A Special Class of Irreducible Matrices—
The Nearly Reducible Matrices

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INTRODUCTION

In [2] and [3] the concept of the nearly decomposable matrix is introduced. Such a matrix has a canonical form which is readily adaptable to many inductive type arguments. For example, with the aid of this form, it is proved in [2] that the permanent of an \( n \times n \) (0, 1)-matrix with exactly three ones in each row and column is at least \( n \).

In this paper we study an analogous concept—that of the nearly reducible matrix. The development in [3] of the above mentioned canonical form is made to depend upon the fact that a fully indecomposable matrix has a doubly stochastic pattern. However, an irreducible matrix need not have a doubly stochastic pattern; in fact, it is shown in this paper that the only nearly reducible doubly stochastic matrix is a full cycle permutation! Nevertheless, a nearly reducible matrix does have a corresponding canonical form. Any question concerning the permanent of a nearly reducible (0, 1)-matrix is answered by the consequence that such a matrix can have at most one positive diagonal.

In our presentation we shall require the following notions and definitions.

An \( n \times n \) matrix \( A \) is said to be doubly stochastic if \( a_{ij} \geq 0 \) and if

\[
\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{kj} = 1
\]

for all \( i \) and \( j \). The set of \( n \times n \) doubly stochastic matrices is denoted by \( \Omega_n \).

An \( n \times n \) matrix \( A \) is said to have doubly stochastic pattern if there is a doubly stochastic matrix \( B \) such that \( a_{ij} = 0 \) if and only if \( b_{ij} = 0 \).

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An \( n \times n \) matrix \( A \) is said to be reducible if there exists a permutation matrix \( P \) such that \( P^TAP \) has the form

\[
\begin{pmatrix}
B & 0 \\
C & D
\end{pmatrix}
\]

where \( B \) and \( D \) are square and \( P^T \) denotes the transpose of \( P \). Otherwise \( A \) is said to be irreducible. In this paper it is convenient to regard the \( 1 \times 1 \) matrix \((0)\) as irreducible.

\( E_{ij} \) is the \( n \times n \) matrix with a 1 in the \((i,j)\) position and zeros elsewhere.

If \( A \) is an \( n \times n \) irreducible matrix such that whenever \( a_{ij} \neq 0 \), \( A - a_{ij}E_{ij} \) is reducible, then \( A \) is said to be nearly reducible.

A diagonal of an \( n \times n \) matrix \( A \) is a collection of entries from the matrix, one from each row and one from each column. If \( \sigma \) is a permutation of \( \{1, 2, \ldots, n\} \), then the diagonal associated with \( \sigma \) is \( a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)} \).

When \( \sigma \) is the identity permutation, the diagonal is called the main diagonal. A positive diagonal is a diagonal in which every \( a_{i,j} > 0 \).

Let \( A \) be an \( m \times n \) matrix and let \( u \) and \( v \) be positive integers such that \( 1 \leq u \leq m, 1 \leq v \leq n \). Let \( \alpha \) denote a strictly increasing sequence of \( u \) integers \((i_1, \ldots, i_u)\) chosen from \( 1, \ldots, m \), and let \( \beta \) denote a strictly increasing sequence of \( v \) integers \((j_1, \ldots, j_v)\) chosen from \( 1, \ldots, n \). Then \( A[\alpha \mid \beta] \) is that submatrix of \( A \) with rows indexed by \( \alpha \) and columns indexed by \( \beta \). \( A[\alpha \mid \beta] \) is the submatrix of \( A \) with rows indexed by \( \alpha \) and columns indexed by the complement of \( \beta \) in \( \{1, 2, \ldots, n\} \). \( A[\alpha \mid \beta] \) and \( A[\alpha \mid \beta] \) are defined analogously.

If

\[
A = \begin{pmatrix}
A_1 & F_{12} & \cdots & F_{1s} \\
F_{21} & A_2 & \cdots & F_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
F_{s1} & F_{s2} & \cdots & A_s
\end{pmatrix}
\]

is such that \( s > 1 \) and each \( A_k \) is irreducible, \( A \) is said to be in normal form. If \( A \) is in normal form and for some \( k \) there is an integer \( j \neq k \) such that \( F_{kj} \neq 0 \) while \( F_{ki} = 0 \) for \( i \neq k, i \neq j \), \( A_k \) is said to be nearly isolated in \( A \).

If

\[
A = \begin{pmatrix}
B_1 & 0 & 0 & \cdots & 0 & 0 & F_1 \\
E_2 & B_2 & 0 & \cdots & 0 & 0 & 0 \\
0 & F_3 & R_3 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & E_{t-1} & B_{t-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & E_t & B_t
\end{pmatrix}
\]

is such that each \( B_k \) is irreducible and each \( E_k \) has exactly one nonzero entry, \( A \) is said to be in trivial form.
A useful tool in the development of the properties of the nearly reducible matrices is the following famous theorem of G. Birkhoff. A proof may be found in [1, p. 98].

**Theorem.** The set of all $n \times n$ doubly stochastic matrices forms a convex polyhedron with the permutation matrices as vertices.

A consequence of Birkhoff's theorem is that an $n \times n$ nonnegative nonzero matrix has doubly stochastic pattern if and only if each nonzero entry lies on a positive diagonal.

**Results and Consequences**

**Lemma 1.** If $A$ is an $n \times n$ reducible matrix in normal form with $s$ main diagonal blocks, then there exists a nonempty proper subset $\omega$ of $\{1, \ldots, s\}$ such that $F_{ij} = 0$ whenever $i \in \omega$ and $j \notin \omega$.

**Proof.** Since $A$ is reducible there is a nonempty proper subset $\tau$ of $\{1, \ldots, n\}$ such that $A[\tau | \tau] = 0$. Let $u_k$ denote the number of rows in $A_k$ that are indexed in $A$ by $\tau$ and let $v_k$ denote the number of columns in $A_k$ that are indexed in $A$ by the complement of $\tau$ in $\{1, \ldots, n\}$. If $A_k$ is $n_k \times n_k$, $u_k + v_k = n_k$. Since each $A_k$ is irreducible, for each $k$ either $u_k = n_k$ and $v_k = 0$ or $u_k = 0$ and $v_k = n_k$. Since $\tau$ is nonempty, and is a proper subset of $\{1, \ldots, n\}$, each case must occur. Let $\omega = \{k | u_k = n_k, v_k = 0\}$. Then $F_{ij} = 0$ if $i \in \omega$ and $j \notin \omega$.

**Lemma 2.** If $A$ is an $n \times n$ nearly reducible matrix in normal form with $s$ main diagonal blocks, then for each $i \neq j$, $F_{ij}$ contains at most one nonzero entry.

**Proof.** Suppose $F_{pq} \neq 0$. Replace a nonzero entry in $F_{pq}$ by 0. This transforms $A$ into a reducible matrix $A'$. If the blocks in the corresponding normal form of $A'$ are labeled by primes, then, by Lemma 1, there is a nonempty proper subset $\omega$ of $\{1, \ldots, s\}$ such that $F'_{ij} = 0$ if $i \in \omega$ and $j \notin \omega$. Since $A$ is irreducible, $p \in \omega$ and $q \notin \omega$. Thus $F_{pq}$ has exactly one nonzero entry.

The next two lemmas are immediate consequences of the definitions.

**Lemma 3.** The main diagonal of an $n \times n$ nearly reducible matrix $A$ is zero. Thus if $n > 1$, every $n \times n$ nearly reducible matrix has a normal form.

**Lemma 4.** If $A$ is in trivial form, then $A$ is irreducible.
Lemma 5. Let $A$ be an $n \times n$ irreducible matrix in normal form with $s$ main diagonal blocks, and suppose that each $F_{ij}$ has at most one nonzero entry. Suppose also that for each proper subset $\omega = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, s\}$ with at least two elements, the submatrix

$$A_{\omega} = \begin{pmatrix} A_{i_1} & F_{i_1i_2} & F_{i_1i_3} & \cdots & F_{i_1i_k} \\ F_{i_2i_1} & A_{i_2} & F_{i_2i_3} & \cdots & F_{i_2i_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{i_ki_1} & F_{i_ki_2} & F_{i_ki_3} & \cdots & A_{i_k} \end{pmatrix}$$

is reducible. Then the rows and columns of $A$ may be simultaneously permuted to bring $A$ into trivial form where the $s$ main diagonal blocks are $A_1, \ldots, A_s$ in some order.

Proof. If $s = 2$, there is nothing to prove. Thus suppose $s > 2$. Let us observe that when one simultaneously permutes the rows and columns of blocks of $A$, he simply permutes the $A_1, \ldots, A_s$ on the main diagonal, and thus the resulting matrix is still in normal form. Thus throughout the proof whenever we simultaneously permute the rows and columns of blocks of $A$, we shall assume that the $A_i$ have been reindexed in such a manner that the first main diagonal block in the resulting matrix is $A_1$, etc. Since $A$ is irreducible, there is some $F_{12} \neq 0$. If $i \neq 1$, we could simultaneously permute the rows and columns of blocks of $A$ and then renumber the $F_{ij}$ blocks to make $F_{12} \neq 0$ in the resulting matrix. Hence for convenience we shall assume that $F_{12} \neq 0$. Since $A_{\omega}$ is reducible for $\omega = \{1, s\}$, it follows by Lemma 4 that $F_{1s} = 0$. Thus, since $A$ is irreducible, there is some $F_{ii} \neq 0$ for $2 \leq i < s$. We may assume that $F_{12} \neq 0$. Since $A_{\omega}$ is reducible for $\omega = \{1, 2\}$, it follows by Lemma 4 that $F_{12} = 0$. By a similar argument, if $\omega = \{1, 2, s\}$, and $s > 3$, $F_{32} = 0$. Thus, since $A$ is irreducible, there is some $F_{12} \neq 0$ for $2 < i < s$. We may suppose that $F_{13} \neq 0$. If $s = 3$, $F_{12} = 0$ implies that $F_{32} \neq 0$. If $s > 4$, we can argue that $F_{13}, F_{23},$ and $F_{33}$ are each 0 and suppose that $F_{43} \neq 0$. If $s = 4$, we conclude that $F_{13} = 0$, $F_{23} = 0$, and $F_{43} \neq 0$. Continuing in the manner, we can argue that $F_{ij} \neq 0$ for $i = j + 1 \pmod s$. A consideration of $A_{\omega}$ for $\omega = \{i, i+1, \ldots, j\} \pmod s$ shows that $F_{ij} = 0$ for $i \neq j + 1 \pmod s$.

Lemma 6. If $A$ is an $n \times n$ nearly reducible matrix with $n > 1$, there exists a permutation $P$ and an integer $t > 1$ such that $P^TAP$ is in trivial form with $t$ main diagonal blocks, each of which is nearly reducible.

Proof. By Lemma 3, $A$ has a normal form with $n_1 \times 1$ diagonal blocks, $A_1, \ldots, A_n$. If the hypothesis of Lemma 5 is satisfied for this form, the
proof is finished. If not, there is a largest integer \( k, 1 < k < n \), such that there exists an irreducible submatrix

\[
A_{\omega} = \begin{pmatrix}
A_{i_1} & F_{i_1i_2} & F_{i_1i_3} & \cdots & F_{i_1i_k} \\
F_{i_2i_1} & A_{i_2} & F_{i_2i_3} & \cdots & F_{i_2i_k} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
F_{i_ki_1} & F_{i_ki_2} & F_{i_ki_3} & \cdots & A_{i_k}
\end{pmatrix}
\]

in \( A \). A simultaneous permutation of the rows and columns of blocks of \( A \) (the blocks are 1 \times 1 \) at this stage of the argument) allows us to suppose that \( i_j = j, j = 1, \ldots, k \). Put \( A'_1 = A_{\omega} \) and \( A'_i = A_{k+i-1} \) for \( i > 1 \), and consider \( A \) in the normal form

\[
A' = \begin{pmatrix}
A_1' & F'_{12} & F'_{13} & \cdots & F'_{1r} \\
F'_{21} & A_2' & F'_{23} & \cdots & F'_{2r} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
F'_{r1} & F'_{r2} & F'_{r3} & \cdots & A'_r
\end{pmatrix}
\]

where \( r = n - k + 1 \). By Lemma 2, each \( F'_{ij} \) contains at most one nonzero entry. If the hypothesis of Lemma 5 concerning the reducibility of submatrices of \( A' \) is not satisfied, the process may be repeated. After a finite number of steps a normal form will be found in which this hypothesis is satisfied. Then the trivial form will follow. It is clear that there must be at least two main diagonal blocks in this final form. The near reducibility of these final main diagonal blocks is a consequence of Lemma 4.

**Lemma 7.** If \( A \) is an \( n \times n \) nearly reducible matrix in normal form with diagonal blocks \( A_1, \ldots, A_s \), then at least two of the \( A_s \) are nearly isolated in \( A \).

**Proof.** We prove the lemma by induction on \( s \). If \( s = 2 \), the result is clear. Thus let \( s > 2 \) and suppose that the result holds whenever there are at least two but less than \( s \) main diagonal blocks in a nearly reducible normal form.

We bring \( A \) into trivial form as in the proof of Lemma 6. Any diagonal block \( B_m \) in this trivial form which is not one of the \( A_k \) can be partitioned into \( A_k \) and \( F'_{ij} \) blocks. Every \( B_m \) is nearly reducible, and if a \( B_m \) is not some \( A_k \), \( B_m \) is in normal form.

If a \( B_m \) is some \( A_k \), that \( A_k \) is nearly isolated in \( A \). If not, by the induction hypothesis, there are at least two nearly isolated \( A_k \) in \( B_m \). The trivial form of \( A \) makes it clear that at least one of these \( A_k \) is nearly isolated in \( A \). Since
There are at least two $B_m$ in the trivial form of $A$, there are at least two nearly isolated $A_k$ in $A$.

We now deduce the main results.

**Theorem 1.** If $A$ is an $n \times n$ nearly reducible doubly stochastic matrix, then $n > 1$ and $A$ is a full cycle permutation matrix.

**Proof.** The fact that $n > 1$ is clear since if $n = 1$, $A = (0)$ and $A \notin \Omega_1$.

Since $A \in \Omega_n$, it follows by Birkhoff's theorem that $A$ contains a positive diagonal $d$. If $d$ corresponds to a full cycle permutation, then the permutation matrix containing $d$ is irreducible and is thus equal to $A$ since $A$ is nearly reducible. If $d$ corresponds to a permutation $\sigma$ which is not a full cycle, $\sigma$ is a product of $s$ disjoint cycles where $s > 1$. Thus there is a permutation $P$ such that

$$PTAP = \begin{pmatrix} A_1 & F_{12} & \cdots & F_{1s} \\ F_{21} & A_2 & \cdots & F_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ F_{s1} & F_{s2} & \cdots & A_s \end{pmatrix}$$

where each $A_k$ is a square matrix which contains a positive diagonal that corresponds to one of the full cycle factors of $\sigma$. Each $A_k$ is irreducible and the form of $PTAP$ is normal. For convenience we suppose that $A$ is already in this normal form. By Lemma 1, each $A_k$ is nearly reducible. It follows that each $A_k$ is at least $2 \times 2$ since no $A_k = 0$. It also follows that the only positive entries in any $A_k$ occur on the diagonal $d$.

By Lemma 7, some $A_k$ is nearly isolated. Suppose it is $A_1$. Then exactly one $F_{1i} \neq 0$; suppose it is $F_{1r}$, and suppose that the unique positive entry $f$ of $F_{1r}$ lies in the $p$-th row. Consider a positive entry $g$ in some $F_{ir}$. (Such a $g$ exists for some $i$ since $A$ is irreducible.) Suppose $g$ is in the $q$-th column of $F_{ir}$. Since $A \in \Omega_n$, $g$ lies on a positive diagonal in $A$ which includes $f$. Thus, since $A_1$ is at least $2 \times 2$, $A_1(p | q)$ contains a positive diagonal, and therefore, since the only positive entries of $A_1$ lie on a diagonal corresponding to a full cycle permutation, $a_{pq} > 0$. The existence of any positive entry $g'$ in the $q'$-th column of any $F_{ir}$ would likewise imply that $a_{pq'} > 0$. This forces $q' = q$; therefore every positive entry in any $F_{ir}$ must be in the $q$-th column.

Put $A' = A - a_{pq}E_{pq}$. Since $A'$ is reducible, there is a nonempty proper subset $\omega$ of $\{1, \ldots, n\}$ such that $A'[,\omega] = 0$. As in the proof of Lemma 1, we see that if $\omega$ contains an index of a row of $A$ that corresponds to a row of $A_k$, $k = 2, \ldots, s$, it contains the index of every row of $A$ that corresponds to any row of $A_k$; likewise, if the complement of $\omega$ in $\{1, \ldots, n\}$ contains the index of a column of $A$ that corresponds to a column of such an $A_k$, it contains the index of every column of $A$ that corresponds to any column of $A_k$.

Suppose $A_1$ is $n_1 \times n_1$. For $j = 2, \ldots, s$, let $F'_{1j}$ denote the submatrix of $F_{1j}$.
whose rows are indexed by $\omega \cap \{1, \ldots, n_1\}$ and whose columns correspond to the columns of $F_{1j}$ in $A$. Also for each such $j$ let $F'_{1j}$ denote the submatrix of $F_{1j}$ whose rows are indexed by those members of $\{1, \ldots, n_1\}$ which are not in $\omega$, and whose columns correspond to the columns of $F_{1j}$ in $A$. The irreducibility of $A$ makes it clear that $F'_{1j}$ and $F'^*_1j$ exist for $j = 2, \ldots, s$. Similarly, for $i = 2, \ldots, s$, let $F'_{i1}$ denote the submatrix of $F_{i1}$ whose rows correspond to the rows of $F_{i1}$ in $A$ and whose columns are indexed by $\omega \cap \{1, \ldots, n_i\}$. Also for such $i$ let $F'^*_i1$ denote the submatrix of $F_{i1}$ whose rows correspond to the rows of $F_{i1}$ in $A$ and whose columns are indexed by those members of $\{1, \ldots, n_i\}$ which are not in $\omega$. The irreducibility of $A$ shows that $F'^*_i1$ and $F'_{i1}$ exist for $i = 2, \ldots, s$.

We can suppose that the rows and columns of blocks in $A$ have been simultaneously permuted so that the columns of $A$ which are indexed by those members of $\{n_1 + 1, \ldots, n\}$ which are not in $\omega$ (if any) correspond to the columns of $A$, for $k = 2, \ldots, m$, and that the rows of $A$ which are indexed by $\omega \cap \{n_1 + 1, \ldots, n\}$ (if any) correspond to the rows of $A_k$ for $k = m + 1, \ldots, s$.

Since $A'$ is reducible, $F'_{ij} = 0$ for $j = 2, \ldots, m$ if $m \geq 2$. Since $p \in \omega$, $F'^*_1j = 0$, and thus $F_{1j} = 0$ for $j = 2, \ldots, m$. Whence $F_{ij} = 0$ if

$$i = 1, m + 1, \ldots, s, \quad \text{and} \quad j = 2, \ldots, m.$$ 

This implies that $A$ is reducible, which is a contradiction. Hence the column indices of $A$ not in $\omega$ belong to $\{1, \ldots, n_1\}$. Then the reducibility of $A'$ implies that $F'^*_1i = 0$ for $i = 2, \ldots, s$. Since $q$ belongs to the complement of $\omega$ in $\{1, \ldots, n_1\}$, $F'^*_1i = 0$, and thus $F_{ij} = 0$ for $i = 2, \ldots, s$. This also implies the contradiction that $A$ is reducible. Hence we must conclude that $s = 1$, and therefore that $\sigma$ is a full cycle permutation after all.

**Theorem 2.** If $A$ is an $n \times n$ nonnegative nearly reducible matrix, then $A$ has at most one positive diagonal.

**Proof.** Let $A'$ be obtained from $A$ by replacing with zero every positive entry which does not lie on a positive diagonal. If $A' = 0$, $A$ has no positive diagonal. If $A' \neq 0$, $A'$ has doubly stochastic pattern. If $A'$ has doubly stochastic pattern and is irreducible, necessarily $A' = A$, and the result is an immediate consequence of Theorem 1. If $A'$ has doubly stochastic pattern and is reducible, there is a permutation $P$ and an integer $s > 1$ such that $P^T A' P = A'_1 \oplus \cdots \oplus A'_s$ where each $A'_k$ is irreducible and has doubly stochastic pattern. In this case write

$$P^T A P = \begin{pmatrix}
A_1 & F_{12} & \cdots & F_{1s} \\
F_{21} & A_2 & \cdots & F_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
F_{s1} & F_{s2} & \cdots & A_s
\end{pmatrix}$$

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where each $A_k$ has the dimension of the corresponding $A_k'$. Since each $A_k'$ is irreducible, so is each $A_k$, and the form of $PTAP$ is normal. It is then a consequence of Lemma 1 that each $A_k$ is nearly reducible, and thus $A_k = A_k'$ for each $k$. By Theorem 1, each $A_k$ has all its positive entries on one positive diagonal which corresponds to a full cycle permutation. Thus $A'$, and therefore $A$, has exactly one positive diagonal.

The examples

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

show that a nearly reducible matrix $A$ can have either one positive diagonal or no positive diagonals.

**References**