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An analytical investigation on unsteady motion of vertically falling spherical particles in non-Newtonian fluid by Collocation Method

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Abstract An analytical investigation is applied for unsteady motion of a rigid spherical particle in a quiescent shear-thinning power-law fluid. The results were compared with those obtained from Collocation Method (CM) and the established Numerical Method (Fourth order Runge–Kutta) scheme. It was shown that CM gave accurate results. Collocation Method (CM) and Numerical Method are used to solve the present problem. We obtained that the CM which was used to solve such nonlinear differential equation with fractional power is simpler and more accurate than series method such as HPM which was used in some previous works by others but the new method named Akbari-Ganji's Method (AGM) is an accurate and simple method which is slower than CM for solving such problems. The terminal settling velocity—that is the velocity at which the net forces on a falling particle eliminate—for three different spherical particles (made of plastic, glass and steel) and three flow behavior index n, in three sets of power-law non-Newtonian fluids was investigated, based on polynomial solution (CM). Analytical results obtained indicated that the time of reaching the terminal velocity in a falling procedure is significantly increased with growing of the particle size that validated with Numerical Method. Further, with approaching flow behavior to Newtonian behavior from shear-thinning properties of flow ($n \to 1$), the transient time to achieving the terminal settling velocity is decreased.

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1. Introduction

An important natural phenomenon that occurs in many industrial processes is the sedimentation and falling of solid particles in gases and liquids. Primarily, sedimentation results from a tendency of suspended particles in fluids to settle and come to rest, due to the forces acting on them through the fluid [1]. Common examples include separation of liquid–solid mixtures, sprays and atomization, sediment transportation and

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Nomenclature

m	particle mass
u	particle velocity
g	gravity acceleration
D	particle diameter
C_D	drag coefficient
t	time
Re	Reynolds number
n	flow behavior index
V_0	velocity at $t = 0$
k	consistency coefficient
$X(n)$	deviation factor
j	number of polynomial statements
u_m	terminal settling velocity
t_m	time of terminal settling velocity
CM	Collocation Method
HPM	Homotopy Perturbation Method
HAM	Homotopy Analysis Method
VIM	Variational Iteration Method
NuM	Numerical Method

Greek symbols

ρ	fluid density
ρ_s	particle density

deposition in pipe lines [2,3], alluvial channels [4,5], and chemical and powder processing.

In many processes it is often essential to obtain the route of particles that accelerates in the fluid region for designing or improving the processes. The majority of previous studies have considered the steady-state conditions and where the particles achieved their terminal velocity. Also, several works have been done to study the unsteady motion of particles in Newtonian fluids [6–13] due to its applications in classification, centrifugal collection and separation (some of the unit operations which require the trajectories of particles accelerating in fluid). Further, the distance required to reach the terminal velocity is necessary for viscosity measurements of fluid with the falling ball experiment.

Along with the same proposition, many researchers realized the physical significance of some analytical methods such as the Homotopy Perturbation Method (HPM) [14], Variational Iteration Method (VIM) [15,16] and Homotopy Analysis Method (HAM) [17] and its compatibility with the physical problems as the unsteady motion of spherical particles in Newtonian fluids. Hatami and Ganji introduced the equation of the motion for variable-mass particle for the first time and solved by Padé approximation of Differential Transformation Method (DTM-Padé) and numerical Runge–Kutta method [18].

These methods were originally proposed by He [19,20] to achieve the series solution of strongly nonlinear differential equations. Jalaal et al. [21] used HPM to study the unsteady motion of a spherical particle falling in a Newtonian fluid for a range of Reynolds number to obtain a solution for nonlinear equations of a falling spherical with drag coefficient. Then, Jalaal et al. [22] used a series-based method called Homotopy Analysis Method (HAM) in order to solve nonlinear particle equation of motion whose results are very accurate and reliable. Meanwhile, an unsteady rolling motion of spheres in inclined tubes filled with incompressible Newtonian fluids was conducted by Jalaal et al. [22]. Later, Hamidi et al. [23] applied the HPM-Padé to solve the coupled equations of a spherical solid particle's motion in Couette flow. Hatami et al. solved coupled equations of particle's motion in Couette fluid flow by Multi-step Differential Transformation Method (Ms-DTM) considering the rotation and shear effects on lift force and neglecting gravity [24]. Hatami and Ganji investigated coupled equations of the motion of a particle in a fluid forced vortex the differential transformation method (DTM) with the Padé approximation and the differential quadrature method (DQM) [25]. Hatami and Ganji introduced the equation of a particle's motion on a rotating parabolic surface through Lagrange equations and solved by Multi-step Differential Transformation Method (Ms-DTM) [26].

Majority of the above mentioned studies have described the motion of solid particles in Newtonian suspensions only, however, many slurries and concentrated suspensions, which are treated in the materials processing industry, behave as non-Newtonian liquids and proper consideration has to be made [27–32]. The numerical solution of Bagchi and Chhabra [33] is one of the studies in this field. They reported the distance traveled by accelerating spherical particles in downward vertical motion of particles in power law liquids. Malvandi et al. [34] have studied analytically with HPM and VIM scheme on present problem and their results had very good agreement with the older researches. Therefore, Collocation Method [35,36] was used to find efficient, reliable and precise polynomial solutions. In order to consider the non-Newtonian fluid flow, the power-law model was employed. Furthermore, the terminal settling velocity for three rigid spherical particles namely plastic, glass and steel, vertically falling in the quiescent power-law fluids was determined. In terms of obtaining the best accuracy of the analytical results, a comparison was made by a numerical solution via forth order Runge–Kutta Numerical Method.

2. Problem formulations

In Fig. 1 it is shown the consideration on one-dimensional accelerated motion of a rigid spherical particle vertically falling to an infinite extent of a power-law shear-thinning fluid. The forces acting on a falling body are usually gravity, buoyancy, inertia, Basset history force, virtual mass and drag force. From the Lagrangian viewpoint, the dynamic of particles submerged in a fluid could be obtained by integrating the forces balanced on them. According to the studies of Renganathan et al. [37] and Bagchi and Chhabra [33], the Basset force can be assumed to be negligible when the density of the spherical particle is much larger than that of the liquid. Under this condition, the equation of motion describing the falling motion of the particle can be written as [22],

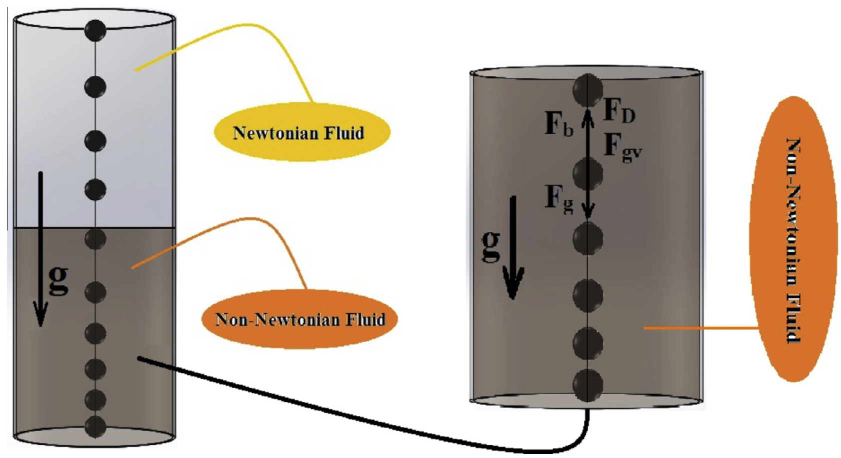


Figure 1 Geometry of physical model.

$$m\frac{du}{dt} = mg\left(1 - \frac{\rho}{\rho_s}\right) - \frac{\pi D^2 \rho C_D}{8}u^2 - \frac{\pi D^3 \rho}{12}\frac{du}{dt} \tag{1}$$

where D, m, ρ_s and C_D are the particle diameter, particle mass, particle density and drag coefficient, respectively. From left to right, the terms represent inertia, gravity–buoyancy, drag and virtual mass (added mass effect due to acceleration of fluid around the particle). The complexity of the above equation arises from the strong non-linear nature of the drag coefficient. The proper formulation of the drag coefficient has routinely been obtained by numerical or experimental results. It is well known today that the drag coefficient for a sphere in a power-law fluid could be expressed as follows:

$$C_D = f(Re, n) \tag{2}$$

For a creeping flow region ($Re \ll 1$), the drag coefficient could be obtained from Stokes law in the following form:

$$C_D = \frac{24}{Re}X(n) \tag{3}$$

where $Re = \rho u^2 - \frac{nDn}{K}$ is the Reynolds number. n and K are the flow behavior index and consistency coefficient, respectively. $X(n)$, a deviation factor, was obtained by researchers via numerical or experimental results. Here, a well-correlated equation of Renaud et al. [38] was used as follows:

$$X(n) = 6^{\frac{n-1}{2}}\left(\frac{3}{n^2 + n + 1}\right)^{n+1} \tag{4}$$

The correlated equation is valid for both shear-thinning ($n < 1$) and shear-thickening ($n > 1$) fluid behaviors. By substituting Eqs. (3) and (4) into Eq. (1) and by rearranging the parameters one could give:

$$a\frac{du}{dt} + b(n)u^n - d = 0, \quad u(0) = v_0 \tag{5}$$

In which

$$a = m + \frac{1}{12}\pi D^3 \rho, \quad b(n) = 3\pi K X(n) D^{2-n},$$
$$d = mg\left(1 - \frac{\rho}{\rho_s}\right) \tag{6}$$

Eq. (5) is classified as an IVP (initial value problem) differential equation, which could be solved with suitable Numerical Methods such as the finite difference scheme. The numerical solution of the problem is not within the scope of this paper, but the analytical solution is described in the following section.

3. Mathematical methods

Before presenting the results, it is necessary to provide some background knowledge about the mathematical methods employed. Therefore, in this section, some basic relationships and theories concerning Collocation Method (CM) and fourth order Runge–Kutta Numerical Method are presented.

There are some simple and accurate approximation techniques for solving differential equations called the Weighted Residuals Methods (WRMs). Collocation (CM), Galerkin (GM) and Least Square (LSM) are examples of the WRMs. Collocation Methods (CMs) were firstly introduced by Ozisik [39] for solving differential equations in heat transfer problems. Stern and Rasmussen [40] used Collocation Method for solving a third order linear differential equation. Vaferi et al. [41] studied the feasibility of applying of Orthogonal Collocation Method to solve diffusivity equation in the radial transient flow system.

Many advantages of CM compared to other analytical make it more valuable and motivate researchers to use it for solving problems. Some of these advantages are listed below [42]:

(a) WRMs solve the equations directly and no simplifications are needed.
(b) They do not need any perturbation, linearization or small parameter versus Homotopy Perturbation Method (HPM) and Parameter Perturbation Method (PPM).
(c) They are simple and powerful compared to Numerical Methods and achieve final results faster than numerical procedures while their results are acceptable and have excellent agreement with numerical outcomes, furthermore their accuracy can be increased by increasing the statements of the trial functions.
(d) They do not need to determine the auxiliary parameter and auxiliary function versus Homotopy Analysis Method (HAM).
(e) They are faster in solving such problems by a bit lower accuracy than new method which is named Akbari-Ganji's Method (AGM).

3.1. Collocation Method (CM)

For conception of the main idea of this method, suppose a differential operator D is acted on a function u to produce a function p [43]:

$$D(u(x)) = p(x) \tag{7}$$

We wish to approximate u by a function $\tilde{u}$, which is a linear combination of basic functions chosen from a linearly independent set. That is,

$$u \cong \tilde{u} = \sum_{i=1}^{n} c_i \varphi_i \tag{8}$$

Now, when substituted into the differential operator, D, the result of the operations is not $p(x)$. Hence an error or residual will exist:

$$E(x) = R(x) = D(\tilde{u}(x)) - p(x) \neq 0 \tag{9}$$

The notion in the Collocation is to force the residual to zero in some average sense over the domain. That is [22],

$$\int R(x) W_i(x) dx = 0 \qquad i = 0, 1, 2, \ldots, n \tag{10}$$

where the number of weight functions W_i is exactly equal to the number of unknown constants c_i in $\tilde{u}$. The result is a set of n algebraic equations for the unknown constants c_i. For Collocation Method, the weighting functions are taken from the family of Dirac δ functions in the domain. That is, $W_i(x) = \delta(x - x_i)$. The Dirac δ function has the property of [43–45]

$$\delta(x - x_i) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

And residual function in Eq. (9) must be forced to be zero at specific point.

3.2. Fourth order Runge–Kutta Method (NUM)

It is obvious that the type of the current problem is initial value problem (IVP) and the appropriate method needs to be chosen. The available sub-methods in the Maple 17.0 are a combination of the base schemes; trapezoid or midpoint method. There are two major considerations when choosing a method for a problem. The trapezoid method is generally efficient for typical problems, but the midpoint method is so capable of handling harmless end-point singularities that the trapezoid method cannot. The midpoint method, also known as the fourth-order Runge–Kutta–Fehlberg method, improves the Euler method by adding a midpoint in the step which increases the accuracy by one order. Thus, the midpoint method is used as a suitable numerical technique in present study [46].

4. Results and discussion

The applicability of the presented methods for the nonlinear equation of motion of settling particles will be illustrated in the following section. In order to measure the accuracy of the results, NUM has been used here for the derived non-linear ODE, given in Eq. (5), where the Fourth order Runge–Kutta is employed for deriving $\frac{du}{dt}$. A Maple code was used to find the numerical solution of the present initial value problem (IVP). The adopted values of the density and consistency coefficient of the non-Newtonian fluid were $\rho = 1050\,\text{kg/m}^3$ and $K = 0.5$, respectively [38]. In addition, the physical properties of particles and corresponding coefficients of Eq. (5) have been tabulated in Table 1.

4.1. Approximate solution with CM

In present study, the fluid is considered non-Newtonian fluid and governing equations for unsteady motion of a rigid spherical particle in a quiescent shear-thinning power-law fluid (non-Newtonian) are solved by CM and NUM. For solving Eq. (5) by WRMs, because trial function must satisfy the initial condition in Eq. (5), so each statement in $u(t)$ should contain t to satisfy initial condition in $t = 0$. In this study, one statement is considered for velocity profile and as explained in above WRMs advantages, accuracy of results can be increased by increasing the number of statements (c_i), so

$$u(t) = c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_j t^j \tag{12}$$

Which satisfies the initial condition in Eq. (5) and by setting it into Eq. (5), residual functions, $R_i(c_1 - c_6, t)$, will be found. On the other hand, the residual functions must be close to zero.

Also we can choose the trial function as the below statement:

$$u(t) = c_1(t - t^2) + c_2(t^2 - t^3) + \ldots + c_j(t^j - t^{j+1}) \tag{12-a}$$

By comparison between Eqs. (12) and (12-a) it is clear that Eq. (12) is simpler and more logical to be used so we decided that it is better to use the Eq. (12) as trial function.

Table 1 Physical properties and corresponding coefficients of Eq. (5) for $n = 0.5$.

Particle type	D (mm)	ρ_s	m(gr)	a	$b(n)$	d
Plastic	0.2	1150	$4.817 * 10^{-6}$	$7.016 * 10^{-9}$	$1.911 * 10^{-5}$	$4.109 * 10^{-9}$
	0.5	1150	$7.526 * 10^{-5}$	$1.096 * 10^{-7}$	$7.552 * 10^{-5}$	$6.421 * 10^{-8}$
	1	1150	$6.021 * 10^{-4}$	$8.770 * 10^{-7}$	$2.136 * 10^{-4}$	$5.137 * 10^{-7}$
Glass	0.2	2500	$1.047 * 10^{-5}$	$1.265 * 10^{-8}$	$1.911 * 10^{-5}$	$5.958 * 10^{-8}$
	0.5	2500	$1.636 * 10^{-4}$	$1.980 * 10^{-7}$	$7.552 * 10^{-5}$	$9.310 * 10^{-7}$
	1	2500	$1.309 * 10^{-3}$	$1.584 * 10^{-6}$	$2.136 * 10^{-4}$	$7.448 * 10^{-6}$
Steel	0.2	7780	$3.259 * 10^{-5}$	$3.479 * 10^{-8}$	$1.911 * 10^{-5}$	$2.765 * 10^{-7}$
	0.5	7780	$5.092 * 10^{-4}$	$5.436 * 10^{-7}$	$7.552 * 10^{-5}$	$4.321 * 10^{-6}$
	1	7780	$4.074 * 10^{-3}$	$4.348 * 10^{-6}$	$2.136 * 10^{-4}$	$3.457 * 10^{-5}$

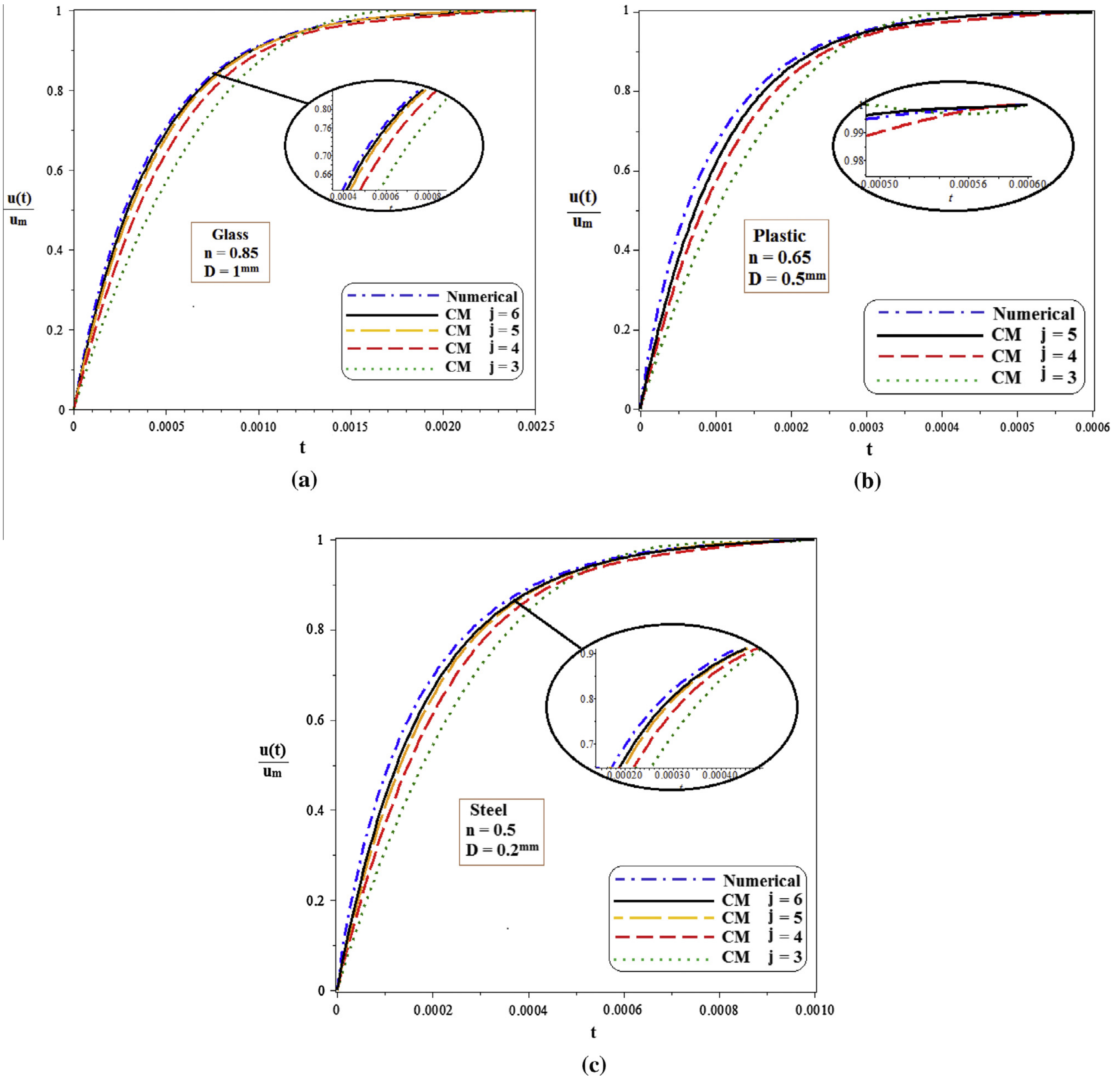


Figure 2 CM solution for different values of j and comparing with numerical results. for (a) $n = 0.85$ (b) $n = 0.65$ (c) $n = 0.5$.

For reaching to this aim, six specific points in the domain $t \in [0, tm]$ should be chosen. These points are:

$$R_i(c_1 - c_j, t) = \frac{i.t_m}{j} \tag{13}$$

Finally by substituting these points into the residual functions, a set of jth equations with jth unknown coefficients will be obtained. After solving these unknown parameters, the velocity concentration equation will be determined. Using Collocation Method, for example when the particle be plastic with $D = 1$ mm for $n = 0.85$ leads to:

$j = 6$:

$$u(t) = -2.155553255 * 10^{13}t^6 + 2.373788034 * 10^{11}t^5 - 1.106742700 * 10^9t^4 + 2.854511380 * 10^6t^3 - 4420.787693t^2 + 4.030708825t \tag{14}$$

$j = 5$:

$$u(t) = 4.810853970 * 10^{10}t^5 - 4.581824359 * 10^8t^4 + 1.767438683 * 10^6t^3 - 3522.987025t^2 + 3.733654742t \tag{15}$$

$j = 4$:

$$u(t) = -1.007673681 * 10^8t^4 + 7.989310560 * 10^5t^3 - 2384.949638t^2 + 3.239977945t \tag{16}$$

$j = 3$:

$$u(t) = 1.927676068 * 10^5t^3 - 1210.692269t^2 + 2.506207614t \tag{17}$$

Settling velocity of a 1 mm-diameter Glass particle with power-law fluids of $n = 0.85$ and a 0.5 mm-diameter plastic particle with power-law fluids of $n = 0.65$ and also a 0.2 mm-diameter Steel particle with power-law fluids of $n = 0.5$ versus time has been depicted in Fig. 2a, b and c, respectively. In all the figures, the velocity is scaled with terminal velocity $u_{terminal}$. As it is obvious, the particle velocity increases until it reaches the terminal velocity where the net force on the particle eliminates. The figures also show that increasing the value of j can enhance the region of convergence and improve the accuracy of the polynomial solution obtained by CM. Thus, most accurate results can be obtained by increasing the value of j in CM.

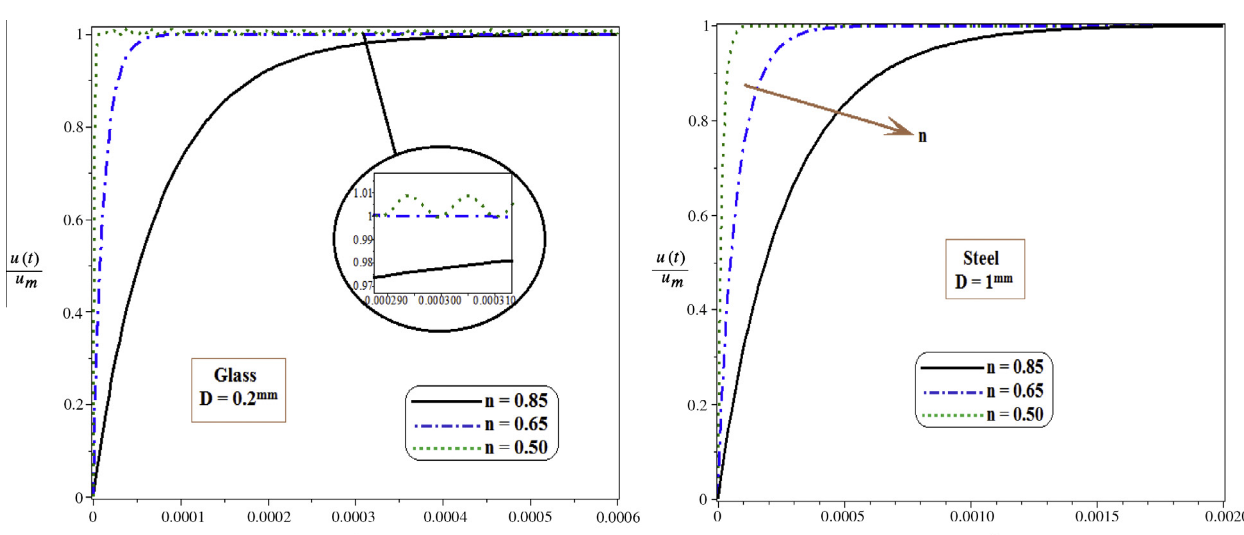
Table 2 shows the effects of different types of particles and power-law fluids on terminal velocity. To show the best accuracy, the numerical results have been added. Because of the highest density of the steel particle, it has the highest terminal velocity, whereas the plastic particle, with the lowest density, achieves the lowest terminal velocity. In addition, terminal velocity has a dramatic upward trend when particle diameter and power-law index parameter increase moderately (because of increasing in the mass of the particle, see Eq. (1).

4.2. Transient motion

Physical quantities of interest, the velocity–time and acceleration–time for different particles, particle diameters, and the flow behavior indexes are shown in Figs. 3–6. Fig. 3a and b shows the effects of flow behavior index n on the velocity and time. Regarding Fig. 3a and b, it is obvious That in a constant time the amount of terminal velocity in a falling procedure for both of glass and steel in constant particle diameter is increased with growing the flow behavior index n. With approaching flow behavior to Newtonian behavior from shear-thinning properties of flow ($n \rightarrow 1$), the time of transient to achieving the terminal velocity is decreased as indicated in Fig. 3a and b. The effects of three different particles namely: plastic, glass, and steel on the falling procedure are shown in Fig. 4a–c. The figures indicate the lowest time of reaching the terminal velocity of plastic particles due to their lower density. As a consequence, the larger the particle density. Regarding Fig. 5, it is obvious that the time of reaching the terminal velocity in a falling procedure is significantly increased with the growing of the particle size. In fig. 6a, b and c, it can be realized that the acceleration of particles is higher for larger

Table 2 Terminal settling velocity ($\times 10^{-7}$ m/s) for different diameters of particles and several fluids.

Particle type	D (mm)	CM results n			Numerical results n			Error%
		0.5	0.65	0.85	0.5	0.65	0.85	$n = 0.5$
Plastic	0.2	1.772	6.781	22.625	1.733	6.753	22.544	2.25
	0.5	28.795	69.589	164.932	28.914	69.756	165.281	0.41
	1	229.736	404.505	747.348	231.047	405.366	745.285	0.57
Glass	0.2	389.187	418.997	521.890	387.877	417.369	522.887	0.34
	0.5	6091.018	4286.642	3859.046	6065.463	4270.318	3842.110	0.42
	1	48617.179	24946.056	17106.862	48346.550	24831.984	17348.164	0.56
Steel	0.2	8328.664	4444.075	3197.794	8327.324	4426.284	3183.089	0.02
	0.5	131345.961	45465.803	23441.211	130731.743	45293.424	23355.788	0.47
	1	1049239.494	26568.143	105560.900	1045030.464	263262.304	105295.159	0.4



(a) (b)

Figure 3 Effects of the flow behavior index (n) with constant diameter on the velocity variation of (a) glass particle (b) steel particle.

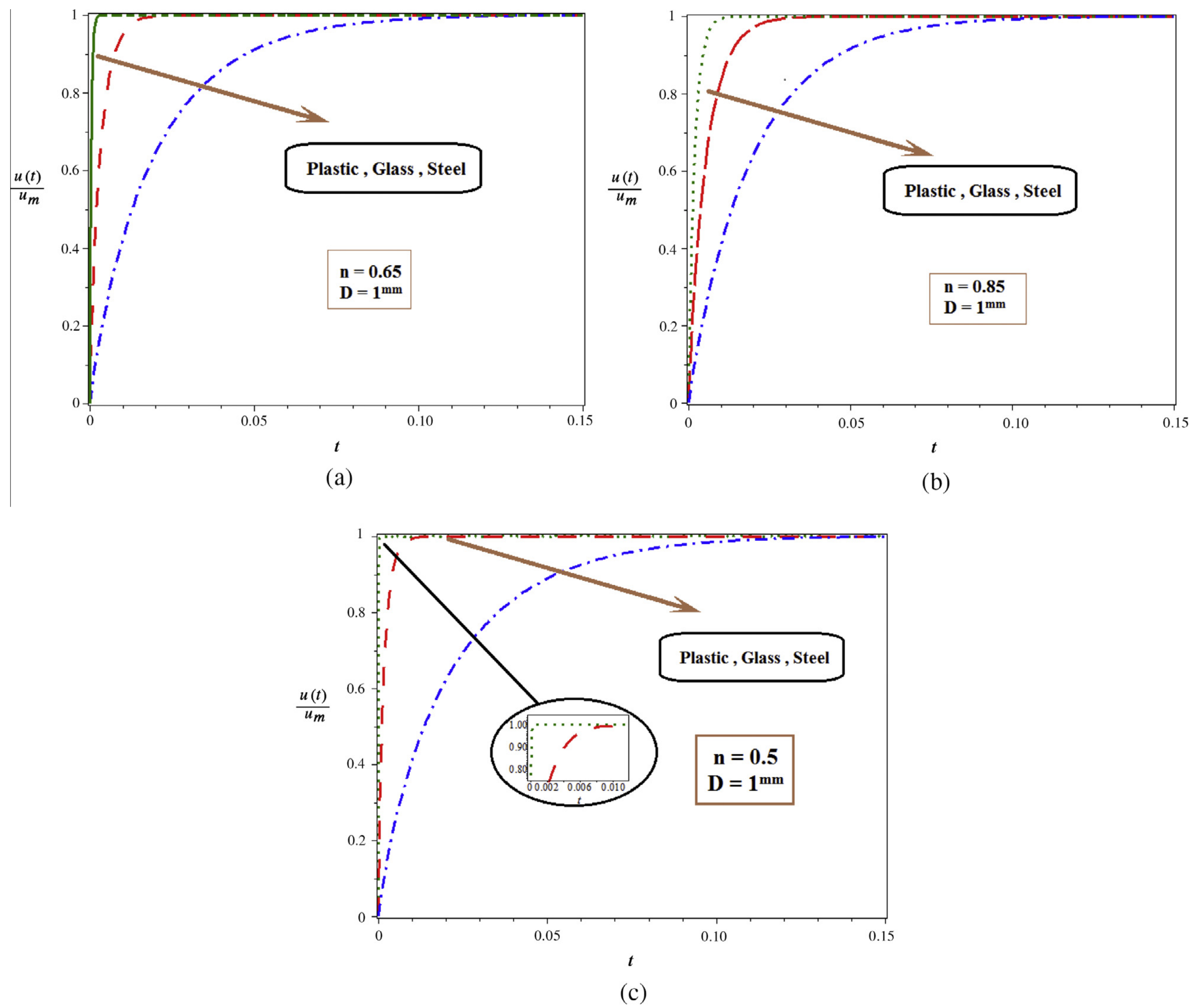


Figure 4 Effects of the particle type on the velocity variation: (a) $n = 0.65$, $D = 1$ mm, (b) $n = 0.85$, $D = 1$ mm, and (c) $n = 0.5$, $D = 1$ mm.

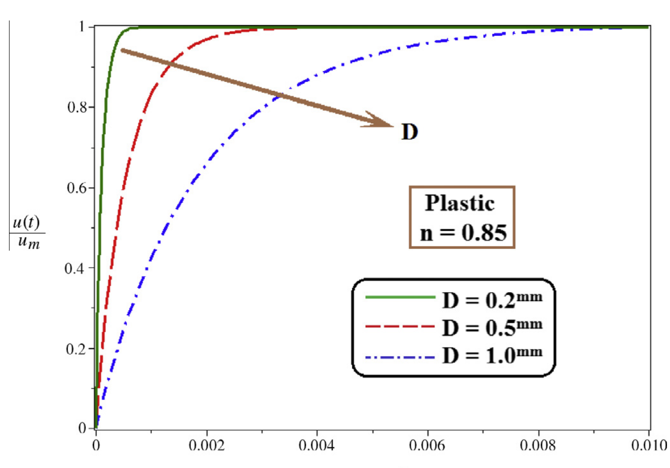


Figure 5 Effects of the particle diameter on the velocity variation of plastic particle.

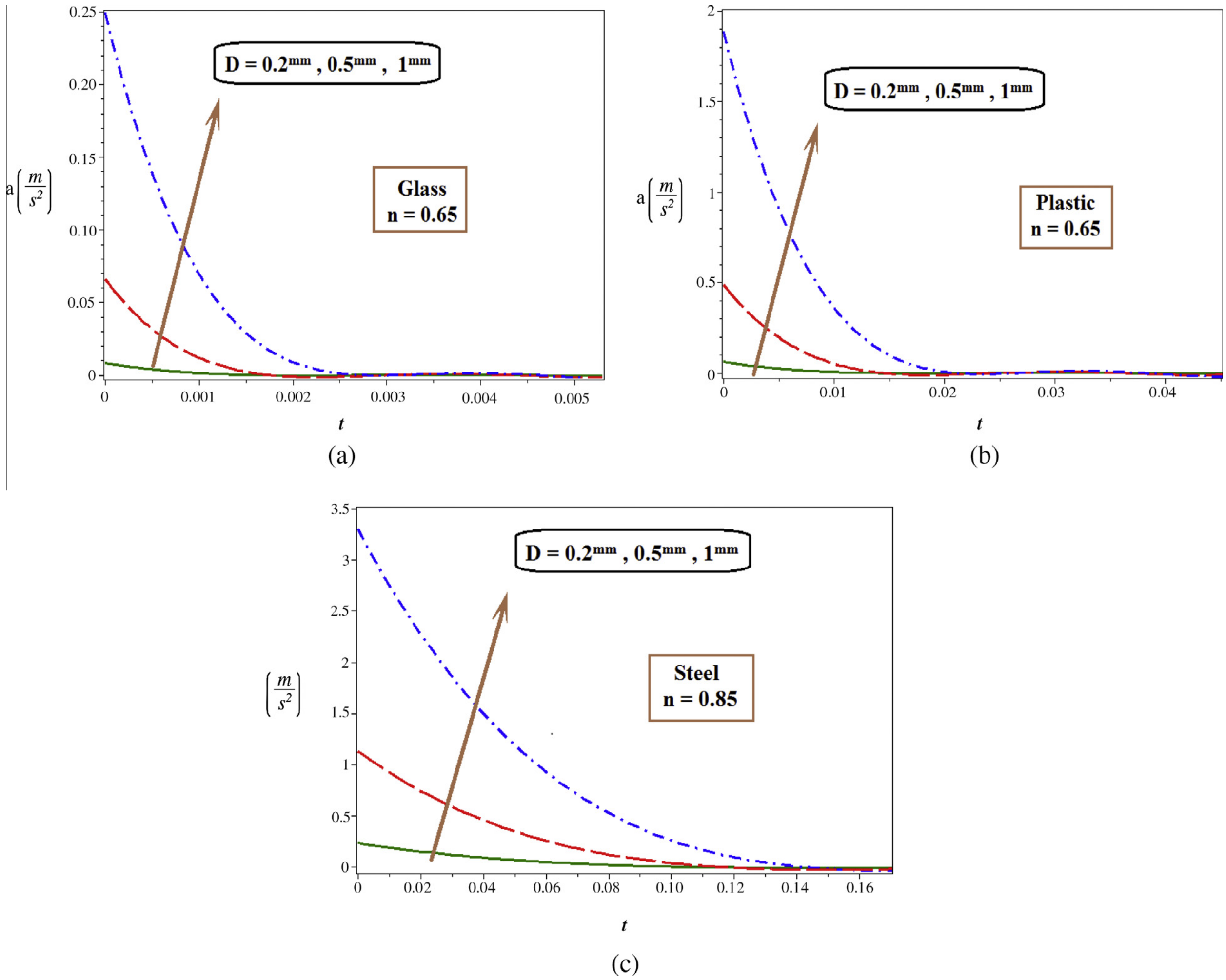


Figure 6 Effects of the particle diameter on the velocity variation.

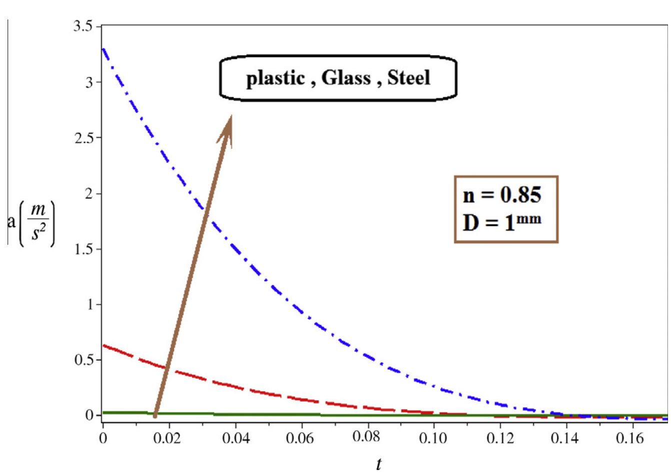


Figure 7 Effects of the particle type on the acceleration variation (n = 0.85, D = 1 mm).

particles. Fig. 7 indicates that initial acceleration time required to reach zero-acceleration state increases by increasing the particle density for a constant particle diameter (D) and flow behavior index (n).

5. Conclusions

The achievement of this work is to apply the CM and NUM in order to study the strongly nonlinear differential equation with fractional power that governed from the unsteady motion of vertically falling spherical particles in a power-law non-Newtonian fluid. The current method was applied without any discretization, restrictive assumptions, or transformation. Also, this method can be used to develop valid solutions even to problems that are highly nonlinear and may be considered as an important and significant refinement of the formerly developed methods. As the results are compared with Numerical Method, it is clear that CM has a good agreement with NUM and provides highly accurate analytical solutions for nonlinear problems and markedly reducing the extent of calculations required. There is a new method named Akbari-Ganji's Method (AGM) that can powerfully solve such complex problem so easily and more accurate but CM is a faster method in order to solve such equations. Moreover, the effects of various materials, sizes and flow behavior indexes on the transient time, terminal velocity and acceleration have been investigated in detail. Analytical results obtained indicated that increasing the particle size and flow behavior index significantly increases the time of reaching the terminal velocity in a falling procedure. However, raising the particle density has the same result in acceleration of the particle. Also by increasing the size and density of the particle it is clear that the time of reaching terminal acceleration in a falling procedure increases.

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