Keller’s Conjecture for Certain $p$-Groups

Kereszthely Corrádi

Department of Computer Tech., Eötvös Loránd University, H-1088 Budapest, Muzeum krt 6-8

AND

Sándor Szabó


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The algebraic form of Keller’s conjecture will be proved for $p$-groups having two direct components. © 1988 Academic Press, Inc.

INTRODUCTION

In 1930 O. H. Keller [4] conjectured that if translates of a closed $n$-dimensional cube tile the $n$-space, then in this cube system there exist two cubes having a common $(n-1)$-dimensional face.

In 1949 G. Hajós [3] gave the following group theoretical equivalent for this conjecture. If $G$ is a finite additive abelian group and

$$G = H + [g_1, q_1] + \cdots + [g_n, q_n]$$

(1)

is a factorization, then

$$\{q_1 g_1, \ldots, q_n g_n\} \cap (H - H) \neq \emptyset.$$ (2)

Here

$$H - H = \{h - h': h, h' \in H\}$$

and

$$[g_i, q_i] = \{0, g_i, 2g_i, \ldots, (q_i - 1) g_i\}.$$
Finally, we say $G$ is factored by its subsets $A_1, ..., A_n$ if each $g$ in $G$ is uniquely expressible in the form

$$g = a_1 + \cdots + a_n, \quad a_i \in A_i, ..., a_n \in A_n.$$ 

If $G$ is the direct sum of cyclic groups of orders $m_1, m_2, ..., m_k$ then the $k$ tuple of integers $(m_1, m_2, ..., m_k)$ is called the type of group $G$.

The generatum of a subset $H$ of $G$ and the order of an element $g$ in $G$ will be denoted by $\langle H \rangle$ and $|g|$, respectively. Keller's conjecture has been proved in the following special cases: when $n < 6$ independently from the structure of $G$ and independently from the value of $n$ when $G$ is one of the types

$$(p^a, q^b), (p^a, p, ..., p), (p^a, q, ..., q),$$

where $p$ and $q$ are different primes. The proofs can be found in order in \cite{L-5, 7, 1, 21}.

The purpose of this paper is to prove Keller's conjecture for groups of type $(p^a, p^b)$. According to \cite{9} it is enough to prove the group theoretical form of Keller's conjecture for $p$-group. So our result represents the first step toward a complete solution.

**RESULT**

We need the next two lemmas. The first one enables us to replace a factor by another one in a factorization while the second one enables us to lift a factoring from the homomorph image of a group to the group.

**Lemma 1** ([\cite{6}, p. 370]). If

$$G = H + [g, q]$$

is a factorization of finite abelian group $G$ and $t$ is an integer prime to $q$, then

$$G = H + [tg, q]$$

is a factorization as well.

**Lemma 2** ([\cite{8}, p. 545]). Let $G$ be the homomorphism image of abelian group $G'$ at homomorphism $f$ and let

$$G = A + B$$
be a factorization of $G$. Assume that $A'$ is a subset of $G'$ such that the restriction of $f$ to $A'$ is a bijection between $A'$ and $A$. Then

$$G' = A' + f^{-1}(B)$$

is a factorization of $G'$.

Now we are ready to prove our main result.

**Theorem.** Keller's conjecture holds for groups of type $(p^a, p^b)$.

**Proof.** Let $G$ be a group of type $(p^a, p^b)$. We should prove that factorization (1) concludes (2). To prove it take a counter example for which $|G|$ is minimal.

As we have already seen in [7] if factorization (1) is a counter example, then it can be replaced by another one in which each $q_i$ is a prime. In addition here we may suppose that $0 \in H$. But after this replacement the number of the factors may be changed.

Thus we will prove that if

$$G = H + [x_1, p] + \cdots + [x_s, p]$$

is a factorization, then

$$\{px_1, ..., px_s\} \cap (H - H) \neq \emptyset.$$

Let us study the structure of $H$. To do this let

$$K = \langle x_1, x_2, ..., x_s \rangle.$$

The minimality of the counter example and the factorization

$$K = G \cap K = (H \cap K) + [x_1, p] + \cdots + [x_s, p]$$

conclude that $\langle x_1, ..., x_s \rangle = K = G$.

Since the elements $x_1, ..., x_s$ generate $G$ which is generated by two elements there are two elements among $x_1, ..., x_s$, say $x_1, x_2$, which generate $G$. From this we can see that $s \geq 2$. Let

$$L = H + [x_3, p] + \cdots + [x_s, p].$$

Then

$$G = L + [x_1, p] + [x_2, p]$$

is a factorization.
Let $G'$ be an abelian group with basis elements $x_1'$ and $x_2'$ of orders $|x_1'|$ and $|x_2'|$, respectively. Consider the homomorphism

$$f: G' \longrightarrow G$$

given by

$$f(a_1 x_1' + a_2 x_2') = a_1 x_1 + a_2 x_2,$$

where $a_1$ and $a_2$ are integers.

It is easy to verify that the restriction of $f$ to

$$[x_1', p] + [x_2', p]$$

is a bijection between

$$[x_1', p] + [x_2', p] \quad \text{and} \quad [x_1, p] + [x_2, p].$$

Setting $L' = f^{-1}(L)$ Lemma 2 gives that

$$G' = L' + [x_1', p] + [x_2', p]$$

(3)

is a factorization.

Now we show that

$$L' \subseteq \langle px_1' \rangle + \langle x_2' \rangle \quad \text{or} \quad L' \subseteq \langle x_1' \rangle + \langle px_2' \rangle$$

(4)

Indeed, let $l_1'$ and $l_2'$ be different elements of $L'$ and let $l_2' - l_1' = a_1 x_1' + a_2 x_2'$. Clearly, $(a_1, a_2) \neq (0, 0)$. If $p \nmid a_1$ and $p \nmid a_2$, then

$$l_2' + 0 \cdot x_1' + 0 \cdot x_2' = l_1' + a_1 x_1' + a_2 x_2'$$

violates the factorization

$$G' = L' + [a_1 x_1', p] + [a_2 x_2', p]$$

which arises from (3) by replacing $[x_i', p]$ by $[a_i x_i', p]$. This replacement is possible because $a_1$ and $a_2$ are prime to $p$. Thus $p \mid a_1$ or $p \mid a_2$.

Apply this fact in the case of $l_1' = 0$. Let $l_2' = b_1 x_1' + b_2 x_2'$ be an element of $L'$. We conclude that $p \mid b_1$ or $p \mid b_2$. Assume that

$$L' \not\subseteq \langle px_1' \rangle + \langle x_2' \rangle \quad \text{and} \quad L' \not\subseteq \langle x_1' \rangle + \langle px_2' \rangle.$$

This means that there are $l_1', l_2' \in L'$ such that

$$l_2' - l_1' = a_1 x_1' + a_2 x_2',$$

where $a_1$ and $a_2$ are prime to $p$. Since this is impossible (4) holds.
For the sake of definiteness suppose that
\[ L' \subseteq \langle x'_1 \rangle + \langle px'_2 \rangle. \]

Let
\[ M = \langle x'_1 \rangle + \langle px'_2 \rangle. \]

Note that
\[ L' \subseteq M, \quad [x'_1, p] \subseteq M, \quad |M| = |L'| \cdot p. \]

So
\[ M = L' + [x'_1, p] \]

is a factorization. Apply \( f \) for this factorization. We have
\[ f(M) = I + [x_1, p] = H + [x_3, p] + \cdots + [x_s, p] + [x_1, p] \]

is a factorization of a proper subgroup of \( G \). This contradiction completes the proof.

REFERENCES