ or $\mathcal{M}_3^*=[SO(2,2)]^2\backslash SO(4,4)$ depending whether the truncation is along a space-like or time-like direction, respectively. Among other things, the relevance of this construction in our approach is that the signature of the metric $\mathcal{M}_3^*$ is also of the type $(8+8)$ (see Refs. [21] and [22] for details).

It is remarkable that the Nambu–Goto action in a flat target "spacetime" with $(\frac{2^{2n+2}}{2}+\frac{2^{2n+2}}{2})$-signature emerges as the underlying motivation for studying the new mathematical structures of qubitoids $Q=(\mathcal{E},E_0,B_0)$ and the corresponding qubitopes.

Acknowledgements

I would like to thank M.C. Marín and A. León for helpful comments and the Mathematical, Computational & Modeling Science Center of the Arizona State University where part of this work was developed.

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