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The complexity probabilistic quasi-metric space

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ABSTRACT

We introduce and study a probabilistic quasi-metric on the set of complexity functions, which provides an efficient framework to measure the distance from a complexity function f to another one g in the case that f is asymptotically more efficient than g. In this context we also obtain a version of the Banach fixed point theorem which allows us to show that the functionals associated both to Divide and Conquer algorithms and Quicksort algorithms have a unique fixed point.

1. Introduction and preliminaries

Our basic references for quasi-uniform spaces and quasi-metric spaces are [6,15], for probabilistic metric spaces they are [13,29], and for probabilistic quasi-metric spaces it is [5].

Following the modern terminology (see [15, Section 11]), by a quasi-metric on a nonempty set X we mean a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$; and (ii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

A quasi-metric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X.

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open d-balls $\{B_d(x, r)\colon\ x \in X,\ r > 0\}$, where $B_d(x, r) = \{y \in X\colon\ d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

A topological space (X, τ) is said to be quasi-metrizable if there is a quasi-metric d on X such that $\tau = \tau_d$. In this case, we say that d is compatible with τ.

Given a quasi-metric d on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-metric on X, called the conjugate of d, and the function d^s defined on $X \times X$ by $d^s(x, y) = d(x, y) \vee d^{-1}(x, y)$ is a metric on X.

In the sequel, the letters $\mathbb{R}$, ω and $\mathbb{N}$ will denote the set of real numbers, the set of nonnegative integer numbers and the set of positive integer numbers, respectively.

M. Schellekens began in [26] the development of a topological approach for the complexity analysis of programs and algorithms by means of the so-called complexity (quasi-metric) space. Some applications of this theory to the complexity

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analysis of Divide and Conquer algorithms were also given in [26, Section 6]. Further contributions to the study of these spaces and of other related ones may be found in [5,9,10,21–23], etc.

Let us recall that the complexity (quasi-metric) space consists of the pair $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \left\{ f : \omega \to (0, +\infty]\colon \sum_{n=0}^{+\infty} 2^{-n} \frac{1}{f(n)} < +\infty \right\},$$

and $d_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ given by

$$d_{\mathcal{C}}(f, g) = \sum_{n=0}^{+\infty} 2^{-n} \left(\left(\frac{1}{g(n)} - \frac{1}{f(n)} \right) \vee 0 \right),$$

for all $f, g \in \mathcal{C}$. (We adopt the convention that $\frac{1}{+\infty} = 0$.) The elements of $\mathcal{C}$ are called complexity functions and $d_{\mathcal{C}}$ is said to be the complexity quasi-metric.

Clearly, condition $d_{\mathcal{C}}(f, g) = 0$ is equivalent to the fact that $f(n) \leqslant g(n)$ for all $n \in \omega$. Hence, if the measure of complexity is the running time of computing, and f and g, with $f \neq g$, represent the running time of two different algorithms P and Q respectively, then $d_{\mathcal{C}}(f, g) = 0$ can be interpreted as "the efficiency of P is better than Q on all inputs", or simply, f is "more efficient" than g on all inputs (see [26, Section 6]).

Following the usual terminology of "asymptotic time", for each $g \in \mathcal{C}$ we define $\mathcal{O}(g) := \{f \in \mathcal{C}\colon$ there exist $c > 0$ and $n_0 \in \omega$ such that $f(n) \leqslant cg(n)$ for all $n \geqslant n_0\}$.

In this context we say that $f \in \mathcal{C}$ is asymptotically more efficient than $g \in \mathcal{C}$ if there is $n_0 \in \omega$ such that $f(n) \leqslant g(n)$ for all $n \geqslant n_0$, and $f(m) < g(m)$ for some $m \geqslant n_0$.

Therefore, it appears in a natural way the interesting problem of constructing a kind of quasi-metrics which provided a suitable measurement of the distance from f to g in the case that f is asymptotically more efficient than g; roughly speaking, if $f \leqslant g$ eventually, then the "distance" from f to g should be equals to zero eventually.

The following example shows that, unfortunately, the complexity quasi-metric $d_{\mathcal{C}}$ is not appropriate to describe this situation, in general.

Example 1. Consider the functions $f, g, h \in \mathcal{C}$ given by $f(n) = n + 2$, $g(n) = 2^n/(n^2 + 1)$ and $h(n) = n + 1$ for all $n \in \omega$. An easy computation shows that $f(n) > g(n)$ for $n = 0, 1, \ldots, 10$, and $f(n) < g(n)$ for $n \geqslant 11$. Hence f is asymptotically more efficient than g. Since $h(n) < f(n)$ for all $n \in \omega$, it is desirable to have $d_{\mathcal{C}}(f, g) < d_{\mathcal{C}}(f, h)$. However $d_{\mathcal{C}}(f, g) > 5/6 > d_{\mathcal{C}}(f, h)$.

Motivated by the above example we here discuss the question of constructing a kind of quasi-metric spaces that provide an appropriate setting to obtain a suitable measurement of the distance from f to g when f is asymptotically more efficient than g and, on the other hand, preserve the efficiency of the complexity space in analyzing the complexity of algorithms by means of techniques of contractive self-mappings and fixed points. In Section 2 we shall show that the notion of a probabilistic quasi-metric, in the sense of [5], provides an appropriate framework to solve this question. This will be done with the help of the additional parameter t introduced by this kind of quasi-metrics and that provides a crucial ingredient in our study. Thus, we shall construct the so-called complexity probabilistic quasi-metric space (actually it is a quasi-Menger space) and we shall discuss several of its properties. Finally, we show that, as in Schellekens' approach, both Divide and Conquer algorithms and Quicksort algorithms give rise to contractive self-mappings of the complexity probabilistic quasi-metric space; then, the complexity of such algorithms is represented via the fixed point of the self-mapping obtained by a probabilistic quasi-metric version of the Banach fixed point theorem which is proved here.

At this point it is interesting to recall that the study of probabilistic (quasi-)metric spaces, probabilistic normed spaces and other related structures have received a lot of attention in the last years (see, for instance [1–5,24,30], etc.).

2. The complexity probabilistic quasi-metric space

We start this section with some pertinent concepts and facts.

According to [28], a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $a * 1 = a$ for every $a \in [0, 1]$; (iii) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$, with $a, b, c, d \in [0, 1]$.

If, in addition, $*$ is continuous on $[0, 1] \times [0, 1]$, we say that it is a continuous t-norm.

Three paradigmatic examples of continuous t-norms are $\wedge$, Prod and $*_L$ (the Lukasiewicz t-norm), which are defined by $a \wedge b = \min\{a, b\}$, $a \operatorname{Prod} b = a \cdot b$ and $a *_L b = \max\{a + b - 1, 0\}$, respectively.

Note that $*_L \leqslant \operatorname{Prod} \leqslant \wedge$. Actually, it is well known and easy to see from condition (iii) above, that $* \leqslant \wedge$ for every t-norm $*$.

A distribution function [29] is a function F defined on the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ that is nondecreasing (i.e., $F(t) \leqslant F(s)$ whenever $t < s$), left continuous on $\mathbb{R}$ and satisfies $F(-\infty) = 0$ and $F(+\infty) = 1$.

As usual, for each $a \in \overline{\mathbb{R}} \backslash \{+\infty\}$ we denote by ε_a the distribution function defined by $\varepsilon_a(t) = 0$ if $t \in [-\infty, a]$, and $\varepsilon_a(t) = 1$ if $t \in (a, +\infty]$, and by $\varepsilon_{+\infty}$ we denote the distribution function defined by $\varepsilon_{+\infty}(t) = 0$ if $t \in [-\infty, +\infty)$ and $\varepsilon_{+\infty}(+\infty) = 1$.

We denote by Δ the set of distribution functions, and by Δ^+ the subset of Δ consisting of those distribution functions F such that $F(0) = 0$. The subset of Δ^+ formed by those functions $F \in \Delta^+$ such that $\lim_{t \to +\infty} F(t) = 1$, is denoted by D^+. Thus $\varepsilon_a \in D^+$ whenever $a \in [0, +\infty)$.

In the following we shall write $t > a$ whenever $t \in (a, +\infty)$.

Cho, Grabiec and Radu defined in [5, Definitions 3.4.1 and 3.4.2] the following nonsymmetric versions of the notion of a probabilistic metric space and of a Menger space, respectively.

Definition 1. (See [5].) A probabilistic quasi-metric space is a pair (X, P) such that X is a nonempty set and P is a mapping from $X \times X$ to D^+ satisfying the following conditions for $x, y, z \in X$ and $t, s > 0$:

(PQM1) $P(x, y) = P(y, x) = \varepsilon_0 \Leftrightarrow x = y$;
(PQM2) $P(x, y)(t) = P(y, z)(s) = 1 \Rightarrow P(x, z)(t + s) = 1$.

In this case, the mapping P is called a probabilistic quasi-metric on X.

Definition 2. (See [5].) A quasi-Menger space is a triple $(X, P, *)$ such that X is a nonempty set, $*$ is a t-norm and P is a mapping from $X \times X$ to D^+ satisfying condition (PQM1) above and, for $x, y, z \in X$ and $t, s > 0$,

(QMe) $P(x, z)(t + s) \geqslant P(x, y)(t) * P(y, z)(s)$.

If the probabilistic quasi-metric space (X, P) (respectively, the quasi-Menger space $(X, P, *)$) satisfies:

(PQM3) $P(x, y) = P(y, x)$,

then, (X, P) (respectively, $(X, P, *)$) is called a probabilistic metric space (respectively, a Menger space) [29].

It is clear that every (quasi-)Menger space is a probabilistic (quasi-)metric space. However, the converse does not hold in general.

Although one usually writes P_{xy} instead of $P(x, y)$, it will be convenient to use the original notation in our context.

If $(X, P, *)$ is a quasi-Menger space, then the triple $(X, P^{-1}, *)$ is also a quasi-Menger space, where $P^{-1}(x, y) = P(y, x)$, and the triple $(X, P^i, *)$ is a Menger space, where $P^i(x, y) = P(x, y) \wedge P(y, x)$ for all $x, y \in X$.

As in the case of Menger spaces, each quasi-Menger space $(X, P, *)$ induces a topology τ_P on X defined as follows

$$\tau_P = \{A \subseteq X\colon \text{ for each } x \in A \text{ there exist } \varepsilon \in (0, 1),\ t > 0, \text{ such that } B_P(x, \varepsilon, t) \subseteq A\},$$

where $B_P(x, \varepsilon, t) = \{y \in X\colon\ P(x, y)(t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in (0, 1)$, $t > 0$.

Furthermore, if the t-norm $*$ is continuous, then (X, τ_P) is a quasi-metrizable topological space because the countable collection

$$\{\{(x, y) \in X \times X\colon\ P(x, y)(1/n) > 1 - 1/n\}\colon n \in \mathbb{N}\},$$

is a base for a quasi-uniformity on X whose induced topology coincides with τ_P (compare [5, Section 7.4], and also [11, Lemma 3.1] where this fact is proved in the realm of fuzzy quasi-metric spaces).

Conversely, let (X, d) be a quasi-metric space and let $P_d : X \times X \to D^+$ defined by $P_d(x, y)(t) = 0$ whenever $t \in [-\infty, 0]$, $P_d(x, y)(+\infty) = 1$, and

$$P_d(x, y)(t) = \frac{t}{t + d(x, y)} \quad \text{whenever } t > 0.$$

It is easily seen (compare [7]) that $(X, P_d, \wedge)$ is a quasi-Menger space, which will be called the standard quasi-Menger space of (X, d). Moreover $\tau_d = \tau_{P_d}$ and $\tau_{d^{-1}} = \tau_{(P_d)^{-1}}$ on X, and hence $\tau_{d^s} = \tau_{(P_d)^i}$ (indeed, it is clear that $\lim_{n \to \infty} d(x, x_n) = 0 \Leftrightarrow \lim_{n \to \infty} P(x, x_n)(t) = 1$ for all $t > 0$, and $\lim_{n \to \infty} d(x_n, x) = 0 \Leftrightarrow \lim_{n \to \infty} P(x_n, x)(t) = 1$ for all $t > 0$).

In order to define the complexity probabilistic quasi-metric space we construct the following auxiliary function.

For each $f, g \in \mathcal{C}$ let $Q_{\mathcal{C}}(f, g) : (0, +\infty) \to [0, +\infty)$ given by

$$Q_{\mathcal{C}}(f, g)(t) = \sum_{k=n}^{+\infty} 2^{-k} \left(\left(\frac{1}{g(k)} - \frac{1}{f(k)} \right) \vee 0 \right),$$

where $t \in (n, n + 1]$, $n \in \omega$.

Remark 1. Note that, for each $f, g \in \mathcal{C}$ and $t > 0$, we have

$$Q_{\mathcal{C}}(f,g)(t) \leqslant \sum_{k=0}^{+\infty} 2^{-k}\left(\left(\frac{1}{g(k)} - \frac{1}{f(k)}\right) \vee 0\right) = d_{\mathcal{C}}(f,g).$$

In particular, $Q_{\mathcal{C}}(f,g)(t) = d_{\mathcal{C}}(f,g)$ whenever $t \in (0,1]$.

Lemma 1. *For each $f, g, h \in \mathcal{C}$ and $t, s > 0$, it follows*

$$Q_{\mathcal{C}}(f,g)(t+s) \leqslant Q_{\mathcal{C}}(f,h)(t) + Q_{\mathcal{C}}(h,g)(s).$$

Proof. Let $t \in (n, n+1]$ and $s \in (m, m+1]$, $n, m \in \omega$. Then $t+s \in (n+m, n+m+1]$ or $t+s \in (n+m+1, n+m+2]$. Consequently

$$\begin{aligned} Q_{\mathcal{C}}(f,g)(t+s) &\leqslant \sum_{k=n+m}^{+\infty} 2^{-k}\left(\left(\frac{1}{g(k)} - \frac{1}{f(k)}\right) \vee 0\right) \\ &\leqslant \sum_{k=n+m}^{+\infty} 2^{-k}\left(\left(\frac{1}{h(k)} - \frac{1}{f(k)}\right) \vee 0\right) + \sum_{k=n+m}^{+\infty} 2^{-k}\left(\left(\frac{1}{g(k)} - \frac{1}{h(k)}\right) \vee 0\right) \\ &\leqslant \sum_{k=n}^{+\infty} 2^{-k}\left(\left(\frac{1}{h(k)} - \frac{1}{f(k)}\right) \vee 0\right) + \sum_{k=m}^{+\infty} 2^{-k}\left(\left(\frac{1}{g(k)} - \frac{1}{h(k)}\right) \vee 0\right) \\ &= Q_{\mathcal{C}}(f,h)(t) + Q_{\mathcal{C}}(h,g)(s). \end{aligned}$$

The proof is finished. $\square$

Lemma 2. *For each $f, g \in \mathcal{C}$ the function $Q_{\mathcal{C}}(f,g)$ is nonincreasing and left continuous on $\mathbb{R}$.*

Proof. Let $f, g \in \mathcal{C}$ and $0 < t < s$. Then, by Lemma 1,

$$Q_{\mathcal{C}}(f,g)(s) \leqslant Q_{\mathcal{C}}(f,f)(s-t) + Q_{\mathcal{C}}(f,g)(t) = Q_{\mathcal{C}}(f,g)(t),$$

so $Q_{\mathcal{C}}(f,g)$ is nonincreasing.

Finally, left continuity of $Q_{\mathcal{C}}(f,g)$ on $(0,+\infty)$ is clear because if $t_m \to t^-$, $t > 0$, there is an m_0 such that $Q_{\mathcal{C}}(f,g)(t_m) = Q_{\mathcal{C}}(f,g)(t)$ for all $m \geqslant m_0$, and thus $\lim_m Q_{\mathcal{C}}(f,g)(t_m) = Q_{\mathcal{C}}(f,g)(t)$. $\square$

Theorem 1. *For each $f, g \in \mathcal{C}$ and each $t \in \overline{\mathbb{R}}$ let $P_{\mathcal{C}}(f,g)(t) = 0$ whenever $t \in [-\infty, 0]$, $P_{\mathcal{C}}(f,g)(+\infty) = 1$, and*

$$P_{\mathcal{C}}(f,g)(t) = \frac{t}{t + Q_{\mathcal{C}}(f,g)(t)} \quad \textit{whenever } t > 0.$$

Then $(\mathcal{C}, P_{\mathcal{C}}, \wedge)$ is a quasi-Menger space.

Furthermore for each $f, g \in \mathcal{C}$, $P_{\mathcal{C}}(f,g)(t) = P_{d_{\mathcal{C}}}(f,g)(t)$ whenever $t \leqslant 1$, and $P_{\mathcal{C}}(f,g)(t) \geqslant P_{d_{\mathcal{C}}}(f,g)(t)$ whenever $t > 1$, where $(\mathcal{C}, P_{d_{\mathcal{C}}}, \wedge)$ is the standard quasi-Menger space of $(\mathcal{C}, d_{\mathcal{C}})$.

Proof. We first show that $P_{\mathcal{C}}(f,g) \in D^+$ for $f, g \in \mathcal{C}$. Indeed, since $P_{\mathcal{C}}(f,g)(t) = 0$ for $t \in [-\infty, 0]$, and, by Lemma 2, $P_{\mathcal{C}}(f,g)$ is nondecreasing and left continuous on $\mathbb{R}$, we deduce that $P_{\mathcal{C}}(f,g) \in \Delta^+$. Moreover, by Remark 1, we obtain

$$P_{\mathcal{C}}(f,g)(t) \geqslant \frac{t}{t + d_{\mathcal{C}}(f,g)},$$

for all $t > 0$, and consequently $\lim_{t \to +\infty} P_{\mathcal{C}}(f,g)(t) = 1$. Hence $P_{\mathcal{C}}(f,g) \in D^+$.

In order to show condition (PQM1) of Definition 1, let $f, g \in \mathcal{C}$ be such that $P_{\mathcal{C}}(f,g)(t) = P_{\mathcal{C}}(g,f)(t) = 1$ for all $t > 0$. In particular, $P_{\mathcal{C}}(f,g)(1) = P_{\mathcal{C}}(g,f)(1) = 1$, and thus $d_{\mathcal{C}}(f,g) = d_{\mathcal{C}}(g,f) = 0$, by Remark 1. Therefore $f = g$ because $d_{\mathcal{C}}$ is a quasi-metric on $\mathcal{C}$. Conversely, it is clear that $P_{\mathcal{C}}(f,f) = \varepsilon_0$.

Now let $f, g, h \in \mathcal{C}$ and $t, s > 0$. We want to show that $P_{\mathcal{C}}(f,g)(t+s) \geqslant P_{\mathcal{C}}(f,h)(t) \wedge P_{\mathcal{C}}(h,g)(s)$. Assume, without loss of generality, that $P_{\mathcal{C}}(f,h)(t) \leqslant P_{\mathcal{C}}(h,g)(s)$. Then $tQ_{\mathcal{C}}(h,g)(s) \leqslant sQ_{\mathcal{C}}(f,h)(t)$. So, from Lemma 1 and the above inequality it follows that

$$tQ_{\mathcal{C}}(f,g)(t+s) \leqslant tQ_{\mathcal{C}}(f,h)(t) + tQ_{\mathcal{C}}(h,g)(s) \leqslant (t+s)Q_{\mathcal{C}}(f,h)(t).$$

Therefore

$$P_{\mathcal{C}}(f,g)(t+s)=\frac{t+s}{t+s+Q_{\mathcal{C}}(f,g)(t+s)}\geqslant\frac{t}{t+Q_{\mathcal{C}}(f,h)(t)}=P_{\mathcal{C}}(f,h)(t).$$

We have shown condition (QMe) of Definition 2 is satisfied.

Consequently $(\mathcal{C},P_{\mathcal{C}},\wedge)$ is a quasi-Menger space.

Finally, we have $P_{\mathcal{C}}(f,g)(t)=0=P_{d_{\mathcal{C}}}(f,g)(t)=0$ for $t\in[-\infty,0]$, and by Remark 1, $P_{\mathcal{C}}(f,g)(t)=P_{d_{\mathcal{C}}}(f,g)(t)$ for $t\in(0,1]$, and $P_{\mathcal{C}}(f,g)(t)\geqslant P_{d_{\mathcal{C}}}(f,g)(t)$ for $t>1$. This concludes the proof. □

Corollary. *For each continuous t-norm $*$, $(\mathcal{C},P_{\mathcal{C}},*)$ is a quasi-Menger space.*

Definition 3. The quasi-Menger space $(\mathcal{C},P_{\mathcal{C}},\wedge)$ is said to be the complexity probabilistic quasi-metric space.

Remark 2. The complexity probabilistic quasi-metric space provides a suitable model to interpret the (asymptotic) efficiency of complexity functions. Indeed, let $f,g\in\mathcal{C}$ such that f is asymptotically more efficient than g; then, there is $n_0\in\omega$ such that $f(n)\leqslant g(n)$ for $n\geqslant n_0$, and hence $P_{\mathcal{C}}(f,g)(t)=1$ for all $t>n_0$. Conversely, if we have computed that $P_{\mathcal{C}}(f,g)(t)=1$ for some $t\in(n_0,n_0+1]$, then the following precise information is automatically obtained: $f(n)\leqslant g(n)$ for all $n\geqslant n_0$.

Remark 3. In the light of Remark 2, observe that if f,g,h, are the complexity functions of Example 1, then $P_{\mathcal{C}}(f,g)(t)=1$ for all $t>11$, while $P_{\mathcal{C}}(f,h)(t)<1$ for all $t>0$, which agrees with the facts that f is asymptotically more efficient than g (more exactly, $f(n)<g(n)$ for $n\geqslant 11$), and that $P_{\mathcal{C}}(f,h)(t)<P_{\mathcal{C}}(f,g)(t)$ for all $t>11$, as its was desirable.

We illustrate Theorem 1 and Remark 2 with the following example.

Example 2. Let $f,g\in\mathcal{C}$ be such that $\mathcal{O}(f)\subset\mathcal{O}(g)$, with $f(0)=f(1)=+\infty$ and $f(n)<g(n)$ eventually (in fact, there exist many interesting situations of this type: for instance, the solution of the recurrence equation associated to Quicksort algorithm for the average case, is in $\mathcal{O}(n\log n)$, while for the worst case, it is in $\mathcal{O}(n^2)$ [14]; of course, $\mathcal{O}(n\log n)\subset\mathcal{O}(n^2)$).

Now take $h\in\mathcal{C}\backslash\{f\}$ such that $h(0)=h(1)=+\infty$ and $h(n)\leqslant f(n)$ for all $n\geqslant 2$.

We shall construct a complexity function $u\in\mathcal{O}(g)$ such that, as in Example 1 above, f is asymptotically more efficient than u but $d_{\mathcal{C}}(f,h)<d_{\mathcal{C}}(f,u)$.

Indeed, since $h\in\mathcal{C}$, there exists $n_0\geqslant 2$ such that $\sum_{n=n_0+1}^{\infty}2^{-n}(1/h(n))<1$. Then, define $u\in\mathcal{C}$ by $u(0)=u(1)=1$, $u(n)=h(n)$ for $2\leqslant n\leqslant n_0$, and $u(n)=g(n)$ for $n>n_0$. Clearly $u\in\mathcal{O}(g)$, and f is asymptotically more efficient than u.

Furthermore we have

$$\begin{aligned}d_{\mathcal{C}}(f,h)&=\sum_{n=2}^{+\infty}2^{-n}\left(\frac{1}{h(n)}-\frac{1}{f(n)}\right)<1+\sum_{n=2}^{n_0}2^{-n}\left(\frac{1}{h(n)}-\frac{1}{f(n)}\right)\\&<1+\frac{1}{2}+\sum_{n=2}^{n_0}2^{-n}\left(\frac{1}{u(n)}-\frac{1}{f(n)}\right)\leqslant d_{\mathcal{C}}(f,u),\end{aligned}$$

so, $d_{\mathcal{C}}$ is not suitable to measure relative progress made in lowering when the complexity function f is replaced by u. However, the probabilistic quasi-metric $P_{\mathcal{C}}$ constructed in Theorem 1 avoids this disadvantage because from the obvious fact that $Q_{\mathcal{C}}(f,h)(t)>0$ for all $t>0$, it follows that $P_{\mathcal{C}}(f,h)(t)<1$ for all $t>0$, while by construction of the complexity function u, it follows that $Q_{\mathcal{C}}(f,u)(t)=0$ eventually, so $P_{\mathcal{C}}(f,u)(t)=1$ eventually, which agrees with the fact that f is asymptotically more efficient than u but it is not more efficient than h (in fact, h is more efficient than f).

Proposition 1. *The topology $\tau_{P_{\mathcal{C}}}$ generated by the complexity probabilistic quasi-metric space $(\mathcal{C},P_{\mathcal{C}},\wedge)$ and the topology $\tau_{P_{d_{\mathcal{C}}}}$ generated by the standard quasi-Menger space $(\mathcal{C},P_{d_{\mathcal{C}}},\wedge)$ coincide on $\mathcal{C}$. Moreover $\tau_{(P_{\mathcal{C}})^{-1}}=\tau_{(P_{d_{\mathcal{C}}})^{-1}}$ on $\mathcal{C}$.*

Proof. Let $f\in\mathcal{C}$. From Theorem 1 it follows $B_{P_{\mathcal{C}}}(f,\varepsilon,t)=B_{P_{d_{\mathcal{C}}}}(f,\varepsilon,t)$ and $B_{(P_{\mathcal{C}})^{-1}}(f,\varepsilon,t)=B_{(P_{d_{\mathcal{C}}})^{-1}}(f,\varepsilon,t)$ for all $\varepsilon\in(0,1)$ and $t\in(0,1]$. Hence $\tau_{P_{\mathcal{C}}}=\tau_{P_{d_{\mathcal{C}}}}$ and $\tau_{(P_{\mathcal{C}})^{-1}}=\tau_{(P_{d_{\mathcal{C}}})^{-1}}$ on $\mathcal{C}$. □

Remark 4. Since $(\mathcal{C},P_{d_{\mathcal{C}}},\wedge)$ is the standard quasi-Menger space of $(\mathcal{C},d_{\mathcal{C}})$, then $\tau_{P_{d_{\mathcal{C}}}}=\tau_{d_{\mathcal{C}}}$ and $\tau_{(P_{d_{\mathcal{C}}})^{-1}}=\tau_{(P_{\mathcal{C}})^{-1}}$. We deduce from Proposition 1, that $\tau_{P_{\mathcal{C}}}=\tau_{d_{\mathcal{C}}}$ and $\tau_{(P_{\mathcal{C}})^{-1}}=\tau_{(d_{\mathcal{C}})^{-1}}$ on $\mathcal{C}$, so $\tau_{(P_{\mathcal{C}})^i}=\tau_{(d_{\mathcal{C}})^s}$ on $\mathcal{C}$.

In the last part of the paper we illustrate our constructions by obtaining a version of the Banach fixed point theorem which will be applied to the complexity analysis of Divide and Conquer algorithms and Quicksort algorithms, respectively. To this end, a result on completeness of the complexity probabilistic quasi-metric space will be needed.

Let us recall that a sequence $(x_n)_n$ in a probabilistic metric space (X, P) is said to be a Cauchy sequence if for each $\varepsilon \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $P_{x_n x_m}(t) > 1-\varepsilon$ for all $n, m \geqslant n_0$. Then, (X, P) is called complete if for each Cauchy sequence $(x_n)_n$ there exists $x \in X$ satisfying the following condition: for each $\varepsilon \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $P(x, x_n)(t) > 1-\varepsilon$ for all $n \geqslant n_0$, i.e., if for each $t > 0$, $\lim_n P(x, x_n)(t) = 1$.

Since every Menger space can be considered as a probabilistic metric space, the notions of a Cauchy sequence in Menger spaces and of a complete Menger space are defined in the obvious manner.

We say that a quasi-Menger space $(X, M, *)$ is bicomplete provided that $(X, P^i, *)$ is a complete Menger space.

Following [19], a sequence $(x_n)_n$ in a quasi-metric space (X, d) is said to be left K-Cauchy if for each $\varepsilon > 0$ there is an n_0 such that $d(x_n, x_m) < \varepsilon$ whenever $n_0 \leqslant n \leqslant m$. The quasi-metric space (X, d) is Smyth completable if and only if every left K-Cauchy sequence is a Cauchy sequence in the metric space (X, d^s), and (X, d) is Smyth complete if and only if every left K-Cauchy sequence is convergent in (X, d^s) [16]. Clearly, every Smyth complete quasi-metric space is bicomplete.

Smyth completability provides an efficient tool to give a topological foundation for many kinds of spaces which arise naturally in Theoretical Computer Science (e.g. [9,16,17,21,27]). In particular, it was proved in [21] the following result.

Theorem 2. *The complexity quasi-metric space $(\mathcal{C}, d_{\mathcal{C}})$ is Smyth complete.*

In the light of these facts we introduce the following probabilistic counterpart to the notion of Smyth completeness.

Definition 4. A sequence $(x_n)_n$ in a probabilistic quasi-metric space (X, P) is called left K-Cauchy if for each $\varepsilon \in (0,1)$ and each $t > 0$ there is an n_0 such that $P(x_n, x_m)(t) > 1-\varepsilon$ whenever $n_0 \leqslant n \leqslant m$. We say that (X, P) is Smyth complete for each left K-Cauchy sequence $(x_n)_n$ in X there exists $x \in X$ such that $\lim_n P^i(x, x_n)(t) = 1$ for all $t > 0$.

Theorem 3. *The complexity probabilistic quasi-metric space $(\mathcal{C}, P_{\mathcal{C}}, \wedge)$ is Smyth complete.*

Proof. Let $(f_n)_n$ be a left K-Cauchy sequence in $(\mathcal{C}, P_{\mathcal{C}}, \wedge)$. Choose $\varepsilon \in (0, 1/2)$. Then, there is an n_0 such that $P_{\mathcal{C}}(f_n, f_m)(1) > 1-\varepsilon$ whenever $n_0 \leqslant n \leqslant m$. Since, by Theorem 1, $P_{\mathcal{C}}(f_n, f_m)(1) = P_{d_{\mathcal{C}}}(f_n, f_m)(1)$, it follows that $1/(1 + d_{\mathcal{C}}(f_n, f_m)) > 1-\varepsilon$ whenever $n_0 \leqslant n \leqslant m$, and thus $d_{\mathcal{C}}(f_n, f_m) < \varepsilon/(1-\varepsilon) < 2\varepsilon$ whenever $n_0 \leqslant n \leqslant m$. We have shown that $(f_n)_n$ is a left K-Cauchy sequence in $(\mathcal{C}, d_{\mathcal{C}})$. It follows from Theorem 2, that $(f_n)_n$ converges to some $f \in \mathcal{C}$ in the metric space $(\mathcal{C}, (d_{\mathcal{C}})^s)$. Therefore $(f_n)_n$ converges to f in the Menger space $(\mathcal{C}, (P_{\mathcal{C}})^i, \wedge)$ by Remark 4. We conclude that $(\mathcal{C}, P_{\mathcal{C}}, \wedge)$ is Smyth complete. $\square$

Corollary. *The complexity probabilistic quasi-metric space $(\mathcal{C}, P_{\mathcal{C}}, \wedge)$ is bicomplete.*

In [26, Section 6], Schellekens proved that Divide and Conquer algorithms induce contractive self-mappings on the complexity space $(\mathcal{C}, d_{\mathcal{C}})$. For applications, the complexity quasi-metric space is typically restricted, in this case, to functions with range over positive integers which are a power of a given integer b. Thus, let $a, b, c \in \mathbb{N}$ with $a, b \geqslant 2$, let n range over the set $\{b^k\colon\ k \in \omega\}$ and let $h \in \mathcal{C}$ with $h(n) < +\infty$ if $n \in \{b^k\colon\ k \in \omega\}$. A functional Φ associated to a Divide and Conquer algorithm is then defined by $\Phi(f)(1) = c$, $\Phi(f)(n) = af(n/b) + h(n)$ if $n \in \{b^k\colon\ k \in \mathbb{N}\}$, and $\Phi(f)(n) = +\infty$ if $n \notin \{b^k\colon\ k \in \mathbb{N}\}$.

On the other hand, Gregori and Sapena introduced in [12] a kind of contractive self-mappings for fuzzy metric spaces in the sense of George and Veeramani [7,8] and obtained an interesting version of the Banach fixed point theorem (see also [18]). In our next result we shall give a version of Gregori and Sapena's theorem for quasi-Menger spaces.

Definition 5. (Compare [12, Definition 3.8] and [18, Definition 3.2].) A sequence $(x_n)_n$ in a quasi-Menger space $(X, P, *)$ is called contractive if there is $\alpha \in (0,1)$ such that

$$P(x_{n+1}, x_{n+2})(t) \geqslant \frac{P(x_n, x_{n+1})(t)}{P(x_n, x_{n+1})(t) + \alpha(1 - P(x_n, x_{n+1})(t))},$$

for all $n \in \mathbb{N}$ and $t > 0$.

If the above inequality holds for all $t \in (0, 1]$, we say that $(x_n)_n$ is $(0, 1]$-contractive.

Definition 6. A self-mapping f of a quasi-Menger space $(X, P, *)$ is called contractive if there is $\alpha \in (0,1)$ such that

$$P(fx, fy)(t) \geqslant \frac{P(x, y)(t)}{P(x, y)(t) + \alpha(1 - P(x, y)(t))}$$

for all $x, y \in X$ and $t > 0$.

Theorem 4. *Let f be a contractive self-mapping of a bicomplete quasi-Menger space $(X, P, *)$, with $*$ a continuous t-norm. If every contractive sequence in $(X, P, *)$ is a Cauchy sequence in $(X, P^i, *)$, then f has a unique fixed point. If, in addition, $P(x, y)(t) > 0$ for all $x, y \in X$, $t > 0$, then the fixed point of f is unique.*

Proof. Choose $x \in X$. Since f is contractive, it is clear that the sequence $(x_n)_n$ is contractive, where $x_n = f^n x$ for all $n \in \omega$. So, by hypothesis, it is a Cauchy sequence in $(X, P^i, *)$. Then, there exists $u \in X$ such that $(x_n)_n$ converges to u for τ_{P^i}, i.e., $\lim_n P(u, x_n)(t) = \lim_n P(x_n, u)(t) = 1$ for all $t > 0$. Therefore, $\lim_n P(fu, fx_n)(t) = \lim_n P(fx_n, fu)(t) = 1$ for all $t > 0$, by the contraction condition. Consequently $(x_n)_n$ converges to fu for τ_{P^i}, so $u = fu$.

Finally, suppose $P(x, y)(t) > 0$ for all $x, y \in X$, $t > 0$, and let $v \in X$ be a fixed point of f. Then

$$P(u,v)(t) = P(fu, fv)(t) \geqslant \frac{P(u,v)(t)}{P(u,v)(t) + \alpha(1 - P(u,v)(t))},$$

for all $t > 0$. Therefore $P(u, v)(t) + \alpha(1 - P(u, v)(t)) \geqslant 1$. Since $\alpha \in (0, 1)$, it follows that $P(u, v)(t) = 1$ for all $t > 0$. Similarly, we have that $P(v, u)(t) = 1$ for all $t > 0$. Hence $u = v$. This concludes the proof. □

The following easy example shows that, unfortunately, the functional Φ associated to a Divide and Conquer algorithm as given above, is not contractive in the sense of Definition 6, and thus Theorem 4 does not work in this case.

Example 3. Let $a = b = c = 2$, and let $f, g \in \mathcal{C}$ given by $f(n) = 2$ for all $n \in \omega$, $g(n) = 2$ for all $n \in \omega \backslash \{1\}$ and $g(2) = 1$. Since $f(n) = g(n)$ for all $n \in \omega \backslash \{1\}$, we deduce that, in particular, $P_{\mathcal{C}}(f, g)(4) = 1$. Hence, for any $\alpha \in (0, 1)$,

$$\frac{P_{\mathcal{C}}(f,g)(4)}{P_{\mathcal{C}}(f,g)(4) + \alpha(1 - P_{\mathcal{C}}(f,g)(4))} = 1.$$

However

$$P_{\mathcal{C}}(\Phi f, \Phi g)(4) = \frac{4}{4 + Q_{\mathcal{C}}(\Phi f, \Phi g)(4)} < 1,$$

because $g(2) < f(2)$, and thus

$$\begin{aligned} Q_{\mathcal{C}}(\Phi f, \Phi g)(4) &= \sum_{n=3}^{\infty} 2^{-n}\left(\left(\frac{1}{\Phi g(n)} - \frac{1}{\Phi f(n)}\right) \vee 0\right) \\ &= 2^{-4}\left(\frac{1}{\Phi g(4)} - \frac{1}{\Phi f(4)}\right) \\ &= 2^{-4}\left(\frac{1}{2g(2) + h(2)} - \frac{1}{2f(2) + h(2)}\right) > 0. \end{aligned}$$

Motivated by the preceding example, we shall prove a variant of Theorem 4, by means of a suitable modification of the contraction condition of Definition 6, which can be successfully applied both to the functional associated to Divide and Conquer algorithms and to the functional associated to Quicksort algorithms, as we shall show.

Theorem 5. *Let f be a self-mapping of a bicomplete quasi-Menger space $(X, P, *)$, with $*$ a continuous t-norm, such that there is $\alpha \in (0, 1)$ satisfying for each $x, y \in X$ and $t > 0$, the following condition*

$$P(fx, fy)(t) \geqslant \frac{tP(x,y)(t - n_t)}{tP(x,y)(t - n_t) + \alpha(t - n_t)(1 - P(x,y)(t - n_t))},$$

where $t \in (n_t, n_t + 1]$, $n_t \in \omega$. If each $(0, 1]$-contractive sequence in $(X, P, *)$ is a Cauchy sequence in $(X, P^i, *)$, then f has a fixed point. If, in addition, $P(x, y)(t) > 0$ for all $x, y \in X$, $t > 0$, then the fixed point of f is unique.

Proof. Choose $x \in X$. Since for each $t \in (0, 1]$, $n_t = 0$, it immediately follows that the sequence $(f^n x)_n$ is $(0, 1]$-contractive. So, by hypothesis, it is a Cauchy sequence in $(X, P^i, *)$. Then, there exists $u \in X$ such that $(f^n x)_n$ converges to u for τ_{P^i}. Thus $\lim_n P(u, f^n x)(t - n_t) = \lim_n P(f^n x, u)(t - n_t) = 1$ for all $t > 0$. Therefore, $\lim_n P(fu, f^n x)(t) = \lim_n P(f^n x, fu)(t) = 1$ for all $t > 0$, by the contraction condition. Consequently $(f^n x)_n$ converges to fu for τ_{P^i}, so $u = fu$.

Finally, suppose $P(x, y)(t) > 0$ for all $x, y \in X$, $t > 0$, and let $v \in X$ be a fixed point of f. Then, as in the proof of Theorem 4 we obtain $P(u, v)(t) = 1$ for all $t \in (0, 1]$. Hence $P(u, v)(t) = 1$ for all $t > 0$, and thus $u = v$. This concludes the proof. □

Next we shall apply Theorem 5 to deduce the existence and uniqueness of solution for the functional Φ associated to a Divide and Conquer algorithm.

Indeed, we have shown (corollary of Theorem 3) that the complexity probabilistic quasi-metric space $(\mathcal{C}, P_{\mathcal{C}}, \wedge)$ is bicomplete. Note also that $P_{\mathcal{C}}(f,g)(t) > 0$ for all $f, g \in \mathcal{C}$ and $t > 0$.

Now we show that if $(f_n)_n$ is a $(0,1]$-contractive sequence in $(\mathcal{C}, P_{\mathcal{C}}, \wedge)$, then it is a Cauchy sequence in the Menger space $(\mathcal{C}, (P_{\mathcal{C}})^i, \wedge)$. Indeed, let $\alpha \in (0,1)$ such that

$$P_{\mathcal{C}}(f_{n+1}, f_{n+2})(t) \geqslant \frac{P_{\mathcal{C}}(f_n, f_{n+1})(t)}{P_{\mathcal{C}}(f_n, f_{n+1})(t) + \alpha(1 - P_{\mathcal{C}}(f_n, f_{n+1})(t))},$$

for all $n \in \omega$ and $t \in (0,1]$. Then, an easy computation shows that

$$Q_{\mathcal{C}}(f_{n+1}, f_{n+2})(t) \leqslant \alpha Q_{\mathcal{C}}(f_n, f_{n+1})(t),$$

for all $n \in \omega$ and $t \in (0,1]$. So, by Remark 1, $d_{\mathcal{C}}(f_{n+1}, f_{n+2}) \leqslant \alpha d_{\mathcal{C}}(f_n, f_{n+1})$ for all $n \in \omega$. From the triangle inequality it easily follows that $d_{\mathcal{C}}(f_n, f_{n+k}) \leqslant (\alpha^n/(1-\alpha)) d_{\mathcal{C}}(f_0, f_1)$ for all $n, k \in \omega$. So $(f_n)_n$ is a left K-Cauchy sequence in $(\mathcal{C}, d_{\mathcal{C}})$. Hence, by Theorem 2 and Remark 4, the sequence $(f_n)_n$ converges for τ_{P^i}, and consequently it is a Cauchy sequence in $(\mathcal{C}, (P_{\mathcal{C}})^i, \wedge)$.

Finally, we show that Φ satisfies the contraction condition of Theorem 5 for $\alpha = 1/a$. Indeed, Schellekens proved in [26, Theorem 6.1] that $d_{\mathcal{C}}(\Phi f, \Phi g) \leqslant \alpha d_{\mathcal{C}}(f,g)$ for all $f, g \in \mathcal{C}$, where $\alpha = 1/a$. Then, by Remark 1 and the fact that $t - n_t \in (0,1]$ for all $t > 0$, we have the following relations

$$\begin{aligned}
\frac{tP_{\mathcal{C}}(f,g)(t-n_t)}{tP_{\mathcal{C}}(f,g)(t-n_t) + \alpha(t-n_t)(1 - P_{\mathcal{C}}(f,g)(t-n_t))} &= \frac{t}{t + \alpha Q_{\mathcal{C}}(f,g)(t-n_t)} \\
&= \frac{t}{t + \alpha d_{\mathcal{C}}(f,g)} \\
&\leqslant \frac{t}{t + d_{\mathcal{C}}(\Phi f, \Phi g)} \\
&\leqslant \frac{t}{t + Q_{\mathcal{C}}(\Phi f, \Phi g)(t)} \\
&= P_{\mathcal{C}}(\Phi f, \Phi g)(t).
\end{aligned}$$

Hence, we can apply Theorem 5 and thus Φ has a unique fixed point $g_0 \in \mathcal{C}$ which is the solution of the recurrence equation corresponding to the Divide and Conquer algorithm.

We conclude the paper by applying Theorem 5 to show the existence and uniqueness of solution for the recurrence equation T given by $T(1) = 0$, and

$$T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} T(n-1),$$

for $n > 1$. (This recurrence equation was obtained by Kruse [14, Section 4.8.4] in discussing the average case analysis of Quicksort algorithms.)

In this case, the associated functional $\Phi : \mathcal{C} \to \mathcal{C}$ is defined by

$$\Phi f(0) = \Phi f(1) = +\infty, \qquad \Phi f(2) = 1,$$

and

$$\Phi f(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} f(n-1)$$

for all $n > 2$, $f \in \mathcal{C}$ (see for instance [20,25]), and we have $d_{\mathcal{C}}(\Phi f, \Phi g) \leqslant d_{\mathcal{C}}(f,g)/2$ for all $f, g \in \mathcal{C}$.

Hence, it suffices to show that the contraction condition of Theorem 5 holds. Indeed, take $\alpha = 1/2$. Then, as in the case of Divide and Conquer algorithms, we obtain, for each $t > 0$,

$$\begin{aligned}
\frac{tP_{\mathcal{C}}(f,g)(t-n_t)}{tP_{\mathcal{C}}(f,g)(t-n_t) + \alpha(t-n_t)(1 - P_{\mathcal{C}}(f,g)(t-n_t))} &= \frac{t}{t + \alpha d_{\mathcal{C}}(f,g)} \\
&\leqslant \frac{t}{t + d_{\mathcal{C}}(\Phi f, \Phi g)} \\
&\leqslant P_{\mathcal{C}}(\Phi f, \Phi g)(t).
\end{aligned}$$

Consequently Φ has a unique fixed point $g_0 \in \mathcal{C}$. We conclude that the function $h : \mathbb{N} \to [0, +\infty)$ given by $h(1) = 0$ and $h(n) = g_0(n)$ for $n > 1$, is the (unique) solution for the recurrence equation T.

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