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A Fundamental Approach to the Generalized Eigenvalue Problem for Self-Adjoint Operators

S. J. L. VAN EIJNDHOVEN AND J. DE GRAAF

Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

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The generalized eigenvalue problem for an arbitrary self-adjoint operator is solved in a Gelfand triple consisting of three Hilbert spaces. The proof is based on a measure theoretical version of the Sobolev lemma, and the multiplicity theory for self-adjoint operators. As an application necessary and sufficient conditions are mentioned such that a self-adjoint operator in $L_2(\mathbb{R})$ has (generalized) eigenfunctions which are tempered distributions. © 1985 Academic Press, Inc.

INTRODUCTION

A natural problem in a theory of generalized functions is the so-called generalized eigenvalue problem. A simplified version of this problem can be formulated as follows. Consider the Gelfand triple $\Phi \subset X \subset \Psi$, in which X is a Hilbert space, Φ is a test space, and Ψ the space of generalized functions. Let $\mathscr{P}$ be a self-adjoint operator in X, and let λ be a number in the spectrum of $\mathscr{P}$ with multiplicity m_λ. The question is whether there exist m_λ (generalized) eigenfunctions in Ψ.

Such a problem has been studied by Gelfand and Shilov (cf. [8]) in the framework of countable Hilbert spaces and also by the authors of the present paper in the setting of analyticity spaces and trajectory spaces (cf. [4]). Here we discuss a more general approach. In a separate section we show how the theory of this paper fits into the functional analytic set-up of a special type of countable Hilbert spaces. In fact, this set-up is a generalization of the theory of tempered distributions.

The present paper is built up in such a way that each section contains one fundamental topic and finally those separate topics culminate in the proof of the main result in Section 4. This result can be loosely formulated as follows:

Let there be given n commuting self-adjoint operators $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_n$ in a

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separable Hilbert space X. Then there exists a positive self-adjoint Hilbert–Schmidt operator $\mathscr{R}$ such that the operators $\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}$, $l=1,\dots,n$, are closable. Denote their closures by $\overline{\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}}$. Then to almost all $\lambda=(\lambda_1,\dots,\lambda_n)$ in the joint spectrum $\sigma(\mathscr{P}_1,\dots,\mathscr{P}_n)$ with multiplicity m_λ there exist m_λ vectors in X, say $e_{\lambda,1},\dots,e_{\lambda,m_\lambda}$ such that

$$\overline{(\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1})}\, e_{\lambda,j}=\lambda_l e_{\lambda,j},$$

where $j=1,\dots,m$ and $l=1,\dots,n$.

The proof of the above stated theorem involves three important ingredients, discussed in three separate sections. In the first section the commutative multiplicity theory for self-adjoint operators is discussed. By this theory the spectrum of a self-adjoint operator is split into components each of which is of uniform multiplicity. In the second section we show that there exists an orthonormal basis in X such that each operator of the commuting n-set $(\mathscr{P}_1,\dots,\mathscr{P}_n)$ has a column finite matrix with respect to this basis. Section 3 contains a general Sobolev lemma. The greater part of this section follows from our paper [5], but also some new results are derived.

As already remarked the last section contains a discussion of generalized eigenfunctions. It is worthwhile to mention one of its consequences here. Let $\mathscr{T}$ be a self-adjoint operator in $\mathfrak{L}_2(\mathbb{R})$. Suppose the operator $\mathscr{H}^{-\alpha}\mathscr{T}\mathscr{H}^{\alpha}$ is closable for some $\alpha>\frac{1}{2}$, where $\mathscr{H}=-d^2/dx^2+x^2$. Then $\mathscr{T}$ has a complete set of generalized eigenfunctions in $\mathscr{H}^{\alpha}(\mathfrak{L}_2(\mathbb{R}))$ and is closable in $\mathscr{H}^{\alpha}(\mathfrak{L}_2(\mathbb{R}))\subset\mathscr{S}'(\mathbb{R})$.

1. Commutative Multiplicity Theory

This section gives the commutative multiplicity theory for a finite number of commuting self-adjoint operators. In the case of bounded operators the proof can be found in [1 or 9]. The unbounded case is a trivial generalization.

Throughout this paper $n\in\mathbb{N}$ will be taken fixed. Let μ denote a finite nonnegative Borel measure on $\mathbb{R}^n$. Then the support of μ, notation $\operatorname{supp}(\mu)$, is the complement of the largest open set in $\mathbb{R}^n$ of μ-measure zero. It can be shown that

$$\operatorname{supp}(\mu)=\{x\in\mathbb{R}^n \mid \forall_{r>0}\colon \mu(B(x,r))>0\},$$

where $B(x,r)$ denotes the closed ball $\{y\in\mathbb{R}^n \mid \|x-y\|_n\leqslant r\}$. For the proof of this statement, see [3, p. 153]. As usual, two Borel measures $\mu^{(1)}$ and $\mu^{(2)}$ are called equivalent, $\mu^{(1)}\sim\mu^{(2)}$, if for all Borel sets N $\mu^{(1)}(N)=0$ iff $\mu^{(2)}(N)=0$. The measures $\mu^{(1)}$ and $\mu^{(2)}$ are disjoint, $\mu^{(1)}\perp\mu^{(2)}$, if $\mu^{(1)}$ $(\operatorname{supp}(\mu^{(1)})\cap\operatorname{supp}(\mu^{(2)}))=\mu^{(2)}$ $(\operatorname{supp}(\mu^{(1)})\cap\operatorname{supp}(\mu^{(2)}))=0$. Now let

$(\mathscr{P}_1,\ldots,\mathscr{P}_n)$ be an n-set of commuting self adjoint operators in a separable Hilbert space X, i.e., their spectral projections mutually commute. The notion of uniform multiplicity for $(\mathscr{P}_1,\ldots,\mathscr{P}_n)$ is defined as follows.

1.1. DEFINITION. The n-set $(\mathscr{P}_1,\ldots,\mathscr{P}_n)$ is of uniform multiplicity m if there exist m finite nonnegative equivalent Borel measures $\mu^{(1)},\ldots,\mu^{(m)}$ on $\mathbb{R}^n$ and a unitary operator $\mathscr{U}: X \to \mathfrak{L}_2(\mathbb{R}^n, \mu^{(1)}) \oplus \cdots \oplus \mathfrak{L}_2(\mathbb{R}^n, \mu^{(m)})$ such that each self-adjoint operator $\mathscr{U}\mathscr{P}_l\mathscr{U}^*$, $l=1,\ldots,n$., equals m-times multiplication by the function $\eta_l: x \mapsto x_l$ in this direct sum.

In a finite dimensional Hilbert space E each commuting n-set of self-adjoint operators $(\mathscr{B}_1, \mathscr{B}_2,\ldots,\mathscr{B}_n)$ has a complete set of simultaneous eigenvectors. An element $\lambda \in \mathbb{R}^n$ is called an eigentuple of the n-set $(\mathscr{B}_1,\ldots,\mathscr{B}_n)$ if there exists a vector $e_\lambda \in E$ such that

$$\mathscr{B}_l e_\lambda = \lambda_l e_\lambda, \qquad l=1,\ldots,n.$$

The set of all eigentuples of $\mathscr{B}_1,\ldots,\mathscr{B}_n$ may be called the joint spectrum of $(\mathscr{B}_1,\ldots,\mathscr{B}_n)$ denoted by $\sigma(\mathscr{B}_1,\ldots,\mathscr{B}_n)$. In order to list all eigentuples in a well-ordered manner one can list all eigentuples of multiplicity one, two, etc. In fact, this is precisely the outcome of the following theorem.

1.2. THEOREM (Commutative Multiplicity Theorem). *The Hilbert space X can be split into a (countable) direct sum*

$$X = X_\infty \oplus X_1 \oplus X_2 \oplus \cdots,$$

such that the following assertions are valid

— *The n-set $(\mathscr{P}_1,\ldots,\mathscr{P}_n)$ restricted to X_m, $m=\infty, 1, 2,\ldots,$ acts invariantly in X_m and has uniform multiplicity m.*

— *The equivalence classes $\langle \mu_m \rangle$ of finite nonnegative Borel measures corresponding to each X_m* (cf. *Definition* 1.1) *are mutually disjoint, i.e., for all $\mu_1 \in \langle \mu_1 \rangle$ and $\mu_2 \in \langle \mu_2 \rangle$ we have $\mu_1 \perp \mu_2$, etc.*

In this paper we always consider the following standard splitting of X with respect to $(\mathscr{P}_1,\ldots,\mathscr{P}_n)$. As in the previous theorem $X = X_\infty \oplus X_1 \oplus X_2 \oplus \cdots$. In each equivalence class $\langle \mu_m \rangle$ we choose a fixed measure μ_m. By $\mathscr{U}_m$ we denote the unitary operator from X_m onto $\mathfrak{L}_2(\mathbb{R}^n, \mu_m) \oplus \cdots \oplus \mathfrak{L}_2(\mathbb{R}^n, \mu_m)$ (m-times). Then $\mathscr{U}_m(\mathscr{P}_l \upharpoonright X_m)\,\mathscr{U}_m^*$ equals m-times multiplication by the function η_l. Let $(\mathscr{E}_\lambda^{(l)})_{\lambda \in \mathbb{R}}$ denote the respective spectral resolution of the identity corresponding to $\mathscr{P}_l$. Then we define the joint spectrum as

$$\sigma(\mathscr{P}_1,\ldots,\mathscr{P}_n) = \left\{ \lambda \in \mathbb{R}^n \mid \forall_{\varepsilon>0}\colon \prod_{l=1}^{n} \mathscr{E}^{(l)}([\lambda_l - \varepsilon, \lambda_l + \varepsilon]) \neq O \right\},$$

where O denotes the null operator. We mention the following simple assertion,

$$\sigma(\mathscr{P}_1,\ldots,\mathscr{P}_n)=\overline{\bigcup_{m=1}^{\infty}\operatorname{supp}(\mu_m)}.$$

2. Column Finite Matrices

Let $\mathscr{L}_1,\ldots,\mathscr{L}_n$ be n densely defined linear operators in X. Then we define their joint $\mathfrak{C}^\infty$-domain as follows.

2.1. Definition. The joint $\mathfrak{C}^\infty$-domain of $\mathscr{L}_1,\ldots,\mathscr{L}_n$, denoted by $\mathfrak{C}^\infty(\mathscr{L}_1,\ldots,\mathscr{L}_n)$ is defined as

$$\mathfrak{C}^\infty(\mathscr{L}_1,\ldots,\mathscr{L}_n)=\{v\in X\mid \forall_{s\in\mathbb{N}}\forall_{\pi\in\{1,\ldots,n\}^{\mathbb{N}}}: v\in\mathfrak{D}(\mathscr{L}_{\pi(1)}\mathscr{L}_{\pi(2)}\cdots\mathscr{L}_{\pi(s)})\}.$$

(The set $\{1,2,\ldots,n\}^{\mathbb{N}}$ consists of all mappings from $\mathbb{N}$ into $\{1,2,\ldots,n\}$; $\mathfrak{D}(\mathscr{L}_{\pi(1)}\mathscr{L}_{\pi(2)}\cdots\mathscr{L}_{\pi(s)})$ denotes the domain of the operator between ( )).

For the commuting n-set $(\mathscr{P}_1,\ldots,\mathscr{P}_n)$ the joint $\mathfrak{C}^\infty$-domain is given by the intersection $\mathfrak{C}^\infty(\mathscr{P}_1)\cap\cdots\cap\mathfrak{C}^\infty(\mathscr{P}_n)$. Hence $\mathfrak{C}^\infty(\mathscr{P}_1,\ldots,\mathscr{P}_n)$ is dense in X.

2.2. Theorem. *Suppose that $\mathfrak{C}^\infty(\mathscr{L}_1,\ldots,\mathscr{L}_n)$ is a dense subspace of X. Then there exists an orthonormal basis in X such that each operator $\mathscr{L}_l$, $l=1,\ldots,n$, has a column finite matrix representation with respect to this basis.*

Proof. Since $\mathfrak{C}^\infty(\mathscr{L}_1,\ldots,\mathscr{L}_n)$ is dense in X and since X is separable, there exists an orthonormal basis $(u_k)_{k\in\mathbb{N}}$ in X which is contained in $\mathfrak{C}^\infty(\mathscr{L}_1,\ldots,\mathscr{L}_n)$. We introduce the orthonormal basis $(v_k)_{k\in\mathbb{N}}$ as follows.

Set $v_1=u_1$. Then there exists an orthonormal set $\{v_2,\ldots,v_{n_1}\}\perp v_1$ with $n_1\leqslant n+2$ such that the span

$$\langle\mathscr{L}_1v_1,\ldots,\mathscr{L}_nv_1,u_2\rangle\subset\langle v_1,\ldots,v_{n_1}\rangle.$$

There exists an orthonormal set $\{v_{n_1+1},\ldots,v_{n_2}\}\perp\{v_1,\ldots,v_{n_1}\}$ with $n_2\leqslant 2n+3$ such that

$$\langle\mathscr{L}_1v_2,\ldots,\mathscr{L}_nv_2,u_3\rangle\subset\langle v_1,\ldots,v_{n_2}\rangle.$$

In general, for $k\in\mathbb{N}$ having produced $\{v_1,\ldots,v_{n_k}\}$, $n_k\leqslant k(n+1)+1$, there exists an orthonormal set $\{v_{n_k+1},\ldots,v_{n_{k+1}}\}\perp\{v_1,\ldots,v_{n_k}\}$ with $n_{k+1}\leqslant(k+1)(n+1)+1$ such that

$$\langle\mathscr{L}_1v_k,\ldots,\mathscr{L}_nv_k,u_{k+1}\rangle\subset\langle v_1,\ldots,v_{n_{k+1}}\rangle.$$

We thus obtain inductively an orthonormal basis $(v_k)_{k\in\mathbb{N}}$. The basis $(v_k)_{k\in\mathbb{N}}$ is complete since $u_k \in \langle v_1,..., v_{n_k}\rangle$. Furthermore, by construction for each $l=1,..., n$, the matrix $(\mathscr{L}_l v_j, v_i)_{i,j\in\mathbb{N}}$ is column finite. ∎

The n-set $(\mathscr{P}_1,..., \mathscr{P}_n)$ has a dense joint $\mathfrak{C}^\infty$-domain. So by Theorem 2.2, there exists an orthonormal basis $(v_k)_{k\in\mathbb{N}}$ such that the operators $\mathscr{P}_1,..., \mathscr{P}_n$ have a column finite (and hence row finite) matrix representation with respect to this basis.

We define the positive Hilbert–Schmidt operator $\mathscr{R}$ by $\mathscr{R}v_k=\rho_k v_k$, where $(\rho_k)_{k\in\mathbb{N}}$ can be any fixed l_2-sequence with positive components. Then the operator $\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}$, $l=1,..., n$ is well defined on the span $\langle\{v_k \mid k\in\mathbb{N}\}\rangle$. Let $l=1,..., n$. For the domain of $\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}$ we take $\mathfrak{D}(\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1})=\mathscr{R}(\mathfrak{D}(\mathscr{P}_l))$. Since $\mathscr{R}$ is injective and self-adjoint, $\mathfrak{D}(\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1})$ is dense in X. Further, for all $f\in\mathfrak{D}(\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1})$ and all $k\in\mathbb{N}$ we have

$$(\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}f, v_k)=(f, \mathscr{R}^{-1}\mathscr{P}_l\mathscr{R}v_k).$$

Hence, the adjoint of $\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}$ is densely defined. Recapitulated,

2.3. THEOREM. *Let $\mathscr{T}_1,..., \mathscr{T}_n$ be n mutually commuting self-adjoint operators in the separable Hilbert space X. Then there exists a positive Hilbert–Schmidt operator $\mathscr{R}$ such that each operator $\mathscr{R}\mathscr{T}_l\mathscr{R}^{-1}$, $l=1,..., n$, is densely defined and closable.*

3. A Measure Theoretical Sobolev Lemma with Applications

Our paper [5] contains a generalization of the well-known Sobolev lemma. Here we use the results of that paper in the following concrete case: the measure space M is the disjoint countable union of copies $\mathbb{R}^n_p$ of $\mathbb{R}^n$, i.e., $M=\bigcup_{p=1}^\infty \mathbb{R}^n_p$; the nonnegative Borel measure on M is given by $\nu=\bigoplus_{p=1}^\infty \nu_p$ where each ν_p is a finite nonnegative Borel measure on $\mathbb{R}^n$. We note that $\mathfrak{L}_2(M,\nu)=\bigoplus_{p=1}^\infty \mathfrak{L}_2(\mathbb{R}^n, \nu_p)$. On M we introduce the metric d as follows

$$d((x,p_1),(y,p_2))=1 \qquad \text{if } p_1\neq p_2,$$
$$d((x,p),(y,p))=\|x-y\|_{\mathbb{R}^n}.$$

Now, let $B((x,p),r)$ denote the closed ball in M with center $(x,p)\in\mathbb{R}^n_p$ and radius $r>0$, and let $B(x,r)$ denote the closed ball in $\mathbb{R}^n$ with center x and radius $r>0$. Then from [6, Theorem 2.8.18], we obtain

3.1. THEOREM. *Let $f\colon M \to \mathbb{C}$ be integrable on bounded Borel sets. Then there exists a null set N_f such that the limit*

$$\tilde{f}(x,p)=\lim_{r\downarrow 0} \nu(B((x,p),r))^{-1}\int_{B((x,p),r)} f\,d\nu$$

exists for all $(x,p)\in \operatorname{supp}(\nu)\backslash N_f$. Moreover, $f=\tilde{f}$ a.e. (μ).

(For convenience we note that $f=(f_1,f_2,\ldots)$ and $f(x,p)=f_p(x)$. Moreover, $\lim_{r\downarrow 0}\nu(B((x,p),r))^{-1}\int_{B((x,p),r)} f\,d\nu = \lim_{r\downarrow 0}\nu_p(B(x,r))^{-1}\times \int_{B(x,r)} f_p\,d\nu_p$.) Let X denote a separable Hilbert space and $\mathscr{R}$ a positive Hilbert–Schmidt operator on X. Let $(v_k)_{k\in\mathbb{N}}$ be the orthonormal basis of eigenvectors of $\mathscr{R}$ with eigenvalues $\rho_k>0$, $k\in\mathbb{N}$. Further, let $\mathscr{U}$ denote a unitary operator from X onto $\bigoplus_{p=1}^{\infty}\mathfrak{L}_2(\mathbb{R}^n,\nu_p)$. The series $\sum_{k=1}^{\infty}\rho_k^2\|\mathscr{U}v_k\|^2$ is convergent. So it follows that $\sum_{k=1}^{\infty}\rho_k^2|\mathscr{U}v_k|^2\in\mathfrak{L}_1(M,\nu)$.

Following Theorem 3.1 there exists a ν-null set $N=\bigcup_{p=1}^{\infty}N_p$ (disjoint union), i.e., each set N_p is a ν_p-null set, with the following properties: There exist functions $\varphi_{k,p}\in(\mathscr{U}v_k)_p$, $k\in\mathbb{N}$, $p\in\mathbb{N}$, such that

(i) $\forall_{k\in\mathbb{N}}\forall_{p\in\mathbb{N}}\forall_{x\in\operatorname{supp}(\nu_p)\backslash N_p}$,

$$\varphi_{k,p}(x)=\lim_{r\downarrow 0}\nu_p(B(x,r))^{-1}\int_{B(x,r)}(\mathscr{U}v_k)_p\,d\nu_p.$$

(ii) $\forall_{k\in\mathbb{N}}\forall_{p\in\mathbb{N}}\forall_{x\in\operatorname{supp}(\nu_p)\backslash N_p}$,

$$|\varphi_{k,p}(x)|^2=\lim_{r\downarrow 0}\nu_p(B(x,r))^{-1}\int_{B(x,r)}|(\mathscr{U}v_k)_p|^2\,d\nu_p.$$

(iii) $\forall_{p\in\mathbb{N}}\forall_{x\in\operatorname{supp}(\nu_p)\backslash N_p}$,

$$\sum_{k=1}^{\infty}\rho_k^2|\varphi_{k,p}(x)|^2=\lim_{r\downarrow 0}\nu_p(B(x,r))^{-1}\int_{B(x,r)}\left(\sum_{k=1}^{\infty}\rho_k^2|(\mathscr{U}v_k)_p|^2\right)d\nu_p.$$

These conditions on $\bigcup_{p=1}^{\infty}N_p$ lead to the proof of the following result (see [5, Lemma 2]).

3.2. THEOREM. *Let $p\in\mathbb{N}$ and let $x\in\operatorname{supp}(\nu_p)\backslash N_p$. Set $e_x^{(p)}=\sum_{k=1}^{\infty}\rho_k\times\overline{\varphi_{k,p}(x)}\,v_k$ and $e_x^{(p)}(r)=\nu_p(B(x,r))^{-1}\sum_{k=1}^{\infty}\rho_k(\int_{B(x,r)}\overline{(\mathscr{U}v_k)_p}\,d\nu_p)\,v_k$, $r>0$. Then we have*

— *$e_x^{(p)}$, $e_x^{(p)}(r)$ are members of X,*
— $\lim_{r\downarrow 0}\|e_x^{(p)}-e_x^{(p)}(r)\|_X=0$.

Let $\mathcal{Q}_l$, $l = 1,..., n$, denote the multiplication operator, formally defined by

$$\mathcal{Q}_l(f_1, f_2,...) = (\eta_l f_1, \eta_l f_2,...).$$

We recall that η_l denotes the function $\eta_l(x) = x_l$, $x \in \mathbb{R}^n$. It is clear that $\mathcal{Q}_l$ is a self-adjoint operator in $\bigoplus_{p=1}^{\infty} \mathfrak{L}_2(\mathbb{R}^n, v_p)$. So the operator $\mathcal{P}_l = \mathcal{U}^* \mathcal{Q}_l \mathcal{U}$ is self-adjoint in X. The following lemma says that the vectors $e_x^{(p)}$ are candidate eigenvectors of the operator $\mathcal{R}\mathcal{P}_l\mathcal{R}^{-1}$.

3.3. LEMMA. *Let $l = 1,..., n$. Then the linear span $\langle \{e_x^{(p)}(r) \mid r > 0, p \in \mathbb{N}, x \in \operatorname{supp}(v_p) \backslash N_p\} \rangle$ is contained in $\mathfrak{D}(\mathcal{R}\mathcal{P}_l\mathcal{R}^{-1})$. Further we have for all $x \in \operatorname{supp}(v_p) \backslash N_p$*

$$\lim_{r \downarrow 0} (\|(\mathcal{R}\mathcal{P}_l\mathcal{R}^{-1}) \; e_x^{(p)}(r) - x_l e_x^{(p)}\|_X) = 0.$$

Proof. Let $p \in \mathbb{N}$ and let $x \in \operatorname{supp}(v_p) \backslash N_p$. Then for each $r > 0$ the series $\sum_{k=1}^{\infty} (\int_{B(x,r)} \overline{(\mathcal{U}v_k)_p} \, dv_p) \, v_k = \sum_{k=1}^{\infty} (\int_{\mathbb{R}^n} (\Delta_{B(x,r)}^{(p)} \overline{(\mathcal{U}v_k)_p}) dv_p) \, v_p$ represents the elements $\mathcal{U}^* \Delta_{B(x,r)}^{(p)}$ in X, where $\Delta_{B(x,r)}^{(p)}$ denotes the characteristic function of the ball $B(x, r)$ as an element of $\mathfrak{L}_2(\mathbb{R}^n, v_p)$. Hence $\mathcal{U}^* \Delta_{B(x,r)}^{(p)} \in \mathfrak{D}(\mathcal{P}_l)$ and $e_x^{(p)}(r) = v_p(B(x, r))^{-1} \, \mathcal{R}\mathcal{U}^* \Delta_{B(x,r)}^{(p)} \in \mathcal{R}(\mathfrak{D}(\mathcal{P}_l)) = \mathfrak{D}(\mathcal{R}\mathcal{P}_l\mathcal{R}^{-1})$ for all $l = 1,..., n$. Next we prove that $\lim_{r \downarrow 0} (\mathcal{R}\mathcal{P}_l\mathcal{R}^{-1}) \, e_x^{(p)}(r) - x_l e_x^{(p)}(r) = 0$. Then by Theorem 3.2 the proof is complete. To this end, observe that for all $r > 0$,

$$(\mathcal{R}\mathcal{P}_l\mathcal{R}^{-1}) \, e_x^{(p)}(r) - x_l e_x^{(p)}(r) =$$

$$= \sum_{k=1}^{\infty} \rho_k \left(v_p(B(x, r))^{-1} \int_{B(x,r)} (y_l - x_l) \, \overline{\varphi_{k,p}(y)} \, dv_p(y) \right) v_k.$$

We estimate as follows

$$\|(\mathcal{R}\mathcal{P}_l\mathcal{R}^{-1}) \, e_x^{(p)}(r) - x_l e_x^{(p)}(r)\|^2$$

$$\leqslant \left(v_p(B(x, r))^{-1} \int_{B(x,r)} |y_l - x_l|^2 \, dv_p(y) \right)$$

$$\times \left(v_p(B(x, r))^{-1} \int_{B(x,r)} \left(\sum_{k=1}^{\infty} \rho_k^2 |\varphi_{k,p}|^2 \right) dv_p \right).$$

The first factor in the last expression tends to zero as $r \downarrow 0$. Because of assumption (iii) the second factor is bounded by $\sum_{k=1}^{\infty} \rho_k^2 |\varphi_{k,p}(x)|^2 + 1$ for sufficiently small $r > 0$. ∎

4. The Main Result

In the introduction we have given a nonrigorous formulation of the main theorem of the present paper. The results of the previous sections culminate in the proof of this theorem.

Again, let $(\mathscr{P}_1,..., \mathscr{P}_n)$ denote an n-set of commuting self-adjoint operators in X. Following Section 1 there exists a standard splitting

$$X = X_\infty \oplus X_1 \oplus X_2 \oplus \cdots,$$

and disjoint finite nonnegative Borel measures $\mu_\infty, \mu_1, \mu_2,...,$ on $\mathbb{R}^n$ such that each X_m is unitarily equivalent to the direct sum $\bigoplus_{j=1}^m \mathfrak{L}_2(\mathbb{R}^n, \mu_m)$. Furthermore, the n-set $(\mathscr{P}_1,..., \mathscr{P}_n)$ acts invariantly in each Hilbert summand X_m and is unitarily equivalent to the n-set $(\mathscr{Q}_1,..., \mathscr{Q}_n)$. Here $\mathscr{Q}_l$ restricted to $\bigoplus_{j=1}^m \mathfrak{L}_2(\mathbb{R}^n, \mu_m)$ is the operator of m-times multiplication by the function η_l.

Following Section 2, there exists an orthonormal basis $(v_k)_{k\in\mathbb{N}}$ such that each operator in $(\mathscr{P}_1,..., \mathscr{P}_n)$ has a column finite matrix representation with respect to $(v_k)_{k\in\mathbb{N}}$. Let $\mathscr{R}$ denote the positive Hilbert Schmidt operator defined by $\mathscr{R}v_k = \rho_k v_k$, where $(\rho_k)_{k\in\mathbb{N}}$ is a fixed sequence in l_2 with $\rho_k > 0$. Then the operators $\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}$ are closable in X for each $l = 1,..., n$. We denote the respective closures by $\overline{\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}}$.

Following Section 3 there exist null-sets $N_{m,j}$, $j = 1,..., m$, with respect to μ_m, $m = \infty, 1, 2,...,$ such that the limit

$$\varphi_{k,j}^{(m)}(x) = \lim_{r\downarrow 0} \mu_m(B(x, r))^{-1} \int_{B(x,r)} (\mathscr{U}v_k)_j^{(m)}\, d\mu_m$$

exists for all $x \in \operatorname{supp}(\mu_m)\backslash N_{m,j}$. Moreover, $\varphi_{k,j}^{(m)} \in (\mathscr{U}v_k)_j^{(m)}$. Here

$$(\mathscr{U}v_k)^{(m)} = ((\mathscr{U}v_k)_1^{(m)}, (\mathscr{U}v_k)_2^{(m)},..., (\mathscr{U}v_k)_m^{(m)}) \in \bigoplus_{j=1}^{m} \mathfrak{L}_2(\mathbb{R}^n, \mu_m).$$

In addition, for all $m = \infty, 1, 2,...,$ and $1 \leqslant j < m+1$, the series

$$e_{x,j}^{(m)} = \sum_{k=1}^{\infty} \rho_k \overline{\varphi_{k,j}^{(m)}(x)}\, v_k$$

and

$$e_{x,j}^{(m)}(r) = \sum_{k=1}^{\infty} \rho_k \left(\mu_m(B(x, r))^{-1} \int_{B(x,r)} (\mathscr{U}v_k)_j^{(m)}\, d\mu_m \right) v_k$$

with $r > 0$, $x \in \operatorname{supp}(\mu_m)\backslash N_{m,j}$, converges in X.

By Theorem 3.2 and Lemma 3.3, Lemma 4.1 follows.

4.1. LEMMA. *Let $m=\infty, 1, 2, \ldots$, and let $j \in \mathbb{N}$ with $1 \leqslant j < m+1$. Then for all $x \in \operatorname{supp}(\mu_m) \backslash N_{m,j}$,*

— $\lim_{r \downarrow 0}(\|e_{x,j}^{(m)}(r) - e_{x,j}^{(m)}\|) = 0$;

— *for each $l = 1, \ldots, n$ and all $r > 0$, $e_{x,j}^{(m)}(r) \in \mathfrak{D}(\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1})$, and*

$$\lim_{r \downarrow 0} (\|(\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1})\, e_{x,j}^{(m)}(r) - x_l e_{x,j}^{(m)}\|) = 0.$$

Since the operator $\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}$ is closable, we obtain from the previous lemma,

4.2. COROLLARY. *Let $m = \infty, 1, 2, \ldots$, and let $j \in \mathbb{N}$, $1 \leqslant j < m+1$. Then for all $x \in \operatorname{supp}(\mu_m) \backslash N_{m,j}$ and for all $l = 1, \ldots, n$,*

— *$e_{x,j}^{(m)}$ is in the domain of $\overline{\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}}$;*

— $\overline{\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}}\, e_{x,j}^{(m)} = x_l e_{x,j}^{(m)}$.

We observe that for each $m = \infty, 1, 2, \ldots$, the set $N_m = \bigcup_{j=1}^{m} N_{m,j}$ is a μ_m-null set. Now we are in a position to formulate the main theorem.

4.3. THEOREM. *Let $(\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_n)$ be an n-set of commuting self-adjoint operators. Then there exists a positive Hilbert–Schmidt operator such that the operators $\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}$ are closable.*

Let $X = X_\infty \oplus X_1 \oplus X_2 \oplus \cdots$, be the standard splitting of X, and $\mu_\infty, \mu_1, \mu_2, \ldots$, be the corresponding multiplicity measures. Let $m = \infty, 1, 2, \ldots$. Then there is a μ_m-null set N_m with the following property: for all $x \in \operatorname{supp}(\mu_m) \backslash N_m$ there exist m independent vectors $e_{x,j}^{(m)} \in X$, $1 \leqslant j < m+1$, satisfying $\overline{\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}} e_{x,j}^{(m)} = x_l e_{x,j}^{(m)}$, $l = 1, \ldots, n$.

Proof. The proof of this theorem is a compilation of the results given in the beginning of this section. ▌

Remark. Let $\mathscr{P}$ be a self-adjoint operator in X and let $\mathscr{R}$ be a positive Hilbert–Schmidt operator such that $\mathscr{R}\mathscr{P}\mathscr{R}^{-1}$ is closable. The spectrum of $\mathscr{R}\mathscr{P}\mathscr{R}^{-1}$ can be larger than the spectrum of $\mathscr{P}$. An interesting example is the following. In $\mathfrak{L}_2(\mathbb{R})$ take $\mathscr{P} = i(d/dx)$, then $\sigma(\mathscr{P}) = \mathbb{R}$. Further take $\mathscr{R} = e^{-\tau\mathscr{H}}$ with $\mathscr{H} = \frac{1}{2}(x^2 - d^2/dx^2 + 1)$ and $\tau > 0$. Then $\mathscr{R}\mathscr{P}\mathscr{R}^{-1} = i \cosh \tau (d/dx) + ix \sinh \tau$. Each $\lambda \in \mathbb{C}$ is an eigenvalue of this operator. Its eigenvector is $x \mapsto \exp((-i\lambda/\cosh \tau)\, x - \frac{1}{2}(\tanh \tau)\, x^2)$ which is an $\mathfrak{L}_2(\mathbb{R})$-function. This continuous set of eigenvectors is closely related to the so-called coherent states.

4.4. COROLLARY. *Let $\mathscr{R}^{-1}(X)$ denote the completion of X with respect to the inner-product $(u, v)_{-1} = (\mathscr{R}u, \mathscr{R}v)_X$. Employ the notation of the previous theorem:*

— *Each operator $\mathscr{P}_l$ is closable in $\mathscr{R}^{-1}(X)$.*

— *$\mathscr{R}^{-1}e_{x,j}^{(m)} = \sum_{k=1}^{\infty} \overline{\varphi_{k,j}^{(m)}(x)}\, v_k \in \mathscr{R}^{-1}(X)$ is a simultaneous generalized eigenvector of the n-set $(\bar{\mathscr{P}}_1,..., \bar{\mathscr{P}}_n)$ with eigentuple $x = (x_1,..., x_n)$, where $\bar{\mathscr{P}}_l$ denotes the $\mathscr{R}^{-1}(X)$-closure of $\mathscr{P}_l$, $l = 1,..., n$.*

Proof. We only prove the closability of $\mathscr{P}_l$. We define the domain $\mathrm{Dom}(\bar{\mathscr{P}}_l)$ of the operator $\bar{\mathscr{P}}_l$ in $\mathscr{R}^{-1}(X)$ by

$$F \in \mathrm{Dom}(\bar{\mathscr{P}}_l) \qquad \text{iff } \mathscr{R}F \in \mathfrak{D}(\overline{\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}}).$$

Further, $\bar{\mathscr{P}}_l F = \mathscr{R}^{-1}(\overline{\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}})\,\mathscr{R}F$, $F \in \mathrm{Dom}(\bar{\mathscr{P}}_l)$. It is clear that $\bar{\mathscr{P}}_l$ extends in $\mathscr{R}^{-1}(X)$. We prove that $\bar{\mathscr{P}}_l$ is a closed operator in $\mathscr{R}^{-1}(X)$. To this end let $(F_s)_{s \in \mathbb{N}}$ be a sequence in $\mathscr{R}^{-1}(X)$ with

$$\lim_{s \to \infty} F_s = F \in \mathscr{R}^{-1}(X) \qquad \text{and} \qquad \lim_{s \to \infty} \bar{\mathscr{P}}_l F_s = G \in \mathscr{R}^{-1}(X).$$

Then $\mathscr{R}F_s \to \mathscr{R}F$ and $(\overline{\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}})\,\mathscr{R}F_s \to \mathscr{R}G$ as $s \to \infty$. Hence $F \in \mathrm{Dom}(\bar{\mathscr{P}}_l)$ and $G = \bar{\mathscr{P}}_l F$, becuase $\overline{\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}}$ is closed in X. ∎

Remark. Since for the eigenvalues of $\mathscr{R}$ any positive l_2-sequence can be taken it is clear that the improper eigenvectors of the operators $\mathscr{P}_l,..., \mathscr{P}_n$ lie at the "periphery" of the Hilbert space X.

5. GENERALIZED EIGENFUNCTIONS IN THE DUAL OF A COUNTABLE HILBERT SPACE

An application of the previous sections is in the field of generalized eigenfunctions and countable Hilbert spaces.

As in Section 4, $(\mathscr{P}_1,..., \mathscr{P}_n)$ is an n-set of commuting self-adjoint operators in X, and $\mathscr{R}$ denotes a positive Hilbert–Schmidt operator with the property that the operators $\mathscr{R}\mathscr{P}_l\mathscr{R}^{-1}$, $l = 1,..., n$, are closable in X. The countable Hilbert space $\Phi_{X,\mathscr{R}}$ is defined by

$$\Phi_{X,\mathscr{R}} = \bigcap_{s=1}^{\infty} \mathscr{R}^s(X),$$

where in each $\mathscr{R}^s(X)$ the norm is defined by $\|\varphi\|_s = \|\mathscr{R}^{-s}\varphi\|_X$, $\varphi \in \mathscr{R}^s(X)$.

With its natural topology, the space $\Phi_{X,\mathscr{R}}$ is a nuclear Frechet space. The strong dual of $\Phi_{X,\mathscr{R}}$ can be represented by the inductive limit

$$\Psi_{X,\mathscr{R}} = \bigcup_{S \in \mathbb{N}} \mathscr{R}^{-s}(X).$$

Here $\mathscr{R}^{-s}(X)$ denotes the completion of X with respect to the norm $\|\mathscr{R}^s w\|$, $w \in X$. On $\Psi_{X,\mathscr{R}}$ the inductive limit topology is imposed. In [2] a set of seminorms has been produced which generates a locally convex topology equivalent to the inductive limit topology. The Gelfand tripel $\Phi_{X,\mathscr{R}} \subset X \subset \Psi_{X,\mathscr{R}}$ places the theory of tempered distributions in a functional analytic framework. (Take $X = \mathfrak{L}_2(\mathbb{R})$ and $\mathscr{R}^{-1} = -d^2/dx^2 + x^2$.)

From Corollary (4.4) it follows that the operators $\mathscr{P}_1, ..., \mathscr{P}_n$ have closed extensions in $\mathscr{R}^{-1}(X)$. Also, it follows that to almost each eigentuple $x = (x_1, ..., x_n) \in \sigma(\mathscr{P}_1, ..., \mathscr{P}_n)$ with multiplicity m_x there are m_x simultaneous generalized eigenvectors $E_{x,j}^{(m)}$, $j = 1, ..., m_x$ in $\mathscr{R}^{-1}(X) \subset \Psi_{X,\mathscr{R}}$, i.e., $\bar{\mathscr{P}}_l E_{x,j}^{(m)} = x_l E_{x,j}^{(m)}$, $l = 1, ..., n$. We observe that $\sigma(\mathscr{P}_1, ..., \mathscr{P}_n) = \bigcup_{m=1}^{\infty} \text{supp}(\mu_m)$ and that $\sigma(\mathscr{P}_1, ..., \mathscr{P}_n) \backslash (\bigcup_{m=1}^{\infty} \text{supp}(\mu_m))$ has μ_m-measure zero, $m = \infty, 1, 2, ...$, (cf. Sect. 1).

Remark. The generalized eigenvectors $E_{x,j}^{(m)}$ as constructed in this paper can be embedded in a trajectory space [7]. There they constitute a Dirac basis. For these concepts and for a rigorous foundation of the genuine Dirac-formalism, see [4 and 3].

Remark. A self-adjoint operator $\mathscr{P}$ in $\mathfrak{L}_2([-1, 1])$ has generalized eigenfunctions which are hyperfunctions on $[-1, 1]$, if $e^{-t\mathscr{L}} \mathscr{P} e^{t\mathscr{L}}$ is densely defined and closable for each $t > 0$. Here $\mathscr{L}$ denotes the positive square root of the Legendre operator, i.e., $\mathscr{L} = \{-(d/dx)(1 - x^2)(d/dx)\}^{1/2}$, cf. [10].

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