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A Special Class of Irreducible Matrices—The Nearly Reducible Matrices

MARK HEDRICK* AND RICHARD SINKHORN†

University of Houston, Houston, Texas 77004

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INTRODUCTION

In [2] and [3] the concept of the nearly decomposable matrix is introduced. Such a matrix has a canonical form which is readily adaptable to many inductive type arguments. For example, with the aid of this form, it is proved in [2] that the permanent of an $n \times n$ (0, 1)-matrix with exactly three ones in each row and column is at least n.

In this paper we study an analogous concept—that of the nearly reducible matrix. The development in [3] of the above mentioned canonical form is made to depend upon the fact that a fully indecomposable matrix has a doubly stochastic pattern. However, an irreducible matrix need not have a doubly stochastic pattern; in fact, it is shown in this paper that the only nearly reducible doubly stochastic matrix is a full cycle permutation! Nevertheless, a nearly reducible matrix does have a corresponding canonical form. Any question concerning the permanent of a nearly reducible (0, 1)-matrix is answered by the consequence that such a matrix can have at most one positive diagonal.

In our presentation we shall require the following notions and definitions.

An $n \times n$ matrix A is said to be doubly stochastic if $a_{ij} \geqslant 0$ and if

$$\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{kj} = 1$$

for all i and j. The set of $n \times n$ doubly stochastic matrices is denoted by Ω_n.

An $n \times n$ matrix A is said to have doubly stochastic pattern if there is a doubly stochastic matrix B such that $a_{ij} = 0$ if and only if $b_{ij} = 0$.

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An $n \times n$ matrix A is said to be reducible if there exists a permutation matrix P such that P^TAP has the form

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B and D are square and P^T denotes the transpose of P. Otherwise A is said to be irreducible. In this paper it is convenient to regard the 1×1 matrix (0) as irreducible.

E_{ij} is the $n \times n$ matrix with a 1 in the (i, j) position and zeros elsewhere.

If A is an $n \times n$ irreducible matrix such that whenever $a_{ij} \neq 0$, $A - a_{ij}E_{ij}$ is reducible, then A is said to be nearly reducible.

A diagonal of an $n \times n$ matrix A is a collection of entries from the matrix, one from each row and one from each column. If σ is a permutation of $\{1, 2,..., n\}$, then the diagonal associated with σ is $a_{1\sigma(1)}, a_{2\sigma(2)},..., a_{n\sigma(n)}$. When σ is the identity permutation, the diagonal is called the main diagonal. A positive diagonal is a diagonal in which every $a_{i\sigma(i)} > 0$.

Let A be an $m \times n$ matrix and let u and v be positive integers such that $1 \leqslant u \leqslant m$, $1 \leqslant v \leqslant n$. Let α denote a strictly increasing sequence of u integers $(i_1,..., i_u)$ chosen from $1,..., m$, and let β denote a strictly increasing sequence of v integers $(j_1,..., j_v)$ chosen from $1,..., n$. Then $A[\alpha \mid \beta]$ is that submatrix of A with rows indexed by α and columns indexed by β. $A[\alpha \mid \beta)$ is the submatrix of A with rows indexed by α and columns indexed by the complement of β in $\{1, 2,..., n\}$. $A(\alpha \mid \beta]$ and $A(\alpha \mid \beta)$ are defined analogously.

If

$$A = \begin{pmatrix} A_1 & F_{12} & \cdots & F_{1s} \\ F_{21} & A_2 & \cdots & F_{2s} \\ \cdot & \cdot & \cdot & \cdot \\ F_{s1} & F_{s2} & \cdots & A_s \end{pmatrix}$$

is such that $s > 1$ and each A_k is irreducible, A is said to be in *normal form*. If A is in normal form and for some k there is an integer $j \neq k$ such that $F_{kj} \neq 0$ while $F_{ki} = 0$ for $i \neq k$, $i \neq j$, A_k is said to be *nearly isolated* in A.

If

$$A = \begin{pmatrix} B_1 & 0 & 0 & \cdots & 0 & 0 & E_1 \\ E_2 & B_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & E_3 & B_3 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & E_{t-1} & B_{t-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & E_t & B_t \end{pmatrix}$$

is such that each B_k is irreducible and each E_k has exactly one nonzero entry, A is said to be in *trivial form*.

A useful tool in the development of the properties of the nearly reducible matrices is the following famous theorem of G. Birkhoff. A proof may be found in [1, p. 98].

THEOREM. *The set of all $n \times n$ doubly stochastic matrices forms a convex polyhedron with the permutation matrices as vertices.*

A consequence of Birkhoff's theorem is that an $n \times n$ nonnegative nonzero matrix has doubly stochastic pattern if and only if each nonzero entry lies on a positive diagonal.

RESULTS AND CONSEQUENCES

LEMMA 1. *If A is an $n \times n$ reducible matrix in normal form with s main diagonal blocks, then there exists a nonempty proper subset ω of $\{1,..., s\}$ such that $F_{ij} = 0$ whenever $i \in \omega$ and $j \notin \omega$.*

Proof. Since A is reducible there is a nonempty proper subset τ of $\{1,..., n\}$ such that $A[\tau \mid \tau) = 0$. Let u_k denote the number of rows in A_k that are indexed in A by τ and let v_k denote the number of columns in A_k that are indexed in A by the complement of τ in $\{1,..., n\}$. If A_k is $n_k \times n_k$, $u_k + v_k = n_k$. Since each A_k is irreducible, for each k either $u_k = n_k$ and $v_k = 0$ or $u_k = 0$ and $v_k = n_k$. Since τ is nonempty, and is a proper subset of $\{1,..., n\}$, each case must occur. Let $\omega = \{k \mid u_k = n_k, v_k = 0\}$. Then $F_{ij} = 0$ if $i \in \omega$ and $j \notin \omega$.

LEMMA 2. *If A is an $n \times n$ nearly reducible matrix in normal form with s main diagonal blocks, then for each $i \neq j$, F_{ij} contains at most one nonzero entry.*

Proof. Suppose $F_{pq} \neq 0$. Replace a nonzero entry in F_{pq} by 0. This transforms A into a reducible matrix A'. If the blocks in the corresponding normal form of A' are labeled by primes, then, by Lemma 1, there is a nonempty proper subset ω of $\{1,..., s\}$ such that $F'_{ij} = 0$ if $i \in \omega$ and $j \notin \omega$. Since A is irreducible, $p \in \omega$ and $q \notin \omega$. Thus F_{pq} has exactly one nonzero entry.

The next two lemmas are immediate consequences of the definitions.

LEMMA 3. *The main diagonal of an $n \times n$ nearly reducible matrix A is zero. Thus if $n > 1$, every $n \times n$ nearly reducible matrix has a normal form.*

LEMMA 4. *If A is in trivial form, then A is irreducible.*

LEMMA 5. *Let A be an $n \times n$ irreducible matrix in normal form with s main diagonal blocks, and suppose that each F_{ij} has at most one nonzero entry. Suppose also that for each proper subset $\omega = \{i_1, ..., i_k\}$ of $\{1, ..., s\}$ with at least two elements, the submatrix*

$$A_\omega = \begin{pmatrix} A_{i_1} & F_{i_1 i_2} & F_{i_1 i_3} & \cdots & F_{i_1 i_k} \\ F_{i_2 i_1} & A_{i_2} & F_{i_2 i_3} & \cdots & F_{i_2 i_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F_{i_k i_1} & F_{i_k i_2} & F_{i_k i_3} & \cdots & A_{i_k} \end{pmatrix}$$

is reducible. Then the rows and columns of A may be simultaneously permuted to bring A into trivial form where the s main diagonal blocks are $A_1, ..., A_s$ in some order.

Proof. If $s = 2$, there is nothing to prove. Thus suppose $s > 2$. Let us observe that when one simultaneously permutes the rows and columns of blocks of A, he simple permutes the $A_1, ..., A_s$ on the main diagonal, and thus the resulting matrix is still in normal form. Thus throughout the proof whenever we simultaneously permute the rows and columns of blocks of A, we shall assume that the A_i have been reindexed in such a manner that the first main diagonal block in the resulting matrix is A_1, etc. Since A is irreducible, there is some $F_{is} \neq 0$. If $i \neq 1$, we could simultaneously permute the rows and columns of blocks of A and then renumber the F_{ij} blocks to make $F_{1s} \neq 0$ in the resulting matrix. Hence for convenience we shall assume that $F_{1s} \neq 0$. Since A_ω is reducible for $\omega = \{1, s\}$, it follows by Lemma 4 that $F_{s1} = 0$. Thus, since A is irreducible, there is some $F_{i1} \neq 0$ for $2 \leqslant i < s$. We may assume that $F_{21} \neq 0$. Since A_ω is reducible for $\omega = \{1, 2\}$, it follows by Lemma 4 that $F_{12} = 0$. By a similar argument, if $\omega = \{1, 2, s\}$, and $s > 3$, $F_{s2} = 0$. Thus, since A is irreducible, there is some $F_{i2} \neq 0$ for $2 < i < s$. We may suppose that $F_{32} \neq 0$. If $s = 3$, $F_{12} = 0$ implies that $F_{32} \neq 0$. If $s > 4$, we can argue that F_{13}, F_{23}, and F_{s3} are each 0 and suppose that $F_{43} \neq 0$. If $s = 4$, we conclude that $F_{13} = 0$, $F_{23} = 0$, and $F_{43} \neq 0$. Continuing in the manner, we can argue that $F_{ij} \neq 0$ for $i = j + 1 \pmod{s}$. A consideration of A_ω for $\omega = \{i, i + 1, ..., j\} \pmod{s}$ shows that $F_{ij} = 0$ for $i \neq j + 1 \pmod{s}$.

LEMMA 6. *If A is an $n \times n$ nearly reducible matrix with $n > 1$, there exists a permutation P and an integer $t > 1$ such that $P^T A P$ is in trivial form with t main diagonal blocks, each of which is nearly reducible.*

Proof. By Lemma 3, A has a normal form with n 1×1 diagonal blocks, $A_1, ..., A_n$. If the hypothesis of Lemma 5 is satisfied for this form, the proof is finished. If not, there is a largest integer k, $1 < k < n$, such that there exists an irreducible submatrix

$$A_\omega = \begin{pmatrix} A_{i_1} & F_{i_1 i_2} & F_{i_1 i_3} & \cdots & F_{i_1 i_k} \\ F_{i_2 i_1} & A_{i_2} & F_{i_2 i_3} & \cdots & F_{i_2 i_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F_{i_k i_1} & F_{i_k i_2} & F_{i_k i_3} & \cdots & A_{i_k} \end{pmatrix}$$

in A. A simultaneous permutation of the rows and columns of blocks of A (the blocks are 1×1 at this stage of the argument) allows us to suppose that $i_j = j$, $j = 1, \ldots, k$. Put $A_1' = A_\omega$ and $A_i' = A_{k+i-1}$ for $i > 1$, and consider A in the normal form

$$A' = \begin{pmatrix} A_1' & F'_{12} & F'_{13} & \cdots & F'_{1r} \\ F'_{21} & A_2' & F'_{23} & \cdots & F'_{2r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F'_{r1} & F'_{r2} & F'_{r3} & \cdots & A_r' \end{pmatrix}$$

where $r = n - k + 1$. By Lemma 2, each F'_{ij} contains at most one nonzero entry. If the hypothesis of Lemma 5 concerning the reducibility of submatrices of A' is not satisfied, the process may be repeated. After a finite number of steps a normal form will be found in which this hypothesis is satisfied. Then the trivial form will follow. It is clear that there must be at least two main diagonal blocks in this final form. The near reducibility of these final main diagonal blocks is a consequence of Lemma 4.

LEMMA 7. *If A is an $n \times n$ nearly reducible matrix in normal form with diagonal blocks $A_1, \ldots, A_s$, then at least two of the A_k are nearly isolated in A.*

Proof. We prove the lemma by induction on s. If $s = 2$, the result is clear. Thus let $s > 2$ and suppose that the result holds whenever there are at least two but less than s main diagonal blocks in a nearly reducible normal form.

We bring A into trivial form as in the proof of Lemma 6. Any diagonal block B_m in this trivial form which is not one of the A_k can be partitioned into A_k and F_{ij} blocks. Every B_m is nearly reducible, and if a B_m is not some A_k, B_m is in normal form.

If a B_m is some A_k, that A_k is nearly isolated in A. If not, by the induction hypothesis, there are at least two nearly isolated A_k in B_m. The trivial form of A makes it clear that at least one of these A_k is nearly isolated in A. Since there are at least two B_m in the trivial form of A, there are at least two nearly isolated A_k in A.

We now deduce the main results.

THEOREM 1. *If A is an $n \times n$ nearly reducible doubly stochastic matrix, then $n > 1$ and A is a full cycle permutation matrix.*

Proof. The fact that $n > 1$ is clear since if $n = 1$, $A = (0)$ and $A \notin \Omega_1$. Since $A \in \Omega_n$, it follows by Birkhoff's theorem that A contains a positive diagonal d. If d corresponds to a full cycle permutation, then the permutation matrix containing d is irreducible and is thus equal to A since A is nearly reducible. If d corresponds to a permutation σ which is not a full cycle, σ is a product of s disjoint cycles where $s > 1$. Thus there is a permutation P such that

$$P^T A P = \begin{pmatrix} A_1 & F_{12} & \cdots & F_{1s} \\ F_{21} & A_2 & \cdots & F_{2s} \\ \cdot & \cdot & \cdot & \cdot \\ F_{s1} & F_{s2} & \cdots & A_s \end{pmatrix}$$

where each A_k is a square matrix which contains a positive diagonal that corresponds to one of the full cycle factors of σ. Each A_k is irreducible and the form of $P^T A P$ is normal. For convenience we suppose that A is already in this normal form. By Lemma 1, each A_k is nearly reducible. It follows that each A_k is at least 2×2 since no $A_k = 0$. It also follows that the only positive entries in any A_k occur on the diagonal d.

By Lemma 7, some A_k is nearly isolated. Suppose it is A_1. Then exactly one $F_{1j} \neq 0$; suppose it is F_{1r}, and suppose that the unique positive entry f of F_{1r} lies in the p-th row. Consider a positive entry g in some F_{i1}. (Such a g exists for some i since A is irreducible.) Suppose g is in the q-th column of F_{i1}. Since $A \in \Omega_n$, g lies on a positive diagonal in A which includes f. Thus, since A_1 is at least 2×2, $A_1(p \mid q)$ contains a positive diagonal, and therefore, since the only positive entries of A_1 lie on a diagonal corresponding to a full cycle permutation, $a_{pq} > 0$. The existence of any positive entry g' in the q'-th column of any F_{i1} would likewise imply that $a_{pq'} > 0$. This forces $q' = q$; therefore every positive entry in any F_{i1} must be in the q-th column.

Put $A' = A - a_{pq}E_{pq}$. Since A' is reducible, there is a nonempty proper subset ω of $\{1,..., n\}$ such that $A'[\omega \mid \omega) = 0$. As in the proof of Lemma 1, we see that if ω contains an index of a row of A that corresponds to a row of A_k, $k = 2,..., s$, it contains the index of every row of A that corresponds to any row of A_k; likewise, if the complement of ω in $\{1,..., n\}$ contains the index of a column of A that corresponds to a column of such an A_k, it contains the index of every column of A that corresponds to any column of A_k.

Suppose A_1 is $n_1 \times n_1$. For $j = 2,..., s$, let F'_{1j} denote the submatrix of F_{1j}

whose rows are indexed by $\omega \cap \{1,..., n_1\}$ and whose columns correspond to the columns of F_{1j} in A. Also for each such j let F''_{1j} denote the submatrix of F_{1j} whose rows are indexed by those members of $\{1,..., n_1\}$ which are not in ω, and whose columns correspond to the columns of F_{1j} in A. The irreducibility of A makes it clear that F'_{1j} and F''_{1j} exist for $j = 2,..., s$.

Similarly, for $i = 2,..., s$, let F'_{i1} denote the submatrix of F_{i1} whose rows correspond to the rows of F_{i1} in A and whose columns are indexed by $\omega \cap \{1,..., n_1\}$. Also for such i let F''_{i1} denote the submatrix of F_{i1} whose rows correspond to the rows of F_{i1} in A and whose columns are indexed by those members of $\{1,..., n_1\}$ which are not in ω. The irreducibility of A shows that F'_{i1} and F''_{i1} exist for $i = 2,..., s$.

We can suppose that the rows and columns of blocks in A have been simultaneously permuted so that the columns of A which are indexed by those members of $\{n_1 + 1,..., n\}$ which are not in ω (if any) correspond to the columns of A_k for $k = 2,..., m$, and that the rows of A which are indexed by $\omega \cap \{n_1 + 1,..., n\}$ (if any) correspond to the rows of A_k for $k = m + 1,..., s$.

Since A' is reducible, $F'_{1j} = 0$ for $j = 2,..., m$ if $m \geqslant 2$. Since $p \in \omega$, $F''_{1j} = 0$, and thus $F_{1j} = 0$ for $j = 2,..., m$. Whence $F_{ij} = 0$ if

$$i = 1, m + 1,..., s, \qquad \text{and} \qquad j = 2,..., m.$$

This implies that A is reducible, which is a contradiction. Hence the column indices of A not in ω belong to $\{1,..., n_1\}$. Then the reducibility of A' implies that $F''_{i1} = 0$ for $i = 2,..., s$. Since q belongs to the complement of ω in $\{1,..., n\}$, $F'_{i1} = 0$, and thus $F_{i1} = 0$ for $i = 2,..., s$. This also implies the contradiction that A is reducible. Hence we must conclude that $s = 1$, and therefore that σ is a full cycle permutation after all.

THEOREM 2. *If A is an $n \times n$ nonnegative nearly reducible matrix, then A has at most one positive diagonal.*

Proof. Let A' be obtained from A by replacing with zero every positive entry which does not lie on a positive diagonal. If $A' = 0$, A has no positive diagonal. If $A' \neq 0$, A' has doubly stochastic pattern. If A' has doubly stochastic pattern and is irreducible, necessarily $A' = A$, and the result is an immediate consequence of Theorem 1. If A' has doubly stochastic pattern and is reducible, there is a permutation P and an integer $s > 1$ such that $P^T A' P = A_1' \oplus \cdots \oplus A_s'$ where each A_k' is irreducible and has doubly stochastic pattern. In this case write

$$P^T A P = \begin{pmatrix} A_1 & F_{12} & \cdots & F_{1s} \\ F_{21} & A_2 & \cdots & F_{2s} \\ \cdot & \cdot & \cdot & \cdot \\ F_{s1} & F_{s2} & \cdots & A_s \end{pmatrix}$$

where each A_k has the dimension of the corresponding A_k'. Since each A_k' is irreducible, so is each A_k, and the form of P^TAP is normal. It is then a consequence of Lemma 1 that each A_k is nearly reducible, and thus $A_k = A_k'$ for each k. By Theorem 1, each A_k has all its positive entries on one positive diagonal which corresponds to a full cycle permutation. Thus A', and therefore A, has exactly *one* positive diagonal.

The examples

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

show that a nearly reducible matrix A can have either one positive diagonal or no positive diagonals.

References

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