Critical Point Theory for Nondifferentiable Functionals and Applications

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We consider minimization results for locally Lipschitzian functionals on a Banach space by introducing a compactness condition of the Palais–Smale type which is suggested by the Variational Principle of Ekeland. Applications are given to some semilinear elliptic problems with discontinuous nonlinearities.

1. INTRODUCTION

In this paper we consider some aspects of critical point theory related to minimization results for functionals \( \Phi: X \to \mathbb{R} \) (\( X \) a Banach space) which are nondifferentiable. In fact, we assume that the given functionals are locally Lipschitzian so that their generalized gradients can be defined (cf. Clarke [7, 8]).

A general approach by variational methods for such locally Lipschitzian functionals was developed by Chang [5] with the use of the notion of generalized gradient of Clarke. By extending the concepts of critical point, the Palais–Smale condition, and the deformation lemma, Chang was able to generalize several minimax results existing in the literature and to apply these results to semilinear partial differential equations with discontinuous nonlinearities.

It is our intention to consider only minimization methods and to provide an approach which, instead of being based on the deformation lemma as in Chang [5], has its starting point at the well-known Variational Principle due to Ekeland [10, 11]. In its strong form, Ekeland's Principle can be stated as follows:

Let \( M \) be a complete metric space and \( \Phi: M \to \mathbb{R} \) be a lower semicon-
Continuous function which is bounded from below, say \( c = \inf_M \Phi \). Let \( \varepsilon > 0 \) be given and \( u \in M \) be such that

\[
\Phi(u) \leq c + \varepsilon.
\]

Then, for any \( \lambda > 0 \), there exists \( v \in M \) such that

\[
\Phi(v) \leq \Phi(u) \\
d(v, u) \leq \lambda \\
\Phi(v) < \Phi(w) + \left( \frac{c}{\lambda} \right) d(v, w) \quad \forall w \neq v.
\]

This Principle suggests a quite natural local compactness condition of the Palais–Smale type which we define in the following section and use to state a first minimization lemma.

The organization of this paper is as follows. In Section 2, after introducing the aforementioned (PS) type condition and one of its variants, we use them to state two abstract minimization results. In addition, we recall some elements of convex analysis and subdifferential calculus and compare our (PS) type condition with the one introduced by Chang in [5]. It turns out that these conditions are actually equivalent for locally Lipschitzian functionals \( \Phi: X \to \mathbb{R} \), \( X \) a Banach space. Finally, we also introduce in Section 2 a weak form of our (PS) type condition which is suitable for an important class of locally Lipschitzian functionals (and, for such a class, turns out to be equivalent to our original (PS) condition). In Section 3, we consider some applications to nonlinear boundary value problems of the form

\[
-\Delta u = f(x, u) \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega,
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( B \) denotes either the Dirichlet or the Neumann operator, and \( f: \Omega \times \mathbb{R} \to \mathbb{R} \) is a measurable function with subcritical growth. Such problems have been investigated by several authors (cf. [3–5, 13, 16, and references therein]). Finally, we reserve Section 4 for the proofs of all the abstract results which are stated in Section 2.

In the recent years there has been a renewed and intensified interest in Ekeland's Variational Principle (cf. [12, 14], e.g.). Aside from its intrinsic elegance and depth, as evidenced by its several and varied applications, Ekeland's Variational Principle seems to have become a natural candidate for substitution of Clark's Deformation Lemma [6] in the Variational Methods, by providing a simpler and less technical alternative. This paper is an attempt to implement such an alternative approach at the level of locally Lipschitzian functionals. A continuation of this paper, dealing with results of minimax type, which have been extensively explored in the \( C^1 \) case (cf. Rabinowitz [15]), is under preparation and will appear elsewhere.
2. Abstract Framework: Main Results

We start with a variant of Ekeland's Variational Principle [10] which plays a key role in the proofs of our minimization principles. It involves a type of local Palais–Smale condition which is suggested by Ekeland's Principle and, in fact, is quite natural when one is dealing with functionals $\Phi: X \to \mathbb{R}$ ($X$ a Banach space) which are not necessarily differentiable.

In the more general setting of a function $\Phi: M \to \mathbb{R}$, $M$ a complete metric space, let us state the fact that $\Phi$ satisfies the aforementioned condition from above at the level $c \in \mathbb{R}$ as follows:

$$(PS)_c^+: \text{Whenever } (u_n) \subset M, (\varepsilon_n), (\delta_n) \subset \mathbb{R}^+ \text{ are sequences with } \varepsilon_n \to 0, \delta_n \to 0, \text{ and such that}$$

$$\Phi(u_n) \to c \quad \text{and} \quad \Phi(u_n) \leq \Phi(u) + \varepsilon_n d(u_n, u), \quad \text{if } d(u_n, u) \leq \delta_n,$$

then $(u_n)$ possesses a convergent subsequence: $u_n \to \hat{u}$.

Similarly, we define the $(PS)^*_c$ condition from below, $(PS)^*_c$, by interchanging $u$ and $u_n$ in the above inequality. And we say that $\Phi$ satisfies $(PS)^*_c$ provided it satisfies $(PS)_{c,*}^+$ and $(PS)_{c,*}^-$ (of course, when $M = X$ is a Banach space we let $d(u, u) = \|u - u\|$ in above).

Minimization Lemma. Let $M$ be a complete metric space and let $\Phi: M \to \mathbb{R}$ be a lower semicontinuous function which is bounded from below, with say $c = \inf_M \Phi$. If $\Phi$ satisfies the condition $(PS)^{*,+}_c$, then $c$ is attained; that is, there exists $u_0 \in M$ such that $\Phi(u_0) = c$.

Next we introduce some elements of convex analysis and subdifferential calculus (cf. [4, 7, 8], e.g.) which are needed in the formulation of our minimization results.

Let $X$ be a reflexive real Banach space. We say that a given functional $\Phi: X \to \mathbb{R}$ is locally Lipschitzian ($\Phi \in \text{Lip}_{loc}(X, \mathbb{R})$) if, for each $u \in X$, there exists an open neighborhood $V = V_u \subset X$ of $u$ and a constant $K = K_u > 0$ such that

$$|\Phi(v_1) - \Phi(v_2)| \leq K \|v_1 - v_2\|$$

for all $v_1, v_2 \in V$. For such a $\Phi$ the generalized directional derivative at $u \in X$ in the direction of $v \in X$ is defined by the formula

$$\Phi^0(u; v) = \lim_{h \to 0} \sup_{\lambda \to 0} \left(\frac{1}{\lambda}\right)[\Phi(u + h + \lambda v) - \Phi(u + h)].$$
It follows easily that $\Phi^0(u; \cdot)$ is a subadditive positively homogeneous function; that is,

(i) $\Phi^0(u; v_1 + v_2) \leq \Phi^0(u; v_1) + \Phi^0(u; v_2)$

(ii) $\Phi^0(u; \lambda v) = \lambda \Phi^0(u; v)$ for $\lambda \geq 0$ and any $u, v, v_1, v_2 \in X$. In particular, $\Phi^0(u; \cdot)$ is convex and satisfies

$$|\Phi^0(u; v)| \leq K \|v\|,$$

where $K = K_u > 0$ depends only on a neighborhood of $u$.

The generalized gradient at $u \in X$ of a given $\Phi \in \text{Lip}_{loc}(X, \mathbb{R})$ is the subset $\partial \Phi(u) \subset X^*$ defined by

$$\partial \Phi(u) = \{ \mu \in X^* \mid \Phi^0(u; v) \geq \langle \mu, v \rangle \ \forall v \in X \},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $X^*$ and $X$.

Remark. Since $\Phi^0(u; \cdot)$ is a convex function, its subdifferential at $z \in X$ is well-defined and given by

$$\partial \Phi^0(u; z) = \{ \mu \in X^* \mid \Phi^0(u; v) \geq \langle \mu, v - z \rangle \ \forall v \in X \}.$$

Therefore, since $\Phi^0(u; 0) = 0$, the generalized gradient $\partial \Phi(u)$ is simply the subdifferential of $\Phi^0(u; \cdot)$ at $z = 0$.

The following properties can be shown to hold:

(P1) For each $u \in X$ the set $\partial \Phi(u) \subset X^*$ is convex, nonempty, and compact in the weak*—topology; in particular, there exists $\bar{\mu} = \bar{\mu}(u)$ such that $\|\bar{\mu}\| = \min \{ \|\mu\|_{X^*} \mid \mu \in \partial \Phi(u) \}$;

(P2) $\max \{ \langle \mu, v \rangle \mid \mu \in \partial \Phi(u) \} = \Phi^0(u; v)$ for any $u, v \in X$;

(P3) $\partial(\Phi + \Psi)(u) \subset \partial \Phi(u) + \partial \Psi(u)$ for any $u \in X, \Phi, \Psi \in \text{Lip}_{loc}(X, \mathbb{R})$;

(P4) $\partial(\lambda \Phi)(u) = \lambda \partial \Phi(u)$ for any $\lambda \in \mathbb{R}, u \in X$;

(P5) If $\mu_n \in \partial \Phi(u_n)$ and $u_n \to u$ in $X, \mu_n \to \mu$ in the weak*—topology then $\mu \in \partial \Phi(u)$;

(P6) If $\Phi$ is convex then $\partial \Phi(u) = \{ \mu \in X^* \mid \Phi(w) - \Phi(u) \geq \langle \mu, w - u \rangle \ \forall w \in X \}$;

(P7) If $\Phi$ is continuously Fréchet differentiable in a neighborhood of $u \in X$ then $\partial \Phi(u) = \{ \Phi'(u) \}$.

A point $u_0 \in X$ is said to be a critical point of $\Phi \in \text{Lip}_{loc}(X, \mathbb{R})$ if $0 \in \partial \Phi(u_0)$. And $c \in \mathbb{R}$ is a critical value of $\Phi$ if there exists a critical point
u_0 \in X \text{ with } \Phi(u_0) = c. \text{ We observe that a point of local minimum is a critical point. Indeed, in this case we have}

\[ 0 \leq \limsup_{\lambda \to 0} (1/\lambda)[\Phi(u_0 + \lambda v) - \Phi(u_0)] \leq \Phi^0(u_0; v) \]

for any \( v \in X \); hence \( 0 \in \partial \Phi(u_0) \). We are now ready to state our main results.

**Theorem 1.** Let \( X \) be a reflexive real Banach space and \( \Phi: X \to \mathbb{R} \) a locally Lipschitzian functional which is bounded from below with \( c = \inf X \Phi \). If \( \Phi \) satisfies \((\text{PS})^*_c\), then \( c \) is attained; that is, there exists \( u_0 \in X \) such that \( \Phi(u_0) = c \). In particular, \( u_0 \) is a critical point of \( \Phi \).

Our next result is a version of Theorem 1 which uses a weak Palais–Smale type condition similar to the one introduced by Brézis, Coron, and Nirenberg in [2]; namely

\[ [\text{PS}]^*_c: \text{Whenever } (u_n) \subset X, (\varepsilon_n), (\delta_n) \subset \mathbb{R}_+ \text{ are sequences with } \varepsilon_n \to 0, \delta_n \to 0 \text{ and such that} \]

\[ \Phi(u_n) \to c, \]

\[ \Phi(u_n) \leq \Phi(u) + \varepsilon_n \|u_n - u\| \quad \text{if } \|u_n - u\| \leq \delta_n, \]

then \( c \) is a critical value.

**Theorem 2.** Let \( X \) be a reflexive real Banach space and \( \Phi: X \to \mathbb{R} \) a locally Lipschitzian functional which is bounded from below with \( c = \inf X \Phi \). If \( \Phi \) satisfies \([\text{PS}]^*_c\), then \( c \) is attained.

**Remarks.** (1) It is not hard to see that \((\text{PS})^*_c\) implies \([\text{PS}]^*_c\) (and hence, Theorem 1 is a consequence of Theorem 2). However, it should be noted that the converse implication is not true (for example, consider a periodic Lipschitzian function \( \Phi: \mathbb{R} \to \mathbb{R} \) and \( c = \min \Phi \) or \( c = \max \Phi \)).

(2) In [5] Chang introduced the following Palais–Smale type condition for locally Lipschitzian functionals, which reduces to the usual Palais–Smale condition when \( \Phi \) is a \( C^1 \) functional:

\((\text{PS})_c: \text{Whenever } (u_n) \subset X \text{ is such that} \)

\[ \Phi(u_n) \to c, \]

\[ m(u_n) = \min \{ \| \mu_n \|_{X^*} \mid \mu_n \in \partial \Phi(u_n) \} \to 0, \]

then \( (u_n) \) possesses a convergent subsequence: \( u_n \rightharpoonup \hat{u} \).

In Section 4 we prove Proposition 3 below which says that our seemingly weaker condition \((\text{PS})^*_c\) is equivalent to \((\text{PS})_c\) for locally Lipschitzian func-
tionals. As a result, our Theorem 1 is equivalent to the corresponding theorem of Chang [5, Theorem 3.5] which was proved using a suitable, technical version of Clark's Deformation Lemma for locally Lipschitzian functionals [5, Theorem 3.1].

**PROPOSITION 3.** Let \( X \) be a reflexive real Banach space and let \( \Phi \in \text{Lip}_{\text{loc}}(X, \mathbb{R}) \). Then \( \Phi \) satisfies (PS)\(_c\) if and only if it satisfies (PS)\(_c^*\).

Last, we introduce a weak Palais–Smale condition in the spirit of (PS)\(_c^*\). Namely, we say that \( \Phi \) satisfies (PS)\(_{c,w}^*\) if it satisfies (PS)\(_{c,w,+}^*\) and (PS)\(_{c,w,-}^*\), where

\[
(PS)\(_{c,w,+}^*\) : \text{Whenever } (u_n) \subset X, (\varepsilon_n), (\delta_n) \subset \mathbb{R}_+ \text{ are sequences with } \varepsilon_n \to 0, \delta_n \to 0 \text{ and such that}
\]
\[
\Phi(u_n) \to c, \quad \Phi(u_n) \leq \Phi(u) + \varepsilon_n \| u_n - u \| \quad \text{if } \| u_n - u \| \leq \delta_n;
\]
then \( (u_n) \) possesses a weakly convergent subsequence \( u_n \to \tilde{u} \).

Similarly, (PS)\(_{c,w}^*\) is defined by interchanging \( u \) and \( u_n \) in the above inequality.

It is clear that (PS)\(_c^*\) implies (PS)\(_{c,w}^*\) but not conversely. However, for a special and important class of functionals it turns out that these two conditions are equivalent. This fact is useful in the applications we provide in the next section.

**PROPOSITION 4.** Let \( X \) be a Hilbert space and \( Y \) a reflexive real Banach space with \( X \subset Y \) embedded compactly and such that \( X \) is dense in \( Y \) (as a subset of \( Y \)). Let \( \Phi : X \to \mathbb{R} \) be of the form

\[
\Phi(u) = \frac{1}{2} \| u \|_X^2 - \tilde{\Psi}(u), \quad u \in X,
\]

where \( \tilde{\Psi} \) is the restriction \( \tilde{\Psi} = \Psi \mid X \) if a given locally Lipschitzian functional \( \Psi : Y \to \mathbb{R} \). Then

\( \Phi \) satisfies (PS)\(_c^*\) if and only if \( \Phi \) satisfies (PS)\(_{c,w}^*\).

**Remarks.**

(3) Typically, \( X \) is taken to be \( H^1_0(\Omega) \) or \( H^1(\Omega) \) and \( Y = L^p(\Omega), \ p < 2^* = 2N/(N-2) \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \).

(4) In these typical cases, when \( \Psi \) (hence \( \Phi \)) is a given \( C^1 \) potential \( \int f(x,u) \, dx \) (with \( f \) a Caratheodory function \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) with subcritical growth \( p \) as above) we have that \( \nabla \Phi(u) = u - T(u) \) with \( T : X \to X \) a compact operator, and Proposition 4 expresses the fact that the usual (PS)\(_{c,w} \) implies the usual (PS)\(_c \) in this case.
3. Applications

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with smooth boundary \( \partial \Omega \). We consider nonlinear boundary value problems of the form

\[
-\Delta u = f(x, u) \quad \text{in } \Omega \\
Bu = 0 \quad \text{on } \partial \Omega,
\]

where \( B \) denotes either the Dirichlet or the Neumann boundary condition and \( f: \Omega \times \mathbb{R} \to \mathbb{R} \) is a measurable function with subcritical growth; that is, \( f \) satisfies

\[
|f(x, s)| \leq a + b |s|^\sigma \quad \text{for all } s \in \mathbb{R}, \ a.e. x \in \Omega,
\]

where \( a, b > 0 \) and \( 0 \leq \sigma < (N + 2)/(N - 2) \) if \( N \geq 3 \) \((0 \leq \sigma < \infty \) if \( N = 1, 2 \).

Following Chang [5] we denote

\[
\bar{f}(x, t) = \lim \inf_{s \to t} f(x, s), \quad \underline{f}(x, t) = \lim \sup_{s \to t} f(x, s)
\]

and assume that

\[
\underline{f}, \bar{f}: \Omega \times \mathbb{R} \to \mathbb{R} \text{ are } N\text{-measurable.}
\]

**Remark.** Hypothesis (f2) is satisfied, for example, in the following situations: (i) \( f(x, s) = f(s) \) is independent of \( x \); (ii) \( f(x, s) \) is Baire measurable in \( \Omega \times \mathbb{R} \) and nondecreasing in \( s \). In this latter case we have

\[
\underline{f}(x, t) = \min \{f(x, t + 0), f(x, t - 0)\}, \\
\bar{f}(x, t) = \max \{f(x, t + 0), f(x, t - 0)\},
\]

where \( f(x, t \pm 0) \) denote the side limits \( f(x, t \pm 0) = \lim_{\delta \to 0} f(x, t \pm \delta) \).

A function \( u \in W^{2,p}(\Omega) \), \( p > 1 \), is called a solution of (\( \clubsuit \)) if it satisfies the boundary condition \( Bu = 0 \) on \( \partial \Omega \) (trace sense) and the differential inequality

\[
-\Delta u(x) \in [\underline{f}(x, u(x)), \bar{f}(x, u(x))] \quad \text{a.e. in } \Omega.
\]

On the other hand, we may consider critical points of the locally Lipschitzian functional \( \Phi \) associated to (\( \clubsuit \)), namely,

\[
\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} F(x, u) \, dx = Q(u) - \Phi(u), \quad u \in X,
\]

where \( F(x, s) = \int_{0}^{s} f(x, \sigma) \, d\sigma \) and \( X = H^1_0(\Omega) \) \((H^1(\Omega))\), according to
whether we take Dirichlet or Neumann boundary condition). Indeed, in view of hypothesis \((f_1)\), the function \(F: \Omega \times \mathbb{R} \to \mathbb{R}\) and the corresponding functional \(\Psi\) on \(Y = L^{\sigma+1}(\Omega)\) can be easily seen to be locally Lipschitzian. By the Sobolev embedding \(X \subset Y\) we then obtain that \(\Psi = \Psi | X\) and (hence) \(\Phi\) are locally Lipschitzian on \(X\), so that we may look for critical points \(u_0 \in X\) of \(\Phi: X \to \mathbb{R}\), that is, solutions of

\[
0 \in \partial \Phi(u), \quad u \in X. \tag{3}
\]

With the aid of the following result due to Aubin and Clarke [1] and Chang [5], it can be shown that critical points of \(\Phi\) are indeed solutions of \((\spadesuit)\) as defined above (cf. (1)).

**Theorem 0.** Assume \((f_1), (f_2)\). Then \(\Psi\) is locally Lipschitzian on \(Y = L^{\sigma+1}(\Omega)\) and

\[
(i) \quad \partial \Psi(u) = \partial_x F(\cdot, u(\cdot)) = [f(x, u(x)), \tilde{f}(x, u(x))] \text{ a.e.}
\]

Moreover, if \(\tilde{\Psi} = \Psi | X\), where \(X = H^1_0(\Omega)\) or \(H^1(\Omega)\), then

\[
(ii) \quad \partial \tilde{\Psi}(u) \subset \partial \Psi(u) \text{ for all } u \in X.
\]

For the proof of Theorem 0 above we refer the reader to Chang [5, Theorems 2.1 and 2.2]. Let us only verify here how it can be used to show that (3) indeed implies (1) in this case, namely

**Corollary.** If \((f_1), (f_2)\) hold then any critical point \(u_0\) of \(\Phi\) is a solution of \((\spadesuit)\).

**Proof.** If \(u_0\) is a critical point of \(\Phi\) then

\[
0 \in \partial \Phi(u_0) \subset \partial Q(u_0) + \partial(-\tilde{\Psi})(u_0) \quad \text{ (4)}
\]

in view of property \((P3)\). Therefore, since \(\partial Q(u_0) = \{Q'(u_0)\}\), where

\[
\left\langle Q'(u_0), v \right\rangle = \int_\Omega \nabla u_0 \cdot \nabla v, \tag{4}
\]

means that

\[
0 \leq \int_\Omega \nabla u_0 \cdot \nabla v \, dx + (-\tilde{\Psi})^0(u_0; v) \quad \forall v \in X,
\]

that is,

\[
\int_\Omega \nabla u_0 \cdot \nabla(-v) \, dx \leq (-\tilde{\Psi})^0(u_0; v) \quad \forall v \in X.
\]

But then, as \((-\tilde{\Psi})^0(u_0; v) = \tilde{\Psi}(u_0; -v)\), we obtain that

\[
\left\langle \mu_0, v \right\rangle = \int_\Omega \nabla u_0 \cdot \nabla v \, dx \leq (\tilde{\Psi})^0(u_0; v) \quad \forall v \in X.
\]
hence $\mu_0 = -\Delta u_0 \in \partial \mathcal{P}(u_0) \subset \partial \mathcal{P}(u_0)$ in view of Theorem 0(ii). Finally, from $\partial \mathcal{P}(u_0) \subset (L^{\sigma+1})^* = L^{(\sigma+1)/\sigma}$ and Theorem 0(i), it follows that $-\Delta u_0 \in L^{(\sigma+1)/\sigma}$, so that $u_0 \in W^{2,(\sigma+1)/\sigma}$ and 

$$-\Delta u_0(x) \in \left[ f(x, u_0(x)), f(x, u_0(x)) \right] \text{ a.e. in } \Omega;$$

that is, $u_0$ is a solution of $(\Phi)$. 

Next we state and prove two basic results which give examples of locally Lipschitzian functionals satisfying the Palais–Smale conditions $(\text{PS})_c^*$ and $[\text{PS}]_c^*$.

**Proposition 5.** Let $X = H^1_0(\Omega)$ and $\Phi: X \to \mathbb{R}$ be of the form 

$$\Phi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(x, u) \, dx,$$

where $G(\cdot, u)$ is measurable for all $u \in \mathbb{R}$ and $G(x, \cdot)$ is globally Lipschitzian uniformly for $x$ in $\Omega$, that is, $|G(x, s) - G(x, t)| \leq M |s - t|$ for all $s, t \in \mathbb{R}$, a.e. $x \in \Omega$, and some constant $M > 0$. Then $\Phi$ satisfies condition $(\text{PS})_c^*$ for any $c \in \mathbb{R}$.

**Proposition 6.** Let $X = H^1(\Omega)$ and $\Phi: X \to \mathbb{R}$ be of the form 

$$\Phi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(u) \, dx - \int_\Omega h u \, dx,$$

where $G: \mathbb{R} \to \mathbb{R}$ is Lipschitzian, periodic of period $T$, and $h \in L^2(\Omega)$ satisfies $\int_\Omega h \, dx = 0$. Then $\Phi$ satisfies condition $[\text{PS}]_c^*$ for any $c \in \mathbb{R}$.

**Proof of Proposition 5.** We show that $\Phi$ satisfies $(\text{PS})_c^*$ since the proof of $(\text{PS})_c^*$ is entirely similar. Let $(u_n) \subset X$ and $(\epsilon_n), (\delta_n) \subset \mathbb{R}_+$ be such that $\Phi(u_n) \to c$ and 

$$\Phi(u_n) \leq \Phi(u) + \epsilon_n \|u_n - u\| \quad \text{if} \quad \|u_n - u\| \leq \delta_n.$$ 

Then, if $\delta \|h\| \leq \delta_n$, we obtain 

$$\Phi(u_n) - \Phi(u_n - \delta h) \leq \delta \epsilon_n \|h\|.$$ 

Taking $h = u_n$ and $\delta > 0$ such that $\delta \|u_n\| \leq \delta_n$ yields 

$$\Phi(u_n) - \Phi((1 - \delta) u_n) \leq \delta \epsilon_n \|u_n\|,$$

that is, 

$$(\delta - \delta^2/2) \|u_n\|^2 + \int_\Omega \left[ G(x, (1 - \delta) u_n) - G(x, u_n) \right] \, dx \leq \delta \epsilon_n \|u_n\|.$$
Dividing by $\delta > 0$ and using the fact that $|G(x, s) - G(x, t)| \leq M |s - t|$ we obtain
\[(1 - \delta/2) \|u_n\|^2 - M \int_{\Omega} |u_n| \, dx \leq \epsilon_n \|u_n\|;
\]
hence
\[(1 - \delta/2) \|u_n\|^2 \leq M_1 \|u_n\|_{L^2} + \epsilon_n \|u_n\| \leq (M_2 + \epsilon_n) \|u_n\|
\]
by Poincaré's inequality, where we note that the constants $M_n \equiv M_2 + \epsilon_n$ are uniformly bounded since $\epsilon_n \to 0$. We have shown that there exists $M' > 0$ such that
\[(1 - \delta/2) \|u_n\|^2 \leq M' \|u_n\|
\]
provided $\delta > 0$ satisfies $\delta \|u_n\| \leq \delta_n$. Letting $\delta \to 0$ we obtain $\|u_n\|^2 \leq M' \|u_n\|$, so that $\|u_n\| \leq M'$ and $(u_n)$ possesses a weakly convergent subsequence.

Thus $\Phi$ satisfies the weak Palais–Smale condition (PS)\(_{\epsilon, w, +}\). In view of Proposition 4 with $X = H_0^1(\Omega) \subset L^2(\Omega) = Y$, $\Phi$ also satisfies the stronger condition (PS)\(_{\epsilon, +}\).

**Proof of Proposition 6.** Decompose the space $X = H_0^1(\Omega)$ as the direct sum
\[X = X_0 \oplus X_1,
\]
where $X_1 = \langle 1 \rangle$ is the space of constants functions and $X_0 = X_1^\perp$ is the space of $H^1$-functions with mean value zero. Again, we only show that $\Phi$ satisfies [PS]\(_{\epsilon, +}\) since the proof of [PS]\(_{\epsilon, -}\) is similar. Let $(u_n) \subset X$ and $(\epsilon_n), (\delta_n) \subset \mathbb{R}_+$ be such that $\Phi(u_n) \to c$ and
\[\Phi(u_n) \leq \Phi(u) + \epsilon_n \|u_n - u\| \quad \text{if} \quad \|u_n - u\| \leq \delta_n.
\]

We write $u_n = v_n + c_n$ where $v_n \in X_0$, $c_n \in X_1 = \mathbb{R}$. Then, since $h \in X_0$ and $|G(s)| \leq M$, we have
\[
\Phi(u_n) = \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 \, dx - \int_{\Omega} G(v_n + c_n) \, dx - \int_{\Omega} h v_n \, dx
\geq \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 \, dx - M |\Omega| - \lambda_2^{-1/2} \left( \int_{\Omega} h^2 \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla v_n|^2 \, dx \right)^{1/2}
\]
in view of Poincaré's inequality $\lambda_2 \int_{\Omega} v^2 \leq \int_{\Omega} |\nabla v|^2$, $\forall v \in X_0$. Since $\Phi(u_n) \to c$ it follows that
\[
\int_{\Omega} |\nabla v_n|^2 \, dx \text{ is bounded};
\]
hence \( \|v_n\| \) is also bounded. On the other hand, defining \( \hat{u}_n = v_n + \hat{c}_n \), were \( c_n = \hat{c}_n \pmod{T} \) and \( \hat{c}_n \in [0,T) \), and noting that the functional \( \Phi \) is \( T \)-periodic, we obtain
\[
\Phi(\hat{u}_n) \leq \Phi(u) + \varepsilon_n \|\hat{u}_n - u\| + (c_n - \hat{c}_n) \|\triangledown_1\|
\]
if \( \|\hat{u}_n - (u + \hat{c}_n - c_n)\| \leq \delta_n \); that is,
\[
\Phi(u_n) \leq \Phi(w) + \varepsilon_n \|\hat{u}_n - w\|
\]
if \( \|\hat{u}_n - w\| \leq \delta_n \). Since \( (\hat{u}_n) \) is bounded, some subsequence of \( (\hat{u}_n) \) is weakly convergent, say \( \hat{u}_n \to \hat{u} \). By the proof of Proposition 4, we obtain that \( \hat{u}_n \to \hat{u} \). In particular,
\[
\Phi(\hat{u}_n) \to \Phi(\hat{u}) = c.
\]
Finally, letting \( w = \hat{u}_n + \delta_n v \) with \( \|v\| \leq 1 \) in above gives
\[
\Phi(\hat{u}_n + \delta_n v) - \Phi(\hat{u}_n) \geq -\varepsilon_n \delta_n,
\]
hence
\[
-\varepsilon_n \leq \frac{1}{\delta_n} [\Phi(\hat{u}_n + \delta_n v) - \Phi(\hat{u}_n)];
\]
so that, passing to the limit as \( n \to \infty \) and recalling that \( \hat{u}_n \to \hat{u} \), we obtain
\[
0 \leq \Phi(\hat{u}; v)
\]
for all \( \|v\| \leq 1 \). Therefore \( 0 \leq \Phi(\hat{u}; v) \) for all \( v \in X \), that is,
\[
0 \in \partial \Phi(\hat{u}).
\]
Since \( \Phi(\hat{u}) = c \), it follows that \( c \) is a critical value.

**APPLICATION 1.** We consider the nonlinear Dirichlet problem
\[
-\Delta u = g(x, u) \quad \text{in } \Omega \tag{D}
\]
\[
u = 0 \quad \text{on } \partial \Omega
\]
where \( g: \Omega \times \mathbb{R} \to \mathbb{R} \) is a measurable function satisfying condition \((f_1)\) with \( \sigma - 0 \) and \((f_2)\), that is, satisfying
\[
|g(x, s)| \leq M \text{ for all } s \in \mathbb{R}, \text{ a.e. } x \in \Omega, \text{ and } \tag{g_1}
\]
\[
g, g_1: \Omega \times \mathbb{R} \to \mathbb{R} \text{ are } N\text{-measurable.}
\]
Theorem 7. Assume \((g_1)\). Then \((D)\) has a solution \(u \in W^{2,\infty}(\Omega)\) in the sense of \((1)\).

Proof. In view of Corollary 0 it is enough to find a critical point of the locally Lipschitzian functional

\[
\Phi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(x, u) \, dx, \quad u \in X = H^1_0(\Omega).
\]

Since \(G(x, \cdot)\) is globally Lipschitzian uniformly for a.e. \(x \in \Omega\), it follows from Proposition 5 that \(\Phi\) satisfies condition \((PS)_*\) for any \(c \in \mathbb{R}\). Therefore, since \(\Phi\) is clearly bounded from below, the result follows from Theorem 1.

Remark. Of course, the above result could also be obtained by noting that the functional \(\Phi\) is in fact coercive and weakly lower semicontinuous. The point in the proof just given is the fact that \(\Phi\) satisfies condition \((PS)_*\).

Application 2. We consider the nonlinear Neumann problem

\[
-\Delta u = g(u) + h(x) \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega
\]

under the following conditions:

\(g: \mathbb{R} \to \mathbb{R}\) is bounded measurable and \(T\)-periodic with \(\int_0^T g(s) \, ds = 0\); \(g_3)\)

\[h \in L^2(\Omega)\) satisfies \(\int_\Omega h(x) \, dx = 0.\) \(g_4)\)

Theorem 8. If \((g_3), (g_4)\) hold then \((N)\) has a solution.

Proof. We find a critical point of the locally Lipschitzian functional

\[
\Phi(u) - \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(u) \, dx \quad \int_\Omega hu \, dx, \quad u \in X = H^1(\Omega),
\]

where \(G(u) = \int_0^u g\) is Lipschitzian and \(T\)-periodic in view of \((g_3)\). Indeed, by Proposition 6, \(\Phi\) satisfies condition \([PS]_*\) for any \(c \in \mathbb{R}\). Therefore, since \((5)\) shows that \(\Phi\) is bounded from below, we can use Theorem 2 to conclude that the infimum \(c = \inf_X \Phi\) is attained and hence that \(\Phi\) has a critical point.

Remark. Theorem 8 generalizes corresponding results in \([9, 17]\) for \(g\) continuous.
4. PROOFS OF MAIN ABSTRACT RESULTS

We start with a proof of the minimization lemma. Let $\Phi: M \to \mathbb{R}$ be a lower semicontinuous function on the complete metric space $M$ with $c = \inf_M \Phi > -\infty$ and such that $\Phi$ satisfies the condition $(PS)_{\epsilon}^*$. If $(v_n) \subset X$ is a minimizing sequence with $\delta_n = \Phi(v_n) - c > 0$, $\delta_n \to 0$, then letting $\varepsilon = \delta_n$, $\lambda = \delta_n^{1/2}$ in Ekeland's Variational Principle [10, Theorem 1.1], we obtain a sequence $(u_n)$ such that

$$
\Phi(u_n) \leq \Phi(v_n),
$$
$$
d(u_n, v_n) \leq \delta_n^{1/2},
$$
$$
\Phi(u_n) < \Phi(u) + \delta_n^{1/2} d(u, u_n) \quad \text{if} \quad u \neq u_n.
$$

In particular, we have $\Phi(u_n) \to c$ and $\Phi(u_n) \leq \Phi(u) + \delta_n^{1/2} d(u_n, u)$ for all $u \in X$, so that $(u_n)$ possesses a convergent subsequence by the condition $(PS)_{\epsilon}^* : u_n \to \tilde{u}$. But then, since $\Phi$ is lower semicontinuous, we obtain that $\Phi(\tilde{u}) \leq \lim\inf \Phi(u_n) = c$, hence $\Phi(\tilde{u}) = c$.

Proof of Theorem 1. Simply let $M = X$, $d(u, v) = \|u - v\|$ in the minimization lemma and recall that, for a locally Lipschitzian functional $\Phi: X \to \mathbb{R}$, a point of (local) minimum $\tilde{u} \in X$ is a critical point of $\Phi$, that is, $0 \in \partial \Phi(\tilde{u})$.

Proof of Theorem 2. In view of Ekeland's Variational Principle, there exists a sequence $(u_n) \subset X$ such that $\Phi(u_n) \to c$ and $\Phi(u_n) \leq \Phi(u) + \varepsilon_n \|u_n - u\|$ if $\|u_n - u\| \leq \varepsilon_n$, where $(\varepsilon_n) \subset \mathbb{R}_+$, $\varepsilon_n \to 0$, is a given sequence. Therefore, using the condition $(PS)_{\epsilon}^*$ for this sequence $(u_n)$, we conclude that $c$ is a critical value; that is, there exists $\tilde{u} \in X$ such that $\Phi(\tilde{u}) = c$ and $0 \in \partial \Phi(\tilde{u})$.

Next we prove Proposition 3 which says that, for locally Lipschitzian functionals, our condition $(PS)_{\epsilon}^*$ is equivalent to the condition $(PS)_c$ introduced by Chang.

Proof of Proposition 3. $(PS)_c \Rightarrow (PS)_{\epsilon}^*$: Let $(u_n) \subset X$, $(\varepsilon_n), (\delta_n) \subset \mathbb{R}_+$ be such that $\varepsilon_n \to 0$, $\delta_n \to 0$, and

$$
\Phi(u_n) \to c,
$$
$$
\Phi(u_n) \leq \Phi(u) + \varepsilon_n \|u_n - u\| \quad \text{if} \quad \|u_n - u\| \leq \delta_n.
$$

Writing $u = u_n + \delta v$ with $\|v\| = 1$, $0 < \delta \leq \delta_n$, we obtain

$$
- \varepsilon_n \delta \leq \Phi(u_n + \delta v) - \Phi(u_n);
$$
CRITICAL POINT THEORY

\[
- \frac{1}{\delta} \left[ \Phi(u_n + \delta v) - \Phi(u_n) \right] \leq \varepsilon_n, \quad 0 < \delta \leq \delta_n,
\]

so that, letting \( \delta \to 0 \) and denoting by \( w_n \) the element in \( \partial \Phi(u_n) \) such that \( \langle w_n, v \rangle = \Phi^o(u_n; v) \) \( \forall v \in X \), it follows that

\[
- \langle w_n, v \rangle \leq \varepsilon_n \quad \forall \|v\| = 1.
\]

Therefore we obtain \( |\langle w_n, v \rangle| \leq \varepsilon_n \) for all \( \|v\| = 1 \), so that

\[
m(u_n) = \|w_n\|_{X^*} = \min \{ \|\mu_n\|_{X^*} \mid \mu_n \in \partial \Phi(u_n) \} \leq \varepsilon_n \to 0
\]
as \( n \to \infty \), and \((PS)_c\) implies that \( (u_n) \) possesses a convergent subsequence. Thus \((PS)^*_+\) holds and, similarly, we also show that \((PS)_c\) implies \((PS)^*_{c,+}\).

Conversely, let us assume that \((PS)^*_c\) holds and let \( (u_n) \subset X \) be such that

\[
\Phi(u_n) \to c,
\]

\[
m(u_n) = \|w_n\|_{X^*} = \min \{ \|\mu_n\|_{X^*} \mid \mu_n \in \partial \Phi(u_n) \} \to 0.
\]

Then, writing \( \varepsilon_n = m(u_n) \), we have

\[
|\langle w_n, v \rangle| \leq \varepsilon_n \quad \forall \|v\| = 1,
\]

and, since \( \langle w_n, v \rangle = \Phi^o(u_n; v) \), it follows that

\[
\frac{1}{\delta} \left[ \Phi(u_n + \delta v) - \Phi(u_n) \right] \leq \langle w_n, v \rangle + \varepsilon_n
\]
(provided that \( 0 < \delta \leq \delta_n = \delta_n(\varepsilon_n) \)). From the above two inequalities we conclude that

\[
\Phi(u_n + \delta v) \leq \Phi(u_n) + 2 \delta \varepsilon_n.
\]

Therefore, \((6), (7)\), and the fact that \( \Phi \) satisfies \((PS)^*_c\) imply that \( (u_n) \) possesses a convergent subsequence. We have shown that \( \Phi \) satisfies \((PS)_c\), and the proof is complete.

As a consequence of the proof of Proposition 3 we obtain the following result, where the weak Palais–Smale condition \((PS)^*_c\) has already been defined and the weak Palais–Smale condition \((PS)_c\) is the corresponding weak form of the \((PS)_c\) condition introduced by Chang (replacing convergent subsequence by weakly convergent subsequence).
COROLLARY 3. If \( \Phi \in \text{Lip}_{\text{loc}}(X, \mathbb{R}) \), \( X \) a reflexive real Banach space, then \( \Phi \) satisfies (PS)_{c,w} if and only if \( \Phi \) satisfies (PS)_{c,w}^*.

Proof. Simply replace convergent subsequence by weakly convergent subsequence in the proof of Proposition 3.

Finally, we prove Proposition 4 which is the analogue, for locally Lipschitzian functionals, of the fact that a functional \( \Phi: X \to \mathbb{R} \) (\( X = H^1_0(\Omega) \) or \( H^1(\Omega) \)) of the form \( \Phi(u) = \|u\|_{L^2}/2 - \bar{\Psi}(u) \), with \( \bar{\Psi} \) a \( C^1 \) potential with subcritical growth, has a gradient of the form \( \nabla \Phi(u) = u - T(u) \), \( T: X \to X \) a compact operator, and, hence, the usual (PS)_c condition is equivalent to the weak (PS)_c condition.

Proof of Proposition 4. Let \( \Phi: X \to \mathbb{R} \) be of the form

\[
\Phi(u) = \frac{1}{2} \|u\|^2_X - \bar{\Psi}(u), \quad u \in X,
\]

where \( X \) is a Hilbert space and \( \bar{\Psi} \) is the restriction \( \bar{\Psi} = \Psi | X \) of a locally Lipschitzian functional \( \Psi: Y \to \mathbb{R} \), with \( Y \) a reflexive Banach space such that \( X \subset Y \) is embedded compactly and \( X \) is dense in \( Y \) (as a subspace of \( Y \)).

We want to show that (PS)_{c,w}^* implies (PS)_c^* or, in view of Proposition 3, that (PS)_{c,w} implies (PS)_c.

So, assume that \( \Phi \) as above satisfies (PS)_{c,w} and let \( (u_n) \subset X \) be such that

\[
\Phi(u_n) \to c, \quad m(u_n) = \|w_n\|_{X^*} = \min \{ \|\mu_n\|_{X^*} : \mu_n \in \partial \Phi(u_n) \} \to 0.
\]

Then, for a subsequence (still labelled \( u_n \)) and some \( \hat{u} \in X \), we have

\[
u_n \to \hat{u} \quad \text{in } X. \quad (9)
\]

On the other hand, since \( \partial(\|u\|_{L^2}/2) = \{ Au \} \), where \( A: X \to X^* \) is the canonical isomorphism, we obtain

\[
\partial \Phi(u_n) \subset Au_n - \partial \bar{\Psi}(u_n)
\]

in view of properties (P3) and (P4) with \( \lambda = -1 \). Therefore,

\[
w_n = Au_n - \rho_n, \quad (10)
\]

where \( w_n \in \partial \Phi(u_n) \) is the element of minimal norm given in (8) and \( \rho_n \in \partial \bar{\Psi}(u_n) \subset \partial \Psi(u_n) \subset Y^* \). Now, since (9) and the compact embedding \( X \subset Y \) imply that \( u_n \to \hat{u} \) in \( Y \), it is not hard to see from the definition of \( \partial \Psi \) that the set \( \bigcup_n \partial \Psi(u_n) \subset Y^* \) is bounded. Therefore, the set \( \{ \rho_n \} \subset Y^* \) is
bounded and, since the embedding $Y^* \subset X^*$ is compact, we obtain for a subsequence (still labelled $\rho_n$) and some $\hat{\rho} \in X^*$ that

$$\rho_n \to \hat{\rho} \quad \text{in } X^*. \quad (11)$$

It follows from (10), (11), and the fact that $w_n \to 0$ in $X^*$ that

$$u_n = A^{-1}w_n + A^{-1}\rho_n \to 0 + A^{-1}\hat{\rho} \quad \text{in } X.$$ 

We have shown that $u_n \to \hat{u} = A^{-1}\hat{\rho}$ in $X$, so that $\Phi$ satisfies $(PS)_c$. The proof is complete.

REFERENCES

15. P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, in "CBMS Regional Conference, Miami, 1983."