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James' conjecture for Hecke algebras of exceptional type, I

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Abstract

In this paper, and a second part to follow, we complete the programme (initiated more than 15 years ago) of determining the decomposition numbers and verifying James' conjecture for Iwahori–Hecke algebras of exceptional type. The new ingredients which allow us to achieve this aim are:

- the fact, recently proved by the first author, that all Hecke algebras of finite type are cellular in the sense of Graham–Lehrer, and
- the explicit determination of W-graphs for the irreducible (generic) representations of Hecke algebras of type E_7 and E_8 by Howlett and Yin.

Thus, we can reduce the problem of computing decomposition numbers to a manageable size where standard techniques, e.g., Parker's MeatAxe and its variations, can be applied. In this part, we describe the theoretical foundations for this procedure.
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Keywords: Hecke algebra; Decomposition numbers; James' conjecture

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1. Introduction

Let k be a field and q a non-zero element of k. Let $H_n(k,q)$ be the Iwahori–Hecke algebra of type A_{n-1} with parameter q; this is a certain deformation of the group algebra of the symmetric group $\mathfrak{S}_n$. In order to study the representation theory of $H_n(k,q)$, Dipper and James [5] developed a q-version of the classical theory of Specht modules for $\mathfrak{S}_n$. In this framework, one obtains a natural parametrization of $\operatorname{Irr}(H_n(k,q))$ (the set of irreducible representations, up to isomorphism) in terms of e-regular partitions, where the parameter e is defined by

$$e = \min\{i \geqslant 2 \mid 1 + q + q^2 + \cdots + q^{i-1} = 0\}.$$

(We set $e = \infty$ if no such i exists.) If k has characteristic 0, then we also know how to determine the dimensions of the irreducible representations, thanks to the Lascoux–Leclerc–Thibon conjecture [30] and its proof by Ariki [1]. However, the analogous problem for k of positive characteristic is completely open.

Assume now that $e < \infty$ and $\operatorname{char}(k) = \ell > 0$. Based on empirical evidence for $n = 2, 3, \ldots, 10$, James [28] made the remarkable conjecture that if $e\ell > n$, then $\operatorname{Irr}(H_n(k,q))$ only depends on e. More precisely, James predicts that $\operatorname{Irr}(H_n(k,q))$ could be obtained from the $\mathbb{C}$-algebra $H_n(\mathbb{C}, \sqrt[e]{1})$ by a process of ℓ-modular reduction. Shortly afterwards, the first-named author [8] formulated a version of James' conjecture for Iwahori–Hecke algebras associated to finite Weyl groups in general, and proved that it holds in the so-called "defect 1 case." (In type A_{n-1}, this corresponds to the case where e divides exactly one of the numbers $2, 3, \ldots, n$.) The article [8] also contains an argument which shows that the irreducible representations of any Iwahori–Hecke algebra over a field of characteristic $\ell > 0$ can always be obtained by ℓ-modular reduction from an algebra in characteristic 0, as long as ℓ is large enough. Thus, James' conjecture and its generalizations are really about finding the correct bound for ℓ.

By ad hoc computational methods, the general version of James' conjecture has been shown to hold for Iwahori–Hecke algebras of type F_4 and E_6; see [9,18]. These methods, however, turned out to be completely inadequate to deal with algebras of larger rank; in particular, types E_7 and E_8 remained far out of reach.

Using the Kazhdan–Lusztig theory of cells [32] and the Graham–Lehrer concept of abstract "cell data" [23], it was recently shown in [16] that a suitable theory of "Specht modules" exists for Iwahori–Hecke algebras associated to finite Weyl groups in general. First of all, this has the theoretical implication that we can now formulate a general version of James' conjecture which is, perhaps, more natural than the one in [8]. Furthermore, this has the practical implication of leading to an algorithm for verifying the general version of James' conjecture, in which the main issue is the determination of the invariant bilinear form (and its rank) on a "cell representation."

In order to make this work, a number of problems have to be resolved. To begin with, we need explicit models for those "cell representations." For W of exceptional type, we will see that such models are given by the W-graph representations which were recently obtained by Howlett and Yin [26,40] and which are readily accessible through Michel's development version [35] of the computer algebra system CHEVIE [17]. Then the determination of the invariant bilinear form essentially amounts to solving a system of linear equations. This works fine for dimensions of up to around 2500, but some more refined methods are necessary for dealing with the large representations (of dimension up to 7168) in type E_8. The discussion of these finer computational methods is beyond the scope of the present article and can be found in [19].

Still, with all these new tools at hand, the computations required to determine the Gram matrices of the invariant bilinear forms for large representations in type E_8 takes *several months* of CPU time on modern computers. Note, however, that once these matrices have been computed, it is relatively easy to verify that they indeed define invariant bilinear forms and to compute their ranks for various specialisations. It is planned to create a data base which makes these data generally available.

This paper is organised as follows. In Section 2, we recall the construction of "cell data" à la Graham–Lehrer in Iwahori–Hecke algebras associated to finite Weyl groups. We also discuss the example of type G_2, which provides a first illustration for the phenomenon expressed in James' conjecture. In Section 3, we formulate the general version of James' conjecture using the new approach based on cell representations. The equivalent formulation in Corollary 3.6 provides the conceptual basis for the algorithm for verifying James' conjecture. In Section 4, we discuss the main computational issues in this algorithm and show how they can be solved—at least in principle. In particular, in Section 4.2, we prove a general result which allows us to verify that the Howlett–Yin W-graph representations do provide suitable models for the "cell representations." This fact raises a general question about W-graph representations which is formulated as Conjecture 4.5.

2. Cellular bases and cell representations

Let W be an irreducible finite Weyl group with generating set S. Let $R \subseteq \mathbb{C}$ be a subring and $A = R[v, v^{-1}]$ the ring of Laurent polynomials in an indeterminate v. Let $\mathcal{H}$ be the corresponding 1-parameter Iwahori–Hecke algebra over A. As an A-module, $\mathcal{H}$ is free with basis $\{T_w \mid w \in W\}$; the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ uT_{sw} + (u-1)T_w & \text{if } l(sw) = l(w) - 1, \end{cases}$$

where $u = v^2$, $s \in S$ and $w \in W$. Here, $l(w)$ denotes the length of $w \in W$. For the general theory of Iwahori–Hecke algebras, we refer to [20]. These algebras, and their specialisations, play an important role in the representation theory of finite reductive groups; see, for example, [32, Chapter 0], [14].

In order to specify a *cell datum* for $\mathcal{H}$ in the sense of Graham and Lehrer [23, Definition 1.1], we must specify a quadruple $(\Lambda, M, C, *)$ satisfying the following conditions.

(C1) Λ is a partially ordered set, $\{M(\lambda) \mid \lambda \in \Lambda\}$ is a collection of finite sets and

$$C : \coprod_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \to \mathcal{H}$$

is an injective map whose image is an A-basis of $\mathcal{H}$;

(C2) If $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, write $C(\mathfrak{s}, \mathfrak{t}) = C^\lambda_{\mathfrak{s},\mathfrak{t}} \in \mathcal{H}$. Then $*: \mathcal{H} \to \mathcal{H}$ is an A-linear anti-involution such that $(C^\lambda_{\mathfrak{s},\mathfrak{t}})^* = C^\lambda_{\mathfrak{t},\mathfrak{s}}$.

(C3) If $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, then for any element $h \in \mathcal{H}$ we have

$$hC^\lambda_{\mathfrak{s},\mathfrak{t}} \equiv \sum_{\mathfrak{s}' \in M(\lambda)} r_h(\mathfrak{s}', \mathfrak{s}) C^\lambda_{\mathfrak{s}',\mathfrak{t}} \mod \mathcal{H}(< \lambda),$$

where $r_h(\mathfrak{s}', \mathfrak{s}) \in A$ is independent of $\mathfrak{t}$ and where $\mathcal{H}(< \lambda)$ is the A-submodule of $\mathcal{H}$ generated by $\{C^{\mu}_{\mathfrak{s}'',\mathfrak{t}''} \mid \mu < \lambda;\ \mathfrak{s}'', \mathfrak{t}'' \in M(\mu)\}$.

For this purpose, we first need to recall some basic facts about the representations of W and $\mathcal{H}_K = K \otimes_A \mathcal{H}$, where K is the field of fractions of A.

It is known that $\mathbb{Q}$ is a splitting field for W; see, for example, [20, 6.3.8]. We will write

$$\operatorname{Irr}(W) = \{E^{\lambda} \mid \lambda \in \Lambda\}, \qquad d_{\lambda} = \dim E^{\lambda},$$

for the set of irreducible representations of W (up to equivalence), where Λ is some finite indexing set. Now, the algebra $\mathcal{H}_K$ is known to be split semisimple; see [20, 9.3.5]. Furthermore, by Tits' Deformation Theorem, the irreducible representations of $\mathcal{H}_K$ (up to isomorphism) are in bijection with the irreducible representations of W; see [20, 8.1.7]. Thus, we can write

$$\operatorname{Irr}(\mathcal{H}_K) = \{E_v^{\lambda} \mid \lambda \in \Lambda\}.$$

The correspondence $E^{\lambda} \leftrightarrow E_v^{\lambda}$ is uniquely determined by the following condition:

$$\operatorname{trace}(w, E^{\lambda}) = \operatorname{trace}(T_w, E_v^{\lambda})\big|_{v=1} \quad \text{for all } w \in W;$$

note that $\operatorname{trace}(T_w, E_v^{\lambda}) \in A$ for all $w \in W$.

The algebra $\mathcal{H}$ is *symmetric* with respect to the trace form $\tau : \mathcal{H} \to A$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$. Hence we have the following orthogonality relations for the irreducible representations of $\mathcal{H}_K$:

$$\sum_{w \in W} u^{-l(w)} \operatorname{trace}(T_w, E_v^{\lambda}) \operatorname{trace}(T_{w^{-1}}, E_v^{\mu}) = \begin{cases} d_{\lambda}\,\mathbf{c}_{\lambda} & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu, \end{cases}$$

where $0 \neq \mathbf{c}_{\lambda} \in \mathbb{Z}[u, u^{-1}]$; see [20, 8.1.7 and 9.3.6]. Following Lusztig, we write

$$\mathbf{c}_{\lambda} = f_{\lambda}\, u^{-\mathbf{a}_{\lambda}} + \text{combination of strictly higher powers of } u,$$

where $\mathbf{a}_{\lambda}$, f_{λ} are integers such that $\mathbf{a}_{\lambda} \geqslant 0$ and $f_{\lambda} > 0$; see [20, 9.4.7]. These integers are explicitly known for all types of W; see Lusztig [31, Chapter 4] or [32, Chapter 22].

Remark 2.1. Since we are in the equal parameter case, the Laurent polynomials $\mathbf{c}_{\lambda}$ have the following properties: Each $\mathbf{c}_{\lambda}$ divides the Poincaré polynomial $P_W = \sum_{w \in W} u^{l(w)}$ in $\mathbb{Q}[u, u^{-1}]$; furthermore, we have

$$\mathbf{c}_{\lambda} = f_{\lambda} u^{-\mathbf{a}_{\lambda}}\, \tilde{\mathbf{c}}_{\lambda} \quad \text{where } \tilde{\mathbf{c}}_{\lambda} \in \mathbb{Z}[u] \text{ is monic and divides } P_W.$$

(For these facts, see [20, 9.3.6] and the references there.) It is well known (see, for example, [3, §9.4]) that

$$P_W = \prod_{1 \leqslant i \leqslant |S|} \frac{u^{d_i} - 1}{u - 1}$$

where $d_1, \dots, d_{|S|}$ are the so-called *degrees* of W; we have $|W| = d_1 \cdots d_{|S|}$. By [3, §10.2], the degrees for the various types of W are given as follows:

Type	degrees d_i
A_{n-1}	$2, 3, 4, \dots, n$
B_n, C_n	$2, 4, 6, \dots, 2n$
D_n	$2, 4, 6, \dots, 2(n-1), n$

Type	degrees d_i
G_2	$2, 6$
F_4	$2, 6, 8, 12$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$

We are now ready to define a "cell datum" of $\mathcal{H}$. The required quadruple $(\Lambda, M, C, *)$ is given as follows. Let Λ be an indexing set for the irreducible representations of W, as above. For $\lambda \in \Lambda$, we set $M(\lambda) = \{1, \dots, d_\lambda\}$. Using the **a**-invariants, we define a partial order $\preccurlyeq$ on Λ by

$$\lambda \preccurlyeq \mu \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \lambda = \mu \quad \text{or} \quad \mathbf{a}_\lambda > \mathbf{a}_\mu.$$

Thus, Λ is ordered according to *decreasing* **a**-value. Next, we define an A-linear anti-involution $*: \mathcal{H} \to \mathcal{H}$ by $T_w^* = T_{w^{-1}}$ for all $w \in W$. Thus, $T_w^* = T_w^\flat$ in the notation of [32, 3.4].

The trickiest part is, of course, the definition of the basis elements $C^\lambda_{\mathfrak{s},\mathfrak{t}}$ for $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$. Let $\{c_w \mid w \in W\}$ be the Kazdan–Lusztig basis of $\mathcal{H}$, as constructed in [32, Theorem 5.2]. Given $x, y \in W$, we write $c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$ where $h_{x,y,z} \in A$. Following Lusztig [32, 13.6], we use the structure constants $h_{x,y,z}$ to define a function $\mathbf{a}: W \to \mathbb{Z}_{\geqslant 0}$ by

$$\mathbf{a}(z) := \min\bigl\{i \geqslant 0 \bigm| v^i h_{x,y,z} \in \mathbb{Z}[v] \text{ for all } x, y \in W\bigr\} \quad \text{for all } z \in W.$$

As in [32], we usually work with the elements $c_w^\dagger$ obtained by applying the unique A-algebra involution $\mathcal{H} \to \mathcal{H}$, $h \mapsto h^\dagger$ such that $T_s^\dagger = -T_s^{-1}$ for any $s \in S$; see [32, 3.5]. We can now state:

Theorem 2.2. *(See Geck [16, Theorem 3.1].) Assume that the subring $R \subseteq \mathbb{C}$ is chosen such that all bad primes for W are invertible in R. Then there is a cell datum $(\Lambda, M, C, *)$ for $\mathcal{H}$ where Λ, M, $*$ are as specified above and, for each $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, the element $C^\lambda_{\mathfrak{s},\mathfrak{t}}$ is a $\mathbb{Z}$-linear combination of basis elements $c_w^\dagger$ where $\mathbf{a}(w) = \mathbf{a}_\lambda$.*

Here, a prime number p is called *bad* for W if p divides f_λ for some $\lambda \in \Lambda$. Otherwise, p is called *good*. This corresponds to the familiar definition of "bad" primes; see Lusztig [31, Chapter 4]. The conditions for being good for the various types of W are as follows:

$$\begin{aligned} A_n &: \text{no condition}, \\ B_n, C_n, D_n &: p \neq 2, \\ G_2, F_4, E_6, E_7 &: p \neq 2, 3, \\ E_8 &: p \neq 2, 3, 5. \end{aligned}$$

For the rest of this paper, we shall now make the definite choice where the ring R consists of all fractions $a/b \in \mathbb{Q}$ such that $a \in \mathbb{Z}$ and $0 \neq b \in \mathbb{Z}$ is divisible by bad primes only.

Remark 2.3. For future reference, we remark that, if $h \in \mathcal{H}$ is a $\mathbb{Z}[v, v^{-1}]$-linear combination of basis elements $\{T_w \mid w \in W\}$, then we also have

$$r_h(\mathfrak{s}', \mathfrak{s}) \in \mathbb{Z}\big[v, v^{-1}\big] \quad \text{for all } \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{s}' \in M(\lambda);$$

see the explicit formula for $r_h(\mathfrak{s}', \mathfrak{s})$ in Step 3 of the proof of [16, Theorem 3.1].

Following Graham and Lehrer [23], we can perform the following constructions. Given $\lambda \in \Lambda$, let W^λ be a free A-module with basis $\{C_\mathfrak{s} \mid \mathfrak{s} \in M(\lambda)\}$. Then W^λ is a left $\mathcal{H}$-module, where the action is given by

$$h.C_\mathfrak{s} = \sum_{\mathfrak{s}' \in M(\lambda)} r_h(\mathfrak{s}', \mathfrak{s})\, C_{\mathfrak{s}'}.$$

Furthermore, we can define a symmetric bilinear form $\phi^\lambda : W^\lambda \times W^\lambda \to A$ by

$$\phi^\lambda(C_\mathfrak{s}, C_\mathfrak{t}) = r_h(\mathfrak{s}, \mathfrak{s}) \quad \text{where } \mathfrak{s}, \mathfrak{t} \in M(\lambda) \text{ and } h = C^\lambda_{\mathfrak{s},\mathfrak{t}}.$$

We have $\phi^\lambda(T_w.C_\mathfrak{s}, C_\mathfrak{t}) = \phi^\lambda(C_\mathfrak{s}, T_{w^{-1}}.C_\mathfrak{t})$ for all $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$ and $w \in W$; see [23, Proposition 2.4].

The modules $\{W^\lambda \mid \lambda \in \Lambda\}$ are called the *cell representations*, or *cell modules*, of $\mathcal{H}$. Extending scalars from A to K, we obtain modules $W^\lambda_K = K \otimes_A W^\lambda$ for $\mathcal{H}_K$. By the discussion in [16, Example 4.4], we have

$$\operatorname{Irr}(\mathcal{H}_K) = \big\{W^\lambda_K \bigm| \lambda \in \Lambda\big\} \quad \text{and} \quad W^\lambda_K \cong E^\lambda_v \quad \text{for all } \lambda \in \Lambda.$$

Now let $\theta : A \to k$ be a ring homomorphism into a field k; note that the characteristic of k will be either 0 or a prime p which is not bad for W. By extension of scalars, we obtain a k-algebra $\mathcal{H}_k(W, \xi) = k \otimes_A \mathcal{H}$ where $\xi := \theta(u) \in k$. Explicitly, $\mathcal{H}_k(W, \xi)$ has a basis $\{T_w \mid w \in W\}$ and the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ \xi T_{sw} + (\xi - 1) T_w & \text{if } l(sw) = l(w) - 1, \end{cases}$$

where $s \in S$ and $w \in W$. The algebra $\mathcal{H}_k(W, \xi)$ is called a *specialisation* of $\mathcal{H}$. Let $\operatorname{Irr}(\mathcal{H}_k(W, \xi))$ be the set of irreducible representations of $H_k(W, \xi)$, up to isomorphism.

Now, we also obtain cell modules $W^\lambda_\xi = k \otimes_A W^\lambda$ $(\lambda \in \Lambda)$ for $\mathcal{H}_k(W, \xi)$, which may no longer be irreducible. Denoting by ϕ^λ_ξ the induced bilinear form on W^λ_ξ, we set

$$L^\lambda_\xi = W^\lambda_\xi / \operatorname{rad}\big(\phi^\lambda_\xi\big).$$

Then, by the general theory of cellular algebras in [23, §3], each L^λ_ξ is either $\{0\}$ or an absolutely simple $\mathcal{H}_k(W, \xi)$-module, and we have

$$\operatorname{Irr}\big(\mathcal{H}_k(W, \xi)\big) = \big\{L^\mu_\xi \bigm| \mu \in \Lambda^\circ_\xi\big\} \quad \text{where } \Lambda^\circ_\xi := \big\{\lambda \in \Lambda \bigm| L^\lambda_\xi \neq 0\big\}.$$

In particular, this shows that the algebra $\mathcal{H}_k(W,\xi)$ is split. Furthermore, denoting by $(W_\xi^\lambda : L_\xi^\mu)$ the multiplicity of L_ξ^μ as a composition factor of W_ξ^λ, we have

$$\begin{cases} \left(W_\xi^\mu : L_\xi^\mu\right) = 1 & \text{for any } \mu \in \Lambda_\xi^\circ, \\ \left(W_\xi^\lambda : L_\xi^\mu\right) = 0 & \text{unless } \lambda = \mu \text{ or } \mathbf{a}_\mu < \mathbf{a}_\lambda. \end{cases} \tag{Δ}$$

Thus, the theory of cellular algebras provides a general method for constructing the irreducible representations of the specialised algebra $\mathcal{H}_k(W,\xi)$.

Proposition 2.4. *Assume that $P_W(\xi) \neq 0$. Then $\mathcal{H}_k(W,\xi)$ is semisimple, $\Lambda = \Lambda_\xi^\circ$ and $W_\xi^\lambda = L_\xi^\lambda$ for all $\lambda \in \Lambda$.*

Proof. Recall from Remark 2.1 that, for each $\lambda \in \Lambda$, we have $\mathbf{c}_\lambda = f_\lambda u^{-\mathbf{a}_\lambda} \tilde{\mathbf{c}}_\lambda$ where $\tilde{\mathbf{c}}_\lambda \in \mathbb{Z}[u]$ is monic and divides P_W. Hence, since the characteristic of k is either 0 or a good prime for W, our assumption $P_W(\xi) \neq 0$ implies that we also have $\theta(\mathbf{c}_\lambda) \neq 0$ for all $\lambda \in \Lambda$. A general semisimplicity criterion for symmetric algebras (see [20, 7.4.7]) then shows that $\mathcal{H}_k(W,\xi)$ is semisimple, a result first proved by Gyoja and Uno [25]. The remaining statements concerning the cell representations are contained in [23, 3.8]. $\square$

Corollary 2.5. *Let $\lambda \in \Lambda$ and G^λ be the Gram matrix of the invariant bilinear form ϕ^λ with respect to the standard basis of W^λ. Then $0 \neq \det(G^\lambda) \in \mathbb{Z}[v, v^{-1}]$. Furthermore, let $0 \neq q \in \mathbb{Z}[v, v^{-1}]$ be irreducible such that q divides $\det(G^\lambda)$. Then either $\pm q$ is a bad prime number or q divides P_W.*

Proof. First note that, by Remark 2.3, all entries of G^λ lie in $\mathbb{Z}[v, v^{-1}]$. Furthermore, by Proposition 2.4, we have $\det(G^\lambda) \neq 0$. Now consider the prime ideal (q) and let F be the field of fractions of $A/(q)$. Then we have a specialisation $\alpha : A \to F$. Let $\mathcal{H}_F(W, \alpha(u))$ be the specialised algebra. Let G_F^λ be the matrix obtained by applying α to all coefficients of G^λ. Then G_F^λ is the Gram matrix of the induced bilinear form ϕ_F^λ on the specialised cell module W_F^λ. If q divides $\det(G^\lambda)$, then $\det(G_F^\lambda) = 0$ and so $\mathcal{H}_F(W, \alpha(u))$ will not be semisimple; see [23, 3.8]. By the general semisimplicity criterion in [20, 7.4.7], we deduce that $\alpha(\mathbf{c}_\mu) = 0$ for some $\mu \in \Lambda$. Now there are two cases.

If $q \in \mathbb{Z}$, then this implies that q must divide f_μ and so $\pm q$ is a bad prime.

If q is an irreducible non-constant polynomial, then q must divide $\mathbf{c}_\mu$. By Remark 2.1, $\mathbf{c}_\mu$ divides P_W. Hence, we deduce that q divides P_W. $\square$

Example 2.6. Let W be of type A_{n-1}. Then W can be identified with the symmetric group $\mathfrak{S}_n$ and Λ consists of all partitions $\lambda \vdash n$. A special feature of this case is that $f_\lambda = 1$ for all $\lambda \in \Lambda$. By [16, Example 4.2], the linear combinations in Theorem 2.2 will only have one non-zero term, with coefficient 1, i.e., the Kazhdan–Lusztig basis itself is a cellular basis. More precisely, for $\lambda \in \Lambda$, let w_λ be the longest element in the corresponding Young subgroup $\mathfrak{S}_\lambda$ of $W = \mathfrak{S}_n$. Now, by [29, §5], the Kazhdan–Lusztig left and right cells of W are given by the Robinson–Schensted correspondence. This explicit description shows that, if $\mathfrak{C}_\lambda$ denotes the left cell containing w_λ, we have

$$\mathfrak{C}_\lambda = \left\{ d(\mathfrak{s}) w_\lambda \,\middle|\, \mathfrak{s} \in M(\lambda) \right\}$$

where the elements $d(\mathfrak{s})$ ($\mathfrak{s} \in M(\lambda)$) are certain distinguished left coset representatives of $\mathfrak{S}_\lambda$ in $W = \mathfrak{S}_n$. Furthermore, given $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, there is a unique $w_\lambda(\mathfrak{s}, \mathfrak{t}) \in W$ such that $w_\lambda(\mathfrak{s}, \mathfrak{t})$ lies in the same right cell as $d(\mathfrak{s})w_\lambda$ and in the same left cell as $w_\lambda d(\mathfrak{t})^{-1}$. (See also [15, Remark 3.9, Corollary 5.6] for further details.) With this notation, [16, Example 4.2] shows that

$$C^\lambda_{\mathfrak{s},\mathfrak{t}} = c^\dagger_{w_\lambda(\mathfrak{s},\mathfrak{t})} \quad \text{for all } \lambda \vdash n \text{ and } \mathfrak{s}, \mathfrak{t} \in M(\lambda).$$

McDonough and Pallikaros [34] showed that the cell modules W^λ are naturally isomorphic to the Dipper–James Specht modules. The invariant bilinear form on W^λ is given by

$$\phi^\lambda(C_\mathfrak{s}, C_\mathfrak{t}) = h_{w_\lambda d(\mathfrak{s})^{-1}, d(\mathfrak{t})w_\lambda, w_\lambda} \quad \text{for all } \mathfrak{s}, \mathfrak{t} \in M(\lambda).$$

For connections of these bilinear forms with the topology of Springer fibres, see Fung [7].

Thus, for general $\mathcal{H}$, the cell modules W^λ arising from Theorem 2.2 can indeed be regarded as analogues of the Dipper–James Specht modules in type A_{n-1}.

Example 2.7. Let W be the Weyl group of type G_2 where $S = \{s_1, s_2\}$ and $(s_1s_2)^6 = 1$. We have $\mathrm{Irr}(W) = \{\mathbf{1}, \varepsilon_1, \varepsilon_2, \varepsilon, r, r'\}$ where $\mathbf{1}$ is the unit representation, ε is the sign representation, ε_1, ε_2 have dimension one, r is the reflection representation and r' is another representation of dimension two. The invariants $\mathbf{a}_\lambda$ and f_λ are given by

$$\mathbf{a}_\mathbf{1} = 0, \qquad \mathbf{a}_{\varepsilon_1} = \mathbf{a}_{\varepsilon_2} = \mathbf{a}_r = \mathbf{a}_{r'} = 1, \qquad \mathbf{a}_\varepsilon = 6;$$
$$f_\mathbf{1} = f_\varepsilon = 1, \qquad f_{\varepsilon_1} = f_{\varepsilon_2} = 3, \qquad f_r = 6, \qquad f_{r'} = 2.$$

Hence, the bad primes are 2 and 3. A cellular basis as in Theorem 2.2 is given as follows:

$$\begin{aligned}
&C^\mathbf{1}_{1,1} = c^\dagger_1, && C^\varepsilon_{1,1} = c^\dagger_{w_0},\\
&C^{\varepsilon_1}_{1,1} = c^\dagger_{s_2} - c^\dagger_{s_2s_1s_2} + c^\dagger_{s_2s_1s_2s_1s_2}, && C^{\varepsilon_2}_{1,1} = c^\dagger_{s_1} - c^\dagger_{s_1s_2s_1} + c^\dagger_{s_1s_2s_1s_2s_1},\\
&C^r_{1,1} = 3c^\dagger_{s_1} + 6c^\dagger_{s_1s_2s_1} + 3c^\dagger_{s_1s_2s_1s_2s_1}, && C^{r'}_{1,1} = c^\dagger_{s_1} - c^\dagger_{s_1s_2s_1s_2s_1},\\
&C^r_{1,2} = -3c^\dagger_{s_1s_2} - 3c^\dagger_{s_1s_2s_1s_2}, && C^{r'}_{1,2} = -c^\dagger_{s_1s_2} + c^\dagger_{s_1s_2s_1s_2},\\
&C^r_{2,1} = -3c^\dagger_{s_2s_1} - 3c^\dagger_{s_2s_1s_2s_1}, && C^{r'}_{2,1} = -c^\dagger_{s_2s_1} + c^\dagger_{s_2s_1s_2s_1},\\
&C^r_{2,2} = c^\dagger_{s_2} + 2c^\dagger_{s_2s_1s_2} + c^\dagger_{s_2s_1s_2s_1s_2}, && C^{r'}_{2,2} = c^\dagger_{s_2} - c^\dagger_{s_2s_1s_2s_1s_2}.
\end{aligned}$$

To find these expressions, we perform computations similar to those in [16, Example 4.3] (where type B_2 was considered). Once this is done, one can then also check directly that the above elements form a cellular basis. The Gram matrices of the invariant bilinear forms on the cell representations W^λ are given by

$$G^\mathbf{1} = [1], \qquad G^\varepsilon = [\,v^{-6}P_W\,], \qquad G^{\varepsilon_1} = G^{\varepsilon_2} = \big[\,3(v+v^{-1})\,\big],$$
$$G^r = \begin{bmatrix} 18(v+v^{-1}) & -18 \\ -18 & 6(v+v^{-1}) \end{bmatrix}, \qquad G^{r'} = \begin{bmatrix} 2(v+v^{-1}) & -2 \\ -2 & 2(v+v^{-1}) \end{bmatrix},$$

where $P_W = (v^{12}-1)(v^4-1)/(v^2-1)^2$ is the Poincaré polynomial of W.

Now let $\theta: A \to k$ be a specialisation; note that the characteristic of k will be either 0 or a prime $\neq 2, 3$. Let $e \geqslant 2$ be minimal such that $1 + \xi + \xi^2 + \cdots + \xi^{e-1} = 0$. Thus, either $\xi = 1$ and e is the characteristic of k, or e is the multiplicative order of ξ in $k^\times$. We see that the above Gram matrices remain non-singular after specialisation unless $\xi \neq 1$ and $e \in \{2, 3, 6\}$. Thus, we obtain non-trivial decomposition numbers only for $e \in \{2, 3, 6\}$. In these cases, the sets Λ_ξ° and the dimensions of L_ξ^μ for $\mu \in \Lambda_\xi^\circ$ are given as follows.

$e = 2$		
Λ_ξ°	$\mathbf{a}_\mu$	$\dim L_\xi^\mu$
$\mathbf{1}$	0	1
r	1	2
r'	1	2

$e = 3$		
Λ_ξ°	$\mathbf{a}_\mu$	$\dim L_\xi^\mu$
$\mathbf{1}$	0	1
ε_1	1	1
ε_2	1	1
r	1	2
r'	1	1

$e = 6$		
Λ_ξ°	$\mathbf{a}_\mu$	$\dim L_\xi^\mu$
$\mathbf{1}$	0	1
ε_1	1	1
ε_2	1	1
r	1	1
r'	1	2

In particular, we notice that the classification of the irreducible representations and their dimensions only depend on e, but not on the particular value of ξ or the characteristic of k. Thus, we have verified in a particular example the general phenomenon which is expressed in James' conjecture.

Remark 2.8. The decomposition matrix D_ξ can also be interpreted in the framework of Brauer's modular representation theory of associative algebras; see [6, §I.1.17]. Indeed, let us assume that k is the field of fractions of the image of θ. By [20, Exercise 7.8], there exists a discrete valuation ring $\mathcal{O} \subseteq K$ with maximal ideal $\mathfrak{p}$ such that $A \subseteq \mathcal{O}$ and $\mathfrak{p} \cap A = \ker(\theta)$. Let $k_\mathfrak{p} \supseteq k$ be the residue field of $\mathcal{O}$. Since $\mathcal{H}_k(W, \xi)$ is split, the scalar extension from k to $k_\mathfrak{p}$ induces a bijection $\mathrm{Irr}(\mathcal{H}_k(W, \xi)) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_{k_\mathfrak{p}}(W, \xi))$. Identifying $\mathrm{Irr}(\mathcal{H}_k(W, \xi))$ and $\mathrm{Irr}(\mathcal{H}_{k_\mathfrak{p}}(W, \xi))$ via this isomorphism, we obtain a well-defined decomposition map

$$d_\xi : R_0(\mathcal{H}_K) \to R_0\big(\mathcal{H}_k(W, \xi)\big)$$

where $R_0(\mathcal{H}_K)$ and $R_0(\mathcal{H}_k(W, \xi))$ denote the Grothendieck groups of finite-dimensional representations of $\mathcal{H}_K$ and $\mathcal{H}_k(W, \xi)$, respectively. Since each cell representation W^λ is defined over A and $W_K^\lambda \cong E_v^\lambda$, we conclude that

$$d_\xi\big(\big[E_v^\lambda\big]\big) = \sum_{\mu \in \Lambda_\xi^\circ} \big(W_\xi^\lambda : L_\xi^\mu\big)\big[L_\xi^\mu\big] \quad \text{for all } \lambda \in \Lambda,$$

where $[E_v^\lambda]$, $[L_\xi^\mu]$ denote the classes of E_v^λ, L_ξ^μ in the respective Grothendieck groups. (Note that, by [4, Ex. 6.16], we do not need to pass to the completion of $\mathcal{O}$, as is usually done in Brauer's modular representation theory.)

Definition 2.9. The *Brauer graph* of $\mathcal{H}$ with respect to $\theta: A \to k$ is the graph with vertices labelled by the elements of Λ and edges given as follows. Let $\lambda \neq \lambda'$ in Λ. Then the vertices labelled by λ and λ' are joined by an edge if there exists some $\mu \in \Lambda_\xi^\circ$ such that $(W_\xi^\lambda : L^\mu) \neq 0$

and $(W_\xi^{\lambda'} : L^\mu) \neq 0$. The connected components of this graph define a partition of Λ which are called the ξ-*blocks* of Λ (or of $\mathrm{Irr}(\mathcal{H}_K)$ or of $\mathrm{Irr}(W)$).

Let $\Lambda = \Lambda_1 \amalg \Lambda_2 \amalg \cdots \amalg \Lambda_r$ be the partition of Λ into ξ-blocks. Then we also have

$$\Lambda_\xi^\circ = \Lambda_{\xi,1}^\circ \amalg \Lambda_{2,\xi}^\circ \amalg \cdots \amalg \Lambda_{\xi,r}^\circ \quad \text{where } \Lambda_{\xi,i}^\circ := \Lambda_i \cap \Lambda_\xi^\circ.$$

If we order the elements of Λ and of Λ_ξ° accordingly, we obtain a block diagonal shape for D_ξ:

$$D_\xi = \begin{pmatrix} D_{\xi,1} & 0 & \dots & 0 \\ 0 & D_{\xi,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & D_{\xi,r} \end{pmatrix},$$

where $D_{\xi,i}$ has rows and columns labelled by the elements of Λ_i and $\Lambda_{\xi,i}^\circ$, respectively. Thus, in order to describe the set Λ_ξ° and the matrix D_ξ, we can proceed block by block. Note that, by Remark 2.8, the blocks of $\mathcal{H}$ as defined above really correspond to blocks in the sense of Brauer's modular representation theory.

3. The general version of James' conjecture

We keep the general setting of the previous section. Let $\mathcal{H}$ be an Iwahori–Hecke algebra associated with a finite Weyl group W, defined over the ring $A = R[v, v^{-1}]$ where $R \subseteq \mathbb{Q}$ is fixed as in the remarks just after Theorem 2.2. Then we have a cellular basis $\{C_{\mathfrak{s},\mathfrak{t}}^\lambda\}$ and cell representations $\{W^\lambda \mid \lambda \in \Lambda\}$ for $\mathcal{H}$.

Now let $\theta : A \to k$ be a ring homomorphism into a field k. Note that the characteristic of k will be either 0 or a prime p which is not bad for W. We obtain a corresponding specialised algebra $\mathcal{H}_k(W, \xi)$ where $\xi = \theta(u) \in k^\times$. Recall that

$$\mathrm{Irr}\big(\mathcal{H}_k(W,\xi)\big) = \big\{L_\xi^\mu \bigm| \mu \in \Lambda_\xi^\circ\big\}.$$

As in Remark 2.8, we have a decomposition map $d_\xi : R_0(\mathcal{H}_K) \to R_0(\mathcal{H}_k(W,\xi))$ such that

$$d_\xi\big(\big[E_v^\lambda\big]\big) = \sum_{\mu \in \Lambda_\xi^\circ} \big(W_\xi^\lambda : L_\xi^\mu\big)\big[L_\xi^\mu\big] \quad \text{for all } \lambda \in \Lambda.$$

Following Dipper and James [5], we set

$$e = \min\big\{i \geqslant 2 \bigm| 1 + \xi + \xi^2 + \cdots + \xi^{i-1} = 0\big\}.$$

(We set $e = \infty$ if no such i exists.) We assume from now on that $\mathrm{char}(k) = \ell > 0$ and $e < \infty$. Let $\zeta_e = \sqrt[e]{1} \in \mathbb{C}$ and consider the Iwahori–Hecke algebra $\mathcal{H}_\mathbb{C}(W, \zeta_e)$ arising from the specialisation

$$\theta_e : A \to \mathbb{C}, \qquad v \mapsto \zeta_{2e} = \sqrt[2e]{1}.$$

We can apply the previous discussion to the algebra $\mathcal{H}_{\mathbb{C}}(W,\zeta_e)$ as well. Thus, we have

$$\operatorname{Irr}\big(\mathcal{H}_{\mathbb{C}}(W,\zeta_e)\big)=\big\{L^{\mu}_{\zeta_e}\;\big|\;\mu\in\Lambda^{\circ}_{\zeta_e}\big\}.$$

Furthermore, there is a decomposition map $d_{\zeta_e}: R_0(\mathcal{H}_K)\to R_0(\mathcal{H}_{\mathbb{C}}(W,\zeta_e))$ such that

$$d_{\zeta_e}\big(\big[E^{\lambda}_{v}\big]\big)=\sum_{\mu\in\Lambda^{\circ}_{\zeta_e}}\big(W^{\lambda}_{\zeta_e}:L^{\mu}_{\zeta_e}\big)\big[L^{\mu}_{\zeta_e}\big]\quad\text{for all }\lambda\in\Lambda.$$

We will want to compare the representations of $\mathcal{H}_k(W,\xi)$ and $\mathcal{H}_{\mathbb{C}}(W,\zeta_e)$. For this purpose, the following remark will be relevant.

Remark 3.1. For any $d\geqslant 1$, we denote by $\Phi_d\in\mathbb{Z}[u]$ the dth cyclotomic polynomial. Note that we have

$$\Phi_d(v^2)=\begin{cases}\Phi_{2d}(v) & \text{if } d \text{ is even},\\ \Phi_d(v)\Phi_d(-v) & \text{if } d \text{ is odd}.\end{cases}$$

Now, in view of the definition of e, it is clear that $\Phi_e(\xi)=0$. Furthermore, note that $\theta(v)^2=\xi$. Hence, choosing a square root of ξ in $k^{\times}$ appropriately, we can assume that $\Phi_{2e}(\theta(v))=0$. (If $\operatorname{char}(k)\neq 2$, we also have $\Phi_e(\theta(v))\neq 0$.) Consequently, there exists a ring homomorphism $R[\zeta_{2e}]\to k$, $r\mapsto\bar{r}$, such that $\theta(a)=\overline{\theta_e(a)}$ for all $a\in A$. Let $\mathcal{O}\subseteq\mathbb{Q}(\zeta_{2e})$ be the localisation of $R[\zeta_{2e}]$ in the prime ideal $\mathfrak{q}=\{r\in R[\zeta_{2e}]\mid \bar{r}=0\}$. Then $\mathcal{O}$ is a discrete valuation ring whose residue field can be identified with a subfield of k. By "$\mathfrak{q}$-modular reduction" (see [6, §I.1.17]), we obtain a well-defined decomposition map

$$d^{e}_{\xi}: R_0\big(\mathcal{H}_{\mathbb{Q}(\zeta_{2e})}(W,\zeta_e)\big)\to R_0\big(\mathcal{H}_k(W,\xi)\big).$$

Note that the scalar extension from $\mathbb{Q}(\zeta_{2e})$ to $\mathbb{C}$ defines a bijection

$$\operatorname{Irr}\big(\mathcal{H}_{\mathbb{Q}(\zeta_{2e})}(W,\zeta_e)\big)\xrightarrow{\sim}\operatorname{Irr}\big(\mathcal{H}_{\mathbb{C}}(W,\zeta_e)\big).$$

Via this bijection, we can identify $R_0(\mathcal{H}_{\mathbb{Q}(\zeta_{2e})}(W,\zeta_e))$ and $R_0(\mathcal{H}_{\mathbb{C}}(W,\zeta_e))$, and regard d^{e}_{ξ} as a map from $R_0(\mathcal{H}_{\mathbb{C}}(W,\zeta_e))$ to $R_0(\mathcal{H}_k(W,\xi))$. Let us write

$$d^{e}_{\xi}\big(\big[L^{\nu}_{\zeta_e}\big]\big)=\sum_{\mu\in\Lambda^{\circ}_{\xi}}a_{\nu\mu}\big[L^{\mu}_{\xi}\big]\quad\text{for any }\nu\in\Lambda^{\circ}_{\zeta_e},$$

where $a_{\nu\mu}\in\mathbb{Z}_{\geqslant 0}$. Following James [28], the matrix $A^{e}_{\xi}:=(a_{\nu\mu})$ is called the *adjustment matrix* associated to the specialisation θ. By a general factorisation result for decomposition maps, we have $d_{\xi}=d^{e}_{\xi}\circ d_{\zeta_e}$ or, in other words,

$$\boxed{\big(W^{\lambda}_{\xi}:L^{\mu}_{\xi}\big)=\sum_{\nu\in\Lambda_{\zeta_e^{\circ}}}a_{\nu\mu}\big(W^{\lambda}_{\zeta_e}:L^{\nu}_{\zeta_e}\big)\quad\text{for all }\lambda\in\Lambda\text{ and }\mu\in\Lambda^{\circ}_{\xi}.}$$

This result first appeared in [8, Theorem 5.3]; see also [21, Proposition 2.5], [11, Proposition 2.6] for analogous statements in more general situations.

Lemma 3.2. *In the above setting, the following hold.*

(a) *Given $\mu \in \Lambda_\xi^\circ$ and $\nu \in \Lambda_{\zeta_e}^\circ$, we have $a_{\nu\mu} = 0$ unless $\nu = \mu$ or $\mathbf{a}_\mu < \mathbf{a}_\nu$.*
(b) *We have $\Lambda_\xi^\circ \subseteq \Lambda_{\zeta_e}^\circ$ and $a_{\mu\mu} = 1$ for all $\mu \in \Lambda_\xi^\circ$. In particular, we have $\Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ$ if these two sets have the same cardinality.*
(c) *We have $\dim L_\xi^\mu \leqslant \dim L_{\zeta_e}^\mu$ for all $\mu \in \Lambda_\xi^\circ$.*

Proof. Let $\lambda \in \Lambda$, $\mu \in \Lambda_\xi^\circ$ and $\nu \in \Lambda_{\zeta_e}^\circ$. Recall the relations (Δ) from Section 2: if $(W_\xi^\lambda : L_\xi^\mu) \neq 0$, then $\mathbf{a}_\mu \leqslant \mathbf{a}_\lambda$ with equality only for $\lambda = \mu$; furthermore, $(W_\xi^\mu : L_\xi^\mu) = 1$. A similar statement holds for the decomposition numbers $(W_{\zeta_e}^\lambda : L_{\zeta_e}^\nu)$.

(a) Assume that $a_{\nu\mu} \neq 0$. Then, since $(W_{\zeta_e}^\nu : L_{\zeta_e}^\nu) = 1$, we have

$$\left(W_\xi^\nu : L_\xi^\mu\right) = \sum_{\nu' \in \Lambda_{\zeta_e}^\circ} a_{\nu'\mu}\left(W_{\zeta_e}^\nu : L_{\zeta_e}^{\nu'}\right) > 0$$

and so the relations (Δ) imply that $\nu = \mu$ or $\mathbf{a}_\mu < \mathbf{a}_\nu$.

(b) We have $1 = (W_\xi^\mu : L_\xi^\mu) = \sum_{\nu' \in \Lambda_{\zeta_e}^\circ} a_{\nu'\mu}(W_{\zeta_e}^\mu : L_{\zeta_e}^{\nu'})$. So there exists some $\nu' \in \Lambda_{\zeta_e}^\circ$ such that $a_{\nu'\mu} \neq 0$ and $(W_{\zeta_e}^\mu : L_{\zeta_e}^{\nu'}) \neq 0$. Consequently, using (a) and the relations (Δ), we have $\mathbf{a}_\mu \leqslant \mathbf{a}_{\nu'} \leqslant \mathbf{a}_\mu$ and so $\mathbf{a}_\mu = \mathbf{a}_{\nu'}$. Thus, we must have $\mu = \nu' \in \Lambda_{\zeta_e}^\circ$ and $a_{\mu\mu} \neq 0$. Since $(W_\xi^\mu : L_\xi^\mu) = 1$, we then also conclude that $a_{\mu\mu} = 1$.

(c) Since $\dim L_{\zeta_e}^\mu = \sum_{\nu \in \Lambda_\xi^\circ} a_{\mu\nu} \dim L_\xi^\nu \geqslant a_{\mu\mu} \dim L_\xi^\mu$, this follows from (b). □

The observation that Λ_ξ° equals $\Lambda_{\zeta_e}^\circ$ once we know that these two sets have the same cardinality was first made by Jacon [27, Theorem 3.3] (in a slightly different context).

Theorem 3.3. *(See Geck and Rouquier [21, 5.4], [13, 3.2].) Assume that $e\ell$ does not divide any degree of W. Then $|\operatorname{Irr}(\mathcal{H}_k(W, \xi))| = |\operatorname{Irr}(\mathcal{H}_\mathbb{C}(W, \zeta_e))|$.*

Actually, using some explicit computations for W of exceptional type and the results of Ariki and Mathas [2] for W of classical type, one can show that the above conclusion holds under the single assumption that ℓ is a good prime; see [13]. However, we do not need this stronger result here.

Remark 3.4. The significance of the assumption on ℓ in Theorem 3.3 is as follows. One easily checks that if $f \geqslant 2$ is such that $\Phi_f(\xi) = 0$ then $f = e\ell^i$ for some $i \geqslant 0$ (see, for example, [13, 3.1]). Hence, assuming that $e\ell$ does not divide any degree of W, we have the following implication for any $f \geqslant 2$:

$$\Phi_f(\xi) = 0 \quad \text{and} \quad \Phi_f \quad \text{divides } P_W \quad \Rightarrow \quad f = e.$$

Conjecture 3.5 *(General version of James' conjecture). Recall our standing assumption that $e < \infty$ and $\operatorname{char}(k) = \ell > 0$ where ℓ is a good prime for W. Assume also that $e\ell$ does not divide any degree of W. Then the decomposition matrix D_ξ only depends on e. More precisely, the adjustment matrix A_ξ^e is the identity matrix or, in other words*:

$$\left(W_\xi^\lambda : L_\xi^\mu\right) = \left(W_{\zeta_e}^\lambda : L_{\zeta_e}^\mu\right) \quad \textit{for all } \lambda \in \Lambda \textit{ and } \mu \in \Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ. \tag{J}$$

(*Note that we do know that $\Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ$ by Theorem* 3.3 *and Lemma* 3.2.)

Using the factorisation in Remark 3.1 and Lemma 3.2, the above conjecture can be reformulated as follows.

Corollary 3.6 *(Alternative version of James' conjecture). Condition* (J) *in Conjecture* 3.5 *holds if and only if* $\dim \mathrm{rad}(\phi^\lambda_\xi) = \dim \mathrm{rad}(\phi^\lambda_{\zeta_e})$ *for all* $\lambda \in \Lambda$.

Thus, in order to verify James' conjecture, it is sufficient to determine the ranks of the Gram matrices of the bilinear forms ϕ^λ for various specialisations. Recall from Section 2 that the entries of these Gram matrices are certain structure constants of $\mathcal{H}$ with respect to its cellular basis, and these can be expressed in terms of the structure constants of the Kazhdan–Lusztig basis of $\mathcal{H}$. These in turn can be computed in principle (using recursive formulae), but note that this is only feasible for algebras of small rank. In Section 4 and [19], we will see how this problem can be solved effectively.

Proposition 3.7. *(See also [8, Proposition 5.5] and [11, 2.7].) There exists a bound N, depending only on W, such that condition* (J) *in Conjecture* 3.5 *holds for all* $\ell > N$.

Proof. We introduce the following notation. Given any matrix M with entries in A, we denote by M_ξ the matrix obtained by applying θ to all entries of M. Similarly, we define M_{ζ_e} via the map θ_e; the entries of M_{ζ_e} will lie in $R[\zeta_{2e}]$. Finally, if N is a matrix with entries in $R[\zeta_{2e}]$, we denote by $\bar{N}$ the matrix obtained by applying the map $\alpha \mapsto \bar{\alpha}$ to all entries of N (see Remark 3.1). With this notation, we have $M_\xi = \overline{M}_{\zeta_e}$ for any matrix M with entries in A.

Now fix $e \geqslant 2$ and $\lambda \in \Lambda$. Let G^λ be the Gram matrix of ϕ^λ; this is a matrix with entries in $\mathbb{Z}[v, v^{-1}]$. With the above notation, we have $G^\lambda_\xi = \overline{G}^\lambda_{\zeta_e}$. This already implies that $\mathrm{rank}(G^\lambda_\xi) \leqslant r := \mathrm{rank}(G^\lambda_{\zeta_e})$. We can find an $r \times r$-submatrix G of G^λ such that $\det(G_{\zeta_e}) \neq 0$. Now $\det(G_{\zeta_e})$ is an algebraic integer in the ring $\mathbb{Z}[\zeta_{2e}]$; its norm will be a non-zero rational integer. If ℓ does not divide that integer, we have

$$\det(G_\xi) = \det(\overline{G}_{\zeta_e}) = \overline{\det(G_{\zeta_e})} \neq 0.$$

So $r = \mathrm{rank}(G^\lambda_\xi) = \mathrm{rank}(G^\lambda_{\zeta_e})$ for ℓ "large enough." Hence, since Λ is a finite set, there is global bound N such that $\mathrm{rank}(G^\lambda_\xi) = \mathrm{rank}(G^\lambda_{\zeta_e})$ for all $\lambda \in \Lambda$ and all $\ell > N$. Hence, by Corollary 3.6, the conclusion of James' conjecture holds for all $\ell > N$, $\square$

Note that the above proof actually provides a method for finding N, assuming that the Gram matrices G^λ are explicitly known.

Recall from Section 2 the definition of the Brauer graph of $\mathcal{H}$ with respect to $\theta : A \to k$; its connected components are called *ξ-blocks*. Similarly, we define the Brauer graph of $\mathcal{H}$ with respect to $\theta_e : A \to \mathbb{C}$. Its connected components are called *ζ_e-blocks*.

Definition 3.8. Given $\lambda \in \Lambda$, we set

$$\delta_\lambda := \max\{i \geqslant 0 \mid \Phi_e^i \text{ divides } \mathbf{c}_\lambda \text{ in } \mathbb{Q}[u]\}.$$

This number is called the *Φ_e-defect* of λ (or of E^λ).

Proposition 3.9. *(See Geck [8, 7.4 and 7.6].) Assume that $e\ell$ does not divide any degree of W. Then the following hold.*

(a) *The ξ-blocks of $\mathcal{H}$ coincide with the ζ_e-blocks of $\mathcal{H}$.*
(b) *If E^λ and E^μ belong to the same ξ-block, then $\delta_\lambda = \delta_\mu$.*

The above result shows that all irreducible representations in a given ξ-block of $\mathcal{H}$ have the same Φ_e-defect, which will be called the Φ_e-defect of the block. Note that the only known proof of Proposition 3.9(b) relies on an interpretation of D_{ζ_e} in the modular representation theory of a finite group of Lie type with Weyl group W, and on known results on heights of characters in blocks of finite groups with abelian defect groups.

We can now state the main result of this article and its sequel [19].

Theorem 3.10. *Recall our standing assumption that $e < \infty$ and $\operatorname{char}(k) = \ell > 0$ where ℓ is a good prime for W. Assume now that W is of exceptional type and that $e\ell$ does not divide any degree of W. Then James's conjecture holds for $\mathcal{H}$. More precisely, let Λ_1 be a ξ-block of Λ. By Proposition 3.9, Λ_1 has a well-defined Φ_e-defect, δ say.*

(a) *If $\delta = 0$, then $\Lambda_1 = \{\lambda\}$ is a singleton set; we have $W_\xi^\lambda = L_\xi^\lambda$ and $W_{\zeta_e}^\lambda = L_{\zeta_e}^\lambda$.*
(b) *If $\delta = 1$, then the following hold:*
   (i) *We have $\mathbf{a}_\lambda \neq \mathbf{a}_{\lambda'}$ for any $\lambda \neq \lambda'$ in Λ_1. Thus, we have a unique labelling $\Lambda_1 = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $\mathbf{a}_{\lambda_1} < \mathbf{a}_{\lambda_2} < \cdots < \mathbf{a}_{\lambda_n}$.*
   (ii) *With the labelling in (i), we have $\Lambda^\circ_{1,\xi} = \{\lambda_1, \ldots, \lambda_{n-1}\}$ and*

$$\big(W_\xi^{\lambda_i} : L_\xi^{\lambda_j}\big) = \big(W_{\zeta_e}^{\lambda_i} : L_{\zeta_e}^{\lambda_j}\big) = \begin{cases} 1 & \textit{if } i = j \textit{ or } i = j+1, \\ 0 & \textit{otherwise.} \end{cases}$$

(c) *If $\delta \geqslant 2$, then $\Lambda^\circ_{1,\xi}$ and $\dim L_\xi^\mu$ for $\mu \in \Lambda^\circ_{1,\xi}$ are given by Tables 1 and 2.*

Remark 3.11. The ζ_e-blocks (together with their defect) of Iwahori–Hecke algebras of exceptional type are explicitly described in [20, Appendix F]. We have verified all the statements of Theorem 3.10 using an actual implementation of the algorithms presented in Section 4, and their refinements in [19]. Some of these statements are known to hold by theoretical arguments. More precisely:

- The statement in (a) follows from a general result about blocks of defect 0 in symmetric algebras; see [20, 7.5.11].
- The statement about $D_{\zeta_e,1}$ in (b) is proved, using general arguments, by a combination of [8, §10], [12, §4], [22, 4.4]. In [8, §10] it is also shown that these statements apply to D_ξ, if ℓ does not divide the order of W.

Note also that, once James' conjecture is established (in the form of Corollary 3.6), the complete decomposition matrices can be easily determined: it is sufficient to compute them for *one* specialisation $\theta : A \to k$ where $\operatorname{char}(k) = \ell$ is a good prime and $e\ell$ does not divide any degree of W. For the types F_4, E_6, E_7, these matrices were known before and can be found in [9,10,18,36]; for type E_8, see [19].

Table 1
The sets $\Lambda^{\circ}_{\zeta_e}$ for type F_4, E_6, E_7.

$F_4,\ e=2$		
1_1	0	1
2_1	1	2
2_3	1	2
9_1	2	5

$F_4,\ e=2$		
4_2	1	4

$F_4,\ e=3$		
1_1	0	1
2_1	1	1
2_3	1	1
4_1	4	1

$F_4,\ e=3$		
4_2	1	4
8_1	3	4
8_3	3	4
16_1	4	4

$F_4,\ e=4$		
1_1	0	1
4_2	1	4
9_1	2	4
6_1	4	1
12_1	4	4

$F_4,\ e=6$		
1_0	0	1
2_1	1	2
2_3	1	2
8_1	3	5
8_3	3	5

$E_6,\ e=2$		
1_p	0	1
6_p	1	6
20_p	2	14
15_q	3	14
30_p	3	10
60_p	5	46

$E_6,\ e=3$		
1_p	0	1
6_p	1	5
20_p	2	14
15_p	3	10
15_q	3	1
30_p	3	25
64_p	4	10
60_p	5	5
60_s	7	14
80_s	7	25

$E_6,\ e=4$		
1_p	0	1
6_p	1	6
15_p	3	15
15_q	3	8
81_p	6	60
10_s	7	1
80_s	7	6
90_s	7	15

$E_6,\ e=6$		
1_p	0	1
6_p	1	6
20_p	2	13
15_q	3	14
30_p	3	11
60_p	5	32
24_p	6	11
60_s	7	14
80_s	7	13
$60'_p$	11	1
$30'_p$	15	6

$E_7,\ e=2$		
1_a	0	1
$7'_a$	1	6
27_a	2	14
35_b	3	14
$105'_a$	4	78
$189'_b$	5	56
$315'_a$	7	126

$E_7,\ e=2$		
$56'_a$	3	56
120_a	4	64
280_b	7	216

$E_7,\ e=4$		
1_a	0	1
$56'_a$	3	56
105_b	6	48
210_a	6	154
189_a	8	35
405_a	8	147
70_a	16	21
315_a	16	120

$E_7,\ e=4$		
$7'_a$	1	−7
$15'_a$	4	8
$105'_a$	4	105
$189'_c$	7	84
280_b	7	168
$378'_a$	9	21
$210'_b$	13	27

$E_7,\ e=4$		
27_a	2	27
21_a	3	21
35_b	3	8
$216'_a$	8	168
210_b	10	7
105_c	12	84
378_a	14	105

$E_7,\ e=4$		
$21'_b$	3	21
120_a	4	120
$189'_b$	5	48
$35'_a$	7	35
$70'_a$	7	1
$315'_a$	7	147
336_a	13	154
$405'_a$	15	56

$E_7,\ e=3$		
1_a	0	1
21_a	3	21
35_b	3	34
120_a	4	98
105_b	6	7
168_a	6	35
210_a	6	91
280_b	7	14
210_b	10	49
420_a	10	196

$E_7,\ e=3$		
$7'_a$	1	7
$21'_b$	3	14
$56'_a$	3	49
$15'_a$	4	1
$105'_a$	4	35
$70'_a$	7	21
$280'_a$	7	196
$336'_a$	10	91
512_a	11	98
$84'_a$	13	34

$E_7,\ e=6$		
1_a	0	1
$7'_a$	1	7
$21'_b$	3	13
21_a	3	21
35_b	3	27
$15'_a$	4	14
$105'_a$	4	77
105_b	6	43
168_a	6	43
210_a	6	92
$70'_a$	7	42
$280'_a$	7	90
$315'_a$	7	13
84_a	10	14
210_b	10	27
420_a	10	92
$210'_b$	13	1
$420'_a$	13	77
280_a	16	21
315_a	16	7

Each table corresponds to a block of defect $\geqslant 2$. The first column specifies the set $\Lambda^{\circ}_{\zeta_e}$, the second column contains $\mathbf{a}_{\mu}$ and the third column contains $\dim L^{\mu}_{\zeta_e}$ for $\mu \in \Lambda^{\circ}_{\zeta_e}$.

Table 2
The sets $\Lambda^{\circ}_{\zeta_e}$ for type E_8.

$E_8,\ e=2$		
1_x	0	1
8_z	1	8
35_x	2	27
84_x	3	48
50_x	4	42
210_x	4	202
560_z	5	246
700_x	6	126
1400_z	7	792
1050_x	8	651
1400_x	8	378
4200_x	12	1863

$E_8,\ e=2$		
112_z	3	112
160_z	4	160
400_z	6	288
1344_x	7	1184
2240_x	10	1056
3360_z	12	2016

$E_8,\ e=4$		
1_x	0	1
35_x	2	34
112_z	3	77
50_x	4	16
210_x	4	176
567_x	6	280
400_z	6	96
175_x	8	1
350_x	8	70
1050_x	8	336
1575_x	8	946
525_x	12	168
3360_z	12	1654
2800_z	13	1302
2835_x	14	34
6075_x	14	280
3150_y	16	77
4480_y	16	176
5670_y	16	946

$E_8,\ e=4$		
8_z	1	8
560_z	5	560
1344_x	7	784
840_z	10	56
$1400_{z}z$	10	832
4536_z	13	2360
$4200'_z$	21	1008
$2240'_x$	28	1400

$E_8,\ e=4$		
28_x	3	28
160_z	4	160
300_x	6	300
972_x	10	512
840_x	12	28
$700_{x}x$	13	512
1344_w	16	160
$840'_x$	24	300

$E_8,\ e=4$		
56_z	7	56
1008_z	7	1008
1400_z	7	1400
3240_z	9	832
2240_x	10	8
4200_z	15	2360
$3200'_x$	21	784
$4536'_z$	23	560

$E_8,\ e=4$		
84_x	3	84
700_x	6	616
2268_x	10	1652
4200_x	12	1848
2100_x	13	448
2016_w	16	84
5600_w	16	1652
$4200'_x$	24	616

$E_8,\ e=3$		
1_x	0	1
35_x	2	35
28_x	3	28
84_x	3	48
50_x	4	1
210_x	4	147
300_x	6	70
700_x	6	518
1344_x	7	497
175_x	8	28
350_x	8	322
1050_x	8	35
1400_x	8	1225
2240_x	10	322
4096_z	11	1036
4200_x	12	147
$700_{x}x$	13	48
3200_x	15	497
4200_y	16	518
4480_y	16	1225

$E_8,\ e=3$		
8_z	1	8
112_z	3	104
160_z	4	56
560_z	5	384
400_z	6	8
448_z	7	56
1400_z	7	848
840_z	10	448
$1400_{z}z$	10	104
4096_x	11	1896
4200_z	15	384
5600_z	15	1896
7168_w	16	848

$E_8,\ e=6$		
1_x	0	1
8_z	1	8
35_x	2	35
28_x	3	28
84_x	3	40
50_x	4	41
210_x	4	210
560_z	5	279
300_x	6	225
700_x	6	86
56_z	7	56
448_z	7	85
1400_z	7	489
175_x	8	85
350_x	8	266
1050_x	8	660
1400_x	8	259
840_z	10	259
$1400_{z}z$	10	40
840_x	12	41
4200_x	12	1906
2100_x	13	1036
2400_z	15	266
4200_z	15	279
5600_z	15	489
420_y	16	1
1680_y	16	56
4200_y	16	660
4480_y	16	8
4536_y	16	225
5670_y	16	28
$4200'_x$	24	35
$1400'_x$	32	210

$E_8,\ e=6$		
112_z	3	112
160_z	4	160
400_z	6	288
1344_x	7	1072
2240_x	10	768
3360_z	12	2128
3200_x	15	2128
1344_w	16	288
7168_w	16	1072
$3360'_z$	24	160
$2240'_x$	28	112

$E_8,\ e=5$		
1_x	0	1
28_x	3	28
84_x	3	83
567_x	6	539
1344_x	7	722
972_x	10	166
2268_x	10	1729
4096_z	11	1078
168_y	16	1
1134_y	16	28
2688_y	16	722
4536_y	16	1729
$4096'_z$	26	539
$972'_x$	30	83

$E_8,\ e=10$		
1_x	0	1
8_z	1	8
28_x	3	28
84_x	3	75
567_x	6	531
448_z	7	372
1008_z	7	449
1400_z	7	786
972_x	10	897
2268_x	10	502
4536_z	13	2406
1400_y	16	449
3150_y	16	372
4200_y	16	897
4480_y	16	786
$4536'_z$	23	75
$2268'_x$	30	531
$448'_z$	37	1
$1008'_z$	37	28
$1400'_z$	37	8

$E_8,\ e=8$		
1_x	0	1
35_x	2	34
160_z	4	160
567_x	6	373
175_x	8	174
1400_x	8	992
1575_x	8	1042
525_x	12	152
2835_x	14	1668
6075_x	14	3516
2016_w	16	174
5600_w	16	1042
7168_w	16	992
$2835'_x$	22	1
$6075'_x$	22	373
$1400'_x$	32	34
$1575'_x$	32	160

$E_8,\ e=12$		
1_x	0	1
35_x	2	35
112_z	3	76
50_x	4	50
210_x	4	99
400_z	6	349
1050_x	8	651
1400_x	8	974
525_x	12	449
3360_z	12	1386
2800_z	13	1202
1400_y	16	99
2688_y	16	651
4536_y	16	974
2100_y	20	449
$3360'_z$	24	349
$2800'_z$	25	76
$1050'_x$	32	50
$1400'_x$	32	1
$210'_x$	52	35

4. Constructing the invariant bilinear form

We have seen in Proposition 3.7 that James' conjecture can be verified once we have constructed the Gram matrices of the invariant bilinear forms on the cell modules W^λ. If $\mathcal{H}$ is not too large, we could actually do this by explicitly working out a cellular basis as in [16, Example 4.3] (type B_2) or Example 2.7 (type G_2). Using computers, it would also be possible to carry out similar computations in type F_4 and, perhaps, type E_6. However, this becomes totally unfeasible for type E_7 or E_8, where we do have to explore alternative routes. The purpose of this section is to show how this can be done. Eventually, we will have to rely on computer calculations, but our aim is to develop a conceptual reduction of our problem where, at the end, standard programs like Parker's MeatAxe [38] and its variations can be applied. (See also Ringe's package [39] which comes with extensive documentation and a variety of additions to Parker's original programs.)

We keep the general setting of the previous section. Recall that $\mathcal{H}$ is defined over the ring $A = R[v, v^{-1}]$ where $R \subseteq \mathbb{Q}$ consists of all fractions $a/b \in \mathbb{Q}$ such that $a \in \mathbb{Z}$ and $0 \neq b \in \mathbb{Z}$ is divisible by bad primes only. Let K be the field of fractions of A. If M is any A-module, we denote $M_K := K \otimes_A M$.

Let $e \geqslant 2$ and $\theta : A \to k$ a ring homomorphism into a field k; let $\xi = \theta(u) \in k$. As before, if M is any A-module, we denote $M_\xi := k \otimes_A M$ where k is regarded as an A-module via θ. We say that θ is *e-regular* if $\mathrm{char}(k) = \ell > 0$ is a good prime and $e\ell$ does not divide any degree of W. (These are precisely the conditions appearing in James' conjecture.) We will address the following three major issues which are sufficient for verifying that James' conjecture holds for a given algebra $\mathcal{H}$:

Problem 4.1. Let $e \geqslant 2$ be an integer which divides some degree of W.

(a) For any $\lambda \in \Lambda$, construct an explicit model for W^λ, that is, an $\mathcal{H}$-module V^λ which is free of finite rank over A such that $V_K^\lambda \cong W_K^\lambda$. Determine Λ_ξ° and the decomposition matrix D_ξ for at least one e-regular specialisation $\theta : A \to k$.
(b) Show that, for each $\lambda \in \Lambda_{\zeta_e}^\circ$, the model V^λ in (a) has the property that $V_\xi^\lambda \cong W_\xi^\lambda$ for any e-regular specialisation $\theta : A \to k$.
(c) For any $\lambda \in \Lambda_{\zeta_e}^\circ$, determine the Gram matrix Q^λ of an invariant bilinear form on V^λ and show that $\mathrm{rank}(Q_\xi^\lambda) = \mathrm{rank}(G_\xi^\lambda)$ for any e-regular specialisation $\theta : A \to k$.

Finally, compute $\mathrm{rank}(Q_{\zeta_e}^\lambda)$ and find the *finite* set of prime numbers $\mathcal{P}_e$ such that

$$\mathrm{rank}\big(Q_\xi^\lambda\big) = \mathrm{rank}\big(Q_{\zeta_e}^\lambda\big) \quad \text{if } \ell \notin \mathcal{P}_e.$$

4.1. Solving Problem 4.1(a)

Natural candidates for models for the cell representations of $\mathcal{H}$ are the representations afforded by W-graphs. In fact, Gyoja [24] has shown that every irreducible representation of $\mathcal{H}_K$ is afforded by a W-graph. We recall:

Definition 4.2. (See Kazhdan–Lusztig [29].) A W-graph for $\mathcal{H}$ consists of the following data:

(a) a set X together with a map I which assigns to each $x \in X$ a set $I(x) \subseteq S$;
(b) a collection of elements $\mu_{x,y} \in \mathbb{Z}$, where $x, y \in X$, $x \neq y$.

These data are subject to the following requirements. Let V be a free A-module with a basis $\{e_y \mid y \in X\}$. For each $s \in S$, define an A-linear map $\sigma_s : V \to V$ by

$$\sigma_s(e_y) = v^2 e_y + \sum_{\substack{x \in X \\ s \in I(x)}} v\mu_{x,y} e_x \quad \text{if } s \notin I(y),$$
$$\sigma_s(e_y) = -e_y \quad \text{if } s \in I(y).$$

Then we require that the assignment $T_s \mapsto \sigma_s$ defines a representation of $\mathcal{H}$.

Thus, in a representation afforded by a W-graph, each generator T_s ($s \in S$) of $\mathcal{H}$ is represented by a matrix of a particularly simple form. Recently, Howlett and Yin [26], [40] explicitly constructed W-graphs for all irreducible representations for Iwahori–Hecke algebras of type E_7, E_8. In combination with earlier results of Naruse [37] on types F_4 and E_6, we now have W-graphs for all irreducible representations of algebras of exceptional type. These W-graphs are electronically accessible through Michel's development version [35] of the computer algebra system CHEVIE [17]. Thus, we do have a collection of explicitly given $\mathcal{H}$-modules

$$\{V^\lambda \mid \lambda \in \Lambda\}$$

such that each V^λ is free of finite rank over A and $V_K^\lambda \cong E_v^\lambda \cong W_K^\lambda$.

Now let $\theta : A \to k$ be an e-regular specialisation. Using the CHOP function in Ringe's version [39] of the MeatAxe, we can decompose each V_ξ^λ into its irreducible constituents. Thus, we obtain:

- $\mathrm{Irr}(\mathcal{H}_k(W, \xi)) = \{M_1, \ldots, M_r\}$ and
- the decomposition numbers $(V_\xi^\lambda : M_i)$ for $\lambda \in \Lambda$ and $1 \leqslant i \leqslant r$.

Note that, by Remark 2.8, we have $(W_\xi^\lambda : M_i) = (V_\xi^\lambda : M_i)$ for all $\lambda \in \Lambda$ and $1 \leqslant i \leqslant r$. The relations ($\Delta$) in Section 2 immediately imply the following "identification result":

Lemma 4.3. *Let $i \in \{1, \ldots, r\}$. Then the unique $\mu \in \Lambda_\xi^\circ$ such that $M_i = L_\xi^\mu$ is determined by the conditions that $(W_\xi^\mu : M_i) = 1$ and*

$$\mathbf{a}_\mu \leqslant \mathbf{a}_\lambda \quad \textit{for all } \lambda \in \Lambda \textit{ such that } \left(W_\xi^\lambda : M_i\right) \neq 0.$$

By Theorem 3.3 and Lemma 3.2, we have $\Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ$. Thus, we are able to determine the sets $\Lambda_{\zeta_e}^\circ$ for any $e \geqslant 2$. This already yields the information contained in the first columns in Table 1 and 2.

4.2. Solving Problem 4.1(b)

Let us fix $e \geqslant 2$ and an element $\lambda \in \Lambda_{\zeta_e}^\circ$. As discussed above, we have an $\mathcal{H}$-module V^λ such that $W_K^\lambda \cong V_K^\lambda$. Now let $\theta : A \to k$ be any e-regular specialisation. In general, without any further knowledge about V^λ, we cannot expect that we also have $W_\xi^\lambda \cong V_\xi^\lambda$. The following result gives a precise condition for when this is the case.

Proposition 4.4. *Assume that there exists some e-regular specialisation $\theta_0 : A \to k_0$ such that $V_{\xi_0}^\lambda$ (where $\xi_0 = \theta_0(u)$) has a unique maximal submodule U^λ, and we have $V_{\xi_0}^\lambda / U^\lambda \cong L_{\xi_0}^\lambda$. Then $V_{\zeta_e}^\lambda \cong W_{\zeta_e}^\lambda$ and $V_\xi^\lambda \cong W_\xi^\lambda$ for any e-regular specialisation $\theta : A \to k$.*

Proof. The module W^λ has a standard basis $\{C_\mathfrak{s} \mid \mathfrak{s} \in M(\lambda)\}$; let $\rho^\lambda : \mathcal{H} \to M_{d_\lambda}(A)$ be the corresponding matrix representation. The module V^λ also has a standard basis, arising from the underlying W-graph; let $\sigma^\lambda : \mathcal{H} \to M_{d_\lambda}(A)$ be the corresponding matrix representation. Since $V_K^\lambda \cong W_K^\lambda$, there exists an invertible matrix $P^\lambda \in M_{d_\lambda}(K)$ such that

$$\rho^\lambda(T_w) P^\lambda = P^\lambda \sigma^\lambda(T_w) \quad \text{for all } w \in W.$$

Multiplying P^λ by a suitable scalar, we may assume without loss of generality that

- all entries of P^λ lie in $\mathbb{Z}[v]$ and
- the greatest common divisor of all non-zero entries of P^λ is 1.

(Here we use the fact that R was chosen to be contained in $\mathbb{Q}$.) These conditions uniquely determine P^λ up to a sign. Let $\delta := \det(P^\lambda) \neq 0$. We need to obtain some more precise information about the irreducible factors of δ. Let us write

$$\delta = m f_1 f_2 \cdots f_r \quad \text{where } 0 \neq m \in \mathbb{Z} \text{ and } f_1, \ldots, f_r \in \mathbb{Z}[v] \setminus \mathbb{Z} \text{ are irreducible.}$$

First we claim that m is divisible by bad primes only. Indeed, let p be a prime number which is good for W. Then p generates a prime ideal in R; let $F = \mathbb{F}_p(v)$. We obtain a specialisation $\alpha : A \to F$ by reducing the coefficients of polynomials in A modulo p. We have a corresponding specialised algebra $\mathcal{H}_F(W, u)$. Since $\alpha(P_W) \neq 0$, we conclude that $\mathcal{H}_F(W, u)$ is semisimple and the specialised cell modules W_F^λ are all irreducible; see Proposition 2.4. Now note that not all entries of P^λ are divisible by p. Hence, reducing the entries of P^λ modulo p, we obtain a non-zero matrix defining a non-trivial module homomorphism $V_F^\lambda \to W_F^\lambda$. Since W_F^λ is irreducible and $\dim W_F^\lambda = \dim V_F^\lambda$, this homomorphism must be an isomorphism. Consequently, P^λ is invertible modulo p and so p cannot divide m.

A similar argument shows that each f_i divides $P_W(v^2)$. Indeed, assume that $f \in \mathbb{Z}[v]$ is a non-constant irreducible polynomial which does not divide $P_W(v^2)$. Then we have a canonical ring homomorphism $\beta : A \to F$ where $F = \mathbb{Q}[v]/(f)$. Again, the corresponding specialised algebra $\mathcal{H}_F(W, \theta(u))$ is semisimple since $\beta(P_W) \neq 0$. Arguing as above, we conclude that f does not divide $\det(P^\lambda)$. Thus, each f_i must divide $P_W(v^2)$.

Now consider the specialisation $\theta_e : A \to \mathbb{C}$ which sends v to ζ_{2e}. We can actually regard this as a map with image in $\mathbb{Q}(\zeta_{2e})$ and work with $\mathbb{Q}(\zeta_{2e})$ instead of $\mathbb{C}$ as base field. Thus, $W_{\zeta_e}^\lambda$ and $V_{\zeta_e}^\lambda$ can be regarded as $\mathbb{Q}(\zeta_{2e})$-vectorspaces and modules for the specialised algebra $\mathcal{H}_{\mathbb{Q}(\zeta_{2e})}(W, \zeta_e)$. Let $\mathcal{O}$ be a discrete valuation ring as in Remark 3.1 with respect to the specialisation θ_0; we have a corresponding decomposition map

$$d_{\xi_0}^e : R_0\big(\mathcal{H}_{\mathbb{Q}(\zeta_{2e})}(W, \zeta_e)\big) \to R_0\big(\mathcal{H}_{k_0}(W, \xi_0)\big).$$

Once again, since the greatest common divisor of all its entries is 1, the matrix P^λ induces a non-trivial module homomorphism $V_{\zeta_e}^\lambda \to W_{\zeta_e}^\lambda$. We claim that this also is an isomorphism. To prove this, let $M \subseteq V^\lambda_{\zeta_e}$ be the kernel of the map $V^\lambda_{\zeta_e} \to W^\lambda_{\zeta_e}$; then M is a proper submodule of $V^\lambda_{\zeta_e}$. By a standard result (see [4, 23.7]), there exists a proper submodule $N \subseteq V^\lambda_{\xi_0}$ such that

$$d^e_{\xi_0}\bigl([M]\bigr) = [N] \quad \text{and} \quad d_{\xi_0}\bigl(\bigl[V^\lambda_{\zeta_e}/M\bigr]\bigr) = \bigl[V^\lambda_{\xi_0}/N\bigr].$$

If $L^\lambda_{\zeta_e}$ were a composition factor of M, then $L^\lambda_{\xi_0}$ would be a composition factor of N by Lemma 3.2(b). But then, by our assumption on $V^\lambda_{\xi_0}$ and since $N \subseteq U$, the simple module $L^\lambda_{\xi_0}$ would appear at least twice as a composition factor of $V^\lambda_{\xi_0}$, which is absurd. So we conclude that $L^\lambda_{\zeta_e}$ is not a composition factor of M. Hence, $L^\lambda_{\zeta_e}$ will be a composition factor of the image of the map $V^\lambda_{\zeta_e} \to W^\lambda_{\zeta_e}$. But, by [23, Proposition 3.2], $L^\lambda_{\zeta_e}$ is a simple quotient of $W^\lambda_{\zeta_e}$, the kernel of the canonical map $W^\lambda_{\zeta_e} \to L^\lambda_{\zeta_e}$ is the unique maximal submodule of $W^\lambda_{\zeta_e}$, and $L^\lambda_{\zeta_e}$ is not a composition factor of that kernel. So we conclude that the map $V^\lambda_{\zeta_e} \to W^\lambda_{\zeta_e}$ is surjective and, hence, an isomorphism. It follows that δ is not divisible by $\Phi_{2e}(v)$. If e is odd, we can also consider the specialisation $\theta'_e : A \to \mathbb{C}$ sending v to $\zeta_e^{(e+1)/2}$ (the other square root of ζ_e, which is a root of $\Phi_e(v)$). Then a similar argument shows that δ is not divisible by $\Phi_e(v)$. Thus, we have reached the following conclusions:

- m is divisible by bad primes only;
- each f_i divides $P_W(v^2)$;
- each f_i is coprime to $\Phi_e(v^2)$.

We can now complete the proof as follows. Let $\theta : A \to k$ be any e-regular specialisation. Assume that $\theta(\delta) = 0$. Since the characteristic of k is a good prime, we must have $\theta(f_i) = 0$ for some $i \in \{1, \dots, r\}$. Since each f_i divides $P_W(v^2)$, there exists some $d \geqslant 2$ such that $\Phi_d(v^2)$ divides $P_W(v^2)$ and f_i divides $\Phi_d(v^2)$. By Remark 3.4, we conclude that $d = e$. Thus, we see that f_i divides $\Phi_e(v^2)$, a contradiction. Hence, our assumption was wrong and so we do have $\theta(\delta) \neq 0$. Thus, we have shown that P^λ induces an isomorphism $V^\lambda_\xi \xrightarrow{\sim} W^\lambda_\xi$. $\square$

Let $\theta_0 : A \to k_0$ be a specialisation as in Proposition 4.4. Using the `MKSUB` function in Ringe's version [39] of the MeatAxe (see also [33]), we can determine the complete submodule lattice of $V^\lambda_{\xi_0}$. Using the `CHOP` function and Lemma 4.3 as discussed in the previous subsection, we can identify the various irreducible constituents of $V^\lambda_{\xi_0}$ and check that the assumption of Proposition 4.4 is satisfied. Thus, Problem 4.1(b) is solved.

It might actually be true that W^λ and V^λ are isomorphic as $\mathcal{H}$-modules, but we have not been able to prove this. We would like to state this as a conjecture:

Conjecture 4.5. *Assume that, for each $\lambda \in \Lambda$, we are given a W-graph affording an $\mathcal{H}$-module V^λ such that $V^\lambda_K \cong E^\lambda_v$. Then the cellular basis in Theorem* 2.2 *can be chosen such that $W^\lambda \cong V^\lambda$ for all $\lambda \in \Lambda$.*

4.3. Solving Problem 4.1(c)

Let $\lambda \in \lambda^\circ_{\zeta_e}$ and G^λ be the Gram matrix of the invariant bilinear form ϕ^λ with respect to the standard basis of W^λ. Instead of W^λ, we now consider the module V^λ and assume that the hypotheses of Proposition 4.4 are satisfied. Thus, we have $V^\lambda_{\zeta_e} \cong W^\lambda_{\zeta_e}$ and $V^\lambda_\xi \cong W^\lambda_\xi$ for any e-regular specialisation $\theta : A \to k$.

Let $\sigma^\lambda : \mathcal{H} \to M_{d_\lambda}(A)$ be the matrix representation afforded by V^λ with respect to the standard basis arising from the underlying W-graph. Our task now is to find some non-zero matrix $Q^\lambda \in M_{d_\lambda}(A)$ such that

$$Q^\lambda \cdot \sigma^\lambda(T_s) = \sigma^\lambda(T_s)^{\mathrm{tr}} \cdot Q^\lambda \quad \text{for all } s \in S. \tag{$*$}$$

Note that $(*)$ implies that $Q^\lambda \cdot \sigma^\lambda(T_{w^{-1}}) = \sigma^\lambda(T_w)^{\mathrm{tr}} \cdot Q^\lambda$ for all $w \in W$. So any solution to $(*)$ is the Gram matrix of an invariant bilinear form on V^λ. Multiplying Q^λ by a suitable scalar, we may assume without loss of generality that

- all entries of Q^λ lie in $\mathbb{Z}[v]$ and
- the greatest common divisor of all non-zero entries of Q^λ is 1.

Note that, by Schur's lemma, any two matrices satisfying $(*)$ are scalar multiples of each other. Hence, the above conditions uniquely determine Q^λ up to a sign.

Lemma 4.6. *Assume that Q^λ is a solution to $(*)$ satisfying the above conditions. Then*

$$\mathrm{rank}(Q_{\zeta_e}) = \mathrm{rank}\big(G^\lambda_{\zeta_e}\big) \quad \textit{and} \quad \mathrm{rank}(Q_\xi) = \mathrm{rank}\big(G^\lambda_\xi\big)$$

for any e-regular specialisation $\theta : A \to k$.

Proof. We are assuming that $\lambda \in \Lambda^\circ_{\zeta_e} = \Lambda^\circ_\xi$, so we have $G^\lambda_{\zeta_e} \neq 0$ and $G^\lambda_\xi \neq 0$.

Now let P^λ be as in the proof of Proposition 4.4 and set $\tilde{Q}^\lambda := (P^\lambda)^{\mathrm{tr}} G^\lambda P^\lambda$. Then $\tilde{Q}^\lambda$ is a solution to $(*)$ and so there exists some $0 \neq \alpha \in K$ such that $\tilde{Q}^\lambda = \alpha Q^\lambda$. Since all three matrices G^λ, Q^λ and $\tilde{Q}^\lambda$ have all their entries in $\mathbb{Z}[v, v^{-1}]$ and since the greatest common divisor of the entries of Q^λ is 1, we can conclude that $\alpha \in \mathbb{Z}[v, v^{-1}]$.

Now, in the proof of Proposition 4.4, we have actually seen that $P^\lambda_{\zeta_e}$ and P^λ_ξ are invertible. Since we also have $G^\lambda_{\zeta_e} \neq 0$ and $G^\lambda_\xi \neq 0$, it follows that

$$\mathrm{rank}(\tilde{Q}_{\zeta_e}) = \mathrm{rank}\big(G^\lambda_{\zeta_e}\big) > 0 \quad \text{and} \quad \mathrm{rank}(\tilde{Q}_\xi) = \mathrm{rank}\big(G^\lambda_\xi\big) > 0.$$

But then it also follows that $\theta_e(\alpha) \neq 0$ and $\theta(\alpha) \neq 0$. Hence, we have $\mathrm{rank}(\tilde{Q}_{\zeta_e}) = \mathrm{rank}(Q^\lambda_{\zeta_e})$ and $\mathrm{rank}(\tilde{Q}_\xi) = \mathrm{rank}(Q^\lambda_\xi)$, and this yields the desired statement. $\square$

It remains to show how a solution to $(*)$ can actually be computed. Note that $(*)$ constitutes a system of $|S|d_\lambda^2$ linear equations for the d_λ^2 entries of Q^λ. If d_λ is not too large, this can be solved directly. However, in type E_8, we have $d_\lambda = 7168$ for some λ, and our system of linear equations simply becomes too large. In such cases, different techniques are required which are based on the following result:

Theorem 4.7. *(See Benson–Curtis [20, §6.3].) Each $E^\mu \in \mathrm{Irr}(W)$ is of parabolic type, that is, there exists a subset $I \subseteq S$ such that the restriction of E^μ to the parabolic subgroup $W_I \subseteq W$ contains the trivial representation of W_I just once. A similar statement holds when "trivial representation" is replaced by "sign representation."*

Now the main idea is as follows: Since the bijection $\mathrm{Irr}(W) \leftrightarrow \mathrm{Irr}(\mathcal{H}_K)$ arising from Tits' Deformation Theorem is compatible with restriction to parabolic subgroups and subalgebras (see [20, 9.1.9]), the above result means that there exists a subset $I \subseteq S$ such that

$$\dim_K\left(\bigcap_{s\in I}\ker\bigl(\sigma_K^\lambda(T_s+T_1)\bigr)\right)=1;$$

let $e_1 \in K^{d_\lambda}$ be a vector spanning this one-dimensional space. Similarly,

$$\dim_K\left(\bigcap_{s\in I}\ker\bigl(\sigma_K^\lambda(T_s+T_1)^{\mathrm{tr}}\bigr)\right)=1;$$

let $v_1 \in K^{d_\lambda}$ be a vector spanning this one-dimensional space. Now, since σ_K^λ is an irreducible representation of $\mathcal{H}_K$, there exist $w_2, \ldots, w_{d_\lambda} \in W$ such that the vectors

$$e_1, \quad e_2 := \sigma^\lambda(T_{w_2})e_1, \quad e_3 := \sigma^\lambda(T_{w_3})e_1, \quad \ldots, \quad e_{d_\lambda} := \sigma^\lambda(T_{w_{d_\lambda}})e_1,$$

form a basis of K^{d_λ}. Then the vectors

$$v_1, \quad v_2 := \sigma^\lambda(T_{w_2^{-1}})^{\mathrm{tr}}v_1, \quad v_3 := \sigma^\lambda(T_{w_3^{-1}})^{\mathrm{tr}}v_1, \quad \ldots, \quad v_{d_\lambda} := \sigma^\lambda(T_{w_{d_\lambda}^{-1}})^{\mathrm{tr}}v_1,$$

will also form a basis of K^{d_λ}. Hence, there exists a unique invertible matrix $\tilde{Q}^\lambda \in M_{d_\lambda}(K)$ such that $v_i = \tilde{Q}^\lambda e_i$ for $1 \leqslant i \leqslant d_\lambda$. Then $\tilde{Q}^\lambda \cdot \sigma^\lambda(T_w) \cdot (\tilde{Q}^\lambda)^{-1} = \sigma^\lambda(T_{w^{-1}})^{\mathrm{tr}}$ for all $w \in W$ and so $\tilde{Q}^\lambda$ is a solution to $(*)$. Multiplying by a suitable scalar, we obtain Q^λ.

The above technique is known as the "standard base" algorithm; see the ZSB function of Ringe's MeatAxe [39] and its description. In practice, we did not apply it to σ^λ itself but to various specialisations into finite fields such that the specialised algebra remains semisimple. For each such specialisation, we use the ZSB function to find the Gram matrix of an invariant bilinear form. Using interpolation and modular techniques (Chinese Remainder), one can recover Q^λ from these specialisations.

Having computed Q^λ, we substitute $v \mapsto \sqrt[2e]{1}$ and determine the rank of the specialised matrix. Arguing as in the proof of Proposition 3.7, we find the *finite* set of prime numbers $\mathcal{P}_e$ such that $\mathrm{rank}(Q_\xi^\lambda) = \mathrm{rank}(Q_{\zeta_e}^\lambda)$ if $\ell \notin \mathcal{P}_e$. See [19] for further details.

Remark 4.8. Assume we are in the above setting, where $I \subseteq S$ is a subset such that the restriction of E^λ to W_I contains the sign representation exactly once. Then, by the formulas in Definition 4.2, the vector e_1 can be taken to be contained in the standard basis of K^{d_λ}. Since $v_1 = \tilde{Q}^\lambda e_1$, we conclude that v_1 is a column of the matrix $\tilde{Q}^\lambda$. In other words, using Theorem 4.7, one column of the matrix $\tilde{Q}^\lambda$ can be computed by simply determining the intersection of the kernels of the maps $\sigma_K^\lambda(T_s+T_1)^{\mathrm{tr}}$ where s runs over the generators in I.

Example 4.9. In general, the matrix Q^λ is far from being sparse. We just give one example. Let W be of type E_6 with Dynkin diagram

E_6

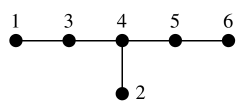
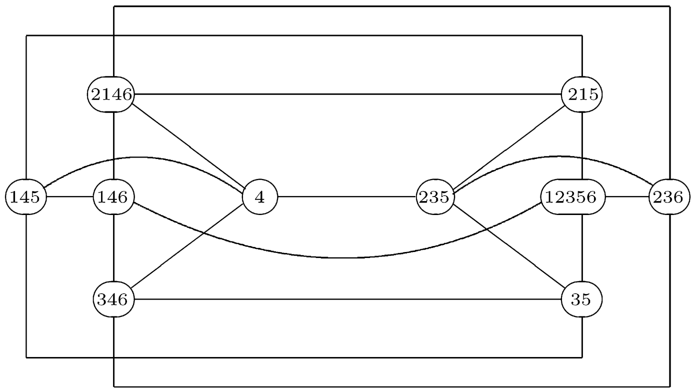


Table 3
W-graph and invariant bilinear form for the representation 10_s in type E_6.



$$Q^{10_s} = \begin{bmatrix}
v^6+3v^4+3v^2+1 & 2v^4+2v^2 & 2v^4+2v^2 & -v^5-2v^3-v & 2v^4+2v^2 & -v^5-2v^3-v & 2v^4+2v^2 & -v^5-2v^3-v & -v^5-2v^3-v & -2v^3 \\
2v^4+2v^2 & v^6+3v^4+3v^2+1 & 2v^4+2v^2 & -v^5-2v^3-v & 2v^4+2v^2 & -v^5-2v^3-v & 2v^4+2v^2 & -v^5-2v^3-v & -2v^3 & -v^5-2v^3-v \\
2v^4+2v^2 & 2v^4+2v^2 & v^6+3v^4+3v^2+1 & -v^5-2v^3-v & 2v^4+2v^2 & -v^5-2v^3-v & 2v^4+2v^2 & -2v^3 & -v^5-2v^3-v & -v^5-2v^3-v \\
-v^5-2v^3-v & -v^5-2v^3-v & -v^5-2v^3-v & v^6+2v^4+2v^2+1 & -v^5-2v^3-v & v^4+v^2 & -2v^3 & v^4+v^2 & v^4+v^2 & v^4+v^2 \\
2v^4+2v^2 & 2v^4+2v^2 & 2v^4+2v^2 & -v^5-2v^3-v & v^6+3v^4+3v^2+1 & -2v^3 & 2v^4+2v^2 & -v^5-2v^3-v & -v^5-2v^3-v & -v^5-2v^3-v \\
-v^5-2v^3-v & -v^5-2v^3-v & -v^5-2v^3-v & v^4+v^2 & -2v^3 & v^6+2v^4+2v^2+1 & -v^5-2v^3-v & v^4+v^2 & v^4+v^2 & v^4+v^2 \\
2v^4+2v^2 & 2v^4+2v^2 & 2v^4+2v^2 & -2v^3 & 2v^4+2v^2 & -v^5-2v^3-v & v^6+3v^4+3v^2+1 & -v^5-2v^3-v & -v^5-2v^3-v & -v^5-2v^3-v \\
-v^5-2v^3-v & -v^5-2v^3-v & -2v^3 & v^4+v^2 & -v^5-2v^3-v & v^4+v^2 & -v^5-2v^3-v & v^6+2v^4+2v^2+1 & v^4+v^2 & v^4+v^2 \\
-v^5-2v^3-v & -2v^3 & -v^5-2v^3-v & v^4+v^2 & -v^5-2v^3-v & v^4+v^2 & -v^5-2v^3-v & v^4+v^2 & v^6+2v^4+2v^2+1 & v^4+v^2 \\
-2v^3 & -v^5-2v^3-v & -v^5-2v^3-v & v^4+v^2 & -v^5-2v^3-v & v^4+v^2 & -v^5-2v^3-v & v^4+v^2 & v^4+v^2 & v^6+2v^4+2v^2+1
\end{bmatrix}$$

Consider the unique 10-dimensional irreducible representation, which is denoted 10_s in [20, Table C.4]. By Naruse [37], a W-graph is given by Table 3. (The numbers inside a circle specify the subset $I(x)$; all $\mu_{x,y}$ are 0 or 1; we have an edge between x and y if and only if $\mu_{x,y} = 1$.) From this graph, we find that the basis vector with $I(x) = \{1, 2, 3, 5, 6\}$ spans the one-dimensional intersection of kernels considered above (in accordance with [20, Table C.4]). This basis vector labels the last row and column in the matrix of Q^{10_s} in Table 3.

In this case, the determination of the bound required by James' conjecture is very easy. By Table 1, we have $10_s \in \Lambda^\circ_{\zeta_4}$. If we specialise $v \mapsto \zeta_8$, we notice that $Q^{10_s}_{\zeta_4}$ has rank 1; all rows become equal to

$$\left[-2+2\zeta_4, -2+2\zeta_4, -2+2\zeta_4, -2\zeta_8^3, -2+2\zeta_4, -2\zeta_8^3, -2+2\zeta_4, -2\zeta_8^3, -2\zeta_8^3, -2\zeta_8^3\right].$$

We see that, if we specialise further into a field of characteristic $\ell > 0$, we will still obtain a matrix of rank 1 unless $\ell = 2$.

Remark 4.10. We have been able to systematically compute the matrices Q^λ (with coefficients in A) for all λ such that $d_\lambda \leqslant 2500$. For those λ in type E_8 where this was not feasible (at least not with the computer power available to us), we nevertheless managed to compute directly the specialised matrices $Q^\lambda_{\zeta_e}$ for all relevant values of e. Note that this is sufficient to find the finite set of prime numbers $\mathcal{P}_e$ as above. (See [19] for details.) There is an on-going project to complete the determination of all "generic" matrices Q^λ and to create a data base for making them generally available.

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