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A degree condition implying that every matching is contained in a hamiltonian cycle

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Abstract

We give a degree sum condition for three independent vertices under which every matching of a graph lies in a hamiltonian cycle. We also show that the bound for the degree sum is almost the best possible.
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1. Introduction

For a graph G, $\mathrm{V}(G)$ denotes its vertex set and $\mathrm{E}(G)$ its edge set. For a vertex x of G, $\mathrm{d}_G(x)$, denotes its degree in G, that is the cardinality of $\mathrm{N}_G(x) = \{y \in \mathrm{V}(G) : xy \in \mathrm{E}(G)\}$, the neighborhood of x in G. The subscript G is omitted when it is clear from the context.

In 1960, Ore [6] proved the following.

Theorem 1. *Let G be a graph on $n \geqslant 3$ vertices. If for any pair of independent vertices $x, y \in \mathrm{V}(G)$ we have*

$$\mathrm{d}(x) + \mathrm{d}(y) \geqslant n, \tag{1}$$

then G is hamiltonian.

Many Ore-type theorems dealing with degree-sum conditions have been proved since. In particular, Bondy [2] showed the following.

Theorem 2. *Let G be a 2-connected graph on $n \geqslant 3$ vertices. If for any independent vertices $x, y, z \in \mathrm{V}(G)$ we have*

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) \geqslant \frac{3n-2}{2},$$

then G is hamiltonian.

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We shall call a set of $k \geqslant 1$ independent edges a *k-matching* and sometimes simply a *matching*. The number of edges in a matching M will occasionally be denoted by $|M|$ and the set of all end vertices of the edges in M will occasionally be denoted by $\mathrm{V}(M)$.

Berman [1] proved the conjecture of Häggkvist [4] about cycles through matchings in general graphs.

Theorem 3. *Let G be a graph on $n \geqslant 3$ vertices. If for any pair of independent vertices $x, y \in \mathrm{V}(G)$ we have*

$$\mathrm{d}(x) + \mathrm{d}(y) \geqslant n + 1,$$

then every matching lies in a cycle.

Theorem 3 has been improved by Jackson and Wormald [5]. Häggkvist [4] also gave a sufficient condition for a general graph to contain any matching in a hamiltonian cycle. We give this theorem below in a slightly improved version obtained by Wojda [8].

Let $\mathcal{G}_n$ be the family of graphs $G = \overline{K}_{\frac{n+2}{3}} * H$, where H is any graph of order $\frac{2n-2}{3}$ containing a perfect matching if $\frac{n+2}{3}$ is an integer, and $\mathcal{G}_n = \emptyset$ otherwise ($*$ denotes the join of graphs).

Theorem 4. *Let G be a graph on $n \geqslant 3$ vertices. If for any pair of independent vertices $x, y \in \mathrm{V}(G)$ we have*

$$\mathrm{d}(x) + \mathrm{d}(y) \geqslant \frac{4n-4}{3},$$

then every matching of G lies in a hamiltonian cycle, unless $G \in \mathcal{G}_n$.

Las Vergnas [7] has a similar result, with the bound for the degree-sum independent of the number of edges of the matching M.

Theorem 5. *Let G be a graph on $n \geqslant 3$ vertices and let k be an integer $0 \leqslant k \leqslant \frac{n}{2}$. If for any pair of independent vertices $x, y \in \mathrm{V}(G)$ we have*

$$\mathrm{d}(x) + \mathrm{d}(y) \geqslant n + k,$$

then every k-matching of G lies in a hamiltonian cycle.

The purpose of this paper is to give new conditions on the degree-sum of three independent vertices under which every matching in a graph G lies in a hamiltonian cycle. First, we state an extension theorem.

Theorem 6. *Let G be a 3-connected graph on $n \geqslant 3$ vertices such that for any independent vertices $x, y, z \in \mathrm{V}(G)$, we have*

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) \geqslant 2n. \tag{2}$$

Let M be a matching in G. If there exists a cycle of G containing M, then there exists a hamiltonian cycle of G containing M.

Theorem 6 shows that if a graph G satisfies (2) and a matching of G lies in a cycle, then this cycle can be extended to a hamiltonian cycle. Using Theorem 6, we prove the following analog of Theorem 2 about hamiltonian cycles through matchings.

Theorem 7. *Let G be a 3-connected graph on $n \geqslant 3$ vertices and let M be a matching in G such that for any independent vertices $x, y, z \in \mathrm{V}(G)$ we have*

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) \geqslant 2n. \tag{3}$$

Then there exists a hamiltonian cycle containing every edge of M or G has a minimal odd M-edge cut-set.

A *minimal odd M-edge cut-set* is a subset of M such that its suppression disconnects the graph G and which has no proper subset being an G-edge cut-set.

Note that the bound $2n$ in Theorem 7 is almost best possible. Let $p \geqslant 2$ and consider a complete graph K_{2p} with a perfect p-matching. We define the graph $G = (p+1)K_1 * K_{2p}$, ($*$ denotes the join of graphs). In this graph,

$n = 3p + 1$ and G is 3-connected. For any independent $x, y, z \in \mathrm{V}(G)$ we have $\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) \geqslant 2n - 2$ and there is no hamiltonian cycle containing the p-matching from K_{2p}. So the bound $2n$ is almost best possible.

Theorem 7 has the following corollary (recall that the *stability number* of a graph G, denoted by $\alpha(G)$ is the cardinality of a maximum independent set of vertices of G).

Corollary 8. *Let G be a 3-connected graph on $n \geqslant 6$ vertices and let M be a matching of G. If $\alpha(G) = 2$, then there is a hamiltonian cycle of G containing M or G has a minimal odd M-edge cut-set.*

2. Notation and preliminary results

Let G be a graph. Let $A \subseteq \mathrm{V}(G)$, $x \in \mathrm{V}(G)$, and define $\mathrm{N}_A(x) = A \cap \mathrm{N}_G(x)$ to be the set of *neighbors of x in A*.

A path or a cycle C in G is usually given as a sequence of vertices from c_0 to c_l such that $c_i c_{i+1} \in \mathrm{E}(G)$ for $i = 0, \ldots, l-1$ (plus the edge $c_l c_0$ if C is a cycle). This induces an orientation on C, say from c_0 to c_l. Thus it makes sense to speak of a *successor* c_{i+1} and a *predecessor* c_{i-1} of a vertex c_i (addition modulo $l+1$). Denote the successor of a vertex x by x^+ and its predecessor by x^-. This notation can be extended to $A^+ = \{x^+ : x \in A\}$, and similarly, to A^- when $A \subseteq \mathrm{V}(G)$.

Let $C = c_0 \ldots c_l$ be a cycle in G with an orientation as above. For any pair of vertices $c_i, c_j \in \mathrm{V}(C)$ we define four intervals (paths) (addition modulo $l+1$). If C is a path, the intervals that make sense are defined similarly.

- $]c_i, c_j[$ is the path $c_i^+ \cdots c_j^-$.
- $[c_i, c_j[$ is the path $c_i c_i^+ \cdots c_j^-$.
- $]c_i, c_j]$ is the path $c_i^+ \cdots c_j^- c_j$.
- $[c_i, c_j]$ is the path $c_i c_i^+ \cdots c_j^- c_j$.

It is useful to define $\epsilon : [V(G)]^2 \longrightarrow \{0, 1\}$ by $\epsilon((u, v)) = 1$ if and only if $uv \in \mathrm{E}(G)$. Of course, we write $\epsilon(uv)$ for $\epsilon((u, v))$ most of the time.

Let W be a property defined for all graphs of order n and let k be a nonnegative integer. The property W is said to be *k-stable* if whenever $G + xy$ has property W and $\mathrm{d}_G(x) + \mathrm{d}_G(y) \geqslant k$ then G itself has property W.

Let $k, s_1, \ldots, s_l$ be positive integers. We call S a *path system of length* k if the components of S are vertex disjoint paths

$$P_1 : \quad x_0^1 x_1^1 \ldots x_{s_1}^1,$$

$$\vdots$$

$$P_l : \quad x_0^l x_1^l \ldots x_{s_l}^l$$

and $\sum_{i=1}^{l} s_i = k$.

Note that a k-matching is a path system of length k with each path of length one.

Bondy and Chvátal [3] proved the following theorem, which we shall need in the proof.

Theorem 9. *Let n and k be positive integers with $k \leqslant n - 3$. Then the property of being k-edge-hamiltonian is $(n + k)$-stable.*

3. Proof of Theorem 6

Let $k = |M|$ and let C be a longest cycle of G containing every edge of M. We assume that C is not hamiltonian. Let $R = \mathrm{V}(G) \setminus \mathrm{V}(C)$ be the set of vertices of G not on C. Let $u \in R$. Since G is 3-connected, we have $P_1[u, a]$, $P_2[u, b]$, $P_3[u, c]$ three internally disjoint paths from u to C, for any distinct $a, b, c \in \mathrm{V}(C)$. Since two consecutive edges of C cannot be in M, there is an orientation of C such that at least at least two edges among aa^+, bb^+, cc^+ are not in M. Without loss of generality we may assume that $aa^+ \notin M$, $bb^+ \notin M$. The three vertices u, a^+, b^+ are independent (since C is the longest cycles containing M), so by the assumption (2) we have

$$\mathrm{d}(u) + \mathrm{d}(a^+) + \mathrm{d}(b^+) \geqslant 2n. \tag{4}$$

From now on the orientation of C is fixed and the vertices on the cycle are implicitly numbered $x_0, \ldots, x_l$ from some arbitrary vertex x_0. This also fixes the intervals on C.

3.1. Neighbors of u, a^+, b^+ in R and C

Since C is the longest cycle, no vertex of R can be adjacent to more than one of a^+, b^+. Thus, since the three vertices are independent, $\mathrm{d}_R(a^+) + \mathrm{d}_R(b^+) + \mathrm{d}_R(u) \leqslant |\mathrm{V}(R)| - 1$.

If a^- is adjacent to u, $a^-a \in M$, otherwise C can be extended, and similarly for b^-. Hence

$$(\mathrm{N}_C(u))^+ \cap [\mathrm{N}_C(a^+) \cup \mathrm{N}_C(b^+)] \subset \{\alpha \in \mathrm{V}(C), \alpha^-\alpha \in M\}$$

and

$$\left|(\mathrm{N}_C(u))^+ \cap [\mathrm{N}_C(a^+) \cup \mathrm{N}_C(b^+)]\right| \leqslant k.$$

As $\left|\mathrm{N}_C(u)^+ \cup \mathrm{N}_C(a^+) \cup \mathrm{N}_C(b^+)\right| \leqslant |\mathrm{V}(C)|$, we have

$$|\mathrm{N}_C(u)| + |\mathrm{N}_C(a^+) \cup \mathrm{N}_C(b^+)| \leqslant |\mathrm{V}(C)| + k.$$

Moreover

$$\left|\mathrm{N}_C(a^+) \cup \mathrm{N}_C(b^+)\right| = \left|\mathrm{N}_C(a^+)\right| + \left|\mathrm{N}_C(b^+)\right| - \left|\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)\right|.$$

To find an upper bound for $\left|\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)\right|$ we shall study vertices of $\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$.

Let $C_1 = C[a, b]$ and $C_2 = C[b, a]$ be the two intervals on the cycle C with endvertices a and b. Note that for any vertex x from the cycle C we have $xx^+ \notin M$ or $x^-x \notin M$.

Let $x \in C_1 \cap \mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$. If $xx^+ \notin M$ and $x^+ \in \mathrm{N}_C(a^+)$, then the cycle

$$P_1[u, a]a^- \cdots b^+xx^- \cdots a^+x^+ \cdots b^- P_2[b, u]$$

is a cycle containing M longer than C, a contradiction.

Hence $x^+ \notin \mathrm{N}_C(a^+)$ and $x^+ \notin \mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$. Similarly if $x^-x \notin M$, then $x^- \notin \mathrm{N}_C(b^+)$ and $x^- \notin \mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$.

Using similar arguments for a vertex $x \in C_2 \cap \mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$, we can show that if $xx^+ \notin M$, then $x^+ \notin \mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$ and if $x^-x \notin M$, then $x^- \notin \mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$.

By removing the edges of the matching M from the cycle C we obtain a sequence of paths P_j such that $\mathrm{V}(C) = \bigcup_j \mathrm{V}(P_j)$.

We have shown that on any path P_j there are no two consecutive vertices from the set $\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$ and thus

$$\left|\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+) \cap \mathrm{V}(P_j)\right| \leqslant \left\lceil \frac{\left|\mathrm{V}(P_j)\right|}{2} \right\rceil.$$

3.2. Relations on degrees of a^+, b^+, u

Recall that P_j are the paths obtained from C by removing the edges of M. For $i \geqslant 2$, let n_i be the number of paths P_j of length $i - 1$. The following relations must be satisfied:

$$k = \sum_{i \geqslant 2} n_i$$

$$|\mathrm{V}(C)| = \sum_{i \geqslant 2} i n_i$$

$$\left|\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)\right| \leqslant \sum_{i \geqslant 2} \left\lceil \frac{i}{2} \right\rceil n_i.$$

As

$$\mathrm{d}_C(a^+) + \mathrm{d}_C(b^+) + \mathrm{d}_C(u) \leqslant |\mathrm{V}(C)| + k + \left|\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)\right|$$

we have

$$\begin{aligned}\mathrm{d}_C(a^+) + \mathrm{d}_C(b^+) + \mathrm{d}_C(u) &\leqslant \sum_{j\geqslant 1}(2jn_{2j} + (2j+1)n_{2j+1}) \\ &\quad + \sum_{j\geqslant 1}(n_{2j} + n_{2j+1}) + \sum_{j\geqslant 1}(jn_{2j} + (j+1)n_{2j+1}) \\ &\leqslant \sum_{j\geqslant 1}(3j+1)n_{2j} + \sum_{j\geqslant 1}(3j+3)n_{2j+1} \\ &\leqslant \sum_{j\geqslant 1}4jn_{2j} + \sum_{j\geqslant 1}(4j+2)n_{2j+1}.\end{aligned}$$

Hence

$$\mathrm{d}_C(a^+) + \mathrm{d}_C(b^+) + \mathrm{d}_C(u) \leqslant 2\,|\mathrm{V}(C)|$$

and

$$\mathrm{d}(a^+) + \mathrm{d}(b^+) + \mathrm{d}(u) \leqslant 2\,|\mathrm{V}(C)| + |\mathrm{V}(R)| - 1 \leqslant 2(|\mathrm{V}(C)| + |\mathrm{V}(R)|) - |\mathrm{V}(R)| - 1 = 2n - |\mathrm{V}(R)| - 1,$$

a contradiction with (4).

This contradiction ends the proof of Theorem 6. □

4. Proof of Theorem 7

Let $k = |M|$.

4.1. Preliminary remarks

Remark 1. For two independent vertices $x, y \in \mathrm{V}(G)$ two cases can occur:

(1) If there exists a vertex z such that x, y, z are independent, then $\mathrm{d}(x) + \mathrm{d}(y) \geqslant 2n - \mathrm{d}(z) \geqslant n + 3$.
(2) If there is no vertex in G independent with x and y, then $\mathrm{N}(x) \cup \mathrm{N}(y) \cup \{x, y\}$ covers $\mathrm{V}(G)$ and $\mathrm{d}(x) + \mathrm{d}(y) \geqslant n - 2$.

Remark 2. If x and y are independent vertices satisfying $\mathrm{d}(x) + \mathrm{d}(y) = n - 2 + \epsilon$, with $0 \leqslant \epsilon \leqslant 3$, then we can assume that $\mathrm{N}(x) \setminus \mathrm{N}(y)$ is a complete graph.

Remark 1 follows from (3).

Proof of Remark 2. Since x and y are independent and $\mathrm{d}(x) + \mathrm{d}(y) = n - 2 + \epsilon$, with $0 \leqslant \epsilon \leqslant 3$, there is no vertex in G independent with x and y. We may assume $\mathrm{d}(y) \leqslant \mathrm{d}(x)$. Note that in this case $\mathrm{d}(y) \leqslant \frac{n-2+\epsilon}{2}$ and if u_1 and u_2 are independent vertices in $\mathrm{N}(x) \setminus \mathrm{N}(y)$, then $\mathrm{d}(u_1) + \mathrm{d}(u_2) \geqslant 2n - \mathrm{d}(y) \geqslant \frac{3n-1}{2} = n + \frac{n-1}{2}$. If n is even, then $\mathrm{d}(u_1) + \mathrm{d}(u_2) \geqslant n + \frac{n}{2} \geqslant n + k$. If n is odd, then any matching of G has at most $\frac{n-1}{2}$ edges, then we have again $\mathrm{d}(u_1) + \mathrm{d}(u_2) \geqslant n + k$. In any case u_1u_2 is in the $(n + k)$-closure of G. From Theorem 9 we can assume that $\mathrm{N}(x) \setminus \mathrm{N}(y)$ is a complete graph. □

We will need the following notion introduced by Berman [1].

Definition 1. A θ-graph through a matching M is the union of two cycles C_1 and C_2 whose intersection is a path of length at least one and such that $M \subset \mathrm{E}(C_1) \cup \mathrm{E}(C_2)$ and every edge of M incident with a vertex of $C_1 \cap C_2$ lies in $C_1 \cap C_2$.

We will prove the theorem by contradiction. We assume that for a matching M there is no hamiltonian cycle containing M and consider a cycle C in G which satisfies the following conditions.

(1) $|\mathrm{E}(C) \cap M|$ is maximum.
(2) Up to condition (1) the length of C is maximum, so by Theorem 6, C is a hamiltonian cycle.

Let $M' = \mathrm{E}(C) \cap M$. By assumption $M' \neq M$ and then there exists an edge $e = xy \in M$, $e \notin \mathrm{E}(C)$. The edge $e = xy$ is a chord of the hamiltonian cycle. Let $C_1 = xx^+ \cdots yx$ and $C_2 = xx^- \cdots yx$. Note that $(C_1 \cup C_2)$ satisfies the definition of a θ-graph through $M' \cup \{e\}$.

Let $\Gamma(C_1, C_2)$ be a θ-graph through $M' \cup \{e\}$ satisfying moreover:

(1) The intersection $C_1 \cap C_2$ is maximum.
(2) Under condition (1) $|\mathrm{V}(\Gamma(C_1, C_2))|$ is maximum.

Define in $\Gamma(C_1, C_2)$, $R' = C_1 \cap C_2 = xr_1r_2 \ldots r_\gamma y$, $R = r_1r_2 \ldots r_\gamma$, $P = C_1 \setminus C_2 = p_1p_2 \ldots p_\alpha$ with $xp_1 \in \mathrm{E}(C_1)$, $Q = C_2 \setminus C_1 = q_1q_2 \ldots q_\beta$ with $xq_1 \in \mathrm{E}(C_2)$. Sometimes we will write Γ instead of $\Gamma(C_1, C_2)$.

Remark 3. From the definition of a θ-graph, the edges $xp_1, xq_1, yp_\alpha, yq_\beta$ are not in M. Hence vertices p_1 and q_β are independent and also q_1 and p_α are independent.

Proof of Remark 3. Suppose that $p_1q_\beta \in \mathrm{E}(G)$, then the cycle $p_1q_\beta q_{\beta-1} \ldots q_1xr_1r_2 \ldots r_\gamma yp_\alpha p_{\alpha-1} \ldots p_1$ is a cycle through $M \cap \mathrm{E}(\Gamma)$, a contradiction. The proof for q_1 and p_α is similar. □

Remark 4. We can use the same arguments as Berman [1] (see inequalities (4)–(12) in [1]) and we have the following inequality:

$$\mathrm{d}(p_1) + \mathrm{d}(q_1) + \mathrm{d}(p_\alpha) + \mathrm{d}(q_\beta) \leqslant 2n. \tag{5}$$

Since the graph G satisfies the condition (3) (i.e. for any independent vertices $w_1, w_2, w_3 \in \mathrm{V}(G)$ we have $\mathrm{d}(w_1) + \mathrm{d}(w_2) + \mathrm{d}(w_3) \geqslant 2n$) and by Remark 1 we have the following inequalities.

$$\mathrm{d}(p_1) + \mathrm{d}(q_\beta) \geqslant n - 2,$$
$$\mathrm{d}(q_1) + \mathrm{d}(p_\alpha) \geqslant n - 2.$$

Hence, from (5) we have

$$\mathrm{d}(q_1) + \mathrm{d}(p_\alpha) \leqslant n + 2,$$
$$\mathrm{d}(p_1) + \mathrm{d}(q_\beta) \leqslant n + 2$$

and from Remark 1 there is no vertex independent of p_1 and q_β and no vertex independent of q_1 and p_α.

Remark 5. From (5), without loss of generality, we may assume that $\mathrm{d}(p_1) + \mathrm{d}(q_\beta) \leqslant n$, $\mathrm{d}(q_\beta) \leqslant \frac{n}{2}$ and so, by Remark 2, $\mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$ is a complete graph.

The following lemmas involve the neighbors of the vertices p_1, q_1, p_α, and q_β on the paths R, P, Q.

Lemma 1. (1) *If uv is an edge of R not in M, then two cases can occur.*
  (a) *Vertices p_1 and q_1 are both adjacent to u and v, and vertices p_α and q_β are independent of u and v, and there is no path internally disjoint with Γ, from u and v to p_α and q_β.*
  (b) *Vertices p_α and q_β are both adjacent to u and v, and vertices p_1 and q_1 are independent of u and v, and even there is no path internally disjoint with Γ, from u or v to p_1 or q_1.*

(2) *Consequently for any $r \in \mathrm{V}(R)$ we have two possibilities.*
  (a) *Vertices p_1 and q_1 are both adjacent to r, and vertices p_α and q_β are independent of r.*
  (b) *Vertices p_α and q_β are both adjacent to r, and vertices p_1 and q_1 are independent of r.*

(3) *If $xr_1 \notin M$, then $r_1p_1, r_1q_1 \in \mathrm{E}(G)$ and $r_1p_\alpha, r_1q_\beta \notin \mathrm{E}(G)$, and if $yr_\gamma \notin M$, then $r_\gamma p_\alpha, r_\gamma q_\beta \in \mathrm{E}(G)$ and $r_\gamma p_1, r_\gamma q_1 \notin \mathrm{E}(G)$.*

Proof of Lemma 1. We shall prove first 1. As $\mathrm{N}(p_1) \cup \mathrm{N}(q_\beta) = \mathrm{V}(G) \setminus \{p_1, q_\beta\}$ and $\mathrm{N}(q_1) \cup \mathrm{N}(p_\alpha) = \mathrm{V}(G) \setminus \{q_1, p_\alpha\}$, the vertex u is adjacent to at least one of the vertices p_1 or q_β. Recall that we prove Theorem 7 and we have supposed that there is no cycle containing every edge of $M \cap \mathrm{E}(\Gamma)$. Suppose that $up_1 \in \mathrm{E}(G)$. Then since there is no cycle through $M \cap \mathrm{E}(\Gamma)$, we have $p_\alpha v \notin \mathrm{E}(G)$ and $q_\beta v \notin \mathrm{E}(G)$. That implies $q_1 v \in \mathrm{E}(G)$ and $p_1 v \in \mathrm{E}(G)$. Hence $q_\beta u \notin \mathrm{E}(G)$ and $p_\alpha v \notin \mathrm{E}(G)$, that implies $q_1 v \in \mathrm{E}(G)$. Suppose now that $up_1 \notin \mathrm{E}(G)$. In this case $q_\beta u \in \mathrm{E}(G)$. That implies $q_1 v \notin \mathrm{E}(G)$ and then $p_\alpha v \in \mathrm{E}(G)$. Hence $q_1 u \notin \mathrm{E}(G)$ and $p_\alpha u \in \mathrm{E}(G)$. From the above $p_1 v \notin \mathrm{E}(G)$

and $q_\beta v \in \mathrm{E}(G)$. Moreover we can replace the condition $wt \notin \mathrm{E}(G)$ by "there is no path from w to t, internally disjoint of Γ, where w may be u or v, and t may be $p_1, p_\alpha, q_1, q_\beta$".

Using similar arguments we can show 2 and 3. □

Note that from Lemma 1, we have $\mathrm{d}_R(p_1) = \mathrm{d}_R(q_1)$ and similarly $\mathrm{d}_R(p_\alpha) = \mathrm{d}_R(q_\beta)$.

Lemma 2. *If $p_i p_{i+1}$ is an edge from $\mathrm{E}(P) \setminus M$, then $q_\beta p_{i+1}, q_1 p_i, q_\beta p_i, q_1 p_{i+1} \notin \mathrm{E}(G)$ and $p_1 p_i, p_1 p_{i+1}, p_\alpha p_i, p_\alpha p_{i+1}$ are edges of G. Similarly, if $q_i q_{i+1}$ is an edge from $\mathrm{E}(Q) \setminus M$, then $p_1 q_i, p_\alpha q_{i+1}, p_1 q_{i+1}, p_\alpha q_i \notin \mathrm{E}(G)$ and $q_1 q_i, q_1 q_{i+1}, q_\beta q_i, q_\beta q_{i+1}$ are edges of G.*

Proof of Lemma 2. We will give a detailed proof showing that if $p_i p_{i+1} \in \mathrm{E}(P)$, then $q_\beta p_i \notin \mathrm{E}(G)$ and $p_1 p_i \in \mathrm{E}(G)$. The proofs for the other vertices are similar.

The hypothesis of maximality of $C_1 \cap C_2$ implies that the edges $q_1 p_i, q_\beta p_{i+1}, p_1 q_i, p_\alpha q_{i+1}$ are not in $\mathrm{E}(G)$. As $\mathrm{N}(p_1) \cup \mathrm{N}(q_\beta) \cup \{p_1, q_\beta\}$ or $\mathrm{N}(q_1) \cup \mathrm{N}(p_\alpha) \cup \{q_1, p_\alpha\}$ cover $\mathrm{V}(G)$ and the edges $p_1 p_{i+1}, p_\alpha p_i, q_1 q_{i+1}, q_\beta q_i$ are in $\mathrm{E}(G)$. If $p_1 p_{i+1} \in \mathrm{E}(G)$, then $q_\beta p_i \notin \mathrm{E}(G)$ since elsewhere

$$x r_1 \cdots r_\gamma y p_\alpha \cdots p_{i+1} p_1 p_2 \cdots p_i q_\beta \cdots q_1 x$$

is a cycle through $M' \cup \{e\}$, a contradiction. Hence $p_1 p_i \in \mathrm{E}(G)$. □

With the preliminary remarks and definitions out of the way, we can proceed with the proof of Theorem 7. We will first study the case where $\alpha = \beta = 2$ and obtain the existence of a minimal odd M-edge cut-set. Then we will assume that $\alpha \geqslant 3$ or $\beta \geqslant 3$ and use the structure of the neighborhood of the vertices $p_1, q_1, p_\alpha, q_\beta$ to obtain a contradiction.

4.2. Proof of Theorem 7 for $\alpha = \beta = 2$

We prove a series of claims. Let $S = G \setminus \Gamma$.

Claim 1. *The vertex p_1 has no neighbor in S.*

Proof of Claim 1. Suppose that $w \in \mathrm{V}(S)$ is adjacent to p_1. Since G is 3-connected, we have a vertex $t \in \mathrm{V}(\Gamma) \setminus \{p_1\}$ and a path $\pi[w, t]$ from w to Γ internally disjoint from Γ. Note that $t \neq q_2$, since elsewhere we obtain a cycle through $M' \cup \{e\}$. Because of the maximality of $|\mathrm{V}(\Gamma)|$, $t \neq x$. For the same reason, $wq_2 \notin \mathrm{E}(G)$ and $wx \notin \mathrm{E}(G)$. If $t = q_1$, then $xq_2 \notin \mathrm{E}(G)$ and thus $x, w \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$. From the above, $wx \in \mathrm{E}(G)$, a contradiction. Note that also $wq_1 \notin \mathrm{E}(G)$, and by Remark 4, $wp_2 \in \mathrm{E}(G)$. By the maximality of $|\mathrm{V}(\Gamma)|$, $t \neq y$. It is possible that $t = p_2$, but in this case, since G is 3-connected, there exists a path, say $\pi[w, r]$ from w to Γ to $r \in \mathrm{V}(R)$, other than the edges wp_1 and wp_2. At least one of the edges rr^+ and r^-r is not in M and either r^+ in the first case or r^- in the second case is adjacent to one of p_1 or p_2. These edges allow us to construct a cycle through $M' \cup \{e\}$, a contradiction. □

Claim 2. *The edge $p_2 q_2$ is in* $\mathrm{E}(G)$.

Proof of Claim 2. *Case* 1: $p_1 q_1 \in \mathrm{E}(G)$ or there exists a path $\pi[p_1, q_1]$ internally disjoint with Γ.

Then $xp_2, xq_2 \notin \mathrm{E}(G)$ elsewhere we obtain a cycle through $M' \cup \{e\}$. The conditions $x \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$, $xp_2 \notin \mathrm{E}(G)$ imply $p_2 \in \mathrm{N}(q_2)$ i.e. $p_2 q_2 \in \mathrm{E}(G)$.

Case 2: $p_1 q_1 \notin \mathrm{E}(G)$ and there exists no path $\pi[p_1, q_1]$ internally disjoint with Γ.

Suppose that $p_2 q_2 \notin \mathrm{E}(G)$. Then $p_2 \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$. We have $\mathrm{N}(p_1) \subset \mathrm{V}(R) \cup \{x, y, p_2\}$.

Let $r \in \mathrm{V}(R)$ be a neighbor of p_1. We have $r, p_2 \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$, that implies $rp_2 \in \mathrm{E}(G)$, a contradiction with Lemma 1. So $\mathrm{N}_R(p_1) = \emptyset$, and $\mathrm{N}(p_1) \subset \{x, y, p_2\}$.

Since G is 3-connected and $\mathrm{N}(p_1) = \{x, y, p_2\}$, the condition $\mathrm{d}(p_1) \geqslant \mathrm{d}(q_2)$ implies that $|V(R)| \leqslant 1$ and so $R = \emptyset$ or $R = \{r_1\}$. If $R = \emptyset$, it is easy to see that if we remove the vertices x and y, the graph is disconnected. Since G is 3-connected, it is a contradiction. Let $R = \{r_1\}$. Note that $xr_1 \notin M$ or $yr_1 \notin M$. If $R = \{r_1\}$ and $xr_1 \notin M$, then $xp_1 p_2 r_1 y q_2 q_1 x$ is a cycle through $M' \cup \{e\}$, a contradiction. If $R = \{r_1\}$ and $xr_1 \notin M$, then $xr_1 p_2 p_1 y q_2 q_1 x$ is a cycle through $M' \cup \{e\}$, a contradiction and Claim 2 is proved. □

Note that we have also the following corollaries from Claim 2.

Corollary 1. *Both pairs of vertices* $\{y, p_1\}$ *and* $\{y, q_1\}$ *are independent and have no common neighbors in* S.

Corollary 2. *If the vertices* $\{y, p_1\}$ *(or* $\{y, q_1\}$*) have no common neighbors on* R*, then* $p_1q_1 \in \mathrm{E}(G)$ *and* y *is adjacent to every neighbor of* p_2 *(or* q_2*) on* R.

Proof of Corollary 2. If there exists a set of three independent vertices containing y and p_1 (or q_1), then $\mathrm{d}(y) + \mathrm{d}(p_1) \geqslant n + 3$. Note that we have $\mathrm{N}(p_1) \cap \mathrm{N}(y) \subset \mathrm{V}(R) \cup \{x, p_2\}$, $|\mathrm{N}_R(p_1) \cap \mathrm{N}_R(y)| \geqslant 3$.

Hence, if $\mathrm{N}_R(p_1) \cap \mathrm{N}_R(y) = \emptyset$, then there is no independent set of three vertices containing p_1 and y, and $p_1q_1 \in \mathrm{E}(G)$. As $\mathrm{N}_R(y) \cup \mathrm{N}_R(p_1) = \mathrm{V}(R)$, by Lemma 1, y is adjacent to every vertex of $\mathrm{N}_R(p_2) = \mathrm{N}_R(q_2)$. □

We can now complete the proof of Theorem 7 for $\alpha = \beta = 2$.

By Lemma 1, the sets $\mathrm{N}_R(p_1) = \mathrm{N}_R(q_1)$ and $\mathrm{N}_R(p_2) = \mathrm{N}_R(q_2)$ define a partition of R and by Remark 2 we may assume that $\mathrm{N}_R(p_1)$ is a complete graph. If an edge $ab \in \mathrm{E}(R)$ is such that a is adjacent to p_1 (and q_1) and b is adjacent to p_2 (and q_2), then, by Lemma 1, $ab \in M$. Let $\{e_j = a_jb_j : a_j \in \mathrm{N}_R(p_1), b_j \in \mathrm{N}_R(p_2)\}$ be the set of these edges. The path R can be partitioned into subpaths: $R_0 = R[x, a_1](= \{x\}$ if $a_1 = x)$, $R_1 = R[b_1 \cdots b_2], \dots R_s = R[b_s, y](= \{y\}$ if $b_s = y)$. Every vertex of $R_0, R_2, \dots, R_{2j} \dots$ is adjacent to p_1 (and q_1), and every vertex of $R_1, R_3, \dots, R_s$ is adjacent to p_2 (and q_2). Note that s is odd. If no other edge exists between $\mathrm{N}(p_1) \cup \{p_1, q_1\}$ and $\mathrm{N}(p_2) \cup \{p_2, q_2\}$, then the set

$$\{e_j = a_jb_j : a_j \in \mathrm{N}_R(p_1), b_j \in \mathrm{N}_R(p_2), 1 \leqslant j \leqslant s\} \cup \{p_1p_2, q_1q_2\}$$

is a minimal odd M-edge cut-set.

Otherwise there exists an edge $cd \in \mathrm{E}(G)$, with $c \in \mathrm{N}(p_1)$, $d \in \mathrm{N}(p_2)$.

Case 1: There is an edge r_ty, with $r_t \in \mathrm{N}_R(p_1)$.

Note that in this case $c = r_t$ and $y = d$. We shall consider two cases $r_tr_{t+1} \notin M$ and $r_tr_{t+1} \in M$. Recall that from Claim 2 $p_2q_2 \in \mathrm{E}(G)$.

Subcase 1.1: $r_tr_{t+1} \notin M$.

By Lemma 1, $r_{t+1}q_1 \in \mathrm{E}(G)$ and $xr_1 \dots r_tyr_\gamma \cdots r_{t+1}q_1q_2p_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 1.2: $r_tr_{t+1} \in M$.

Since $\mathrm{N}_R(p_1)$ and $\mathrm{N}_R(p_2)$ define a partition of R, we have $r_{t+1} \in \mathrm{N}_R(p_1)$ or $r_{t+1} \in \mathrm{N}_R(p_2)$. If $r_{t+1} \in \mathrm{N}_R(p_1)$, then, from Lemma 1, $r_{t-1} \in \mathrm{N}_R(p_1)$, $r_{t+2} \in \mathrm{N}_R(p_1)$ and $r_{t-1}r_{t+2} \in \mathrm{E}(G)$. In this case $xr_1 \dots r_{t-1}r_{t+2\dots}r_\gamma yr_tr_{t+1}q_1q_2p_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction.

If $r_{t+1} \in \mathrm{N}_R(p_2)$, then, by Lemma 1, $r_{t+2} \in \mathrm{N}_R(p_2)$. Note that, since $r_tr_{t+1} \in M$, we have $r_{t-1}t, t_{t+1}r_{t+2} \notin M$. Hence $xr_1 \dots r_{t-1}p_1p_2r_{t+2} \dots r_\gamma yr_tr_{t+1}q_2q_1x$ is a cycle through $M' \cup \{e\}$, a contradiction.

Case 2: The vertex y is not adjacent to any vertex of $\mathrm{N}_R(p_1)$.

By Corollary 2, y is adjacent to any vertex of $\mathrm{N}_R(p_2)$. Let $r_t \in \mathrm{N}_R(p_1)$, $r_m \in \mathrm{N}_R(p_2)$ such that $r_tr_m \in \mathrm{E}(G)$.

Subcase 2.1: $r_tr_{t+1}, r_mr_{m+1} \notin M$ or $r_{t-1}r_t, r_{m-1}r_m \notin M$.

If $t < m$ and $r_tr_{t+1}, r_mr_{m+1} \notin M$, then, from Lemma 1, $q_1r_{t+1}, q_2r_{m+1} \in \mathrm{E}(G)$ and, hence, $xr_1 \dots r_tr_mr_{m-1} \dots r_{t+1}q_1q_2r_{m+1} \dots yp_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction. If $t < m$ and $r_{t-1}r_t, r_{m-1}r_m \notin M$, then, from Lemma 1, $r_{t-1}q_1, r_{m-1}p_2 \in \mathrm{E}(G)$ and, hence, $xr_1 \dots r_{t-1}q_1q_2yr_\gamma \dots r_mr_tr_{t+1} \dots r_{m-1}p_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction.

If $t > m$ and $r_tr_{t+1}, r_mr_{m+1} \notin M$, then, from Lemma 1, $r_{m+1}q_2, q_1r_{t+1} \in \mathrm{E}(G)$ and, hence, $xr_1 \dots r_mr_tr_{t-1} \dots r_{m+1}q_2q_1r_{t+1} \dots r_\gamma yp_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction. If $t > m$ and $r_{t-1}r_t, r_{m-1}r_m \notin M$, then, from Lemma 1, $r_{m-1}q_2, q_1r_{t-1} \in \mathrm{E}(G)$ and, hence, $xr_1 \dots r_{m-1}q_2q_1r_{t-1} \dots r_mr_t \dots yp_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 2.2: $r_tr_{t+1} \in M$ and $r_{m-1}r_m \in M$ if $t < m$, $r_{t-1}r_t \in M$ and $r_mr_{m+1} \in M$ if $t > m$.

There exists i, i between t and m, such that $r_ir_{i+1} \notin M$. The vertices r_i and r_{i+1} are both adjacent to p_1 and q_1 or to p_2 and q_2.

Subcase 2.2.1: The vertices r_i and r_{i+1} are both adjacent to p_1 and q_1.

If $t < m$, then since $r_{t-1}, r_{i+1} \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$, we have $r_{t-1}r_{i+1} \in \mathrm{E}(G)$ and since $r_m \in \mathrm{N}_R(p_2)$ from Lemma 1 $q_2r_{m+1} \in \mathrm{E}(G)$. Hence $xr_1 \dots r_{t-1}r_{i+1} \dots r_mr_tr_{t+1} \dots r_iq_1q_2r_{m+1} \dots yp_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction. If $t > m$, then $r_{t+1}, r_{i+1} \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$, we have $r_{t+1}r_{i+1} \in \mathrm{E}(G)$ and since $r_m \in \mathrm{N}_R(p_2)$ from

Lemma 1 $q_2r_{m-1} \in E(G)$. Hence $xr_1 \dots r_{m-1}q_2q_1r_ir_{i-1} \dots r_mr_t \dots r_{i+1}r_{t+1} \dots yp_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 2.2.2: The vertices r_i and r_{i+1} are both adjacent to p_2 and q_2.

In this case, from Corollary 2, r_i and r_{i+1} are adjacent to y. If $t < m$, then, from Lemma 1, $r_{t-1}q_1 \in E(G)$ and $xr_1 \dots r_{t-1}q_1q_2r_ir_{i-1} \dots r_tr_mr_{m-1} \dots r_{i+1}yr_\gamma \dots r_{m+1}p_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction. If $t > m$, then from Lemma 1 $r_{m-1}p_2$, $p_1r_{t+1} \in E(G)$ and $xr_1 \dots r_{m-1}p_2p_1r_{t+1} \dots yr_{i+1} \dots r_tr_m \dots r_iq_2q_1x$ is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 2.3: $r_{t-1}r_t \in M$ and $r_mr_{m+1} \in M$ if $t < m$ or $r_tr_{t+1} \in M$ and $r_{m-1}r_m \in M$ if $t > m$.

Recall that from Claim 2, $q_2p_2 \in E(G)$. From Corollary 2, if $t < m$, then $r_{m-1}y \in E(G)$ and if $t > m$, then $r_{m+1}y \in E(G)$. Hence if $t < m$, then $xr_1 \dots r_tr_m \dots yr_{m-1} \dots r_{t+1}q_1q_2p_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction. If $t > m$, $xr_1 \dots r_mr_tr_{t+1} \dots yr_{m+1} \dots r_{t-1}q_1q_2p_2p_1x$ is a cycle through $M' \cup \{e\}$, a contradiction.

This completes the proof of Theorem 7 for $\alpha = \beta = 2$. □

4.3. *Proof of Theorem 7 for $\alpha \geqslant 3$ or $\beta \geqslant 3$*

Case 1: $p_1q_1 \in E(G)$.

Remark 6. The hypothesis of maximality of the intersection $C_1 \cap C_2$ implies that the edges p_1p_2 and q_1q_2 are in M.

Remark 7. Since there is no cycle through $M' \cup \{e\}$ we have $xq_\beta \notin E(G)$, $xp_\alpha \notin E(G)$ and there is no path $\pi[x, q_\beta]$ or $\pi[x, p_\alpha]$ internally disjoint of Γ.

Remark 8. Since $x \in N(p_1) \setminus N(q_\beta)$, $p_\alpha \in N(p_1) \cup N(q_\beta)$ and $xp_\alpha \notin E(G)$, $p_\alpha \notin N(p_1) \setminus N(q_\beta)$, we have $p_\alpha q_\beta \in E(G)$, that implies $p_\alpha p_{\alpha-1}$, $q_\beta q_{\beta-1} \in M$ and yp_1, $yq_1 \notin E(G)$.

Remark 9. If $w \in N_S(p_1)$ and $w \notin N(q_\beta)$, then $w \in N(p_1) \setminus N(q_\beta)$, that implies $wx \in E(G)$, a contradiction with the hypothesis of maximality of $|V(\Gamma)|$. Hence $N_S(p_1) = \emptyset$.

By Lemma 2 and the property that p_1p_2, q_1q_2, $p_\alpha p_{\alpha-1}$, $q_{\beta-1}q_\beta$ are in M, we deduce the following lemma.

Lemma 3. (1) *The vertex p_1 is independent of $q_2, , \dots, q_\beta$ and adjacent to $p_2, \dots, p_{\alpha-1}$.*
(2) *The vertex q_1 is independent of $p_2, \dots, p_\alpha$ and adjacent to $q_2, \dots, q_{\beta-1}$.*
(3) *The vertex p_α is independent of $q_1, \dots, q_{\beta-1}$ and adjacent to $p_2, \dots, p_{\alpha-1}$.*
(4) *The vertex q_β is independent of $p_1, \dots, p_{\alpha-1}$ and adjacent to $q_2, \dots, q_{\beta-1}$.*

We recall that we consider the case $\alpha \geqslant 3$ or $\beta \geqslant 3$.

Subcase 1.1: $\alpha \geqslant 3$.

By Lemma 2, $p_{\alpha-1} \in N(p_1) \setminus N(q_\beta)$. As $x \in N(p_1) \setminus N(q_\beta)$, $xp_{\alpha-1} \in E(G)$, the edges p_1p_2 and $p_{\alpha-1}p_\alpha$ are in M and the condition $\alpha > 2$ implies $\alpha \geqslant 4$. By Lemma 3, $p_{\alpha-2}p_\alpha \in E(G)$, and then $xr_1 \dots r_\gamma yq_\beta \dots q_1p_1p_2 \dots p_{\alpha-2}p_\alpha p_{\alpha-1}x$ is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 1.2: $\alpha = 2$ and $\beta \geqslant 3$.

The vertex p_2 is a common neighbor of p_1 and q_β, thus $d(p_1)+d(q_\beta) \geqslant n-1$ and that implies $d(q_1)+d(p_2) \leqslant n+1$ and $\min\{d(q_1), d(p_2)\} \leqslant \frac{n+1}{2}$.

When $d(q_1) \geqslant d(p_2)$, the $(n+k)$-closure of $N(q_1) \setminus N(p_2)$ is a complete graph. If it is not, then $d(p_2) > d(q_1)$. We shall examine both cases.

Subcase 1.2.1: $d(q_1) \geqslant d(p_2)$.

As observed above, $N(q_1) \setminus N(p_2)$ induces a complete graph. As $\beta \geqslant 3$, $q_2q_3 \notin M$, $q_3 \in N(q_1) \setminus N(p_2)$ and $xq_3 \in E(G)$. Then $x \dots yp_2p_1q_1q_2q_\beta \cdots q_3x$ is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 1.2.2: $d(p_2) > d(q_1)$.

The following inequalities are satisfied: $d(p_1) \geqslant d(q_\beta)$, $d(p_2) \geqslant d(q_1)$, $d(p_2) + d(q_1) \geqslant d(p_1) + d(q_\beta)$.

They imply that $d(p_1)+d(p_2) \geqslant n-1$. We have $N(p_1) = N_R(p_1) \cup \{p_2, x, q_1\}$ and $N(p_2) = N_R(p_2) \cup \{p_1, y, q_\beta\} \cup N_S(p_1)$. By Lemma 1, $d_R(p_1) + d_R(p_2) = |V(R)| = \gamma$. $d(p_1) + d(p_2) = d_R(p_1) + d_R(p_2) + 6 + d_S(p_1) + d_S(p_2) \leqslant$

$\gamma+6+|\mathrm{V}(S)|$. Since $n=\gamma+\beta+4+|\mathrm{V}(S)|$, we obtain $n-1=\gamma+\beta+4+|\mathrm{V}(S)|-1 \leqslant \mathrm{d}(p_1)+\mathrm{d}(p_2) \leqslant \gamma+6+|\mathrm{V}(S)|$ and this implies $\beta \leqslant 3$. We have $q_1q_2 \in M$ and $q_{\beta-1}q_\beta \in M$, then if $\beta \leqslant 3$, $q_1q_2 = q_{\beta-1}q_\beta$ and $\beta = 2$, a contradiction.

Case 2: $p_1q_1 \notin \mathrm{E}(G)$.

Lemma 4. (1) *The vertex p_1 is independent of $q_1, q_2, \dots, q_\beta$ and adjacent to $p_2, \dots, p_\alpha$.*
(2) *The vertex q_1 is independent of $p_1, \dots, p_\alpha$ and adjacent to $q_2, \dots, q_\beta$.*
(3) *The vertex p_α is independent of $q_1, \dots, q_{\beta-1}$ and adjacent to $p_1, \dots, p_{\alpha-1}$.*
(4) *The vertex q_β is independent of $p_1, \dots, p_{\alpha-1}$ and adjacent to $q_1, \dots, q_{\beta-1}$.*

Proof of Lemma 4. The condition $q_1 \notin \mathrm{N}(p_1)$ implies that $q_1 \in \mathrm{N}(q_\beta)$, the condition $p_1 \notin \mathrm{N}(q_1)$ implies that $p_1 \in \mathrm{N}(p_\alpha)$ i.e., the edges p_1p_α and q_1q_β are in $\mathrm{E}(G)$. Let i be a minimal integer such that $p_1q_i \in \mathrm{E}(G)$. For $1 \leqslant j \leqslant i-1$, $p_1q_j \notin \mathrm{E}(G)$, and so $q_\beta q_j \in \mathrm{E}(G)$. The hypothesis of maximality of $C_1 \cap C_2$ implies that $q_iq_{i+1} \in M$ and then $q_{i-1}q_i \notin M$. The cycle $xr_1 \dots r_\gamma yp_\alpha \dots p_1q_i \dots q_\beta q_{i-1} \dots q_1x$ is a cycle through $M' \cup \{e\}$, a contradiction. The vertex p_1 is independent of $q_1, q_2, \dots, q_\beta$, and hence q_β is adjacent to $q_1, q_2, \dots, q_{\beta-1}$.

The proofs for the other vertices are similar. □

Subcase 2.1: $p_\alpha q_\beta \notin \mathrm{E}(G)$.

Claim 3. *If $p_\alpha q_\beta \notin \mathrm{E}(G)$, then $\mathrm{N}_R(p_1) = \mathrm{N}_R(q_1) = \emptyset$.*

Proof of Claim 3. If $p_\alpha \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$ and $u \in \mathrm{N}_R(p_1)$, then $uq_\beta \notin \mathrm{E}(G)$, $u \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$ and hence $u \in \mathrm{N}_R(p_1) \cap \mathrm{N}_R(p_\alpha)$, a contradiction with Lemma 1. □

Claim 4. *At least one of the edges xp_α or xq_β is in* $\mathrm{E}(G)$.

Proof of Claim 4. If $x \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$, x is adjacent to every vertex of $\mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$, then $xp_\alpha \in \mathrm{E}(G)$. □

Corollary 3. $\mathrm{N}_S(p_1) \cap \mathrm{N}_S(q_1) = \emptyset$.

Claim 5. *At least one of the edges yp_1 or yq_1 is in* $\mathrm{E}(G)$.

Proof of Claim 5. Vertices p_1 and q_1 have no common neighbor in S. The following inequality is satisfied:

$$\mathrm{d}(p_1) + \mathrm{d}(q_1) \leqslant \alpha + \beta + |\mathrm{V}(S)| + \epsilon(yp_1) + \epsilon(yq_1)$$

and since $n = \alpha + \beta + \gamma + 2 + |\mathrm{V}(S)|$ we have

$$\mathrm{d}(p_1) + \mathrm{d}(q_1) \leqslant n.$$

The vertices p_1 and q_1 are not in any set of three independent vertices and so Claim 5 is proved. □

Subsubcase 2.1.1: $\gamma = |\mathrm{V}(R)| = 0$.

In this case $xy \in M$. As G is 3-connected, $G \setminus \{x, y\}$ is connected. The conditions $\epsilon(xp_\alpha) + \epsilon(xq_\beta) \geqslant 1$, $\epsilon(yp_1) + \epsilon(yq_1) \geqslant 1$ imply that there is no path $\pi[p_1, q_1], \pi[p_1, q_\beta], \pi[p_\alpha, q_1], \pi[p_\alpha, q_\beta]$ otherwise there is a cycle through $M' \cup \{e\}$. As G is 3-connected, there exists a path $\pi[p_i, q_j]$, with $2 \leqslant i \leqslant \alpha - 1$, $2 \leqslant j \leqslant \beta - 1$. We can easily construct a cycle through $M' \cup \{e\}$.

Subcase 2.1.2: $\gamma \geqslant 1$, $\mathrm{d}(q_1) \geqslant \mathrm{d}(p_\alpha)$.

By Claims 4 and 5, $\epsilon(xp_\alpha) + \epsilon(xq_\beta) \geqslant 1$ and $\epsilon(yp_1) + \epsilon(yq_1) \geqslant 1$. Hence we have $\mathrm{d}(p_1) + \mathrm{d}(q_1) + \mathrm{d}(p_\alpha) + \mathrm{d}(q_\beta) \geqslant 2n - 2$ and this implies $\mathrm{d}(p_1) + \mathrm{d}(q_1) \geqslant n - 1$. We have $\mathrm{N}(p_1) \subset \{x, y\} \cup \{p_2, \dots, p_\alpha\} \cup S$, $\mathrm{N}(q_1) \subset \{x, y\} \cup \{q_2, \dots, q_\beta\} \cup S$, $\mathrm{N}_S(p_1) \cap \mathrm{N}_S(q_1) = \emptyset$ and so $\mathrm{d}(p_1) + \mathrm{d}(q_1) \leqslant \alpha + \beta + |\mathrm{V}(S)| + \epsilon(yp_1) + \epsilon(yq_1)$. Moreover $n = \alpha + \beta + \gamma + 2 + |\mathrm{V}(S)|$.

The inequality $\mathrm{d}(p_1) + \mathrm{d}(q_1) \geqslant n - 1$ gives $\gamma + 1 \leqslant \epsilon(yp_1) + \epsilon(yq_1)$. Hence $\gamma = 1 = \epsilon(yp_1) = \epsilon(yq_1)$. If $xr_1 \in M$, then $xr_1q_\beta \dots q_1yp_\alpha \dots p_1x$ is a cycle through $M' \cup \{e\}$. If $r_1y \in M$, then $xp_1 \dots p_\alpha r_1yq_\beta \dots q_1x$ is a cycle through $M' \cup \{e\}$. In both cases we have a contradiction.

Subcase 2.1.3: $\gamma \geqslant 1$, $\mathrm{d}(q_1) < \mathrm{d}(p_\alpha)$.

Note that $d(p_\alpha)+d(q_1) = \alpha+\beta+\gamma+|V(S)|+\epsilon(yq_1)+\epsilon(xp_\alpha) \leqslant n$. Hence the $(n+k)$-closure of $N(p_\alpha)\setminus N(q_1)$ is a complete graph. Let $u \in N_R(p_\alpha)$, $u \in N(p_\alpha)\setminus N(q_1)$ and $p_1 \in N(p_\alpha)\setminus N(q_1)$. This implies $up_1 \in E(G)$, a contradiction with Lemma 1. Hence $N_R(p_1) = N_R(p_\alpha) = \emptyset$, $\gamma = 0$, a contradiction with the hypothesis of Subcase 2.1.3.

Subcase 2.2: $p_\alpha q_\beta \in E(G)$.

Claim 6. *If $p_\alpha q_\beta \in E(G)$, then $yp_1 \notin E(G)$ and $yq_1 \notin E(G)$.*

By Claim 6 $d(p_1) + d(q_\beta) = \alpha + \beta + \gamma + \epsilon(xq_\beta) + 1 + |V(S)| = n - 1 + \epsilon(xq_\beta) \leqslant n$. $d(q_1) + d(p_\alpha) = \alpha + \beta + \gamma + \epsilon(xp_\alpha) + 1 + |V(S)| = n - 1 + \epsilon(xp_\alpha) \leqslant n$.

Subcase 2.2.1: $d(q_1) \geqslant d(p_\alpha)$.

The $(n+k)$-closure of $N(q_1)\setminus N(p_\alpha)$ is a complete graph, so we may assume that $N(q_1)\setminus N(p_\alpha)$ is complete. Vertices p_1, q_1, y are independent and thus $d(p_1) + d(q_1) \geqslant n + 3$. Recall that $d_R(q_1) = d_R(p_1)$.

The two equalities:

$$d(p_1) = \alpha + d_R(p_1) + d_S(p_1)$$
$$d(q_1) = \beta + d_R(q_1) + d_S(q_1)$$

imply that

$$\alpha + \beta + 2d_R(p_1) + d_S(p_1) + d_S(q_1) \geqslant n + 3.$$

If $xp_\alpha \in E(G)$ or $xq_\beta \in E(G)$, then $N_S(p_1) \cap N_S(q_1) = \emptyset$. If $xp_\alpha \notin E(G)$ and $xq_\beta \notin E(G)$, then $x \in N(p_1)\setminus N(q_\beta)$; if $w \in N_S(p_1)$, then $w \in N(p_1)\setminus N(q_\beta)$ and $xw \in E(G)$, a contradiction with the hypothesis of maximality of $|V(\Gamma)|$. Hence $d_S(p_1)+d_S(q_1) \leqslant |V(S)|$. Note that $n+3 \leqslant d(p_1)+d(q_1) \leqslant \alpha+\beta+|V(S)|+2d_R(p_1)$. This implies that $\alpha+\beta+\gamma+|V(S)|+5 \leqslant \alpha+\beta+|V(S)|+2d_R(p_1)$. Since $2d_R(p_1) \geqslant \gamma+5$, we have $d_R(p_1) \geqslant 5$. If $\alpha > 2$, then $p_{\alpha-1}p_\alpha \in M$ and $p_{\alpha-2}p_{\alpha-1} \notin M$. Let $r_i r_{i+1}$ be an edge of R not in M, with r_i and r_{i+1} adjacent to p_1. Vertices r_i and r_{i+1} are adjacent to $p_{\alpha-1}$ and $p_{\alpha-2}$. Hence $xr_1 \ldots r_i p_{\alpha-2} \ldots p_1 p_\alpha p_{\alpha-1} r_{i+1} \ldots r_\gamma y q_\beta \ldots q_1 x$ is a cycle through $M' \cup \{e\}$, a contradiction.

If $\beta > 2$, the argument is similar.

Subcase 2.2.2: $d(p_\alpha) > d(q_1)$.

If $p_1, y \in N(p_\alpha)\setminus N(q_1)$, then $yp_1 \in E(G)$, a contradiction with Claim 6.

The proof of Theorem 7 is complete. □

References

[1] K.A. Berman, Proof of a conjecture of Häggkvist on cycles and independent edges, Discrete Math. 46 (1983) 9–13.
[2] J.A. Bondy, Longest paths and cycles in graphs of high degree, Research Raport CORR, 1980, pp. 80–16.
[3] J.A. Bondy, V. Chvátal, A method in graph theory, Discrete Math. 15 (1976) 111–135.
[4] R. Häggkvist, On F-hamiltonian graphs, in: J.A. Bondy, U.S.R. Murty (Eds.), Graph Theory and Related Topics, Academic Press, New York, 1979, pp. 219–231.
[5] B. Jackson, N.C. Wormald, Cycles containing matchings and pairwise compatible Euler tours, J. Graph Theory 14 (1990) 127–138.
[6] O. Ore, Note on hamiltonian circuits, Amer. Math. Monthly 67 (1960) 55.
[7] M.Las Vergnas, Problèmes de couplages et problemes hamiltoniens en théorie des graphes, Ph.D. Thesis, Université Paris XI, 1972.
[8] A.P. Wojda, Hamiltonian cycles through matchings, Demonstratio Math. XXI (2) (1983) 547–553.