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Straight-ahead walks in Eulerian graphs

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Abstract

A *straight-ahead walk* in an embedded Eulerian graph G always passes from an edge to the opposite edge in the rotation at the same vertex. A straight-ahead walk is called *Eulerian* if all the edges of the embedded graph G are traversed in this way starting from an arbitrary edge. An embedding that contains an Eulerian straight-ahead walk is called an *Eulerian embedding.* In this article, we characterize some properties of Eulerian embeddings of graphs and of embeddings of graphs such that the corresponding medial graph is Eulerian embedded. We prove that in the case of 4-valent planar graphs, the number of straight-ahead walks does not depend on the actual embedding in the plane. Finally, we show that the minimal genus over Eulerian embeddings of a graph can be quite close to the minimal genus over all embeddings.

Keywords: Straight-ahead walk; Eulerian embedding; Medial graph; Genus of a graph

1. Introduction

Given an Eulerian graph, any matching of edges at each vertex results in a circuit decomposition of the graph. Since there are so many matchings, it would be nice to look at matchings that arise in some natural way or are connected to other properties of the graph. Embeddings of the graph provide an interesting source of matchings. The purpose of this paper is to study the relationship between the embeddings of an Eulerian graph and the circuit decomposition of the graph induced by the embedding by a "straight-ahead" matching. In the other direction, we also show that an Eulerian circuit in a graph can be used to construct interesting embeddings of the graph.

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A *straight-ahead walk* or a SAW in the embedded Eulerian graph G always passes from an edge to the opposite edge adjacent to the same vertex; two edges are "opposite" at a vertex of valence $2d$ in an embedded graph if they are d edges apart in the cyclic ordering (*rotation*) of the edges at that vertex induced by the embedding.

In this paper we assume the graphs to be finite and connected and the embeddings to be 2-cell. Let us now introduce some terminology and notation. A *circuit* is a closed walk with no repeated edges. The straight-ahead walks, the SAWs, of an embedded Eulerian graph G induce a circuit partition of the edges. Let us denote by $s(G \to S)$ the number of components of an SAW decomposition of G. Notice that this number depends not only on the surface S but also on the given embedding in that surface; it is not hard, for example, to give two embeddings of K_5 in the torus, such that one embedding has two SAWs and the other has three. An embedding of an Eulerian graph G in a surface S is *Eulerian*, if it contains exactly one SAW, i.e., $s(G \to S) = 1$. The *medial graph* of an embedded graph G, $\mathrm{Me}(G)$, is a graph, embedded in the same surface as G and is obtained from G as follows: the vertices of $\mathrm{Me}(G)$ are the edges of G and two vertices of $\mathrm{Me}(G)$ are adjacent if they are adjacent edges in the rotation of a vertex in G. The medial graph of a graph G can have multiple edges (if G has vertices of degree 2 or double edges, which bound a face) and loops (if G has vertices of degree 1 or loops, that bound a face). Note that embedded graphs, which are dual to each other, have the same medial graphs. The medial graph of any graph is 4-valent and thus Eulerian. An embedded graph is *Eulerian medial embedded* if its medial is Eulerian embedded.

Eulerian embeddings of 4-valent graphs in the plane are just knot projections (without a specification of which parts of the knot are over or under other parts) and hence are related to Gauss's coding of knot projections (see [7]). An Eulerian embedding of a 4-valent graph in a surface of genus g can be viewed as a knot projection on a genus g Heegard splitting surface for a closed 3-manifold. Unfortunately, the Reidermeister moves for such knot projections include moves across solid handles of the splitting and make knot theory, say, for knot polynomials, too complicated. Planar Eulerian graphs are discussed in [4]. Works of Bouchet and others [1–3,6,11] are also related to this paper.

2. Counting SAWs in graphs and medial graphs: some examples

In this section, we give some examples of Eulerian embedded plane graphs and of plane graphs whose medial graph is Eulerian embedded. The most obvious examples of Eulerian embedded graphs are cycles C_n. The medial graphs of odd cycles, which are odd cycles with double edges, are also Eulerian embedded. There exist less trivial infinite families of plane graphs, whose medial graphs are Eulerian embedded, too. For $n \geqslant 3$, the *wheel* of n spokes W_n is the graph obtained from the cycle C_n by adding a new vertex and joining it to all vertices of C_n. It can be embedded in the plane in the obvious way. The medial graph of the wheel W_n embedded in the plane is the so-called *antiprism* A_n, embedded in the plane, which can be described as follows: A_n consists of two cycles on vertices $v_0 = v_n, v_1, \ldots, v_{n-1}$ and $u_0 = u_n, u_1, \ldots, u_{n-1}$, where

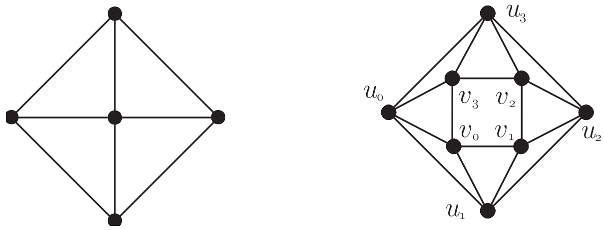

Fig. 1. The wheel W_4 and its medial graph—the antiprism A_4.

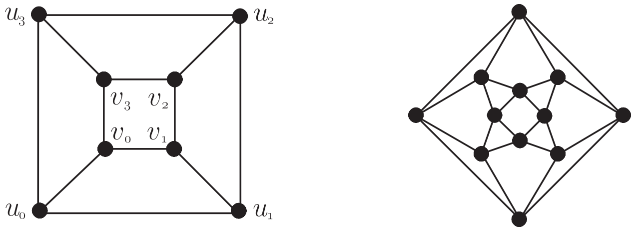

Fig. 2. The prism Π_4 and its medial graph.

u_i is connected by an edge to v_{i-1} and v_i, $i = 1,\ldots,n$. The wheel W_4 and its medial A_4 are shown in Fig. 1. The *prism* Π_n is a graph which consists of two cycles on vertices $v_0, v_1,\ldots, v_{n-1}$ and $u_0, u_1,\ldots, u_{n-1}$, where u_i is connected by an edge to v_i, $i = 0,\ldots,n-1$. The prisms can also be embedded in the plane. The prism Π_4 and its medial are shown in Fig. 2. It is easy to see, that the medial of the wheel graph, the antiprism in Fig. 1, is Eulerian embedded. We used the computer system VEGA, see [9], to verify whether this property holds for all the wheels. We also checked the number of SAWs in medial graphs of prisms Π_n and antiprisms A_n. The results gave us the following theorem:

Theorem 1.

$$s(A_n \to \text{Sphere}) = \begin{cases} 3 & n = 3k, \\ 1 & n \neq 3k. \end{cases}$$

$$s(\text{Me}(\Pi_n) \to \text{Sphere}) = \begin{cases} 1 & n = 2k+1, \\ 4 & n = 4k, \\ 2 & n = 4k+2. \end{cases}$$

$$s(\text{Me}(A_n) \to \text{Sphere}) = \begin{cases} 4 & n = 3k, \\ 2 & n \neq 3k. \end{cases}$$

Proof. We only prove the first part of the theorem; the other two parts are similar and are left to the reader.

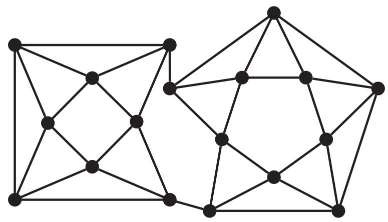

Fig. 3. The connected sum of the antiprisms A_4 and A_5.

Let $u_0, u_1, \ldots, u_{n-1}$ and $v_0, v_1, \ldots, v_{n-1}$ denote the vertices of the antiprism A_n and the indices will be taken by modulo n. Since A_n is a simple graph, the rotation at each vertex can be given by its neighbors. In the planar embedding of A_n, the cyclic order of vertices around u_i is $u_{i-1}, u_{i+1}, v_i, v_{i-1}$ and around v_i, the cyclic order of neighbors is $v_{i+1}, v_{i-1}, u_i, u_{i+1}$, $i = 1, \ldots, n$. The SAWs consist of walks w_i with length 4, given by sequences of vertices $u_i, v_i, v_{i+1}, u_{i+2}, u_{i+3}$. If $n = 3k+1$, we get one straight-ahead walk with sequence of vertices $w_0 w_3 w_6 \cdots w_{n-1} w_2\ w_5 \cdots w_{n-2} w_1 w_4 \cdots w_{n-3}$ and similarly for $n = 3k+2$, we get one SAW $w_0 w_3 w_6 \cdots w_{n-2} w_1 w_4 \cdots w_{n-1} w_2 w_5 \cdots w_{n-3}$. For $n = 3k$, we obviously get 3 SAWs $w_0 w_3 \cdots w_{n-3}$, $w_1 w_4 \cdots w_{n-2}$ and $w_2 w_5 \cdots w_{n-1}$. $\square$

Let G_1 and G_2 be graphs, 2-cell embedded in orientable surfaces S_{k_1} and S_{k_2}, respectively, where S_k denotes the sphere with $k \geqslant 0$ handles. Let (u_1, v_1) be an edge in G_1 and (u_2, v_2) be an edge in G_2. If these edges are not both bridges, we can define the *connected sum* $G_1 \# G_2$ of graphs G_1 and G_2 with respect to the (directed) edges (u_1, v_1) and (u_2, v_2) as follows: take the union of graphs G_1 and G_2 and substitute the edges (u_1, v_1) and (u_2, v_2) by the edges (u_1, u_2) and (v_1, v_2). The rotation scheme is inherited from the embeddings of G_1 and G_2, except for the vertices u_1, u_2, v_1 and v_2. In the rotation around u_1, v_1 is substituted by u_2, in the rotation around v_1, u_1 is substituted by v_2, and in the rotation around u_2, v_2 is substituted by u_1, in the rotation around v_2, u_2 is substituted by v_1. The connected sum of G_1 and G_2 is therefore a connected graph, and if at least one of the edges lies on the boundary of two different faces, the graph $G_1 \# G_2$ is 2-cell embedded in the surface $S_{k_1+k_2}$. The following theorem is very useful for constructing infinite families of Eulerian embedded graphs:

Theorem 2. *Let $G = G_1 \# G_2$. Then $s(G \to S) = s(G_1 \to S_{k_1}) + s(G_2 \to S_{k_2}) - 1$. In particular, if G_1 and G_2 are Eulerian embedded, then G is Eulerian embedded as well.*

Proof. SAWs that contain edges (v_1, u_1) in G_1 and (v_2, u_2) in G_2 merge to a single SAW in G. Other SAWs remain the same as in G_1 and G_2. $\square$

In Fig. 3, the connected sum of the antiprisms A_4 and A_5 is shown. Both A_4 and A_5 are Eulerian embedded and so is their connected sum.

Given an embedded graph, we substitute every k-valent vertex by a cycle on k vertices. The obtained graph is cubic and embedded in the same surface. It is called the *truncation* of the embedded graph. There are two types of faces in a truncated graph: the ones that correspond to former vertices and the ones that correspond to the faces with the boundary twice as long as in the original graph. In [10], the following theorem is proved:

Theorem 3. *The truncations of cubic maps preserve the number of SAWs in their medials.*

So we obtain some other infinite families of Eulerian embedded plane graphs—the medials of all the truncations of the "odd" prisms, medials of their truncations and so on.

3. Number of SAWs in 4-valent plane graphs

Every Eulerian graph has an Eulerian embedding, orientable and nonorientable. To obtain such an embedding just choose any embedding where the SAW is a given Eulerian circuit—at each vertex the opposite edges are consecutive in the Eulerian circuit.

But it is not at all obvious how to embed a graph in a given surface with the minimal possible number of SAWs or to find the surface of minimal genus in which a graph G can be embedded so as to have only one SAW. These questions seem to be very difficult and are still open. Nevertheless, for the plane the following result holds:

Theorem 4. *Let G be a planar 4-valent graph. Then the number of SAWs is the same for any embedding of G in the plane.*

Proof. For 3-connected graphs the theorem trivially holds, since they have essentialy unique embeddings in the plane.

For graphs of connectivity 2 the proof depends on the well-known theorem, that any embedding of a planar 2-connected graph can be obtained from another by a sequence of operations dual to the Whitney's 2-switchings. This operation is defined as follows: if we have a separation pair $\{x, y\}$, we turn around one component of a graph, adjacent to x and y; so the orders of neighbors of x and y in this component are reversed. This procedure is illustrated in Fig. 4.

The proof consists of considering all possible cases of how SAWs can pass through a separation pair. As an example, let us consider the case, where there is only one SAW passing through x and y, and it passes first twice through x and then twice through y. After the dual 2-switching, the SAW through x and y is changed, but the number of SAWs in G remains the same, see Fig. 4, where the SAWs through x and y are depicted in bold lines and the rest of the graph, in which the dual 2-switching does not affect the SAWs, is depicted in gray.

If G is not 2-connected, it has a cut-vertex, say v. Through the cut-vertex v, only one SAW can pass. Changing the rotation at v such that the embedding remains planar does not change the number of SAWs through v. □

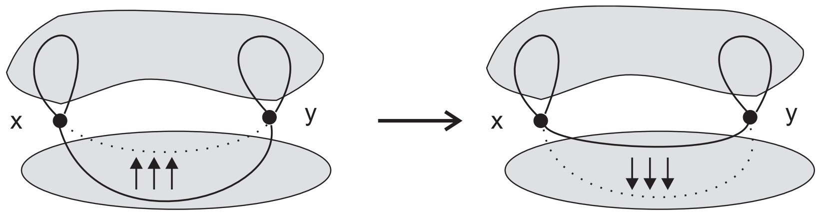

Fig. 4. An example of a dual 2-switching.

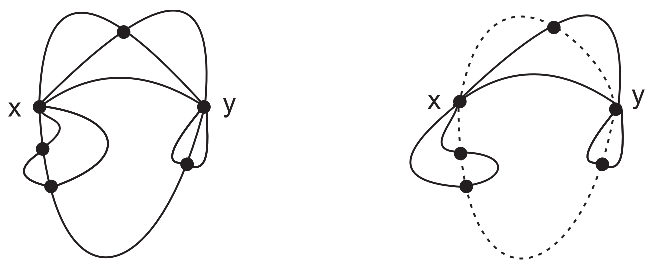

Fig. 5. An example of a planar graph having different number of SAWs in different embeddings in the plane.

This theorem does not hold for all planar Eulerian graphs. In Fig. 5 two embeddings in the plane of the same graph are shown, which contain different numbers of SAWs.

But from the proof of Theorem 4 it can easily be seen that the Theorem holds for a more general class of graphs, namely, the planar Eulerian graphs with cut-vertices and separation pairs of degree not different from 4.

Corollary 5. *Let G be a planar Eulerian graph with possible cut-vertices and separation pairs of degree* 4. *Then the number of SAWs is independent of the embedding of G in the plane*.

4. Eulerian medial embeddings

Given a 2-cell embedding of a connected graph G in a surface S, a local rotation around each vertex can be selected, which can be declared as clockwise orientation. This orientation induces a cyclic ordering—rotation π_v—of edges at each vertex v of G. The anticlockwise ordering at v is then π_v^{-1}. The set $P = \{\pi_v; v \in V(G)\}$ is called the *rotation system* of the given embedding of G in S. Any 2-cell embedding of a connected graph G can be represented by a triple (G, P, λ), where P is the rotation system of G and $\lambda : E(G) \to \{-1, 1\}$ assigns *signatures* to the edges, which tell us how the local rotations of two adjacent vertices fit together: an edge is labeled $+1$, if its endvertices have consistent orientation (the edge is orientation preserving) and -1 otherwise (the edge is orientation reversing). For details, see [12] or [8]. The faces of G are determined by the following procedure: we start with an arbitrary vertex v and an edge $e=(v,u)$ incident with v. Traverse the edge from v to u. We continue the walk along the edge $e'=\pi_u(e)$. Whenever we traverse an edge with signature -1, the anticlockwise rotation is used to determine the next edge in the walk. We continue using anticlockwise ordering until the next edge with signature -1 is traversed and so forth. The walk is completed when the initial edge is encountered in the same direction from v to u and we are in the same mode (clockwise ordering). Changing the signature of an edge between two different faces f_1 and f_2 thus results in f_1 and f_2 being merged to one face.

Given an embedding of a graph G, we change the signatures of the edges such that the orientation preserving edges become orientation reversing and vice versa. A different embedding of G is obtained, which is called the *Petrie dual* of (the embedded) graph G. The faces of the Petrie dual are called *Petrie walks* of the original embedding of G. It is not hard to see that SAWs of medial graphs correspond to the Petrie walks of the original map. See, for example, [7], where the Petrie walks are called left–right paths. That means, that an Eulerian medial embedding of a graph is equivalent to the Petrie dual being 1-face embedded.

Theorem 6. *Every graph embedding can be subdivided to give an Eulerian medial embedding.*

Proof. The proof depends on the following idea: If SAWs of a 4-valent graph have two circuits at a vertex the other two matchings at a vertex give one circuit through that vertex. Subdividing an edge of the original graph can be viewed as changing the matching of the corresponding vertex of the medial graph by means of a double edge. At each step we subdivide an edge, whose corresponding vertex of the medial graph is contained in two different SAWs, and at the end we obtain an Eulerian medial embedded graph. □

The following corollary is an easy consequence of the theorem and the fact that for every surface there exist medial graphs.

Corollary 7. *Every surface admits Eulerian embeddings.*

The question arises, whether every graph has an Eulerian medial embedding. If we consider only orientable surfaces, the answer is "no". The simplest example of graphs having no orientable Eulerian medial embedding are even cycles. The embedding of an even cycle in the sphere is unique and the corresponding medial graph has two SAWs. Let us define *a cactus* as a graph, in which every vertex belongs to at most one cycle. By using the formula of Xuong [13] for calculating the maximum genus of a graph it is easy to see, that any embedding of a cactus is planar.

Theorem 8. *For each embedding of a cactus having c even cycles, the number of SAWs in the corresponding medial is equal to $c+1$.*

Proof. By induction on the number of cycles of the cactus. □

Note, that Theorem 8 is not valid for a more general class of graphs with the property that each *edge* belongs to at most one cycle. A counterexample is given by the toroidal embedding of a graph consisting of two even cycles with a single common vertex. Its medial has only one SAW.

Attaching a graph G_1 to a graph G_2 by an edge is the following procedure: we choose vertices v_1 in G_1 and v_2 in G_2, and join the vertices v_1 and v_2 by an edge. Let G be a graph, obtained by attaching graphs G_1 and G_2 by an edge e. Suppose G is embedded in an orientable surface S. By deleting the edge e, this embedding induces embeddings of G_1 and G_2 into two surfaces, say S_1 and S_2. It is easy to verify, that $s(\mathrm{Me}(G) \to S) = s(\mathrm{Me}(G_1) \to S_1) + s(\mathrm{Me}(G_2) \to S_2) - 1$. Since the medial graph of any embedding of a cactus with even cycles has more than one SAW, the following holds:

Corollary 9. *If a cactus with at least one even cycle is attached by an edge to an arbitrary graph G, then the resulting graph does not have an Eulerian medial embedding.*

These examples of graphs are not even 2-connected. The graph of a three-dimensional cube, usually denoted by Q_3, is a 3-connected cubic graph. It has 2^8 different embeddings (many of them are equivalent). We have counted the numbers of SAWs in the medials of all these embeddings of Q_3 with the help of a computer and found out, that they always have more than one SAW. The question arises, which 3-connected graphs do have an Eulerian medial embedding. In particular, is it true that a graph with a 1-face embedding has an Eulerian medial embedding?

If we also allow nonorientable embeddings, every graph has an Eulerian medial embedding.

Theorem 10. *For every rotation scheme, there is an assignment of signatures to edges that gives an Eulerian medial embedding (possibly nonorientable).*

Proof. The proof is divided in two steps.

- Change the signatures of edges between distinct faces until a one-face embedding is obtained.
- The Petrie dual of the so-obtained graph has the medial with the required property. □

5. Bounds on Eulerian genus

Every Eulerian graph has an Eulerian embedding, orientable and nonorientable. To obtain such an embedding just choose any embedding where the SAW is a given Eulerian circuit—at each vertex the opposite edges are consecutive in the Eulerian circuit. We can define the *Eulerian genus* of a graph G as the smallest possible genus of an orientable surface, in which G can be Eulerian embedded. In Section 2, we have

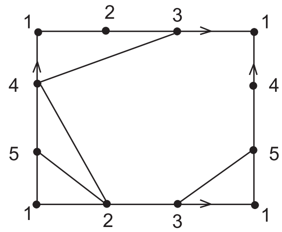

Fig. 6. An Eulerian embedding of K_5 in the torus.

seen some examples of planar graphs which are Eulerian embedded in the plane. In Fig. 6 one of the embeddings of K_5 in the torus is shown. It only has one SAW, which means, that the Eulerian genus of K_5 is equal to its ordinary genus.

Lemma 11. *Let G be an Eulerian graph, embedded in a surface of genus g with* $s(G \to S_g) = k$. *Then the Eulerian genus of G is less or equal to* $g + k - 1$.

Proof. Let e and f be two edges, adjacent in the rotation at a vertex v, and let them belong to different SAWs (if there is more than 1 SAW, this must happen). Switching e and f at v causes the SAWs through e and f to be joined into one SAW. We repeat this procedure until there is only one SAW left. Switching the rotation at a vertex can only increase the genus by one (see, for example [5]). So after $k - 1$ switches, the genus is increased by at most $k - 1$. □

Remark. Let $(\dots, e_1, e_2, \dots, e_k, \dots.)$ be the rotation at a vertex v and let the edges $e_1, \dots, e_k$ belong to distinct SAWs. Then changing the rotation at v to $(\dots, e_2, \dots, e_k, e_1, \dots)$ causes all these SAWs to join.

Corollary 12. *The Eulerian genus of* $C_m \times C_n$ *is less or equal to* $m + n$.

Proof. The graph $C_m \times C_n$ can be embedded in the torus in an obvious way such that it contains $n + m$ SAWs. It follows from the Lemma, that the Eulerian genus must be at most $1 + (m + n - 1) = m + n$. □

6. Conclusion and open problems

The natural question is which Eulerian graphs have their Eulerian genus equal to the ordinary genus. Another question that can be posed is the following: which 2-cell embeddings of graphs have their connected and four-valent medial graphs Eulerian embedded? Finally, which graphs have at least one orientable embedding such that the corresponding medial graph is Eulerian embedded?

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