Superconvergence of Coupling Techniques in Combined Methods for Elliptic Equations with Singularities

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(Received and accepted May 2000)

Abstract—Several coupling techniques, such as the nonconforming constraints, penalty, and hybrid integrals, of the Ritz-Galerkin and finite difference methods are presented for solving elliptic boundary value problems with singularities. Based on suitable norms involving discrete solutions at specific points, superconvergence rates on solution derivatives are exploited by using five combinations, e.g., the nonconforming combination, the penalty combination, Combinations I and II, and symmetric combination. For quasi-uniform rectangular grids, the superconvergence rates, \(O(h^{2-\delta})\), of solution derivatives by all five combinations can be achieved, where \(h\) is the maximal mesh length of difference grids used in the finite difference method, and \(\delta > 0\) is an arbitrarily small number.

Superconvergence analysis in this paper lies in estimates on error bounds caused by the coupling techniques and their incorporation with finite difference methods. Therefore, a similar analysis and conclusions may be extended to linear finite element methods using triangulation by referring to existing references. Moreover, the five combinations having \(O(h^{2-\delta})\) of solution derivatives are well suited to solving engineering problems with multiple singularities and multiple interfaces. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Elliptic equation, Singularity problem, Superconvergence, Combined method, Coupling technique, Finite difference method, Finite element method, Ritz-Galerkin method, Penalty method, Hybrid method.

1. INTRODUCTION

As a continued study of [1,2], in this paper, superconvergence rates of solution derivatives are pursued by several combinations of the Ritz-Galerkin and finite difference methods (simply written as RG-FDMs). In [1], only the penalty combination is discussed; in this paper, five combinations are explored together for comparisons and deep insight of algorithmic nature; in [2], only optimal convergence rates are derived for three combinations. There exist many reports on superconvergence of single methods, for instance, finite element methods, finite difference method, and the finite volume element method, see [3–12].

Since the solution domains of elliptic problems often involve concave corners, and since they may also be composed of different materials, solution derivatives are unbounded at the singular points. Traditional finite difference methods (or finite element methods) based on discrete approximation of derivatives by difference quotients, therefore, incur a reduction of convergence rates of the approximate solutions. Hence, new finite element methods and combined methods are...
developed to deal with singularity problems, to regain optimal convergence or even superconvergence rates, such as those of the local refinements [13]. Here we should mention other techniques to deal with the solutions with singularities, such as singular elements in [14,15], infinite elements [16–18], singular function method [19–22], the $p$-version of FEM [23,24], combination of finite element and boundary element methods [25], the $T$-complete method [26], natural boundary element methods [27,28], conformal transformation methods [29,30], and many others [31–35]. A systematic review on these techniques is given in [36, Chapter 2] with several hundred references.

In this paper, we employ the combined methods of [37–40] for solving singularity problems. In combined methods, different numerical methods are used in different subdomains simultaneously. Let the solution domain $S$ be divided into a singular domain $S_2$, where there exists a singular point of the solution, and a subdomain $S_1$ where there does not. The Ritz-Galerkin method using singular functions is used in $S_2$ to best approximate the solution singularity, but the traditional finite difference method can also be used in $S_1$. Some additional integrals along their common boundary $\Gamma_0$ play an important role in matching the two different methods. For coupling different admissible functions, the penalty combination along $\Gamma_0$ employs the penalty integral involving the solutions. On the other hand, generalized combinations employ the hybrid integrals involving also the solution derivatives. The coupling techniques in combinations are important to maintain optimal convergence, superconvergence, and numerical stability.

Optimal convergence rates of combinations of the Ritz-Galerkin and finite element methods (RG-FEMs) are reported in [2,39]. In this paper, we will focus on superconvergence analysis of the coupling strategies and their incorporation with the finite difference method. Since the finite difference method is a special kind of finite element method, the analysis and conclusions in this paper may be extended to finite element methods using triangulation. For simplicity, quasi-uniform rectangular grids are taken into account so that a systematic analysis of different coupling strategies in various combinations can be briefly presented together. The superconvergence rates $O(h^{2-\delta})$ of solution derivatives by all five combinations of RG-FDMs can be achieved, where $h$ is the maximal mesh length of difference grids used in the finite difference method, and $\delta(>0)$ is an arbitrarily small number. It is worth noting that suitable discrete norms are important in achieving superconvergence rates.

While using finite difference methods, usually triangular elements are necessarily employed near the exterior and interior slant boundary, we can prove that only the lower convergence rate $O(h^{3/2})$ can be obtained by the traditional treatments of the finite difference methods given in [1].

Below, we first describe five combinations of the RG-FDMs in Section 2, then estimate error bounds of the solutions, and derive superconvergence rates in Sections 3–5. In Section 6, the results of numerical experiments of Motz's problem are given to support the theoretical analysis made in Sections 3–5. Also, we address the importance of the error norms chosen for convergence analysis, and describe the methods to compute the error norms.

2. COMBINATIONS OF RG-FDMS

Consider the Poisson equation with the Dirichlet boundary condition

\[
\begin{align*}
-\Delta u &= -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x,y), \quad (x,y) \in S, \\
u &= 0, \quad (x,y) \in \Gamma,
\end{align*}
\]

where $S$ is a polygonal domain, $\Gamma(=\partial S)$ is the exterior boundary of $S$, and the function $f$ is to be sufficiently smooth. Let the solution domain $S$ be divided by a piecewise straight line $\Gamma_0$ into two subdomains $S_1$ and $S_2$. The Ritz-Galerkin method is used in $S_2$ where there may exist a singular point, and the finite difference method is used in $S_1$. For simplicity, the subdomain $S_1$ is again split by difference grids into small quasi-uniform rectangles $\Box_{ij}$, where $\Box_{ij} = \{(x,y), x_i \leq x \leq x_{i+1}, y_j \leq y \leq y_{j+1}\}$.
Denote \( u_{i,j} = u(x_i, y_j) \), \( h_i = x_{i+1} - x_i \), \( k_j = y_{j+1} - y_j \), and the maximal mesh spacing \( h = \max_{i,j}(h_i, k_j) \). The quasi-uniform difference grids imply that there exists a bounded constant \( C \) independent of \( h_i \) and \( k_j \) such that \( h / \min_{i,j}(h_i, k_j) \leq C \). The boundary difference nodes \((i, j)\) are always placed on \( \partial S_1 \).

The conventional finite difference method can be regarded as a special kind of finite element method, by using piecewise bilinear interpolatory functions \( v_1(x, y) \) on \( \square_{ij} \),

\[
v_1(x, y) = \frac{1}{h_i k_j} \left\{ (x_{i+1} - x)(y_{j+1} - y)v_{ij} + (x - x_i)(y_{j+1} - y)v_{i+1,j} + (x_{i+1} - x)(y - y_j)v_{i,j+1} + (x - x_i)(y - y_j)v_{i+1,j+1} \right\},
\]

(2.3)

and by approximating integrals by the following specific rules (see [38]):

\[
\iint_{S_1} \nabla u \nabla v \, ds = \sum_{ij} \iint_{\square_{ij}} \nabla u \nabla v \, ds = \sum_{ij} \left\{ \iint_{\square_{ij}} u_x v_x \, ds + \iint_{\square_{ij}} u_y v_y \, ds \right\},
\]

(2.4)

\[
\iint_{\square_{ij}} u_x v_x \, ds = \frac{h_i k_j}{2} \left[ u_x \left( i + \frac{1}{2}, j \right) v_x \left( i + \frac{1}{2}, j + 1 \right) + u_x \left( i + \frac{1}{2}, j + 1 \right) v_x \left( i + \frac{1}{2}, j + 1 \right) \right],
\]

(2.5)

\[
\sum_{ij} \iint_{\square_{ij}} f v \, ds = \sum_{ij} \left\{ \frac{h_i k_j}{4} \left[ f_{ij} v_{ij} + f_{i+1,j} v_{i+1,j} + f_{i,j+1} v_{i,j+1} + f_{i+1,j+1} v_{i+1,j+1} \right] \right\},
\]

(2.6)

where \( u_x(i + (1/2), j) = u_x(x_{i+(1/2)}, y_j) \) and \( x_{i+(1/2)} = (1/2)(x_i + x_{i+1}) \).

In \( S_2 \), we assume that the solution \( u \) can be spanned by \( u = \sum_{i=1}^{\infty} a_i \psi_i \), where \( a_i \) are the expansion coefficients, and \( \psi_i(i = 1, 2, \ldots) \) are complete and linearly independent basis functions in \( L^2(S_2) \). \( \{\psi_i\} \) may be chosen as analytical and singular functions. Then the admissible functions of combinations of the RG-FDMs are written as

\[
v = \begin{cases} 
v^- = v_1, & \text{in } S_1, \\
v^+ = f_L(\tilde{a}_i), & \text{in } S_2, \end{cases}
\]

(2.7)

where \( v_1 \) is given in (2.3), \( \tilde{a}_i \) are unknown coefficients to be sought, and \( f_L(a_i) = \sum_{i=1}^{L} a_i \psi_i \).

If the particular solutions of (2.1) and (2.2) can be chosen as \( \psi_i \), the total number of \( \psi_i \) used will be greatly reduced for a given accuracy of solutions (2.7). Considering the discontinuity of solutions on \( \Gamma_0 \), i.e.,

\[
v^+ \neq v^-, \quad \text{on } \Gamma_0,
\]

(2.8)

we define the space

\[
H = \left\{ v \mid v \in L^2(S), \ v \in H^1(S_1), \ \text{and} \ v \in H^1(S_2) \right\}.
\]

(2.9)

where \( H^1(S_1) \) is the usual Sobolev space.

Let \( V_h(\subset H) \) denote a finite dimensional collection of functions \( v \) in (2.7) satisfying (2.2). The combinations of RG-FDMs involving integral approximation on \( \Gamma_0 \) can be expressed by

\[
\tilde{a}_h(u_h, v) = \hat{f}_h(v), \quad \forall v \in V_h,
\]

(2.9)

where

\[
\tilde{a}_h(u, v) = \iint_{S_1} \nabla u \nabla v \, ds + \iint_{S_2} \nabla u \nabla v \, ds + \hat{D}(u, v),
\]

(2.10)

\[
\hat{f}_h(v) = \iint_{S_1} f v \, ds = \sum_{ij} \iint_{\square_{ij}} f v \, ds,
\]

(2.11)
\[ D(u,v) = \frac{P_c}{h^\sigma} \int_{\Gamma_0} (u^+ - u^-) (v^+ - v^-) \, dl - \int_{\Gamma_0} \left( \alpha \frac{\partial u^+}{\partial n} + \beta \frac{\Delta u^-}{\Delta n} \right) \times (v^+ - v^-) \, dl - \int_{\Gamma_0} \left( \alpha \frac{\partial v^+}{\partial n} + \beta \frac{\Delta v^-}{\Delta n} \right) (u^+ - u^-) \, dl, \] (2.12)

where \( \Delta u^-/\Delta n \mid_{(x_i, y_j) \in \Gamma_0} = \left( u^-(x_i + h_i, y_j) - u^-(x_i, y_j) \right)/h_i. \)

In the coupling integrals (2.12), \( P_c (> 0) \) is the penalty constant, \( \sigma \) is the penalty power, and \( \alpha (\geq 0) \) and \( \beta (\geq 0) \) satisfy \( \alpha + \beta = 1 \) or \( 0. \) The first term on the right side of (2.12) is called the penalty integral, and the second and third terms, the hybrid integrals. Four combinations of (2.9) are obtained from different parameters in (2.12) (see [2, 39]).

(I) COMBINATION I: \( \alpha = 0 \) and \( \beta = 1. \)

(II) COMBINATION II: \( \alpha = 1 \) and \( \beta = 0. \)

(III) SYMMETRIC COMBINATION: \( \alpha = \beta = 1/2. \)

(IV) PENALTY COMBINATION: \( \alpha = \beta = 0. \)

In addition, we consider the nonconforming method, in which \( \tilde{f}(v, v) \equiv 0 \) in (2.10) and a direct continuity constraint,

\[ v^+(Z_k) = v^-(Z_k), \quad \forall Z_k \in \Gamma_0, \] (2.13)

is imposed at all interface nodes \( Z_k \) located on \( \Gamma_0. \) We denote by \( \bar{V}_h \) the subspace of \( V_h \) satisfying (2.13). We then obtain the nonconforming combination

\[ I(u_N, v) = \tilde{f}_h(v), \quad \forall v \in \bar{V}_h, \] (2.14)

where

\[ I(u, v) = \iint_{S_1} \nabla u \nabla v \, ds + \iint_{S_2} \nabla u \nabla v \, ds. \] (2.15)

In this paper, we denote by \( u_N \) and \( u_h \) the solutions of the nonconforming combination (2.14) and other combinations (2.9), respectively.

The approximate integrals \( \tilde{D}(u, v) \) on \( \Gamma_0 \) use the integration rules

\[ \int_{\Gamma_0} \xi \eta \, dl \approx \int_{\Gamma_0} \hat{\xi} \hat{\eta} \, dl = \sum_{k=1}^{N_1} \frac{Z_{k-1} Z_k}{6} \{ 2 \xi (Z_{k-1}) \eta (Z_{k-1}) + \xi (Z_{k-1}) \eta (Z_k) + \xi (Z_k) \eta (Z_{k-1}) + 2 \xi (Z_k) \eta (Z_k) \}, \] (2.16)

where \( \Gamma_0 = \bigcup_{k=1}^{N_1} \Gamma_0^{(k)}, \Gamma_0^{(k)} = Z_{k-1} Z_k, \) \( Z_{k-1} Z_k \) denotes the length of \( Z_{k-1} Z_k, \) and \( \hat{\xi} \) and \( \hat{\eta} \) are the piecewise linear interpolatory functions on \( \Gamma_0. \) For the interior boundary \( \Gamma_0, \) we have

\[ \int_{\Gamma_0} \frac{\partial u^-}{\partial n} (v^+ - v^-) \, dl \approx \int_{\Gamma_0} \frac{\partial u^-}{\partial x} (v^+ - v^-) \, dy + \int_{\Gamma_0} \frac{\partial u^-}{\partial y} (v^+ - v^-) \, dx \]

\[ \approx \int_{\Gamma_0} \frac{\Delta u^-}{\Delta x} (v^+ - v^-) \, dy + \int_{\Gamma_0} \frac{\Delta u^-}{\Delta y} (v^+ - v^-) \, dx = \int_{\Gamma_0} \frac{\Delta u^-}{\Delta n} (v^+ - v^-) \, dl. \] (2.17)

Note that this integration rule is also suited for slant straight lines of boundary \( \Gamma_0 \) when using triangular elements.

Suitable norms should be defined for establishing error bounds for the solutions by combinations. We thus define

\[ \| v \|_h = \left( \| v \|_{1, S_1}^2 + \| v \|_{1, S_2}^2 + \frac{P_c}{h^\sigma} \| v^+ - v^- \|_{0, \Gamma_0}^2 \right)^{1/2}, \] (2.18)

\[ \| v \|_1 = \left( \| v \|_{1, S_1}^2 + \| v \|_{1, S_2}^2 \right)^{1/2}, \] (2.19)
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where \( \|v\|_{1,S_i} \) is the Sobolev norm (see [11,41]). Optimal convergence rates of numerical solutions, \( \|e\|_h = \|u - u_h\|_h = O(h) \), have been obtained in [2,39].

In this paper, we pursue superconvergence based on the new norms

\[
\|v\|_h = \left( \|v\|_{1,S_1}^2 + \|v\|_{1,S_2}^2 + \frac{P_c}{h^2} \|v^+ - v^-\|_{0,\Gamma_0}^2 \right)^{1/2},
\]

\[
\|v\|_1 = \left( \|v\|_{1,S_1}^2 + \|v\|_{1,S_2}^2 \right)^{1/2},
\]

where the norms with discrete summation are defined by

\[
\|v\|_{1,S_1}^2 = \|v\|_{0,S_1}^2 + \|v\|_{0,S_1}^2,
\]

\[
\|v\|_{0,S_1}^2 = \sum_{ij} \iint_{\mathcal{Q}_{ij}} (\nabla v)^2 \, ds,
\]

\[
\|v^+ - v^-\|_{0,\Gamma_0}^2 = \iint_{\Gamma_0} (v^+ - v^-)^2 \, dl.
\]

The discrete formulas, \( \iint_{\mathcal{Q}_{ij}} (\nabla v)^2 \, ds \) and \( \iint_{\mathcal{Q}_{ij}} v^2 \, ds \), are given by (2.4)-(2.6), and the integration \( \iint_{\Gamma_0} v^2 \, dl \) is given in (2.16). Notice that the definition of \( \|v\|_h \) agrees with the integration rules used in the combinations (2.9), and this norm will play an important role in obtaining superconvergence of the solutions.

The superconvergence rates of \( \|e\|_h = O(h^{2-\delta}) \) can be achieved by all five combinations for the quasi-uniform rectangles; detailed proofs are supplied by Theorems 3.2, 4.2, and 5.1 in the subsequent sections.

3. NONCONFORMING COMBINATION

We can easily prove the following theorem and lemmas (see [38,40]).

THEOREM 3.1. The solution \( u_N \) of the nonconforming combination (2.14) has the bounds

\[
\|u_N - u\|_1 \leq C \left\{ \inf_{v \in \mathcal{V}_h} \|u - v\|_1 + \|\partial u\|_{0,\Gamma_0} \sup_{w \in \mathcal{V}_h} \frac{\|w^+ - w^-\|_{0,\Gamma_0}}{\|w\|_1} \right. \\
+ \sup_{w \in \mathcal{V}_h} \frac{\left( \iint_{S_1} - \iint_{S_1} \right) \nabla u \nabla w \, ds}{\|w\|_1} + \sup_{w \in \mathcal{V}_h} \frac{\left( \iint_{S_1} - \iint_{S_1} \right) f \, ds}{\|w\|_1} \right\},
\]

where \( C \) is a bounded constant independent of \( L, h, v, \) and \( u \).

LEMMA 3.1. Assume

\[
|v^+|_{\ell,\Gamma_0} \leq C L^\ell \mu \|v^+\|_{0,\Gamma_0}, \quad \ell = 1,2, \quad \forall v \in \mathcal{V}_h,
\]

where \( \mu(> 0) \) is a bounded constant independent of \( L, h, \) and \( v \). Then

\[
\|w^+ - w^-\|_{0,\Gamma_0} \leq C h^2 L^{2\mu} \|w\|_1, \quad \forall w \in \mathcal{V}_h.
\]

Let \( \Gamma_0 \subset S_1 \) consist of the coordinate grid lines (see Figure 1), and \( S_1 \) be divided by \( \Gamma_0 \) into \( S_1^* \) and \( S_1^* \) (\( S_1 = S_1^* \cup S_1^* \)) such that the middle region \( S_1^* \) is located between \( S_1^* \) and \( S_2 \). Denote \( S_2^* = S_2 \cup S_2^* \), we can also prove the following lemma (see [38]).
LEMMA 3.2. Let
\[ u \in C^3(S_1) \quad (3.4) \]
hold, where \( C^k(S_1) \) denotes the space of functions having \( k \)-order continuous derivatives. Then
\[ \inf_{v \in \mathcal{V}_h} \| u - v \|_1 \leq C h^2 + \| R_L \|_{1,S_1^*}, \quad (3.5) \]
where the remainder, \( R_L = \sum_{\ell=L+1}^{\infty} a \ell \psi_{\ell} \).

Define
\[ M_n(u) = \max_{i+j=k \leq n} \| \frac{\partial^k u}{\partial x^i \partial y^j} \|_1; \]
below we prove the following lemma.

LEMMA 3.3. Let (3.2), (3.4), and
\[ f \in C^2(S_1) \quad (3.6) \]
hold. Then there exist the bounds
\[ \left( \int_{S_1} - \int_{S_1} \right) \nabla u \nabla w \, ds \leq C h^2 L^\mu M_3(u) \| w \|_1, \quad \forall w \in \mathcal{V}_h, \quad (3.7) \]
\[ \left( \int_{S_1} - \int_{S_1} \right) f \, ds \leq C h^2 M_2(f) \| w \|_1, \quad \forall w \in \mathcal{V}_h. \quad (3.8) \]

**PROOF.**
\[ \left( \int_{S_1} - \int_{S_1} \right) \nabla u \nabla w \, ds \leq \left( \int_{S_1} - \int_{S_1} \right) u_x w_x \, ds \left( \int_{S_1} - \int_{S_1} \right) u_y w_y \, ds, \quad (3.9) \]
we only prove bounds of one term in the right side of (3.9), for example,
\[ \left( \int_{S_1} - \int_{S_1} \right) u_x w_x \, ds \leq C h^2 L^\mu M_3(u) \| w \|_1, \quad \forall w \in \mathcal{V}_h. \quad (3.10) \]
The proof for bounds of the other term is the same. By Taylor's formula, we obtain
\[ \int_{\Box_{i,j}} g \, ds = \frac{h_i k_j}{2} \left[ g \left( i + \frac{1}{2}, j \right) + g \left( i + \frac{1}{2}, j + 1 \right) \right] + R_{i,j}^{(1)}, \quad (3.11) \]
where the truncation errors are given by
\[ R_{i,j}^{(1)} = h_i k_j \left\{ \frac{1}{24} \left( h^2 \frac{\partial^2 g_{i,j}}{\partial x^2} - 2 k_j \frac{\partial^2 g_{i,j}}{\partial y^2} \right) + \frac{1}{32} h_i k_j \left[ \frac{\partial^2 g_{i,j}}{\partial x \partial y} - \frac{\partial^2 g_{i,j}}{\partial x \partial y} \right] \right\}; \quad (3.12) \]
\[ \tilde{g}_{i,j}^{(k)} = g \left( \xi_{i,j}^{(k)} \right), \quad \xi_{i,j}^{(k)} \in \Box_{i,j}, \quad k = 1, 2, 3, 4. \quad (3.13) \]
Since $w(\in \tilde{V}_h)$ is a bilinear function on $\square_{ij}$, then we have (2.3),

\begin{align}
& w_{xx} = w_{yy} = 0, \quad \text{in } \square_{ij}, \quad (3.14) \\
& w_{xy} = \frac{1}{h_i k_j} \left[ w_{ij} - w_{i+1,j} - w_{i,j+1} + w_{i+1,j+1} \right], \quad \text{in } \square_{ij}. \quad (3.15)
\end{align}

Letting $g = u_x, w_x$, we have

\begin{align}
& g_{xx} = u_{xxx} w_x, \quad g_{yy} = u_{xyy} w_x + 2u_{xy} w_{xy}, \quad g_{xy} = u_{xxy} w_x + u_{xx} w_{xy}. \quad (3.16)
\end{align}

We can apply (3.12) to the integration rule (2.5) to yield the following bounds:

\begin{align}
\left| \left( \iint_{S_1} - \iint_{S_2} \right) u_x w_x ds \right| &= \left| \sum_{ij} \left( \iint_{\square_{ij}} - \iint_{\square_{ij}} \right) u_x w_x ds \right| \\
&= \sum_{ij} R_{ij}^{(1)} \leq C \left( h^2 M_3(u) \sum_{ij} h_i k_j \left| w_x(\eta_{ij}) \right| + \left| \sum_{ij} h_i k_j^3 \frac{\partial^2 u_{ij}^{(1)}}{\partial x \partial y} w_{xy} \right| \\
&+ \left| \sum_{ij} h_i^2 k_j^2 \left( \frac{\partial^2 u_{ij}^{(2)}}{\partial x^2} - \frac{\partial^2 u_{ij}^{(3)}}{\partial x^2} \right) w_{xy} \right| \right), \quad (3.17)
\end{align}

where $\eta_{ij} \in \square_{ij}, u_{ij}^{(k)} = u(\epsilon_{ij}^{(k)}), \epsilon_{ij}^{(k)} \in \square_{ij}, k = 1, 2, 3$. Bounds of the first term of the right-hand side in (3.17) can be obtained from the Schwarz inequality

\begin{align}
T_1 &= h^2 M_3(u) \sum_{ij} h_i k_j \left| w_x(\eta_{ij}) \right| \\
&\leq h^2 M_3(u) \sum_{ij} h_i k_j \left| w_x \left( i + \frac{1}{2}, j \right) \right| + \left| w_x \left( i + \frac{1}{2}, j + 1 \right) \right| \quad (3.18)
\end{align}

For the third term in (3.17), we can see from (3.14), (3.15), and the Schwarz inequality,

\begin{align}
T_{III} &= \left| \sum_{ij} h_i^2 k_j^2 \left( \frac{\partial^2 u_{ij}^{(2)}}{\partial x^2} - \frac{\partial^2 u_{ij}^{(3)}}{\partial x^2} \right) w_{xy} \right| \leq C M_3(u) h \left| \sum_{ij} h_i^2 k_j^3 w_{xy} \right| \\
&\leq C M_3(u) h \sum_{ij} h_i k_j \left| w_{ij} - w_{i+1,j} - w_{i,j+1} + w_{i+1,j+1} \right| \quad (3.19)
\end{align}

By applying the boundary conditions and coupling relations, the second term of the right-hand side in (3.17) may be estimated by summation by parts,

\begin{align}
T_{II} &= \left| \sum_{ij} h_i k_j \frac{\partial^2 u_{ij}^{(1)}}{\partial x \partial y} w_{xy} \right| \\
&\leq \left| \sum_j k_j^3 \sum_i \frac{\partial^2 u_{ij}^{(1)}}{\partial x \partial y} \left[ w_{ij} - w_{i+1,j} - w_{i,j+1} + w_{i+1,j+1} \right] \right|. \quad (3.20)
\end{align}
Denote by $\partial_y \square_{ij}$, a vertical segment of $\partial \square_{ij}$, between the vertices $(i,j)$ and $(i,j+1)$. From the assumption, we may locate the vertical segments $\{\partial_y \square_{ij}\}$ either inside of $S_1$ or just on the boundary $\partial S_1$.

**CASE I.** $\partial_y \square_{ij} \subset S_1$.

**CASE II.** $\partial_y \square_{ij} \subset \partial S_1$.

Since $\partial S_1 = (\partial S_1 \cap \Gamma) \cup (\partial S_1 \cap \Gamma_0)$, Case II can also be split into two following subcases.

**SUBCASE IIa.** $\partial_y \square_{ij} \subset \partial S_1 \cap \Gamma$. The Dirichlet boundary condition (2.2) for $w \in V_h$ implies that $w_{i,j} = w_{i,j+1} = 0$, so that

$$\sum_j \sum_i |w_{i,j+1} - w_{ij}| = 0. \quad (3.21)$$

**SUBCASE IIb.** $\partial_y \square_{ij} \subset \partial S_1 \cap \Gamma_0$. In this case, we obtain from the coupling condition (2.13) as well as assumption (3.2),

$$\sum_j \sum_i |w_{i,j+1} - w_{ij}| = \sum_j \sum_i |w_{i,j+1}^+ - w_{ij}|$$

$$\leq \sum_j \sum_i k_j \frac{|w_{i,j+1}^+ - w_{ij}|}{k_j} \leq C |w^+|_{1,\Gamma_0} \quad (3.22)$$

Since $\Gamma_0 = (\Gamma_0 \cap \Gamma) \cup (\Gamma_0 \cap \Gamma_0')$, Case II can also be split into two following subcases.

**SUBCASE IIa.** $\partial_y \square_{ij} \subset \Gamma_0 \cap \Gamma$. The coupling condition (2.13) as well as assumption (3.2),

$$\sum_j \sum_i j \sum_i k_j \frac{|w_{i,j+1}^+ - w_{ij}|}{k_j} \leq C |w^+|_{1,\Gamma_0} \quad (3.23)$$

For the first term in the right side of (3.23), we obtain from the Schwarz inequality,

$$\sum_j k_j^2 \sum_i (w_{i,j+1} - w_{ij}) \left\{ \left| \frac{\partial^2 \tilde{u}_{ij}^{(1)}}{\partial x \partial y} - \frac{\partial^2 \tilde{u}_{i-1,j}^{(1)}}{\partial x \partial y} \right| \right\}$$

$$\leq C M_3(u) h \sum_j k_j^2 \sum_i |w_{i,j+1} - w_{ij}| \quad (3.24)$$

Combining (3.21)-(3.23) yields

$$T_{II} \leq C h^2 \left[ M_3(u) + M_2(u) L^\mu \right] \||w||_1 \leq C h^2 L^\mu M_3(u) \||w||_1. \quad (3.25)$$

The desired result (3.10) is obtained from (3.17)-(3.19) and (3.25); this completes the proof of (3.7). The above arguments are different from those in [1] in using different coupling conditions along $\Gamma_0$. The proof of (3.8) is given in [1]. This completes the proof of Lemma 3.3.
Based on Theorem 3.1 and Lemmas 3.1–3.3, we have the following theorem.

**Theorem 3.2.** Let all the conditions in Lemmas 3.3 and 3.4 hold. Then the solution $u_N$ from the nonconforming combination (2.14) has the error bounds

$$
\| u_N - u \|_1 \leq C\left( h^2 + \| R_L \|_{1,S_2^L} + h^2 L^{2\mu} \right).
$$

(3.26)

Also suppose that the number $L$ of the basis functions used for $u^+$ in (2.7) is chosen such that

$$
\| R_L \|_{1,S_2^L} = O(h^2) \quad \text{and} \quad L = O(\lfloor \ln h \rfloor).
$$

(3.27)

Then there exist the superconvergence rates, $\| u_N - u \|_1 = O(h^{2-\delta})$, of the solution $u_N$, where $\delta (> 0)$ is arbitrarily small.

Theorem 3.2 is a development of superconvergence results in [38] to quasi-uniform rectangular grids.

4. COMBINATIONS I, II, AND SYMMETRIC COMBINATION

First, we give the following theorem and lemma without proofs (see [2]).

**Theorem 4.1.** Suppose that there exist two constants $C_0 (> 0)$ and $C_1$ independent of $L$, $h$, and $u$ and $v$ such that

$$
C_0 \|v\|_{h}^2 \leq \delta_h(v,v), \quad \forall v \in V_h,
$$

(4.1)

$$
|\delta_h(u,v)| \leq C_1 \| u \|_h \| v \|_h, \quad \forall u, v \in V_h.
$$

(4.2)

Then the solution $u_h$ of combinations (2.9) has the error bounds

$$
\| u - u_h \|_h \leq C \left\{ \inf_{v \in V_h} \| u - v \|_h + \sup_{w \in V_h} \left| \int_{S_1} - \int_{S_2} \nabla u \nabla w \, ds \right| \| w \|_h + \sup_{w \in V_h} \left| \int_{\Gamma_0} \frac{\partial u}{\partial n} (w^+ - w^-) \, d\ell \right| \| w \|_h + (1 - (\alpha + \beta)) \sup_{w \in V_h} \left| \int_{\Gamma_0} \frac{\partial w}{\partial n} \, d\ell \right| \| w \|_h \\
+ (\alpha + \beta) \sup_{w \in V_h} \left| \int_{\Gamma_0} \frac{\partial u^}{\partial n} \left( \frac{\partial u^-}{\partial n} - \Delta u^- \right) \, d\ell \right| \| w \|_h + \beta \sup_{w \in V_h} \left| \int_{\Gamma_0} \frac{\partial w^+}{\partial n} \left( \Delta u^+ \right) \, d\ell \right| \| w \|_h \right\},
$$

(4.3)

where $u$ is the true solution of (2.1) and (2.2).

**Lemma 4.1.** Let

$$
\left\| \frac{\partial v^+}{\partial n} \right\|_{0,\Gamma_0} \leq C L^\mu \| v^+ \|_{0,\Gamma_0}, \quad \forall v \in V_h,
$$

(4.4)

and suppose that $\sigma \geq 1$ when $\beta > 0$, or $\sigma > 0$ and $h^\sigma L^{2\mu} < C$ when $\alpha > 0$. Then inequalities (4.1) and (4.2) hold when $P_c(> 0)$ is chosen suitably large but independent of $L$, $h$, and $v$.

Below we prove the following lemma involving the coupling relations along $\Gamma_0$. 
LEMMA 4.2. Let (3.2) hold for all $v \in V_h$, (3.4) and $\frac{\partial u}{\partial n} \in H^2(\Gamma_0)$ be given, then for all $w \in V_h$,

\[
\left( \int_{\Gamma_0} \int_{\Gamma_0} \frac{\partial u}{\partial n} (w^+ - w^-) \, dt \right) \leq \left\| \frac{\partial u}{\partial n} \right\|_{0,\Gamma_0} \left\| w^+ - \hat{w}^+ \right\|_{0,\Gamma_0},
\]

(4.5)

\[
\int_{\Gamma_0} \left( \frac{\partial u}{\partial n} - \frac{\Delta u^-}{\Delta n} \right) (w^+ - w^-) \, dt \leq C M_2(u) h^{1+\sigma/2} \|w\|_h.
\]

(4.6)

PROOF. We first have

\[
\left( \int_{\Gamma_0} \int_{\Gamma_0} \xi \eta \, dt \right) = \left( \int_{\Gamma_0} \xi (\eta - \hat{\eta}) \, dt \right) \leq \left( \int_{\Gamma_0} \xi \eta \, dt \right) + \left( \int_{\Gamma_0} \xi \hat{\eta} \, dt \right)
\]

\[
\leq \left\| \xi \right\|_{0,\Gamma_0} \left\| \eta - \hat{\eta} \right\|_{0,\Gamma_0} + \left\| \xi - \hat{\xi} \right\|_{0,\Gamma_0} \left\| \eta \right\|_{0,\Gamma_0}.
\]

Let $\xi = \frac{\partial u}{\partial n}$, $\eta = w^+ - w^-$. Since $w^-$ is the piecewise linear function, then $\eta - \hat{\eta} = w^+ - \hat{w}^+$. Hence, we obtain from (3.2) for all $v \in V_h$,

\[
\left( \int_{\Gamma_0} \int_{\Gamma_0} \frac{\partial u}{\partial n} (w^+ - w^-) \, dt \right) \leq \left\| \frac{\partial u}{\partial n} \right\|_{0,\Gamma_0} \left\| w^+ - \hat{w}^+ \right\|_{0,\Gamma_0},
\]

(4.7)

\[
\left( \int_{\Gamma_0} \int_{\Gamma_0} \frac{\partial u}{\partial n} (w^+ - w^-) \, dt \right) \leq \left\| \frac{\partial u}{\partial n} \right\|_{0,\Gamma_0} \left\| w^+ - w^- \right\|_{0,\Gamma_0}.
\]

(4.8)

This is (4.5). Next, we obtain

\[
\int_{\Gamma_0} \left( \frac{\partial u}{\partial n} - \frac{\Delta u^-}{\Delta n} \right) (w^+ - w^-) \, dt \leq \left\| \frac{\partial u}{\partial n} - \frac{\Delta u^-}{\Delta n} \right\|_{0,\Gamma_0} \left\| w^+ - w^- \right\|_{0,\Gamma_0},
\]

(4.9)

This completes the proof of Lemma 4.2.

Since the norm $\|v^+ - v^-\|_{0,\Gamma_0}$ is defined by the values $(v^+ - v^-)$ only on the nodes $z_k \in \Gamma_0$, the solution $u_N \in V_h$ from the nonconforming combination (2.14) has $\|u_N^+ - u_N^-\|_{0,\Gamma_0} = 0$. Then we have from Lemma 3.2,

\[
\inf_{v \in V_h} \| u - v \|_h \leq \| u - u_N \|_1 = \| u - u_N \|_1, \quad 0 \leq C h^2 + \| R_L \|_{1,S^1}.
\]

(4.10)

Similarly to Lemma 3.3, we have the following lemma.

LEMMA 4.3. Suppose that (3.2) holds for all $v \in V_h$, and that $u \in C^0(S_1)$ and $f \in C^2(S_1)$. Then

\[
\left( \int_{S_1} \int_{S_1} \nabla w \cdot \nabla \phi \, ds \right) \leq C \left\{ h^2 L^+ + h^{1+\sigma/2} \right\} M_3(u) \|w\|_h, \quad \forall w \in V_h,
\]

(4.11)

\[
\left( \int_{S_1} \int_{S_1} f \phi \, ds \right) \leq C h^2 M_2(f) \|w\|_h, \quad \forall w \in V_h.
\]

(4.12)
Theorem 4.2. Let all the conditions in Lemmas 4.1 and 4.2 hold. Then the solution $u_h$ from Combinations I and II and the symmetric combination has the error bounds

$$
\|u_h - u\|_h \leq C \left( h^2 + h^{1+\sigma/2} + \|R_h\|_{1,S_1^2} + h^2 L^2 \right).
$$

Moreover, if $\sigma \geq 2$ and (3.27) hold, then $\|u - u_h\|_h = O(h^{2-\delta}).$

In Lemma 4.2 and Theorem 4.2, assumptions (3.4) and $\frac{\partial u}{\partial n} \in H^2(\Gamma_0)$ are needed to derive the superconvergence rates $O(h^{2-\delta}).$ The assumption $\frac{\partial u}{\partial n} \in H^2(\Gamma_0)$ is not severe, compared to (3.4). Note that $\Gamma_0$ is an artificial interface chosen within $S$ and $\Gamma_0 = S_1 \cap S_2,$ so that there are no jumps of the normal derivatives on $\Gamma_0.$ If (3.4) is given a little stronger as $u \in C^3(S_1),$ then $\frac{\partial u}{\partial n} \in C^2(\Gamma_0)$ leads to $\frac{\partial u}{\partial n} \in H^2(\Gamma_0).$ In applications, $\Gamma_0$ consists of piecewise straight lines, to be located far from the singular points. Usually, the interface $\Gamma_0$ is also chosen to be perpendicular to the outside boundary $\Gamma.$ Hence, the assumption $\frac{\partial u}{\partial n} \in H^2(\Gamma_0)$ may be satisfied (see Figure 1).

5. PENALTY COMBINATION

For the penalty combination, the constants are chosen. We then have from (2.9),

$$
\hat{a}_h(u_h, v) = \hat{j}_h(v), \quad \forall v \in V_h,
$$

where

$$
\hat{a}_h(u, v) = \iint_{S_1} \nabla u \nabla v \, ds + \iint_{S_2} \nabla u \nabla w \, ds + \frac{P_c}{h^2} \int_{\Gamma_0} (u^+ - u^-) (v^+ - v^-) \, dl.
$$

The superconvergence of the penalty combination was first discussed in [1], and can be now derived easily from the analysis in Section 4. It is easy to prove that for any $P_c(>0)$ and $\sigma(\geq 0),$...
both (4.1) and (4.2) hold. Also, the error bounds of solution \( u_h \) are obtained from (4.3) by letting \( \alpha = \beta = 0 \).

\[
\| u - u_h \|_h \leq C \left\{ \inf_{v \in V_h} \| u - u_h \|_h + \sup_{w \in V_h} \left\| \int_{S_1} \nabla u \nabla w \, ds \right\|_h + \sup_{w \in V_h} \left( \int_{S_1} \frac{\partial u}{\partial \mathbf{n}} (w^+ - w^-) \, d\Gamma \right) \right\}. \tag{5.3}
\]

The bounds of the first three terms in the right sides of (5.3) have been provided in Section 4 already; only the last terms need to be estimated. We cite a lemma in [1].

**Lemma 5.1.** Suppose that (3.2) holds for all \( v \in V_h \); then

\[
\| w^+ - w^- \|_{0,\Gamma_0} \leq \| w^+ - w^- \|_{0,\Gamma_0} + C(hL)^2 \| v \|_{1,S_2}, \quad \forall v \in V_h. \tag{5.4}
\]

From Lemma 5.1, we obtain the bounds

\[
\left| \int_{\Gamma_0} \frac{\partial u}{\partial \mathbf{n}} (w^+ - w^-) \, d\Gamma \right| \leq \left\| \frac{\partial u}{\partial \mathbf{n}} \right\|_{0,\Gamma_0} \| w^+ - w^- \|_{0,\Gamma_0} \leq \left\| \frac{\partial u}{\partial \mathbf{n}} \right\|_{0,\Gamma_0} \frac{h^{s/2} + (hL)^2}{\| w \|_h}. \tag{5.5}
\]

Based on (5.3), (5.10), and (4.10)-(4.12), we provide the following theorem.

**Theorem 5.1.** Suppose that \( u \in C^3(S_1) \), \( f \in C^2(S_1) \), and (3.2) hold. Then the solution \( u_h \) from the penalty combination (5.1) has the error bounds

\[
\| u_h - u \|_h \leq C \left\{ h^2 + h^{s/2} + \| R_L \|_{1,S_2} + (Lh)^2 \right\}. \tag{5.6}
\]

Moreover, if \( \sigma \geq 4 \) and (3.27) hold, then \( \| u_h - u \|_h = O(h^{2-\sigma}) \).

6. DISCUSSIONS AND NUMERICAL EXPERIMENTS

6.1. Importance of Error Norms Chosen

The superconvergence rates in the norm \( \| \cdot \|_h \) are significant to the penalty combination, Combinations I, II, and symmetric combination. Note that the limitation of \( \sigma = 2 \) is derived for optimal convergence rates in [39] based on the norm \( \| \cdot \|_h \) given in (2.18). This comparison shows that a suitable choice of error norms is important to evaluation of the proposed algorithms. On the common boundary \( \Gamma_0 \), the norm \( (Pc/h^{s/2}) \| v^+ - v^- \|_{0,\Gamma_0} \) is a discrete solution summation over all the interface difference nodes \( Z_k \) (see (2.24) and (2.16)). This discrete penalty technique plays a coupling role in ensuring that \( v^+ \) and \( v^- \) will be close to each other only at the interface difference nodes \( Z_k \). The larger the ratio \( Pc/h^s \) is, the smaller the differences \( |v^+(Z_k) - v^-(Z_k)| \) are at all \( Z_k \). Therefore, this norm will never cause deterioration of the global error norm \( \| \cdot \|_h \), even when \( Pc/h^s \to \infty \) (i.e., \( P \to \infty \) or \( \sigma \to \infty \) while \( h < 1 \)). In this case, the discrete penalty technique leads to the nodal continuity constraints (2.13) of the solutions at \( Z_k \) given in the nonconforming combination.

By contrast, a continuous penalty integral, \( (Pc/h^s)^{1/2} \| v^+ - v^- \|_{0,\Gamma_0} / h^{s/2} \), also plays a role in coupling \( v^+ \) and \( v^- \) on the **entire** interface boundary \( \Gamma_0 \). Since the admissible functions \( v^+ \) and \( v^- \) in (2.7) are different, the norm \( \| v^+ - v^- \|_{0,\Gamma_0} \), can never diminish except for the null solution. Therefore, when \( Pc/h^s \) is large or infinite, so is the norm \( \| \cdot \|_h \) in (2.18). This leads to a boundedness of \( \sigma \) given in [39].

Besides, a stability analysis is given in [2], to show that the condition number of the associated matrix is \( O(h^{2+\sigma}) \). Hence, from the viewpoint of stability, we should choose the values of \( \sigma \) as small as possible. Obviously, Combinations I, II, and the symmetric combination are superior to the penalty combination in numerical stability, compared condition \( \sigma \geq 2 \) with \( \sigma \geq 4 \).
6.2. Numerical Experiments for Motz’s Problems

We consider the Motz problem on the rectangle $S(-1 \leq x \leq 1, 0 \leq y \leq 1)$ in Figure 1:

$$\Delta u = 0, \quad u|_{x<0 \wedge y=0} = 0, \quad u|_{x=1} = 500, \quad \text{(6.1)}$$

$$u|_{y=1} = \frac{\partial u}{\partial y}|_{y=1} = \frac{\partial u}{\partial x}|_{x=-1} = \frac{\partial u}{\partial y}|_{x>0 \wedge y=0} = 0. \quad \text{(6.2)}$$

The admissible function is (see [38])

$$v = \begin{cases} \phi, & \text{in } S_1, \\ v^+ = \sum_{\ell=0}^{L} \hat{D}_\ell t^{l+1/2} \cos \left( \ell \frac{1}{2} \right) \theta, & \text{in } S_2, \end{cases} \quad \text{(6.3)}$$

where $\phi$ is the piecewise bilinear interpolatory functions on the difference partition, $(r, \theta)$ are the polar coordinates with the origin $(0,0)$, and $\hat{D}_\ell$ are the unknown coefficients to be sought.

### Table 1. Error norms and approximate coefficients from the symmetric combination of RG-FDMs with $P_c = 10$ and $\sigma = 2$

<table>
<thead>
<tr>
<th>Divisions</th>
<th>$|\epsilon^+ - \epsilon^\pm|_{0,F_0}$</th>
<th>$|\epsilon^+ - \epsilon^-|_{\infty,F_0}$</th>
<th>Max</th>
<th>$|\epsilon|_{0,S}$</th>
<th>$|\epsilon|_1$</th>
<th>$|\epsilon|_h$</th>
<th>$|\epsilon|_1$</th>
<th>$\hat{D}_0$</th>
<th>$\hat{D}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MS = 2$</td>
<td>1.963</td>
<td>3.037</td>
<td>3.370</td>
<td>1.1671</td>
<td>21.66</td>
<td>32.95</td>
<td>6.295</td>
<td>399.4392</td>
<td>86.5969</td>
</tr>
<tr>
<td>$L + 1 = 4$</td>
<td></td>
<td></td>
<td>6.294</td>
<td>6.236</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$MS = 4$</td>
<td>0.4680</td>
<td>0.7310</td>
<td>0.9451</td>
<td>0.2930</td>
<td>10.49</td>
<td>15.82</td>
<td>1.696</td>
<td>400.8742</td>
<td>87.3944</td>
</tr>
<tr>
<td>$L + 1 = 5$</td>
<td></td>
<td></td>
<td>10.40</td>
<td>15.82</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$MS = 6$</td>
<td>0.2043</td>
<td>0.3196</td>
<td>0.4531</td>
<td>0.1315</td>
<td>0.6941</td>
<td>10.40</td>
<td>0.7691</td>
<td>401.0323</td>
<td>87.5239</td>
</tr>
<tr>
<td>$L + 1 = 5$</td>
<td></td>
<td></td>
<td>10.40</td>
<td>15.82</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$MS = 8$</td>
<td>0.1139</td>
<td>0.1803</td>
<td>0.2823</td>
<td>0.0572</td>
<td>5.192</td>
<td>7.758</td>
<td>0.4505</td>
<td>401.0887</td>
<td>87.6455</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
<td></td>
<td></td>
<td>5.192</td>
<td>7.758</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$MS = 10$</td>
<td>0.0727</td>
<td>0.1147</td>
<td>0.1859</td>
<td>0.0484</td>
<td>4.148</td>
<td>6.192</td>
<td>0.2944</td>
<td>401.1152</td>
<td>87.6493</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
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<td></td>
<td>4.148</td>
<td>6.192</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The numerical solutions of the nonconforming combination are given in [38]. Numerical solutions have been obtained from Combinations I, II, the symmetric combination, and the penalty combination, and their error norms are listed in Tables 1 and 2. Since results from Combinations I, II, and the symmetric combination are close to each other, we only provide those from the symmetric combination.

### Table 2. Error norms of solutions from the penalty combination of RG-FDMs with $P_c = 1$.

<table>
<thead>
<tr>
<th>Divisions</th>
<th>$|\epsilon|_1$</th>
<th>$|\epsilon|_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MS = 2$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$L + 1 = 4$</td>
<td>34.43</td>
<td>12.84</td>
</tr>
<tr>
<td>$MS = 4$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$L + 1 = 5$</td>
<td>21.15</td>
<td>4.339</td>
</tr>
<tr>
<td>$MS = 6$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$L + 1 = 5$</td>
<td>15.89</td>
<td>2.320</td>
</tr>
<tr>
<td>$MS = 8$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
<td>12.99</td>
<td>1.491</td>
</tr>
<tr>
<td>$MS = 10$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
<td>11.14</td>
<td>1.060</td>
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</table>
Table 2b. The error norms $\|e\|_1$ and $\|e\|_h$.

<table>
<thead>
<tr>
<th>Divisions</th>
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<th>$|e|_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MS = 2$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$L + 1 = 4$</td>
<td>39.07</td>
<td>24.07</td>
</tr>
<tr>
<td>$MS = 4$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$L + 1 = 5$</td>
<td>22.56</td>
<td>11.09</td>
</tr>
<tr>
<td>$MS = 6$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$L + 1 = 5$</td>
<td>16.38</td>
<td>7.204</td>
</tr>
<tr>
<td>$MS = 8$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
<td>13.10</td>
<td>5.337</td>
</tr>
<tr>
<td>$MS = 10$</td>
<td>$\sigma = 1$</td>
<td>$\sigma = 2$</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
<td>11.04</td>
<td>4.239</td>
</tr>
</tbody>
</table>

Table 2c. Other error norms.

<table>
<thead>
<tr>
<th>Divisions</th>
<th>$|e|_{0,S}$</th>
<th>$|e|_{0,S}$</th>
<th>Max</th>
<th>$|e^+ - e^-|_{0,S,0}$</th>
<th>$|e^+ - e^-|_{0,1,0}$</th>
<th>$\bar{D}_0 - \bar{D}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MS = 2$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
<td>$\sigma = 2$</td>
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<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
</tr>
<tr>
<td>$MS = 4$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
</tr>
<tr>
<td>$L + 1 = 5$</td>
<td>0.7687</td>
<td>0.2938</td>
<td>0.7655</td>
<td>0.2445</td>
<td>2.600</td>
<td>0.9666</td>
</tr>
<tr>
<td>$MS = 6$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
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</tr>
<tr>
<td>$L + 1 = 5$</td>
<td>0.3450</td>
<td>0.1317</td>
<td>0.3417</td>
<td>0.1080</td>
<td>1.182</td>
<td>0.4464</td>
</tr>
<tr>
<td>$MS = 8$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
<td>0.1939</td>
<td>0.0753</td>
<td>0.1914</td>
<td>0.0619</td>
<td>0.6243</td>
<td>0.2787</td>
</tr>
<tr>
<td>$MS = 10$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
<td>$\sigma = 2$</td>
<td>$\sigma = 4$</td>
</tr>
<tr>
<td>$L + 1 = 6$</td>
<td>0.1245</td>
<td>0.0485</td>
<td>0.1226</td>
<td>0.0398</td>
<td>0.4035</td>
<td>0.1834</td>
</tr>
</tbody>
</table>

From Table 1, we can see the following empirical, asymptotic relations:

$$
\|e\|_1 = \|u_h - u_1\| = O(h^{2-\delta}), \\
\|e\|_h = O(h), \quad \|e\|_1 = O(h), \quad \|e\|_{0,S} = O(h^2), \quad \text{Max} = \|e\|_{\infty,S} = O(h^{2-\delta}), \\
\|e^+ - e^-\|_{0,S,0} = O(h^2), \quad \|e^+ - e^-\|_{\infty,0} = O(h^2), \quad \|D_i - \bar{D}_i\| = O(h^2), \quad \text{as } i = 0, 1.
$$

Equation (6.4) is consistent with the superconvergence rates. Equations (6.5),(6.6) are all optimal convergence rates.

Take as an example the penalty combination [39]. We will investigate convergence rates in various norm definitions and the influence of $\sigma$ upon convergence rates. From Table 2a and Figure 2a, we find that

$$
\|e\|_h = O\left(h^{2/3}\right), \quad \text{for } \sigma = 1, 2, 3, \quad \|e\|_h = O\left(h^{2-\delta}\right), \quad \text{for } \sigma = 4, 5.
$$

Equations (6.7) perfectly verify Theorem 5.1. Also from Figure 2b, we see that

$$
\|e\|_1 = O\left(h^{2/3}\right), \quad \text{for } \sigma = 1, \\
\|e\|_1 = O\left(h^{3/2}\right), \quad \text{for } \sigma = 2, \\
\|e\|_1 = O\left(h^{2-\delta}\right), \quad \text{for } \sigma = 3, 4, 5.
$$

When $\sigma = 1, 2, 3$, the experimental convergence rates of $\|e\|_1$ are a little higher than those of $\|e\|_h$ (note $\|e\|_1 \leq \|e\|_h$). The error norm $\|e\|_{1,S} = O(h^{2-\delta})$ implies that the average nodal derivatives
in $S_1$ are $O(h^{2-\delta})$, and that the majority of maximal derivatives are $O(h^{2-\delta})$. Under *a posteriori* interpolation as in [12], the global superconvergence rates $O(h^{2-\delta})$ in $S_1$ and $S_2$ can also be achieved in [42].

Next, we can discover from Table 2b and Figure 2c,

\[
\|\varepsilon\|_h = O\left(h^{1/2}\right), \quad \text{for } \sigma = 1, \quad \|\varepsilon\|_h = O(h), \quad \text{for } \sigma = 2, \quad (6.9)
\]
\[
\|\varepsilon\|_h = O\left(h^{3/4}\right), \quad \text{for } \sigma = 3, \quad \|\varepsilon\|_h = O(1), \quad \text{for } \sigma = 4, \quad (6.10)
\]
\[
\|\varepsilon\|_h = O\left(h^{-1/2}\right), \quad \text{for } \sigma = 5. \quad (6.11)
\]

The optimal convergence rates $O(h)$ can be reached only when $\sigma = 2$ (see [39]). However, $\|\varepsilon\|_h \to \infty$ as $\sigma = 5$. From Table 2c and Figure 2d, we can also see that

\[
\|\varepsilon\|_1 = O\left(h^{3/4}\right), \quad \text{for } \sigma = 1, \quad \|\varepsilon\|_1 = O(h), \quad \text{for } \sigma = 2, 3, 4, 5. \quad (6.12)
\]

Surprisingly, the optimal convergence rates $O(h)$ in $\|\varepsilon\|_1$ can always be obtained when $\sigma \geq 2$. Therefore, equation (6.11) does not lead to a real divergence of numerical solutions. This comparison underscores the importance of the norms chosen. The norms $\|\varepsilon\|_1$, in particular, $\|\varepsilon\|_1$ and $\|\varepsilon\|_h$, should be used to replace $\|\varepsilon\|_h$ in both analysis and computation. From Table 2c, the
optimal convergence rates (6.5) and (6.6) also hold when \( \sigma = 2, 4 \). Overall, the penalty combination is simple, the nonconforming combination basic, and Combinations I, II, and symmetric combination are flexible for wide application, and beneficial in better stability.

A comparison on these five combinations are made here; numerical comparisons of the combined methods with other methods such as [16,30] are made in [37], and more comprehensive expositions of combinations and coupling techniques are given in the monograph [36].

6.3. Techniques for Evaluating Error Norms

In this section, we describe the methods for seeking the true coefficients \( D_\ell \) and the techniques for computing the error norms \( \| \varepsilon \|_h \).

The true solution of Motz's problem is given by the expansions

\[
\begin{align*}
  u &= \sum_{\ell=0}^{\infty} D_\ell r^{\ell + 1/2} \cos \left( \ell + \frac{1}{2} \right) \theta, \quad \text{in } S. \quad (6.13)
\end{align*}
\]

The leading coefficients \( D_\ell \) can be obtained from the boundary approximate method (BAM) in [43],

\[
\begin{align*}
  u &= \sum_{\ell=0}^{34} D_\ell r^{\ell + 1/2} \cos \left( \ell + \frac{1}{2} \right) \theta, \quad \text{in } S, \quad (6.14)
\end{align*}
\]

having the maximal errors \( 5.4 \times 10^{-9} \) at \( x = 1 \). I am grateful to Lucas and Oh in [44] for pointing out an error of \( D_{31} \) in [43]: the factor \( 10^{-8} \) should be corrected to \( 10^{-9} \). The correct digits are \( D_{31} = -0.3405273585694 \times 10^{-9} \).
The solution errors of combinations in Tables 1 and 2 are conducted, based on the true coefficients $D_\ell$ given in [43]. The BAM is, indeed, a development of Fox, Henrici and Moler [33], in
the case that local and piecewise particular solutions are used. The BAM and the method of [33] are the most accurate methods for seeking the entire solutions (6.13) in S of Motz's problem. However, the conformal transformation methods (CTM) of Whitman and Papamichael [30] and Rosser and Papamichael [29] are most accurate in seeking the leading coefficients $D_\ell$ in (6.13).

Below, let us briefly describe the techniques to compute the error norms (2.18)–(2.24). Since the discrete norms $\|v\|_{1,S_1}$ are easy to evaluate, we only focus on how to compute the error norms $\|\varepsilon\|_{1,S_2}$ in (2.18)–(2.21) involving singularities, where $\varepsilon = u - u_h$ (or $\varepsilon = u - u_N$). We have

$$\|\varepsilon\|^2_{1,S_2} = \|\varepsilon\|^2_{0,S_2} + |\varepsilon|^2_{1,S_2}.$$  

By using the Green formula and the boundary conditions in (6.1) and (6.2),

$$|\varepsilon|^2_{1,S_2} = \iint_{S_2} (\varepsilon^2 + \varepsilon_y^2) \, ds = \oint_{\partial S_2} \varepsilon \frac{\partial \varepsilon}{\partial n} \, d\ell = \int_{\Gamma_0} \varepsilon^+ \frac{\partial \varepsilon^+}{\partial n} \, d\ell,$$

where

$$\varepsilon^+ = u - u_h = \sum_{\ell=0}^{L} (D_\ell - \tilde{D}_\ell) \phi_\ell + \sum_{\ell=L+1}^{N} D_\ell \phi_\ell,$$

$$\phi_\ell = r^{\ell+1/2} \cos \left( \ell + \frac{1}{2} \right) \theta,$$

$N = 34$ and $L = 3 - 5$ in Tables 1 and 2. $D_\ell$ and $\tilde{D}_\ell$ are the coefficients from the BAM in [43] and the combinations in this paper, respectively. Note that the errors $\varepsilon$ in (6.17) on $\Gamma_0$ are no longer singular because $\Gamma_0$ is far from the singular point (0,0). Hence, we may choose the Gaussian rules of integration to evaluate the right-hand side of (6.16). In computation, we use the Gaussian rules with six nodes on $\partial \square_{ij} \cap \Gamma_0 \cap S_2$.

Next, consider the error norms

$$\|\varepsilon\|^2_{0,S_2} = \iint_{S_2} \varepsilon^2 \, ds.$$  

Since the integrand $\varepsilon^2$ in (6.18) is not singular either, we may choose the traditional integration rules in two dimensions in [41,45]. In fact, even for the leading error

$$\varepsilon_0 = \left( D_0 - \tilde{D}_0 \right) r^{1/2} \cos \frac{\theta}{2},$$

the function

$$\varepsilon_0^2 = \left( D_0 - \tilde{D}_0 \right)^2 r \cos^2 \frac{\theta}{2} = \left( D_0 - \tilde{D}_0 \right)^2 \frac{r + x}{2}$$

is smooth enough, when $D_0$ and $\tilde{D}_0$ are given. In fact, $\varepsilon_0^2 \in H^{2-\delta}(S_2)$, $0 < \delta \ll 1$.

For the division of Figure 1, we may divide $S_2$ into the uniform squares $\square_{ij}^+$ as those in $S_1$: $S_2 = \cup_{ij} \square_{ij}^+$, then

$$\|\varepsilon\|^2_{0,S_2} = \sum_{ij} \iint_{\square_{ij}^+} \varepsilon^2 \, ds.$$  

Furthermore, let $\square_{ij}^+$ be refined again into smaller uniform squares $\square_{ij,kt}^+$, i.e., $\square_{ij} = \cup_{k,t} \square_{ij,kt}^+$. Then we may evaluate the integrals in (6.19) by the simplest composite centroid rule,

$$\iint_{\square_{ij}^+} \varepsilon^2 \, ds = \sum_{k,t} \iint_{\square_{ij,kt}^+} \varepsilon^2 \, ds \approx \sum_{k,t} \varepsilon^2(G_{ij,kt}) \cdot \text{Area} \left( \square_{ij,kt}^+ \right),$$

where $G_{ij,kt}$ is the center of gravity of $\square_{ij,kt}^+$. The error norm $\|\varepsilon\|_{0,S_2}$ in this paper is computed by (6.20); error analysis can be done to show its validity.

In summary, for Motz's problem, the direct evaluation on $\|\varepsilon\|_{0,S_2}$ is not difficult, but the direct evaluation on $|\varepsilon|_{1,S_2}$ must be avoided. Our approach is to compute the rightmost of (6.16) instead.
REFERENCES