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NOTE

Common Fixed Point Theorems for Contractive Maps

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Two common fixed point theorems have been proved by using minimal type commutativity and contractive conditions. The last theorem extends known results on compatible maps to a wider class of mappings. © 1998 Academic Press

INTRODUCTION

In 1986 Jungck [3] generalized the notion of weakly commuting maps by introducing the concept of compatible maps. Since then many interesting fixed point theorems for compatible mappings satisfying contractive type conditions have been obtained by various authors. These common fixed point theorems invariably require a commutativity condition besides a contractive condition and a majority of the successive generalizations are aimed at weakening either or both of these conditions. The present paper is an attempt to obtain common fixed point theorems for a family of maps under minimal type commutativity and contractive conditions.

Two self-maps A and S of a metric space (X, d) are called compatible if $\lim_n d(ASx_n, SAX_n) = 0$ when $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. In a recent work, the present author [7] introduced the notion of R-weakly commuting maps. Two self-maps A and S of a metric space (X, d) are called R-weakly commuting at a point x in X if $d(ASx, SAx) \leq Rd(Ax, Sx)$ for some $R > 0$. The maps A and S are called pointwise R-weakly commuting on X if given x in X there exists $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$. It is obvious that A and S can fail to be pointwise R-weakly commuting only if there is some x in X

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such that $Ax = Sx$ but $ASx \neq SAx$, that is, only if they possess a coincidence point at which they do not commute. This means that (1) a contractive type mapping pair cannot possess a common fixed point without being pointwise R-weakly commuting since a common fixed point is also a coincidence point at which the mappings commute and since contractive conditions exclude the possibility of two types of coincidence points, and (2) compatible maps are necessarily pointwise R-weakly commuting since compatible maps commute at coincidence points. However, pointwise R-weakly commuting maps need not be compatible as shown in the example considered by us.

It is clear from the above discussion that pointwise R-weak commutativity is a necessary, hence minimal, condition for the existence of common fixed points of contractive type maps. As an application of R-weak commutativity, we prove two common fixed point theorems for a family of mappings satisfying a minimal type contractive condition determined by a contractive function ϕ. Prior to this, there is perhaps no common fixed point theorem obtained without assuming additional conditions on the contractive function ϕ and we feel that the present theorems cannot be further simplified except in respect of the condition on the ranges of the mappings.

MAIN RESULTS

If $\{A_i\}$, $i = 1, 2, \ldots, S$, and T are self-mappings of a metric space (X, d), in the sequel for each $i > 1$ we shall denote

$$M_{1i}(x, y) = \max\{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), [d(A_1x, Ty) + d(A_iy, Sx)]/2\}.$$

Also, let ϕ: $R_+ \to R_+$ denote a function such that $\phi(t) < t$ for each $t > 0$.

THEOREM 1. *Let $\{A_i\}$, $i = 1, 2, 3, \ldots, S$, and T be self-mappings of a metric space (X, d) such that $A_iX \subset SX$ when $i > 1$, $A_1X \subset TX$ and*

(i) *pairs (A_1, S) and (A_i, T), $i > 1$, are pointwise R-weakly commuting with at least one pair noncompatible,*

(ii) *$d(A_1x, A_iy) < M_{1i}(x, y)$ whenever $M_{1i}(x, y) > 0$ and $i > 1$,*

(iii) *$d(A_1x, A_2y) \leq \phi(M_{12}(x, y))$.*

If the range of one of the mappings is a complete subspace of X then all the A_i, S, and T have a unique common fixed point.

Proof. Suppose that T is noncompatible with A_k for some $k > 1$. Then there exists a sequence $\{z_n\}$ in X such that $\lim_n A_k z_n = \lim_n Tz_n = t$ for some t in X but $\lim_n d(A_k Tz_n, TA_k z_n)$ is either nonzero or does not exist. Since $A_k X \subset SX$, corresponding to each z_n there exists x_n in X such that $A_k z_n = Sx_n$. Thus $A_k z_n = Sx_n \to t$ and $Tz_n \to t$ as $n \to \infty$. We claim that $A_1 x_n \to t$ as $n \to \infty$. If not, then by virtue of (ii) for sufficiently large values of n we get

$$\begin{aligned} d(A_1 x_n, A_k z_n) &\le M_{1k}(x_n, z_n) \\ &= \max\{d(Sx_n, Tz_n), d(A_1 x_n, Sx_n), d(A_k z_n, Tz_n), \\ &\qquad [d(A_1 x_n, Tz_n) + d(A_k z_n, Sx_n)]/2\} \\ &= d(A_1 x_n, Sx_n) = d(A_1 x_n, A_k z_n), \end{aligned}$$

a contradiction. Hence $A_1 x_n \to t$. Also, since $A_1 X \subset TX$, for each x_n there exists y_n in X such that $A_1 x_n = Ty_n$ and $A_1 x_n = Ty_n \to t$. We show that $A_i y_n \to t$ for each $i > 1$. If not, then using (ii) for sufficiently large values of n we get

$$d(A_1 x_n, A_i y_n) < M_{1i}(x_n, y_n) = d(A_1 x_n, A_i y_n),$$

a contradiction. Thus $A_1 x_n \to t$, $Sx_n \to t$, $Ty_n \to t$, and, for each $i > 1$, $A_i y_n \to t$, where $Ty_n = A_1 x_n$.

Next, suppose that S is noncompatible with A_1. Then there exists a sequence $\{x_n\}$ in X such that $\lim_n A_1 x_n = \lim_n Sx_n = t$ for some t in X but $\lim_n d(A_1 Sx_n, SA_1 x_n)$ is either nonzero or does not exist. Since $A_1 X \subset TX$, corresponding to each x_n there exists y_n in X such that $A_1 x_n = Ty_n$ and $A_1 x_n = Ty_n \to t$. By using (ii), just as in the previous case, for each $i > 1$ we obtain that $\lim_n A_i y_n = t$. Thus, in both cases, we obtain sequences $\{x_n\}$ and $\{y_n\}$ in X such that $A_1 x_n \to t$, $Sx_n \to t$, $Ty_n \to t$, and, for each $i > 1$, $A_i y_n \to t$, where $Ty_n = A_1 x_n$.

Now suppose that SX, the range of S, is a complete subspace of X. Then, since $\lim_n Sx_n = t$, there exists a point u in X such that $t = Su$. If $A_1 u \neq Su$, using (iii) for sufficiently large values of n we get

$$\begin{aligned} d(A_1 u, A_2 y_n) &\le \phi(M_{12}(u, y_n)) \\ &= \phi(\max\{d(Su, Ty_n), d(A_1 u, Su), \\ &\qquad d(A_2 y_n, Ty_n), [d(A_1 u, Ty_n) + d(A_2 y_n, Su)]/2\} \\ &= \phi(d(A_1 u, Su)). \end{aligned}$$

On letting $n \to \infty$ this inequality yields $d(A_1 u, Su) \le \phi(d(A_1 u, Su)) < d(A_1 u, Su)$, a contradiction. Hence $A_1 u = Su$. Since $A_1 X \subset TX$, there exists w in X such that $A_1 u = Tw$. If $A_1 u \neq A_i w$ for any value of $i > 1$, using (ii) we obtain

$$d(A_1 u, A_i w) < M_{1i}(u, w) = d(A_1 u, A_i w),$$

a contradiction. Hence $Su = A_1 u = Tw = A_i w$ for every $i > 1$.

Next, let us assume that TX is a complete subspace of X. Then, since $\lim_n Ty_n = t$, there exists a point w in X such that $t = Tw$. If $A_2 w \neq Tw$, using (iii) for sufficiently large values of n we get

$$d(A_1 x_n, A_2 w) \leq \phi(M_{12}(x_n, w)) = \phi(d(A_2 w, Tw)).$$

On letting $n \to \infty$ this yields $d(Tw, A_2 w) \leq \phi(d(A_2 w, Tw)) < d(A_2 w, Tw)$, a contradiction. Hence $A_2 w = Tw$. Since $A_2 X \subset SX$, there exists u in X such that $Tw = A_2 w = Su$. Using (ii) we get $Tw = A_2 w = Su = A_1 u$. Using (ii) once more we shall get $Su = A_1 u = Tw = A_i w$ for each $i > i$. Thus, irrespective of whether SX is assumed complete or TX is assumed to be so, we get u, w in X such that

$$A_1 u = Su = Tw = A_i w, \qquad i > 1.$$

Pointwise R-weak commutativity of A_1 and S implies that there exists $R_1 > 0$ such that $d(A_1 Su, SA_1 u) \leq R_1 d(A_1 u, Su) = 0$, that is, $A_1 Su = SA_1 u$ and $A_1 A_1 u = A_1 Su = SA_1 u = SSu$. Similarly, for each $i > 1$ there exists $R_i > 0$ such that $d(A_i Tw, TA_i w) \leq R_i d(A_i w, Tw) = 0$, that is, $A_i Tw = TA_i w$ and $A_i A_i w = A_i Tw = TA_i w = TTw$. If $A_1 A_1 u \neq A_1 u$, using (ii) we get

$$d(A_1 A_1 u, A_1 u) = d(A_1 A_1 u, A_2 w) < M_{12}(A_1 u, w) = d(A_1 A_1 u, A_2 w),$$

a contradiction. Hence $A_1 u = A_1 A_1 u = SA_1 u$ and $A_1 u$ is a common fixed point of A_1 and S. Similarly, if $A_i A_i w \neq A_i w$ for some $i > 1$, using (ii) we get

$$d(A_i w, A_i A_i w) = d(A_1 u, A_i A_i w) < M_{1i}(u, A_i w) = d(A_1 u, A_i A_i w),$$

a contradiction. Hence $A_i w = A_i A_i w = TA_i w$ for each $i > 1$, that is, $A_i w = A_1 u$ is a common fixed point of T and A_i for each $i > 1$. Uniqueness of the common fixed point follows easily. The proof is similar when $A_i X$ is assumed complete for some $i \geq 1$ since $A_1 X \subset TX$ and $A_i X \subset SX$ for $i > 1$. This completes the proof of the theorem.

We now slightly modify the above theorem to obtain the following theorem:

THEOREM 2. *Let* $\{A_i\}$, $i = 1, 2, 3, \ldots, S$, *and* T *be self-mappings of a metric space* (X, d) *such that* $A_1 X \subset TX$, $A_i X \subset SX$ *when* $i > 1$ *and*

(i) *pairs* (A_1, S) *and* (A_i, T), $i > 1$, *are pointwise R-weakly commuting with at least one pair compatible and one noncompatible.*

(ii) $d(A_1 x, A_i y) < M_{1i}(x, y)$ *whenever* $M_{1i}(x, y) > 0$ *and* $i > 1$,

(iii) $d(A_1 x, A_2 y) \leq \phi(M_{12}(x, y))$.

If one of the mappings in the compatible pair is continuous then all the A_i, S, and T have a unique common fixed point.

Proof. As in Theorem 1, noncompatibility of one of the R-weak commuting pairs implies the existence of sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$A_1 x_n \to t, \qquad Sx_n \to t, \qquad Ty_n \to t, \qquad A_i y_n \to t, \qquad \text{for each } i > 1,$$

where $Ty_n = A_1 x_n$ and $t \in X$.

Suppose that S is continuous and compatible with A_1. Then $SSx_{2n} \to St$, $SA_1 x_{2n} \to St$ and compatibility of A_1 and S implies that $A_1 Sx_{2n} \to St$. If we write $Sx_{2n} = y_n$ then, since $A_1 X \subset TX$, for each y_n there exists z_n such that $A_1 y_n = Tz_n$ and

$$A_1 Sx_{2n} = A_1 y_n \to St, \qquad SSx_{2n} = Sy_n \to St, \qquad Tz_n \to St.$$

Using (ii) it follows that $A_i z_n \to St$ for each $i > 1$. The remaining part of the proof is similar to that in Theorem 1 when SX was assumed complete.

Next, let A_1 be continuous and compatible with S. Then $A_1 A_1 x_{2n} \to A_1 t$, $A_1 Sx_{2n} \to A_1 t$ and compatibility of A_1 and S implies that $SA_1 x_{2n} \to A_1 t$. Since $A_1 X \subset TX$, there exists w in X such that $A_1 t = Tw$. Thus $A_1 A_1 x_{2n} \to Tw$ and $SA_1 x_{2n} \to Tw$. Now the remaining part of the proof in this case is similar to that when TX was assumed complete in Theorem 1.

Similar arguments apply when T is assumed compatible with A_p for some $p > 1$ and T or A_p is assumed continuous. Hence the theorem.

The following example illustrates our theorems.

EXAMPLE. Let $X = [2, 20)$ with the usual metric d. Define A_i, S, T: $X \to X$, $i = 1, 2, 3, \ldots,$ as follows:

$$A_1 x = 2, \qquad \text{for all } x,$$

$$Sx = x, \quad \text{if } x \le 8, \qquad Sx = 9, \quad \text{if } x > 8,$$

$$T2 = 2, \qquad Tx = 12 + x, \quad \text{if } 2 < x \le 5, \qquad Tx = x - 3, \quad \text{if } x > 5,$$

$$A_3 x = 2, \quad \text{if } x = 2 \text{ or } > 5, \qquad A_3 x = 8, \quad \text{if } 2 < x \le 5,$$

and for $i \ne 1, 3$

$$A_i x = 2, \qquad \text{if } x \le 4 \text{ or } > 5 - (1/i),$$

$$A_i x = 9, \qquad \text{if } 4 < x \le 5 - (1/i).$$

Then $\{A_i\}$, S, and T satisfy all the conditions of Theorems 1 and 2 above and have a unique common fixed point $x = 2$.

It can be verified in the above example that A_1 and S are compatible and also A_i and T are compatible when $i \ne 3$. The mappings A_3 and T

are pointwise R-weak commuting but noncompatible. A_3 and T are pointwise R-weak commuting since they commute at their coincidence points. To see that A_3 and T are noncompatible, let us consider a decreasing sequence $\{x_n\}$ such that $\lim_n x_n = 5$. Then $Tx_n = x_n - 3 \to 2$, $A_3 x_n = 2$, $TA_3 x_n = T2 = 2$, and $A_3 T x_n = A_3(x_n - 3) = 8$, that is, $\lim_n d(A_3 T x_n, TA_3 x_n) = 6$ and hence A_3 and T are noncompatible. It can also be verified that for each $i \neq 3$ the mappings A_1, A_i, S, and T satisfy the contractive condition $d(A_1 x, A_i y) \le \phi_i(M_{1i}(x, y))$ but the contractive function $\phi_i(t)$ fails to be upper semicontinuous at $t = 7$ while the function $g_i(t) = t/(t - \phi_i(t))$ fails to be nonincreasing in every open interval containing $t = 7$. Similarly, A_1, A_3, S, and T satisfy the contractive condition $d(A_1 x, A_3 y) \le \phi_3(M_{13}(x, y))$ but $\phi_3(t)$ fails to be upper semicontinuous at $t = 6$ while $g_3(t) = t/(t - \phi_3(t))$ fails to be nonincreasing in every open interval containing $t = 6$.

DISCUSSION AND AUXILIARY RESULTS

In view of the example given above, it is clear that our theorems apply to a wider class of mappings than the results on compatible maps since compatible maps constitute a proper subclass of pointwise R-weakly commuting maps. A few observations regarding the present results will be in order. We shall also state a theorem that can be routinely established.

(I) In condition (iii) of Theorems 1 and 2 above the only assumption made on the contractive function ϕ is that $\phi(t) < t$ for each $t > 0$. The analogous results using contractive function ϕ assume additional conditions on ϕ. For example, Carbone *et al.* [1] require ϕ to be nondecreasing and $g(t) = t/(t - \phi(t))$ nonincreasing. Theorems 3.3 and 5.1 of Jachymski [2] require ϕ to be upper semicontinuous. Pant [8] also assumes ϕ to be upper semicontinuous. Similarly, Park and Rhoades [10] assume ϕ to be nondecreasing and continuous from the right. In the above example neither ϕ is upper semicontinuous nor is $g(t) = t/(t - \phi(t))$ nonincreasing.

(II) Likewise, as was shown by Rao and Rao [11], the results on common fixed points of four mappings obtained by using Meir and Keeler type (ε, δ)-contractive condition

$$\varepsilon \le M_{12}(x, y) < \varepsilon + \delta \quad \Rightarrow \quad d(A_1 x, A_2 y) < \varepsilon$$

in place of condition (iii) hold only when δ satisfies some additional condition. The main theorem of Rhoades *et al.* [12] (read with Jungck *et al.* [4]) requires δ to be lower semicontinuous. Pant [5, 6] and Pant *et al.* [9]

assume δ to be nondecreasing. On the other hand, Rao and Rao [11] have used an additional inequality. But, as shown in [2], these additional inequalities or conditions on δ imply the existence of a contractive function ϕ which is upper semicontinuous. Hence our contractive condition (iii) is strictly weaker than the corresponding Meir–Keeler type conditions used in analogous results. In the example given above, it may be noted that the mappings A_1, A_i, S, T, $i \neq 3$, do not satisfy an (ε, δ) condition since δ cannot be defined at $\varepsilon = 7$ and A_1, A_3, S, T also do not satisfy an (ε, δ) condition since δ cannot be defined for them at $\varepsilon = 6$. Thus, our theorems apply even in the cases when an (ε, δ) condition may not hold.

It is thus clear that Theorems 1 and 2 have been proved by assuming much weaker conditions than in analogous results. The results concerning compatible mappings, besides being extendable in the spirit of Theorems 1 and 2 above, can be extended verbatim by simply using the property of R-weak commutativity in place of compatibility to the wider class of pointwise R-weakly commuting maps. For example, the following theorem of Jachymski [2]:

THEOREM (5.1 of [2]). *Let S and T be self-maps of a complete metric space (X, d) and either S or T is continuous. Let $\{A_i\}$, $i = 1, 2, \ldots$, be a sequence of self-maps of X satisfying* (i) *$A_1 X \subset TX$ and $A_i X \subset SX$ for $i > 1$,* (ii) *pairs of (A_1, S) and (A_i, T), $i > 1$, are compatible,* (iii) *$d(A_1 x, A_i y) \leq \phi_i(M_{1i}(x, y))$, where ϕ_i: $R_+ \to R_+$ is an upper semicontinuous function such that $\phi_i(t) < t$ for each $t > 0$. Then all the A_i, S, and T have a unique common fixed point.*

can be generalized to obtain the following theorem:

THEOREM 3. *Let $\{A_i\}$, $i = 1, 2, \ldots, S$, and T be self-maps of a complete metric space (X, d) satisfying* (i) *$A_1 X \subset TX$ and $A_i X \subset SX$ for $i > 1$,* (ii) *pairs of (A_1, S) and (A_i, T), $i > 1$, are pointwise R-weakly commuting,* (iii) *$d(A_1 x, A_i y) \leq \phi_i(M_{1i}(x, y))$, where ϕ_i: $R_+ \to R_+$ is an upper semicontinuous function such that $\phi_i(t) < t$ for each $t > 0$. Let one of the R-weakly commuting pairs of mappings be compatible. If one mapping in the compatible pair is continuous then all the A_i, S, and T have a unique common fixed point.*

REFERENCES

1. A. Carbone, B. E. Rhoades, and S. P. Singh, A fixed point theorem for generalized contraction map, *Indian J. Pure Appl. Math.* **20** (1989), 543–548.
2. J. Jachymski, Common fixed point theorems for some families of maps, *Indian J. Pure Appl. Math.* **25** (1994), 925–937.

3. G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.* **9** (1986), 771–779.
4. G. Jungck, K. B. Moon, S. Park, and B. E. Rhoades, On generalizations of the Meir–Keeler type contraction maps: Corrections, *J. Math. Anal. Appl.* **180** (1993), 221–222.
5. R. P. Pant, Common fixed points of two pairs of commuting mappings, *Indian J. Pure Appl. Math.* **17** (1986), 187–192.
6. R. P. Pant, Common fixed points of weakly commuting mappings, *Math. Student* **62** (1993), 97–102.
7. R. P. Pant, Common fixed points of noncommuting mappings, *J. Math. Anal. Appl.* **188** (1994), 436–440.
8. R. P. Pant, Common fixed points of sequences of mappings, *Ganita* **47** (1996), 43–49.
9. R. P. Pant, J. M. C. Joshi, and N. K. Pande, On convergence and common fixed points of sequences of mappings, *J. Natur. Phys. Sci.* **5–8** (1991–94), 167–174.
10. S. Park and B. E. Rhoads, Extension of some fixed point theorems of Hegedus and Kasahara, *Math. Seminar Notes* **9** (1981), 113–118.
11. I. H. N. Rao and K. P. R. Rao, Generalizations of the fixed point theorems of Meir–Keeler type, *Indian J. Pure Appl. Math.* **16** (1985), 1249–1262.
12. B. E. Rhoades, S. Park, and K. B. Moon, On generalizations of the Meir–Keeler type contraction maps, *J. Math. Anal. Appl.* **146** (1990), 482–494.