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A characterization of Q-polynomial association schemes

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ABSTRACT

We prove a necessary and sufficient condition for a symmetric association scheme to be a Q-polynomial scheme.

1. Introduction

A *symmetric association scheme* of class d is a pair $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$, where X is a finite set and each R_i is a nonempty subset of $X \times X$ satisfying the following:

(1) $R_0 = \{(x, x) \mid x \in X\}$,
(2) $X \times X = \bigcup_{i=0}^{d} R_i$ and $R_i \cap R_j$ is empty if $i \neq j$,
(3) ${}^tR_i = R_i$ for any $i \in \{0, 1, \dots, d\}$, where ${}^tR_i = \{(y, x) \mid (x, y) \in R_i\}$,
(4) for all $i, j, k \in \{0, 1, \dots, d\}$, there exist integers p_{ij}^k such that for all $x, y \in X$ with $(x, y) \in R_k$,

$$p_{ij}^k = \left|\left\{z \in X \;\middle|\; (x, z) \in R_i,\ (z, y) \in R_j\right\}\right|.$$

The integers p_{ij}^k are called the *intersection numbers*.

Let $\mathfrak{X}$ be a symmetric association scheme. The i-th *adjacency matrix* A_i of $\mathfrak{X}$ is the matrix with rows and columns indexed by X such that the (x, y)-entry is 1 if $(x, y) \in R_i$ or 0 otherwise. The *Bose–Mesner algebra* of $\mathfrak{X}$ is the algebra generated by the adjacency matrices $\{A_i\}_{i=0}^{d}$ over the complex

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field $\mathbb{C}$. Then $\{A_i\}_{i=0}^d$ is a natural basis of the Bose–Mesner algebra. By [2, p. 59], the Bose–Mesner algebra has a second basis $\{E_i\}_{i=0}^d$ such that

(1) $E_0 = |X|^{-1}J$, where J is the all-ones matrix,
(2) $I = \sum_{i=0}^d E_i$, where I is the identity matrix,
(3) $E_iE_j = \delta_{ij}E_i$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

The basis $\{E_i\}_{i=0}^d$ is called the *primitive idempotents* of $\mathfrak{X}$. We have the following equations:

$$A_i = \sum_{j=0}^d p_i(j)E_j, \tag{1.1}$$

$$E_i = \frac{1}{|X|}\sum_{j=0}^d q_i(j)A_j, \tag{1.2}$$

$$A_iA_j = \sum_{k=0}^d p_{ij}^k A_k, \tag{1.3}$$

$$E_i \circ E_j = \frac{1}{|X|}\sum_{k=0}^d q_{ij}^k E_k, \tag{1.4}$$

where $\circ$ denotes the *Hadamard product*, that is, the entry-wise matrix product. The matrices $P = (p_j(i))_{i,j=0}^d$ and $Q = (q_j(i))_{i,j=0}^d$ are called the first and second *eigenmatrices*, respectively. The numbers q_{ij}^k are called the *Krein parameters*. The Krein parameters are non-negative real numbers (the Krein condition) [10], [2, p. 69].

A symmetric association scheme is called a *P-polynomial scheme* (or a *metric scheme*) with respect to the ordering $\{A_i\}_{i=0}^d$ if for each $i \in \{0, 1, \dots, d\}$, there exists a polynomial v_i of degree i such that $p_i(j) = v_i(p_1(j))$ for any $j \in \{0, 1, \dots, d\}$. We say a symmetric association scheme is a P-polynomial scheme with respect to A_1 if it has the P-polynomial property with respect to some ordering $A_0, A_1, A_{i_2}, A_{i_3}, \dots, A_{i_d}$. Dually a symmetric association scheme is called a *Q-polynomial scheme* (or a *cometric scheme*) with respect to the ordering $\{E_i\}_{i=0}^d$ if for each $i \in \{0, 1, \dots, d\}$, there exists a polynomial v_i^* of degree i such that $q_i(j) = v_i^*(q_1(j))$ for any $j \in \{0, 1, \dots, d\}$. Moreover a symmetric association scheme is called a Q-polynomial scheme with respect to E_1 if it has the Q-polynomial property with respect to some ordering $E_0, E_1, E_{i_2}, E_{i_3}, \dots, E_{i_d}$. Note that both $\{v_i\}_{i=0}^d$ and $\{v_i^*\}_{i=0}^d$ form systems of orthogonal polynomials.

Throughout this paper, we use the notation $m_i = q_i(0)$ and $\theta_i^* = q_1(i)$ for $0 \leqslant i \leqslant d$. If an association scheme is Q-polynomial, then $\{\theta_i^*\}_{i=0}^d$ are mutually distinct because the second eigenmatrix $Q = (v_i^*(\theta_j^*))_{j,i=0}^d$ is non-singular. For a univariate polynomial f and a matrix M, we denote by $f(M^\circ)$ the matrix obtained by substituting M into f with multiplication the Hadamard product. We introduce known equivalent conditions of the Q-polynomial property of symmetric association schemes [2, p. 193]. The following are equivalent:

(1) $\mathfrak{X}$ is a Q-polynomial scheme with respect to the ordering $\{E_i\}_{i=0}^d$.
(2) $(q_{1,i}^j)_{i,j=0}^d$ is an irreducible tridiagonal matrix.
(3) For each $i \in \{0, 1, \dots, d\}$, there exists a polynomial f_i of degree i such that $E_i = f_i(E_1^\circ)$.

In the present paper, we prove a new necessary and sufficient condition for a symmetric association scheme to be Q-polynomial. Since the Q-polynomial property of a symmetric association scheme of class 1 is trivial, we assume that d is greater than 1.

Theorem 1.1. *Let $\mathfrak{X}$ be a symmetric association scheme of class $d \geqslant 2$. Suppose that $\{\theta_j^*\}_{j=0}^d$ are mutually distinct. Then the following are equivalent*:

(1) *$\mathfrak{X}$ is a Q-polynomial scheme with respect to E_1.*
(2) *There exists $l \in \{2, 3, \dots, d\}$ such that for any $i \in \{1, 2, \dots, d\}$,*

$$\prod_{\substack{j=1\\ j\neq i}}^{d} \frac{\theta_0^* - \theta_j^*}{\theta_i^* - \theta_j^*} = -p_i(l).$$

Moreover if (2) *holds, then $l = i_d$.*

Remark 1.2. We call a finite set X in $\mathbb{R}^m$ a *d-distance set* if the number of the Euclidean distances between distinct two points in X is equal to d. Larman, Rogers and Seidel [6] proved that if the size of a two-distance set with the distances a, b $(a < b)$ is greater than $2m+3$, then there exists a positive integer k such that $a^2/b^2 = (k-1)/k$, *i.e.* $k = b^2/(b^2 - a^2)$. Bannai and Bannai [1] proved that the ratio k of the spherical embedding of a primitive association scheme of class 2 coincides with $-p_i(2)$. The research of the present paper is motivated by [1]. For a symmetric association scheme satisfying that $\{\theta_j^*\}_{j=0}^d$ are mutually distinct, the values $K_i := \prod_{j=1,\, j\neq i}^{d} (\theta_0^* - \theta_j^*)(\theta_i^* - \theta_j^*)^{-1}$ $(1 \leqslant i \leqslant d)$ are the generalized Larman–Rogers–Seidel's ratios [9] of the spherical embedding of this association scheme with respect to E_1. Theorem 1.1 is an extension of Bannai–Bannai's result to Q-polynomial schemes of any class. Furthermore Theorem 1.1 is a new characterization of the Q-polynomial property on the spherical embedding of a symmetric association scheme.

At the end of this paper, we give some sufficient conditions for the integrality of K_i.

2. Proof of Theorem 1.1

First we give several lemmas that will be needed to prove Theorem 1.1.

Lemma 2.1. *For mutually distinct $\beta_1, \beta_2, \dots, \beta_s$, the following formal identity holds*:

$$\sum_{i=1}^{s} \beta_i^j \prod_{\substack{k=1\\ k\neq i}}^{s} \frac{x - \beta_k}{\beta_i - \beta_k} = x^j$$

for all $j \in \{0, 1, \dots, s-1\}$.

Proof. For each $j \in \{0, 1, \dots, s-1\}$, the polynomial

$$L_j(x) := \sum_{i=1}^{s} \beta_i^j \prod_{\substack{k=1\\ k\neq i}}^{s} \frac{x - \beta_k}{\beta_i - \beta_k}$$

of degree at most $s-1$ is known as the interpolation polynomial in the Lagrange form (see [3]). Namely, the property $L_j(\beta_i) = \beta_i^j$ holds for any $i \in \{1, 2, \dots, s\}$. Therefore $L_j(x) = x^j$, and the lemma follows. □

We say E_j is a *component* of an element M of the Bose–Mesner algebra if $E_j M \neq 0$. Let N_h denote the set of indices j such that E_j is a component of $E_1^{\circ h}$ but not of $E_1^{\circ l}$ $(0 \leqslant l \leqslant h-1)$. Note that $N_0 = \{0\}$ and $N_1 = \{1\}$.

Lemma 2.2. *Suppose $\mathfrak{X}$ is a symmetric association scheme of class $d \geqslant 2$. Then the following are equivalent*:

(1) *$\mathfrak{X}$ is a Q-polynomial scheme with respect to E_1.*
(2) *The cardinality of N_d is equal to 1.*
(3) *N_d is nonempty.*

Proof. (2) $\Rightarrow$ (3): Clear.

(1) $\Rightarrow$ (2): Without loss of generality, we assume that $\mathfrak{X}$ is a Q-polynomial scheme with respect to $\{E_i\}_{i=0}^d$. By noting that $\{\theta_i^*\}_{i=0}^d$ are mutually distinct, $\{E_1^{\circ i}\}_{i=0}^d$ are linearly independent, and a basis of the Bose–Mesner algebra. We have

$$E_i = f_i\big(E_1^{\circ}\big) = \sum_{j=0}^{i} \alpha_{i,j} E_1^{\circ j},$$

where $\alpha_{i,j} \in \mathbb{R}$ are the coefficients of a polynomial f_i of degree i. The upper triangular matrix $(\alpha_{i,j})_{i,j=0}^d$ is non-singular because $\alpha_{i,i} \neq 0$ for each i. Since the inverse matrix $(\alpha'_{i,j})_{i,j=0}^d$ of $(\alpha_{i,j})_{i,j=0}^d$ is also an upper triangular matrix with $\alpha'_{i,i} \neq 0$ for each i, we can express

$$E_1^{\circ i} = \sum_{j=0}^{i} \alpha'_{i,j} E_j.$$

Therefore (2) follows.

(3) $\Rightarrow$ (1): First we prove that if N_i is empty for some $i \in \{1, 2, \dots, d-1\}$, then N_{i+1} is also empty. Let $\mathcal{I} = \bigcup_{j=0}^{i-1} N_j$. We consider the expression $\sum_{j=0}^{i-1} E_1^{\circ j} = \sum_{j \in \mathcal{I}} \beta_j E_j$. Note that $\beta_j > 0$ for any $j \in \mathcal{I}$ by the Krein condition. Then we have

$$E_1 \circ \left(\sum_{h=0}^{i-1} E_1^{\circ h} \right) = \sum_{j \in \mathcal{I}} \beta_j \sum_{k=0}^{d} q_{1,j}^k E_k = \sum_{k=0}^{d} \sum_{j \in \mathcal{I}} \beta_j q_{1,j}^k E_k.$$

If N_i is empty, then

$$q_{1,j}^k = 0 \quad \text{for any } j \in \mathcal{I} \text{ and any } k \notin \mathcal{I} \tag{2.1}$$

because $\beta_j > 0$ holds for any $j \in \mathcal{I}$. We can express $E_1^{\circ i} = \sum_{j \in \mathcal{I}} \beta'_j E_j$, where β'_j are non-negative integers for any $j \in \mathcal{I}$. By (2.1) and the equalities

$$E_1^{\circ (i+1)} = E_1 \circ E_1^{\circ i} = E_1 \circ \sum_{j \in \mathcal{I}} \beta'_j E_j = \sum_{k=0}^{d} \sum_{j \in \mathcal{I}} \beta'_j q_{1,j}^k E_k,$$

we obtain $\sum_{j \in \mathcal{I}} \beta'_j q_{1,j}^k = 0$ for $k \notin \mathcal{I}$. Hence N_{i+1} is also empty. This means that if N_d is not empty, then the cardinalities of N_h is equal to 1 for any $h \in \{0, 1, \dots, d\}$. Put $N_h = \{i_h\}$ and order $E_0, E_1, E_{i_2}, E_{i_3}, \dots, E_{i_d}$. Then we can construct polynomials f_h of degree h such that $f_h(E_1^{\circ}) = E_{i_h}$ for any $h \in \{0, 1, \dots, d\}$. Hence (1) follows. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. (1) $\Rightarrow$ (2): Without loss of generality, we assume that $\mathfrak{X}$ is a Q-polynomial scheme with respect to $\{E_i\}_{i=0}^d$. For each $i \in \{1, 2, \dots, d\}$, we define the polynomial

$$F_i(t) := \prod_{\substack{j=1 \\ j \neq i}}^{d} \frac{|X|t - \theta_j^*}{\theta_i^* - \theta_j^*}$$

of degree $d-1$. Set $M_i = F_i(E_1^\circ)$. Then $|X|E_1 = \sum_{j=0}^{d} \theta_j^* A_j$ yields that the (x, y)-entries of M_i are

$$M_i(x,y) = \begin{cases} K_i & \text{if } (x,y) \in R_0, \\ 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $K_i := \prod_{j=1,\, j\neq i}^{d} (\theta_0^* - \theta_j^*)(\theta_i^* - \theta_j^*)^{-1}$. Since F_i is a polynomial of degree $d-1$, the matrix M_i is a linear combination of $\{E_i\}_{i=0}^{d-1}$. This means that $M_i E_d = 0$. By (1.1),

$$0 = M_i E_d = (K_i I + A_i) E_d = \big(K_i + p_i(d)\big) E_d$$

for any $i \in \{1, 2, \dots, d\}$. Therefore the desired result follows.

(2) ⇒ (1): From the equation $A_i = \sum_{j=0}^{d} p_i(j) E_j$ and our assumptions, we have

$$A_i E_l = p_i(l) E_l = -K_i E_l.$$

By Lemma 2.1,

$$\big(|X|E_1\big)^{\circ j} E_l = \left(\left(\theta_0^*\right)^j I + \sum_{i=1}^{d} \left(\theta_i^*\right)^j A_i \right) E_l = \left(\left(\theta_0^*\right)^j - \sum_{i=1}^{d} \left(\theta_i^*\right)^j K_i \right) E_l = 0$$

for any $j \leqslant d-1$. This means that l is not an element of N_j for any $j \leqslant d-1$. Note that the following equality holds:

$$\prod_{j=1}^{d} \frac{|X|E_1 - \theta_j^* J}{\theta_0^* - \theta_j^*} = I,$$

where the multiplication is the Hadamard product. Obviously, I has E_l as a component. Since $l \notin N_i$ for any $i \in \{0, 1, \dots, d-1\}$, we have $l \in N_d$. By Lemma 2.2, the desired result follows. □

3. Integrality of K_i

In this section, we consider when $K_i = -p_i(d)$ is an integer for each $i \in \{1, 2, \dots, d\}$ for a Q-polynomial scheme. The following theorem is important in this section.

Theorem 3.1. *(See Suzuki [11].) Let $\mathfrak{X}$ with $m_1 > 2$ be a Q-polynomial scheme with respect to the ordering $\{E_i\}_{i=0}^{d}$. Suppose $\mathfrak{X}$ is Q-polynomial with respect to another ordering. Then the new ordering is one of the following*:

(1) $E_0, E_2, E_4, E_6, \dots, E_5, E_3, E_1$,
(2) $E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, E_3, E_{d-3}, \dots$,
(3) $E_0, E_d, E_2, E_{d-2}, E_4, E_{d-4}, \dots, E_{d-5}, E_5, E_{d-3}, E_3, E_{d-1}, E_1$,
(4) $E_0, E_{d-1}, E_2, E_{d-3}, E_4, E_{d-5}, \dots, E_5, E_{d-4}, E_3, E_{d-2}, E_1, E_d$, *or*
(5) $d = 5$ *and* $E_0, E_5, E_3, E_2, E_4, E_1$.

Note that Q-polynomial schemes with $m_1 = 2$ are the ordinary n-gons as distance-regular graphs.

Proposition 3.2. *Let $\mathfrak{X}$ with $m_1 > 2$ be a Q-polynomial association scheme with respect to the ordering $\{E_i\}_{i=0}^{d}$. If there exists t such that $t \leqslant d/2$, $t \equiv 1 \pmod 2$ and $m_t \neq m_{d-t+1}$, then K_j is an integer for any j.*

Proof. Let $\mathbb{F}$ be the splitting field of the scheme, generated by the entries of the first eigenmatrix P. Then $\mathbb{F}$ is a Galois extension of the rational field. Let G be the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{Q})$. We consider the action of G on the primitive idempotents E_i, where elements of G are applied entry-wise. Then the action of G on $\{E_i\}_{i=0}^{d}$ is faithful and $|G| \leqslant 2$ [8].

Suppose K_j is not an integer for some j. Since $-K_j = p_j(d)$ is an eigenvalue of A_j, K_j is an algebraic integer. By the basic number theory, K_j is irrational. Therefore $|G| \neq 1$ and hence $|G| = 2$. Let σ be the non-identity element of G. From the definition of K_j, E_1 must have an irrational entry, and $E_1^\sigma \neq E_1$. Therefore $\{E_i^\sigma\}_{i=0}^d$ is another Q-polynomial ordering with the same polynomials f_i. Hence $\{E_i^\sigma\}_{i=0}^d$ coincides with one of (1)–(5) in Theorem 3.1.

For $d = 2$, it is known that K_i is an integer for each $i = 1, 2$ if $m_1 \neq m_2$ [1]. For (1) and (2) with $d > 2$, $(E_1^\sigma)^\sigma \neq E_1$, this contradicts that σ^2 is the identity. Since $p_j(d)$ is irrational and $A_j E_d = p_j(d) E_d$, E_d has an irrational entry. Therefore $E_d^\sigma \neq E_d$. For (4), σ fixes E_d, a contradiction. Therefore the ordering $\{E_i^\sigma\}_{i=0}^d$ coincides with (3) or (5).

Suppose that there exists t such that $t \leqslant d/2$, $t \equiv 1 \pmod 2$ and $m_t \neq m_{d-t+1}$. Since $E_t \circ I = (m_t/|X|)I$, we have $E_t^\sigma \circ I^\sigma = (m_t/|X|)I^\sigma$ and hence $E_t^\sigma \circ I = (m_t/|X|)I \neq (m_{d-t+1}/|X|)I$. Therefore $E_t^\sigma \neq E_{d-t+1}$. Thus, the ordering $\{E_i^\sigma\}_{i=0}^d$ does not coincide with (3) for $d \geqslant 2$. If $d = 5$, then $m_1 \neq m_5$ and hence $E_1^\sigma \neq E_5$. Therefore $\{E_i^\sigma\}_{i=0}^5$ does not coincide with (5). Thus the proposition follows. □

Remark that the known Q-polynomial schemes with some irrational K_i and $d > 2$ are the ordinary n-gons and the association scheme obtained from the icosahedron [4,7]. We can give a similar equivalent condition of the P-polynomial property of symmetric association schemes [5]. Let $\theta_i = p_1(i)$ for $0 \leqslant i \leqslant d$.

Theorem 3.3. *Let $\mathfrak{X}$ be a symmetric association scheme of class $d \geqslant 2$. Suppose $\{\theta_j\}_{j=0}^d$ are mutually distinct. Then the following are equivalent:*

(1) *$\mathfrak{X}$ is a P-polynomial association scheme with respect to A_1.*
(2) *There exists $l \in \{2, 3, \ldots, d\}$ such that for any $i \in \{1, 2, \ldots, d\}$,*

$$\prod_{\substack{j=1 \\ j \neq i}}^{d} \frac{\theta_0 - \theta_j}{\theta_i - \theta_j} = -q_i(l).$$

Moreover if (2) holds, then $l = i_d$.

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