Discrete Mathematics 112 (1993) 185–197
North-Holland

A generalized chromatic polynomial, acyclic orientations with prescribed sources and sinks, and network reliability

J. Rodriguez and A. Satyanarayana*

Department of Electrical Engineering and Computer Science, Stevens Institute of Technology, Hoboken, NJ 07030, USA

Received 29 January 1990
Revised 9 April 1991

Abstract

Rodriguez, J. and A. Satyanarayana, A generalized chromatic polynomial, acyclic orientations with prescribed sources and sinks, and network reliability, Discrete Mathematics 112 (1993) 185–197.

Suppose $G=(V,E)$ is a graph and K, K', K'' are subsets of V such that $K\subseteq K'\cap K''$. We introduce and study a polynomial $P(G,K,K',K'';\lambda)$ in λ. This polynomial coincides with the classical chromatic polynomial $P(G;\lambda)$ when $K=V$. The results of this paper generalize Whitney's characterizations of the coefficients of $P(G;\lambda)$ and the work of Stanley on acyclic orientations. Furthermore, we establish a connection between a family of polynomials associated with network reliability and a family of polynomials associated with $P(G,K,K',K'';\lambda)$.

1. Introduction

An orientation of a graph G is an assignment of a direction to each edge of the graph. An orientation is acyclic if the resulting digraph has no directed cycles. If ω is a directed graph and u is a point of ω then we say that u is a source of ω if its indegree is zero but its outdegree is positive. Likewise, u is termed a sink if its indegree is positive and outdegree is zero. For a given pair of subsets $K',K''\subseteq V(G)$, an acyclic orientation ω of G is said to be (K',K'')-proper if the sources of ω are in K' while the sinks are in K''. Suppose $K\subseteq K'\cap K''$ and λ is a positive integer. Let $P(G,K,K',K'';\lambda)$ be the number of pairs (ω,f), where ω is (K',K'')-proper and $f\colon K\to\{1,2,\ldots,\lambda\}$ is

Correspondence to: A. Satyanarayana, Department of Electrical Engineering and Computer Science, Stevens Institute of Technology, Hoboken, NJ 07030, USA

* Work of the second author supported in part by ARO under Grant DAAL03-90-G-0078.

a mapping such that $f(x)<f(y)$ if $x, y \in K$ and there is a directed path from x to y in ω. Likewise, let $\bar{P}(G, K, K', K''; \lambda)$ be the number of pairs (ω, f), where ω is (K', K'')-proper and $f: K \to \{1, 2, \ldots, \lambda\}$ such that $f(x) \leqslant f(y)$ if $x, y \in K$ and there is a directed path from x to y in ω.

Let $\sigma(G: K', K'')$ be the set of spanning subgraphs S of G such that any component of S intersects K' iff it intersects K''. Suppose $H_i \subseteq V(G)$, S is a spanning subgraph of G then define $c(S: H_1, \ldots, H_h, \neg H_{h+1}, \ldots, \neg H_r) = |\{C$: C is a component of $S, C \cap H_i \neq \emptyset$ for all $1 \leqslant i \leqslant h$ and $C \cap H_i = \emptyset$ for all $h+1 \leqslant i \leqslant r\}|$. Define $P(x)$ to be the polynomial

$$(-1)^{|V(G)|+|K|} \sum_{S \in \sigma(G:\, K', K'')} (-1)^{|E(S)|+c(S:K', \neg K)} x^{c(S:K)}$$

over the complex number field. In this paper we show that $P(G, K, K', K''; \lambda) = P(\lambda)$ and $\bar{P}(G, K, K', K''; \lambda) = (-1)^{|K|} P(-\lambda)$ for λ integer and positive. For the case of $K = V(G)$, the polynomial $P(G, K, K', K''; \lambda)$ coincides with the classical chromatic polynomial $P(G; \lambda)$.

The topological characterizations of the coefficients of $P(G, K, K', K''; \lambda)$ generalize the results of Whitney [8] on $P(G; \lambda)$. The extended chromatic polynomial studied by Satyanarayana and Tindell [4] is a special case of $P(G, K, K', K''; \lambda)$ when $K = K' = K''$. Furthermore, the results of this paper generalize the work of Stanley [6] concerning acyclic orientations. Finally, we establish a connection between a family of polynomials associated with network reliability and a family of polynomials associated with $P(G, K, K', K''; \lambda)$.

2. Preliminaries

Unless defined otherwise, graph-theoretic terminology used here follows Harary [3]. One exception is that we allow multiple edges and loops so that by a graph we mean a pseudo graph. The edge set and the point set of a graph G are denoted by $E(G)$ and $V(G)$ respectively. If $x = \{u, v\}$ is an edge of G then $G|x$ is the graph obtained from G by deleting x and identifying the points u and v to a single point. We say that $G|x$ is obtained from G by *contracting the edge* x. Note that all other edges with endpoints u, v become loops of $G|x$. Likewise, if X is a set of edges of G then $G|X$ is the graph obtained from G by successively contracting all the edges of X. Each edge in $E(G)-X$ can be regarded as an edge of $G|X$. This identification, which we henceforth assume, constitutes a bijection between $E(G)-X$ and $E(G|X)$. Furthermore, the number of connected components of a graph remains unchanged upon edge contractions. By $\Gamma(G, X)$, we mean the collection of spanning subgraphs S of G such that $E(S) \subseteq X$. When $X = E(G)$ we simply write $\Gamma(G)$ instead $\Gamma(G, X)$.

If λ is a positive integer, we will denote by $[\lambda]$ the set $\{1, \ldots, \lambda\}$. A *λ-coloring* of a graph $G = (V, E)$ is a mapping $f: V \to [\lambda]$. The integers $1, 2, \ldots, \lambda$ are called *colors*. A λ-coloring is *proper* if no two adjacent points of G are assigned the same color. The number of distinct proper λ-colorings of a given graph G can be expressed as a polynomial in λ, well known as the *chromatic polynomial* of G and denoted by $P(G; \lambda)$. The following topological interpretation for the coefficients of $P(G; \lambda)$ is due to Whitney [8].

Proposition 2.1. *$P(G; \lambda) = \sum_{S \in \Gamma(G)} (-1)^{|E(S)|} \lambda^{c(S)}$, where $c(S)$ is the number of connected components of S.*

We need the following generalization of Proposition 2.1.

Proposition 2.2. *If G is a graph and $H \subseteq E(G)$, then*

$$P(G, \lambda) = \sum_{S \in \Gamma(G,H)} (-1)^{|E(S)|} P((G-(H-E(S)))|E(S); \lambda).$$

Proof. By Proposition 2.1,

$$P(G, \lambda) = \sum_{S \in \Gamma(G)} (-1)^{|E(S)|} \lambda^{c(S)}$$

$$= \sum_{S' \in \Gamma(G,H)} (-1)^{|E(S')|} \sum_{S'' \in \Gamma(G, E(G)-H)} (-1)^{|E(S'')|} \lambda^{c(S' \cup S'')},$$

where $c(X)$ is the number of connected components of X. Consider the subgraphs, say $G_1 \in \Gamma(G)$ such that $E(G_1) = E(S' \cup S'')$, and $G_2 \in \Gamma((G-(H-E(S')))|E(S'))$ such that $E(G_2) = E(S'')$. Clearly $c(G_1) = c(G_2)$. Thus

$$\sum_{S'' \in \Gamma(G, E(G)-H)} (-1)^{|E(S'')|} \lambda^{c(S' \cup S'')} = \sum_{S'' \in \Gamma((G-(H-E(S')))|E(S'))} (-1)^{|E(S'')|} \lambda^{c(S'')}.$$

Now, by Proposition 2.1,

$$\sum_{S'' \in \Gamma((G-(H-E(S')))|E(S'))} (-1)^{|E(S'')|} \lambda^{c(S'')} = P((G-(H-E(S')))|E(S'); \lambda),$$

and the proposition follows. □

Definition 2.1 (acyclic orientations, sources. sinks, extensions and restrictions). An *orientation* of a graph G is an assignment of a direction to each edge of the graph. An orientation is *acyclic* if the resulting digraph has no directed cycles. If ω is a directed graph and u is a point of ω then we say that u is a *source* of ω if its indegree is zero but its outdegree is positive. Likewise, u is termed a *sink* if its indegree is positive and outdegree is zero. Suppose ω and ω' are orientations of a graph G and its subgraph G' respectively. If the digraph ω' is a subgraph of ω then ω is an *extension* of ω', and ω' is a *restriction* of ω.

The following proposition is due to Greene [2].

Proposition 2.3. *Let $\Omega(G: u, v) = \{\omega \mid \omega$ is an acyclic orientation of G with u being the only source and v the only sink of $\omega\}$. If $\{u, v\}$ is an edge of a graph G containing i isolated points, then*

$$|\Omega(G: u, v)| = (-1)^{|V(G)|+i} (P(G; \lambda)/(\lambda - 1))|_{\lambda=1}.$$

Let $N(G)$ be the number of acyclic orientations of a graph G. The following proposition is due to Stanley [6].

Proposition 2.4. *For any graph G, $N(G) = (-1)^{|V(G)|} P(G; -1)$.*

An immediate consequence of Propositions 2.1 and 2.4 is the following.

Proposition 2.5. *For any graph G, $N(G) = (-1)^{|V(G)|} \sum_{S \in \Gamma(G)} (-1)^{|E(S)|+c(S)}$.*

The following proposition is well known.

Proposition 2.6. *If u is a point of G such that the neighborhood of u induces a clique in G, then $P(G; \lambda) = (\lambda - \deg(u))\, P(G - u; \lambda)$.*

3. The polynomial $P(G, K, K', K''; \lambda)$

Let G be a graph and $K', K'' \subseteq V(G)$. An acyclic orientation ω of G is said to be (K', K'')-*proper* if the sources of ω are in K' while the sinks are in K''. Suppose $K \subseteq K' \cap K''$ and λ is a positive integer. Let $P(G, K, K', K''; \lambda)$ be the number of pairs (ω, f), where ω is (K', K'')-proper and $f: K \to [\lambda]$ such that $f(x) < f(y)$ if $x, y \in K$ and there is a directed path from x to y in ω. Likewise, let $\bar{P}(G, K, K', K''; \lambda)$ be the number of pairs (ω, f), where ω is (K', K'')-proper and $f: K \to [\lambda]$ such that $f(x) \leqslant f(y)$ if $x, y \in K$ and there is a directed path from x to y in ω. Let $\sigma(G: K', K'')$ be the set of spanning subgraphs S of G such that any component of S intersects K' iff it intersects K''. Suppose $H_i \subseteq V(G)$, S is a spanning subgraph of G then define $C(S: H_1, \ldots, H_h, \neg H_{h+1}, \ldots, \neg H_r) = \{C: C$ is a component of S, $C \cap H_i \neq \emptyset$ for all $1 \leqslant i \leqslant h$ and $C \cap H_i = \emptyset$ for all $h+1 \leqslant i \leqslant r\}$. Let $c(S: H_1, \ldots, H_h, \neg H_{h+1}, \ldots, \neg H_r) = |C(S: H_1, \ldots, H_h, \neg H_{h+1}, \ldots, \neg H_r)|$.

Theorem 3.1. *Let G be a graph with $K, K', K'' \subseteq V(G)$ such that $K \subseteq K' \cap K''$ and if u is an isolated point of G then $u \in K' \cap K''$. Then*

$$\bar{P}(G, K, K', K''; \lambda) = (-1)^{|V|} \sum_{S \in \sigma(G:\, K', K'')} (-1)^{|E(S)| + c(S: K', \neg K)} (-\lambda)^{c(S: K)}.$$

Proof. For a given integer $n \geqslant 1$, we construct the graph $G(n)$ as follows: $V(G(n)) = V(G) \cup \{u_0, u_1, \ldots, u_n\}$ where the u_i are new vertices not in $V(G)$ and $E(G(n)) = E(G) \cup \{\{u_i, u_j\}\colon 0 \leqslant i < j \leqslant n\} \cup \{\{u_0, x\}\colon x \in K'\} \cup \{\{u_1, x\}\colon x \in K''\} \cup \{\{u_i, x\}\colon 2 \leqslant i \leqslant n, x \in K\}$. Graph $G(n)$ is illustrated in Fig. 1.

By Proposition 2.2,

$$P(G(n); \lambda) = \sum_{S \in \Gamma(G(n), E(G))} (-1)^{|E(S)|} P((G(n) - (E(G) - E(S))) | E(S); \lambda)$$

$$= \sum_{S \in \Gamma(G)} (-1)^{|E(S)|} P((G(n) - (E(G) - E(S))) | E(S); \lambda).$$

Consider a term in the summation corresponding to a subgraph $S \in \Gamma(G)$. Clearly, the set of connected components of S can be partitioned into $\{C(S\colon \neg(K' \cup K'')), C(S\colon K', \neg K''), C(S\colon K'', \neg K'), C(S\colon K', K'', \neg K), C(S\colon K)\}$. Let $G'(n)$ be the graph obtained from $(G(n) - (E(G) - E(S))) | E(S)$ by replacing multiple edges by single ones. Every component in $C(S\colon \neg(K' \cup K''))$ becomes an isolated point in $G'(n)$. Likewise, the components in $C(S\colon K', \neg K'')$ and in $C(S\colon K'', \neg K')$ become degree-1 points adjacent to u_0 and u_1 respectively. The components in $C(S\colon K', K'', \neg K)$ correspond to degree-2 points adjacent to both u_0 and u_1. Finally, those in $C(S\colon K)$ become points of degree $n+1$ adjacent to $u_0, \ldots, u_n$. Since the graph induced by $\{u_0, \ldots, u_n\}$ in $G'(n)$ is the complete graph K_{n+1}, by repeated application of Proposition 2.6, we get

$$P(G'(n); \lambda) = \lambda^{c(S\colon \neg(K' \cup K''))} (\lambda - 1)^{c(S\colon K', \neg K'') + c(S\colon K'', \neg K')} (\lambda - 2)^{c(S\colon K', K'', \neg K)}$$

$$\times (\lambda - n - 1)^{c(S\colon K)} P(K_{n+1}; \lambda).$$

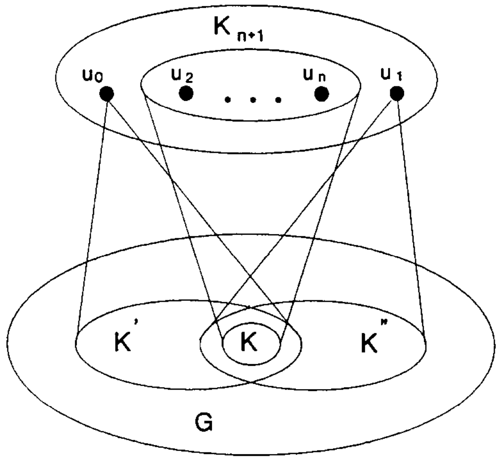

Fig. 1. The graph $G(n)$ of Theorem 3.1.

As $G'(n)$ is obtained from $(G(n)-(E(G)-E(S)))|E(S)$ by replacing multiple edges by single ones, we have $P((G(n)-\{E(G)-E(S)\})|E(S);\ \lambda)=P(G'(n);\ \lambda)$. Thus

$$P(G(n);\ \lambda)=\sum_{S\in\Gamma(G)}(-1)^{|E(S)|}\lambda^{c(S:\neg(K'\cup K''))}(\lambda-1)^{c(S:K',\neg K'')+c(S:K'',\neg K')}$$
$$\times(\lambda-2)^{c(S:K',K'',\neg K)}(\lambda-n-1)^{c(S:K)}P(K_{n+1};\ \lambda).$$

By Proposition 2.3,

$$|\Omega(G(n)\colon u_0,u_1)|=(-1)^{|V(G)|+n+1}\sum_{S\in\Gamma(G)}(-1)^{|E(S)|}$$
$$\times P((G(n)-(E(G)-E(S)))|E(S);\ \lambda)/(\lambda-1)|_{\lambda=1}.$$

However, a simple computation shows that

$$P((G(n)-(E(G)-E(S)))|E(S);\ \lambda)/(\lambda-1)|_{\lambda=1}=0$$

if $c(S:K',\neg K'')>0$ or $c(S:K'',\neg K')>0$, and

$$P((G(n)-(E(G)-E(S)))|E(S);\ \lambda)/(\lambda-1)|_{\lambda=1}=(-1)^{n-1+c(S:K',K'',\neg K)}$$
$$\times(-n)^{c(S:K)}(n-1)!$$

otherwise.

Hence

$$|\Omega(G(n),u_0,u_1)|=(-1)^{|V(G)|}(n-1)!\sum_{S\in\sigma(G:K',K'')}(-1)^{|E(S)|+c(S:K',\neg K)}$$
$$\times(-n)^{c(S:K)}.$$

It remains to show that $|\Omega(G(n),u_0,u_1)|=(n-1)!\ \bar{P}(G,K,K',K'';n)$. Consider the clique Q induced by $\{u_2,u_3,\dots,u_n\}$ in $G(n)$. Let ω_0 be one of the $(n-1)!$ acyclic orientations of Q. Let $\Omega(G(n),u_0,u_1,\omega_0)=\{\omega\in\Omega(G(n),u_0,u_1)|\ \omega$ is an extension of $\omega_0)$. It suffices to show that $|\Omega(G(n),u_0,u_1,\omega_0)|=\bar{P}(G,K,K',K'';n)$.

Let $\Omega F(G,K,K',K'';n)=\{(\omega,f)|\,\omega$ is (K',K'')-proper and $f\colon K\to[n]$ such that $f(x)\leqslant f(y)$ if $x,y\in K$ and there is a directed path from x to y in $\omega\}$. We construct a bijection $\Phi:\Omega(G(n),u_0,u_1,\omega_0)\to\Omega F(G,K,K',K'';n)$. For $\omega\in\Omega(G(n),u_0,u_1,\omega_0)$, $\Phi(\omega)=(\omega_G,f)$ where ω_G is the restriction of ω to G. For a given $v\in K$, ω induces a total ordering $<_\omega$ on the set $\{v,u_2,\dots,u_n)$ in the sense that if $x,y\in\{v,u_2,\dots,u_n\}$ then $x<y$ iff the edge $\{x,y\}$ is oriented from x to y in ω. Let φ be the unique strict order preserving map from $(\{v,u_2,\dots,u_n),<_\omega)$ to $([n],<)$. Define $f(v)=\varphi(v)$. We first show that $(\omega_G,f)\in\Omega F(G,K,K',K'';\ n)$. Clearly ω_G is (K',K'')-proper. Suppose $x,y\in K$ and there is a directed path π from x to y in ω_G. If $f(x)>f(y)$ then it follows that there exists a point $u_j, 2\leqslant j\leqslant n,$ such that the edge $\{u_j,x\}$ is oriented from u_j to x and the edge $\{y,u_j\}$ is oriented from y to u_j in ω. Then ω is not acyclic. Hence $f(x)\leqslant f(y)$ and $(\omega_G,f)\in\Omega F(G,K,K',K'';\ n)$.

To show that Φ is 1:1, suppose $\omega, \omega' \in \Omega(G(n), u_0, u_1, \omega_0)$ such that $\Phi(\omega) = \Phi(\omega')$, where $\Phi(\omega) = (\omega_G, f)$ and $\Phi(\omega') = (\omega'_G, f')$. Let $X = \{\{v, u_j)\,|\, v \in K \text{ and } u_j \in Q\}$. Since $\omega_G = \omega'_G$, any edge $\{x, y\} \notin X$ is oriented from x to y in ω iff it is oriented from x to y in ω'. Furthermore, since $f = f'$, $\{x, y\} \in X$ is oriented from x to y in ω iff it is oriented from x to y in ω'. Thus $\omega = \omega'$. To show that Φ is onto, pick any $(\omega_G, f) \in \Omega F(G, K, K', K''; n)$. We extend ω_G to an orientation ω such that $\omega \in \Omega(G(n), u_0, u_1, \omega_0)$. Consider a partition of the edge set $E(G(n)) = \{E(G), E(Q), X, Y\}$, where $X = \{\{v, u_j\}\,|\, v \in K$ and $u_j \in Q\}$ and Y is the set of edges incident on u_0 or u_1. The orientation ω is constructed as follows. Edges $\{u_0, y\} \in Y$ are oriented from u_0 to y while the edges $\{u_1, y\} \in Y$ are oriented from y to u_1. To orient the edges of X, consider the total ordering $u_{i_1} < \cdots < u_{i_{n-1}}$ on the set $\{u_2, u_3, \ldots, u_n\}$ induced by ω_0; the ordering is in the sense that $x < y$ iff the edge $\{x, y\}$ is oriented from x to y in ω_0. If $\{x, y\} \in X$ such that $x \in K$ and $y = u_{i_j}$, then the orientation of $\{x, u_{i_j}\}$ is from u_{i_j} to x whenever $j < f(x)$, and it is oriented from x to u_{i_j} otherwise. The edges of $E(G)$ and $E(Q)$ are oriented exactly as in ω_G and ω_0 respectively. Clearly ω is an acyclic orientation. Since G has no isolated points outside $K' \cap K''$, we conclude that u_0 is the only source and u_1 is the only sink of ω and $\omega \in \Omega(G(n), u_0, u_1, \omega_0)$. This completes the proof. □

A consequence of Theorem 3.1 is the following generalization of Proposition 2.5.

Corollary 3.1. *Let G be a graph with $K', K'' \subseteq V(G)$ and if u is an isolated point of G then $u \in K' \cap K''$. If $N(G, K', K'')$ is the number of acyclic orientations of G in which all sources are in K' and the sinks are in K'', then*

$$N(G, K', K'') = (-1)^{|V|} \sum_{S \in \sigma(G:\ K', K'')} (-1)^{|E(S)| + c(S:K')}.$$

Let $(X, \leqslant)$ be a partially ordered set. Let $\bar{O}(X, \leqslant; \lambda)$ be the number of order preserving mappings $f\colon (X, \leqslant) \to ([\lambda], \leqslant)$ and $O(X, \leqslant; \lambda)$ be the number of strict order preserving mappings $f\colon (X, \leqslant) \to ([\lambda], \leqslant)$. A mapping f is *order preserving* if $x \leqslant y$ implies $f(x) \leqslant f(y)$ and it is *strict order preserving* if $x < y$ implies $f(x) < f(y)$. It is well known that $\bar{O}(X, \leqslant; \lambda)$ and $O(X, \leqslant; \lambda)$ are polynomials in λ. The following result of Stanley [5] establishes an important connection between these two polynomials.

Proposition 3.1. $O(X, \leqslant; \lambda) = (-1)^{|X|}\, \bar{O}(X, \leqslant; -\lambda)$.

Theorem 3.2. *Let G be a graph with $K, K', K'' \subseteq V(G)$ such that $K \subseteq K' \cap K''$ and if u is an isolated point of G then $u \in K' \cap K''$. Then*

$$P(G, K, K', K''; \lambda) = (-1)^{|V| + |K|} \sum_{S \in \sigma(G:\ K', K'')} (-1)^{|E(S)| + c(S:K', \neg K)}\, \lambda^{c(S:K)}.$$

Proof. If F is the collection of (K',K'')-proper acyclic orientations of G, then it is clear that each $\omega\in F$ induces a partial ordering $\leqslant_\omega$ on K. Thus $P(G,K,K',K'';\lambda)=\sum_{\omega\in F} O(K,\leqslant_\omega;\lambda)$, $\bar{P}(G,K,K',K'';\lambda)=\sum_{\omega\in F}\bar{O}(K,\leqslant_\omega;\lambda)$. The theorem follows from Theorem 3.1 and Proposition 3.1. □

If isolated points are allowed outside of $K'\cap K''$ then Theorems 3.1 and 3.2 yield the following.

Corollary 3.2. *Suppose G is a graph and $K,K',K''\subseteq V(G)$ such that $K\subseteq K'\cap K''$. Let I be the set of isolated points of G that are not in $K'\cap K''$. Then*

$$P(G,K,K',K'';\lambda)=(-1)^{|V|+|K|-|I|} \times \sum_{S\in\sigma(G-I:\ K'-I,K''-I)} (-1)^{|E(S)|+c(S:K'-I,\neg K)}\lambda^{c(S:K)},$$

$$\bar{P}(G,K,K',K'';\lambda)=(-1)^{|V|-|I|} \times \sum_{S\in\sigma(G-I:\ K'-I,K''-I)} (-1)^{|E(S)|+c(S:K'-I,\neg K)}(-\lambda)^{c(S:K)}.$$

Definition 3.1 (external activity). A *spanning forest* of a graph G is an acyclic spanning subgraph of G. Let G be a graph and $<$ a strict linear order on $E(G)$. Let F be a spanning forest of G. An edge $x=\{u,v\}$, $x\notin F$, is said to be *externally active* relative to F if there is a path π between u and v in F such that $x<y$ for all edges y on π. The *external activity* of F is the number of externally active edges of F.

The following interpretation for the coefficients of $P(G;\lambda)$ is due to Whitney [8].

Proposition 3.2. *Let $G=(V,E)$ be a graph. If $m_j(G)$ is the number of spanning forests of G having j connected components and external activity zero, then*

$$P(G;\lambda)=\sum_{j=1}^{|V|}(-1)^{|V|-j}\, m_j(G)\,\lambda^j.$$

We extend the characterization of Whitney to $P(G,K,K',K'';\lambda)$. For this, we need the following notion of internal activity. Tutte [7] introduced the notion of internal activity relative to a set of edges and showed that $P(G;\lambda)$ can be expressed in terms of certain spanning trees of G. However, the internal activity introduced here is not a generalization of Tutte's notion of internal activity.

Definition 3.2 (internal activity). Let $G=(V,E)$ be a graph with $K,K',K''\subseteq V$ such that $K\subseteq K'\cap K''$ and if u is an isolated point of G then $u\in K'\cap K''$. Let $\sigma(G\colon K',K'')$ be the set of spanning subgraphs S of G such that any component of S intersects K' iff it intersects K''. Let $\sigma_{K'}(G)$ be the set of spanning subgraphs S of G such that every component of S intersects K'. An edge x of a spanning forest $F \in \sigma(G\colon K', K'') \cap \sigma_{K'}(G)$ is said to be *internally active* relative to F if $F - x \in \sigma(G\colon K', K'') - \sigma_{K'}(G)$ and whenever y is an edge of G such that $F - x + y \in \sigma_{K'}(G)$ then $x \leqslant y$. The *internal activity* of F is the number of internally active edges of F.

Theorem 3.3. *Let G be a graph with $K, K', K'' \subseteq V(G)$ such that $K \subseteq K' \cap K''$, and if u is an isolated point of G then $u \in K' \cap K''$. Let $<$ be a strict linear order on E. If $m_j(G, K, K', K'')$ is the number of spanning forests $F \in \sigma(G\colon K', K'') \cap \sigma_{K'}(G)$ with $c(F:K) = j$ and with external and internal activity zero, then*

$$P(G, K, K', K''; \lambda) = (-1)^{|V| + |K|} \sum_{j=1}^{|K|} (-1)^{|V| - j} m_j(G, K, K', K'') \lambda^j.$$

Proof. We can assume that every connected component of G has a non-empty intersection with K' and K'', for otherwise there would be no acyclic orientations for G with all sinks in K' and all sources in K'', and we would have $P(G, K, K', K''; \lambda) = 0$. Let $\gamma(G\colon K', K'')$ be the set of spanning forests F of G such that $F \in \sigma(G\colon K', K'') \cap \sigma_{K'}(G)$. For any $S \in \sigma(G, K', K'')$, with respect to the strict linear order $<$, pick the maximum weight spanning forest, say F', of S. Let F be the smallest weight forest in $\gamma(G\colon K', K'')$ such that $F' \subseteq F$ and $c(F:K') = c(F':K')$. Define a mapping $\Phi\colon \sigma(G, K', K'') \to \gamma(G\colon K', K'')$ where $\Phi(S) = F$.

By Theorem 3.2,

$$(-1)^{|V| + |K|} P(G, K, K', K''; \lambda) = \sum_{S \in \sigma(G\colon K', K'')} (-1)^{|E(S)| + c(S:K', \neg K)} \lambda^{c(S:K)}$$

$$= \sum_{F \in \gamma(G\colon K', K'')} \left(\sum_{S \in \Phi^{-1}(F)} (-1)^{|E(S)|} (-1)^{c(S:K', \neg K)} \lambda^{c(S:K)} \right).$$

Since $c(S:K', \neg K) = c(F:K', \neg K)$ and $c(S:K) = c(F:K)$ for all $S \in \Phi^{-1}(F)$,

$$(-1)^{|V| + |K|} P(G, K, K', K''; \lambda) = \sum_{F \in \gamma(G\colon K', K'')} \left(\sum_{S \in \Phi^{-1}(F)} (-1)^{|E(S)|} \right) \times (-1)^{c(F:K', \neg K)} \lambda^{c(F:K)}.$$

Let E_F and I_F be the set of externally active and internally active edges relative to the spanning forest $F \in \gamma(G\colon K', K'')$. We first show that $\Phi^{-1}(F) = \{S\colon F - I_F \subseteq S \subseteq F \cup E_F\}$. If an edge $x \in E(S) - E(F)$ is not externally active relative to $F = \Phi(S)$, then there exists an edge y in the unique cycle, say C, of $F + x$ such that $y < x$. As before, let F' be the maximum weight spanning forest of S. Since F' is obtained from S by the deletion of nonbridges of S it follows that C is in S. Then x is not externally active relative to F'. However, $F' - y + x$ is a spanning forest of S with a weight larger than that of F', a contradiction. Thus all edges in $E(S) - E(F)$ are externally active relative to F. Next, the edges of $E(F) - E(S)$ are internally active relative to F. To see this, pick an edge

$x \in E(F) - E(S)$. If x is not internally active, there exists an edge y such that $F - x + y \in \gamma(G\colon K', K'')$ and $y < x$. Clearly $F' \subseteq F - x + y$ and $c(F'\colon K') = c(F - x + y\colon K')$. However the weight of $F - x + y$ is smaller than that of F, contradicting the choice of F.

Hence $\Phi^{-1}(F) \subseteq \{S\colon F - I_F \subseteq S \subseteq F \cup E_F\}$. Next consider an $S = (F - Y) \cup X$ where $Y \subseteq I_F$ and $X \subseteq E_F$. We show $\Phi(S) = F$ so that $\{S\colon F - I_F \subseteq S \subseteq F \cup E_F\} \subseteq \Phi^{-1}(F)$. Suppose F' is the maximum weight forest of S. Clearly, F' is obtained by repeatedly deleting, say $x_{i_1} < \cdots < x_{i_h}$, smallest weight edges that are nonbridges. If $x_{i_j} \notin X$ for some j, then it follows that there is a cycle C in $S - \{x_{i_1}, \ldots, x_{i_{j-1}}\}$ such that edges x and x_{i_j} are in C and $x < x_{i_j}$. This is impossible since F' is the maximum weight forest. Thus $F' = S - X = F - Y$. Let $F'' \in \gamma(G\colon K', K'')$ be the minimum weight spanning forest of S such that $F - Y \subseteq F''$ and $c(F - Y\colon K') = c(F''\colon K')$. Clearly F'' is obtained from $F - Y$ by repeatedly adding smallest weight edges, say $y_{i_1}, \ldots, y_{i_m}$, so that $F'' \in \gamma(G\colon K', K'')$. If edge $y_{i_j} \notin Y$ for some j then there exists an edge y such that $F'' - y_{i_j} + y \in \gamma(G\colon K', K'')$ and $y < y_{i_j}$. Clearly $F - Y \subseteq F'' - y_{i_j} + y$ and $c(F - Y\colon K') = c(F'' - y_{i_j} + y\colon K')$. However the weight of $F'' - y_{i_j} + y$ is smaller than that of F'', contradicting the choice of F''. Hence $F'' = F$.

Since $\Phi^{-1}(F) = \{S\colon F - I_F \subseteq S \subseteq F \cup E_F\}$, we conclude that $\sum_{S \in \Phi^{-1}(F)} (-1)^{|E(S)|} = (-1)^{|E(F)|}$ if $E_F = I_F = \phi$, and $\sum_{S \in \Phi^{-1}(F)} (-1)^{|E(S)|} = 0$, otherwise. If $\gamma'(G\colon K', K'') = \{F \in \gamma(G\colon K', K'')\colon F$ has external and internal activity equal to zero$\}$, then

$$(-1)^{|V| + |K|} P(G, K, K', K''; \lambda) = \sum_{F \in \gamma'(G\colon K', K'')} (-1)^{|E(F)| + c(F : \neg K)} \lambda^{c(F : K)}.$$

Since $|E(F)| = |V(G)| - c(F : \neg K) - c(F : K)$,

$$(-1)^{|V| + |K|} P(G, K, K', K''; \lambda) = \sum_{F \in \gamma'(G\colon K', K'')} (-1)^{|V(G)| - c(F : K)} \lambda^{c(F : K)}.$$

Collecting the terms with $c(F : K) = j$, we have

$$P(G, K, K', K''; \lambda) = (-1)^{|V| + |K|} \sum_{j=1}^{|K|} (-1)^{|V| - j} m_j(G, K, K', K'') \lambda^j. \qquad \square$$

4. Reliability polynomials and coloring polynomials

Consider the following network reliability measure. Suppose G is a probabilistic graph such that the points of G do not fail but the edges of G fail independently of each other with equal failure probabilities, say q. Let $R_i(G; q)$ be the probability that the spanning subgraph induced by the surviving edges contains exactly i connected components. Suppose $P_j(G; \lambda)$ denotes the number of λ-colorings of $G = (V, E)$ such that exactly j edges are monochromatic. If $f\colon V\to[\lambda]$ is a λ-coloring and $x=\{u,v\}$ is an edge of G, then x is *monochromatic* iff $f(u)=f(v)$. Clearly $P_0(G;\lambda)=P(G;\lambda)$. We prove that

$$\sum_{i=1}^{|V|} R_i(G;q)\,\lambda^i=\sum_{j=0}^{|E|} P_j(G;\lambda)\,q^{|E|-j}$$

in a more general setting.

Let G be a graph such that $K'\subseteq K''\subseteq V(G)$. Recall that if X is a set of edges of G then $G|X$ is the graph obtained from G by successively contracting all the edges of X. If K is a subset of the point set of G, we shall denote by $K|X$ the corresponding subset of the point set of $G|X$.

Define

$$P_j(G,K',K'';\lambda)=\sum_{\{H\subseteq E(G):|H|=j\}}(-1)^{|V(G|H)|+|(K'|H)|}\times P(G|H,K'|H,K'|H,K''|H;\lambda).$$

Note that $P_0(G,K',K'';\lambda)=(-1)^{|V(G)|+|K'|}\,P(G,K',K',K'';\lambda)$ and if $K'=V$ then $P_j(G,V,V;\lambda)=P_j(G;\lambda)$.

As before, let $\sigma(G\colon K',K'')$ be the set of spanning subgraphs S of G such that each component of S intersects K' iff it intersects K''. Also, let $c(S\colon K')$ be the number of connected components of S that intersect K'.

Suppose that $G=(V,E)$ is a probabilistic graph such that the points of G do not fail but the edges fail independently of each other. Assume that all the edges of G have the same failure probability q, where $0\leqslant q\leqslant 1$. Let $R_i(G,K',K'';q)$ be the probability that the spanning subgraph S containing the surviving edges of G is in $\sigma(G\colon K',K'')$ and $c(S\colon K')=i$. For the case of $i=1$, the reliability measure $R_1(G,K',K'';q)$ reduces to the well known K-terminal reliability of G [1]. Furthermore, if $K'=V$ then $R_i(G,V,V;q)=R_i(G;q)$.

If $S_j^{(i)}(G,K',K'')$ is the number of spanning subgraphs S of G such that $S\in\sigma(G\colon K',K'')$, $c(S\colon K')=i$, and $|E(S)|=j$, then $R_i(G,K',K'';q)$ may be written as

$$R_i(G,K',K'';q)=\sum_{j=0}^{|E(G)|} S_j^{(i)}(G,K',K'')(1-q)^j\,q^{|E(G)|-j}.$$

We now explore the interplay between the polynomials $R_i(G,K',K'';q)$ and $P_j(G,K',K'';\lambda)$.

Theorem 4.1. *Suppose $G=(V,E)$ is a graph and $K'\subseteq K''\subseteq V$, then*

$$\sum_{i=1}^{|K'|} R_i(G,K',K'';q)\,\lambda^i=\sum_{j=0}^{|E|} P_j(G,K',K'';\lambda)\,q^{|E|-j}.$$

Proof. Since $R_i(G,K',K'';q)=\sum_{j=0}^{|E|} S_j^{(i)}(G,K',K'')(1-q)^j q^{|E|-j}$, we have

$$\sum_{i=1}^{|K'|} R_i(G,K',K'';q)\,\lambda^i=\sum_{i=1}^{|K'|}\sum_{j=0}^{|E|} S_j^{(i)}(G,K',K'')(1-q)^j q^{|E|-j}\,\lambda^i.$$

By the binomial theorem and the fact that $\binom{i}{k}=0$ whenever $k>i$,

$$\sum_{i=1}^{|K'|} R_i(G,K',K'';q)\,\lambda^i=\sum_{i=1}^{|K'|}\sum_{j=0}^{|E|}\sum_{k=0}^{|E|} S_j^{(i)}(G,K',K'')(-1)^k\binom{j}{k} q^{|E|-j+k}\,\lambda^i.$$

Letting $j=k+m$ and using the fact that $S_j^{(i)}(G,K',K'')=0$ whenever $j>|E|$, we get

$$\sum_{i=1}^{|K'|} R_i(G,K',K'';q)\,\lambda^i=$$

$$\sum_{i=1}^{|K'|}\sum_{k=0}^{|E|}\sum_{m=0}^{|E|} S_{k+m}^{(i)}(G,K',K'')(-1)^k\binom{k+m}{m} q^{|E|-m}\,\lambda^i.$$

Interchanging the order of summation we get

$$\sum_{i=1}^{|K'|} R_i(G,K',K'';q)\,\lambda^i=\sum_{m=0}^{|E|} q^{|E|-m} P_m,$$

where

$$P_m=\sum_{i=1}^{|K'|}\sum_{k=0}^{|E|} S_{k+m}^{(i)}(G,K',K'')(-1)^k\binom{k+m}{m}\lambda^i.$$

We need only to show that $P_m=P_m(G,K',K'';\lambda)$. Clearly

$$\begin{aligned}P_m&=\sum_{i=1}^{|K'|}\lambda^i\sum_{k=0}^{|E|} S_{k+m}^{(i)}(G,K',K'')(-1)^{k+m}(-1)^{-m}\binom{k+m}{m}\\&=\sum_{i=1}^{|K'|}\lambda^i\sum_{\{S\in\sigma(G:\,K',K'')|c(S:K')=i\}}(-1)^{|E(S)|-m}\binom{|E(S)|}{m}\\&=\sum_{S\in\sigma(G:\,K',K'')}(-1)^{|E(S)|-m}\binom{|E(S)|}{m}\lambda^{c(S:K')}.\end{aligned}$$

For any $H\subseteq E$, there exists a bijection

$$\Phi:\{S\in\sigma(G,K',K''):\ H\subseteq E(S)\}\to\sigma(G|H,K'|H,K''|H)$$

such that $\Phi(S)=S|H$. Moreover, $c(S:K')=c(\Phi(s):K'|H)$. Using this bijection in conjunction with Theorem 3.2,

$$P(G|H,K'|H,K'|H,K''|H;\lambda)$$

$$=(-1)^{|V(G|H)|+|(K'|H)|}\sum_{\{S\in\sigma(G:\,K',K''):\ H\subseteq E(S)\}}(-1)^{|E(S)|-|H|}\,\lambda^{c(S:K')}.$$

Thus

$$P_m(G,K',K'';\lambda)$$
$$= \sum_{\{H\subseteq E:|H|=m\}} (-1)^{|V(G|H)|+|(K'|H)|} P(G|H, K'|H, K'|H, K''|H;\lambda)$$
$$= \sum_{\{H\subseteq E:\ |H|=m\}} \sum_{\{S\in\sigma(G:\ K',K''):\ H\subseteq E(S)\}} (-1)^{|E(S)|-m}\lambda^{c(S:K')}.$$

By interchanging sums,

$$P_m(G,K',K'';\lambda) = \sum_{S\in\sigma(G:\ K',K'')} \sum_{\{H\subseteq E(S):\ |H|=m\}} (-1)^{|E(S)|-m}\lambda^{c(S:K')}.$$
$$= \sum_{S\in\sigma(G:\ K',K'')} (-1)^{|E(S)|-m}\binom{|E(S)|}{m}\lambda^{c(S:K')} = P_m. \qquad \square$$

References

[1] C.J. Colbourn, The Combinatorics of Network Reliability (Oxford Univ. Press, New York, 1987).

[2] C. Greene, Acyclic orientations, in: Aigner, ed., Higher Combinatorics, NATO Advanced Study Institute Series (Reidel, Dordrecht, 1977).

[3] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).

[4] A. Satyanarayana and R. Tindell, Chromatic polynomials and network reliability, Discrete Math. 67 (1987) 57–79.

[5] R. Stanley, Ordered structures and partitions, Mem. Amer. Math. Soc. 119 (1972).

[6] R. Stanley, Acyclic orientations of graphs, Discrete Math. 5 (1973) 171–178.

[7] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6 (1954) 80–91.

[8] H. Whitney, A logical expansion in mathematics, Bull. Amer. Math. Soc. 38 (1932) 572–579.