Basic deformation theory of smooth formal schemes

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A R T I C L E   I N F O

A B S T R A C T

We provide the main results of a deformation theory of smooth formal schemes as defined in [L. Alonso Tarrío, A. Jeremías López, M. Pérez Rodríguez, Infinitesimal lifting and Jacobi criterion for smoothness on formal schemes, Comm. Algebra 35 (2007) 1341–1367]. Smoothness is defined by the local existence of infinitesimal liftings. Our first result is the existence of an obstruction in a certain Ext^1 group whose vanishing guarantees the existence of global liftings of morphisms. Next, given a smooth morphism $f_0 \colon \mathfrak{X}_0 \to \mathfrak{Y}_0$ of noetherian formal schemes and a closed immersion $\mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}$ given by a square zero ideal $\mathcal{I}$, we prove that the set of isomorphism classes of smooth formal schemes lifting $\mathfrak{X}_0$ over $\mathfrak{Y}$ is classified by $\mathrm{Ext}^1(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^* \mathcal{I})$ and that there exists an element in $\mathrm{Ext}^2(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^* \mathcal{I})$ which vanishes if and only if there exists a smooth formal scheme lifting $\mathfrak{X}_0$ over $\mathfrak{Y}$.

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0. Introduction

We provide here a further step in the program of the study of infinitesimal conditions in the category of formal schemes developed, among others, in the recent papers [2,3]. These previous works have systematically studied the infinitesimal conditions of locally noetherian formal schemes together with a hypothesis of finiteness, namely, the pseudo-finite type condition. In [2], the fundamental properties of the infinitesimal conditions of usual schemes are generalized to formal schemes. One of the main tools is the sheaf of differentials, which is coherent for a pseudo-finite type map of formal schemes. The latter is concentrated on the study of properties that are noteworthy in the category of formal schemes, obtaining a structure theorem for smooth morphisms and focusing on the relationship between the infinitesimal conditions of a map of formal schemes and those of the underlying maps of usual schemes. We have to mention that some basics of smoothness of formal schemes have also been studied by Yekutieli in [12] under the assumption that the base of the map is a usual noetherian scheme, and in Nayak's thesis for essentially pseudo-finite type maps, whose results have been included in [10].

This background motivates our interest of obtaining a deformation theory in the context of locally noetherian formal schemes. This needs the development of a suitable version of the cotangent complex. The problem is difficult because it involves the use of the derived category of complexes with coherent cohomology associated to a formal scheme, whose behavior is not straightforward, as is clear from looking at [1]. We concentrate here on the case of smooth morphisms—a particular situation that arises quite often. The problem consists in constructing morphisms that extend a given morphism over a smooth formal scheme to a base which is an "infinitesimal neighborhood" of the original. Questions of existence and uniqueness should be analyzed. We want to express the answer via cohomological invariants that are explicitly computed using the Čech complex. Another group of questions that we treat are the construction of formal schemes over an infinitesimal neighborhood of the base lifting a given relative formal scheme. The existence of such lifting will be controlled by an element belonging to a 2nd-order cohomology group. We prefer to use the more down-to-earth Čech viewpoint, which has the minor drawback of requiring separateness, but which suffices for a large class of applications. Although our exposition generalizes the well-known analogous statements for smooth schemes (*cf.* [7, III] and [4, VII, Section 1]), we have

not been able to deduce from them, even in the case of a map of formal schemes such that the underlying morphisms of usual schemes are all smooth (see [3]). For our argument, we require main results related to smoothness of formal schemes such as the universal property of the module of differentials [2, Theorem 3.5], some lifting property [2, Proposition 2.3] and the matrix Jacobian criterion for the affine formal disc [3, Corollary 5.13]. We expect that our results would be applied to the cohomological study of singular varieties.

Let us describe briefly the organization of this paper. The first section deals with preliminary material, pointing to precise references in the literature. The second section treats the case of global lifting of smooth morphisms. We prove that the obstruction to the existence of a global lifting lies in a H^1 group.

The setup for the remaining sections is a smooth morphism $f_0 \colon \mathfrak{X}_0 \to \mathfrak{Y}_0$ of noetherian formal schemes and a closed immersion $\mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}$ given by a square zero ideal $\mathcal{I}$. We deal first with the uniqueness of a lifting of *smooth formal schemes*. We prove that the set of isomorphism classes lifting $\mathfrak{X}_0$ over $\mathfrak{Y}$ is classified by $H^1(\mathfrak{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$, in the sense that they form an affine space over this module. In the last section, we study the existence of liftings of smooth formal schemes. There exists an obstruction, lying in $H^2(\mathfrak{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$, whose vanishing characterizes the existence of a smooth formal scheme lifting $\mathfrak{X}_0$ over $\mathfrak{Y}$.

All the results in Sections 2–4 generalize the corresponding results in the category of schemes. We have followed the outline for this case given in [9, pp. 111–113].

1. Preliminaries

We denote by NFS the category of locally noetherian formal schemes together with morphisms of formal schemes. The affine noetherian formal schemes are a full subcategory of NFS, denoted NFS$_{\mathbf{af}}$.

We assume the basics of the theory of formal schemes as explained in [6, Section 10]. Also, this work rests on the theory of smoothness in NFS as studied in the papers [2,3].

1.1. Let $\mathfrak{X}$ be in NFS. If $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$ is a coherent ideal, $\mathcal{X}'$ the corresponding closed subset and $(\mathcal{X}', (\mathcal{O}_{\mathfrak{X}}/\mathcal{I})|_{\mathcal{X}'})$ the induced formal scheme on it, then we say that $\mathcal{X}'$ is the *closed (formal) subscheme* of $\mathfrak{X}$ defined by $\mathcal{I}$. A morphism $f : \mathfrak{Z} \to \mathfrak{X}$ is a *closed immersion* if there exists a closed subset $\mathfrak{Y} \subset \mathfrak{X}$ such that f factors as $\mathfrak{Z} \xrightarrow{g} \mathfrak{Y} \hookrightarrow \mathfrak{X}$ where g is an isomorphism [6, Section 10.14.].

1.2. Given $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism in NFS and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an ideal of definition, there exists an ideal of definition $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ (see [6, (10.5.4) and (10.6.10)]). The map f induces the morphism of locally noetherian (usual) schemes $f_0 \colon (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}) \to (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K})$ (see [6, (10.5.6)]). The morphism f is of *pseudo-finite type* [1, p. 7] (*separated* [6, Section 10.15] and [1, 1.2.2]) if for any such pair of ideals the induced morphism of schemes, f_0, is of finite type (separated). A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is of *finite type* if it is adic and of pseudo-finite type [6, (10.13.1)].

1.3. A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in NFS is *smooth (unramified, étale)* [2, Definitions 2.1 and 2.6] if it is of pseudo-finite type and satisfies the following lifting condition:

For all affine $\mathfrak{Y}$-schemes Z and for each closed subscheme $T \hookrightarrow Z$ given by a square zero ideal $\mathcal{I} \subset \mathcal{O}_Z$, the induced map

$$\mathrm{Hom}_{\mathfrak{Y}}(Z, \mathfrak{X}) \longrightarrow \mathrm{Hom}_{\mathfrak{Y}}(T, \mathfrak{X})$$

is surjective (injective, bijective; respectively).

1.4. Given $f : \mathfrak{X} \to \mathfrak{Y}$ in NFS the *differential pair of $\mathfrak{X}$ over $\mathfrak{Y}$*, $(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \widehat{d}_{\mathfrak{X}/\mathfrak{Y}})$, is locally given by $((\widehat{\Omega}^1_{A/B})^{\triangle}, \mathcal{O}_{\mathfrak{U}} = A^{\triangle} \xrightarrow{\text{via } \widehat{d}_{A/B}} (\widehat{\Omega}^1_{A/B})^{\triangle})$ for all open subsets $\mathfrak{U} = \mathrm{Spf}(A) \subset \mathfrak{X}$ and $\mathfrak{V} = \mathrm{Spf}(B) \subset \mathfrak{Y}$ with $f(\mathfrak{U}) \subset \mathfrak{V}$ The $\mathcal{O}_{\mathfrak{X}}$-module $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ is called the *module of 1-differentials of $\mathfrak{X}$ over $\mathfrak{Y}$* and the continuous $\mathfrak{Y}$-derivation $\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}$ is called the *canonical derivation of $\mathfrak{X}$ over $\mathfrak{Y}$*. The basic properties of the differential pair in NFS are treated, for instance, in [2, Section 3].

1.5. Let $\mathfrak{Y} = \mathrm{Spf}(A)$ be in NFS$_{\mathbf{af}}$, $\mathbf{T} = T_1, T_2, \ldots, T_r$ and $\mathbf{Z} = Z_1, Z_2, \ldots, Z_s$ finite numbers of indeterminates and $\mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}}} = \mathrm{Spf}(A\{\mathbf{T}\}[[\mathbf{Z}]])$ (*cf.* [2, Example 1.6]). Then $\widehat{\Omega}^1_{\mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}}}/\mathfrak{Y}} = (\widehat{\Omega}^1_{A\{\mathbf{T}\}[[\mathbf{Z}]]/A})^{\triangle}$ and in [2, 3.14] it is shown that $\widehat{\Omega}^1_{A\{\mathbf{T}\}[[\mathbf{Z}]]/A}$ is a free $A\{\mathbf{T}\}[[\mathbf{Z}]]$-module, with basis $\{\widehat{d}T_1, \ldots, \widehat{d}T_r, \widehat{d}Z_1, \ldots, \widehat{d}Z_s\}$ where $\widehat{d} = \widehat{d}_{A\{\mathbf{T}\}[[\mathbf{Z}]]/A}$. Furthermore, given $g \in A\{\mathbf{T}\}[[\mathbf{Z}]]$ it holds that:

$$\widehat{d}g = \sum_{i=1}^{r} \frac{\partial g}{\partial T_i} \widehat{d}T_i + \sum_{j=1}^{s} \frac{\partial g}{\partial Z_j} \widehat{d}Z_j.$$

1.6. Given $f : \mathfrak{X} = \mathrm{Spf}(A) \to \mathfrak{Y} = \mathrm{Spf}(B)$ a morphism in NFS$_{\mathbf{af}}$ of pseudo-finite type, there exists a factorization of f as

$$\mathfrak{X} = \mathrm{Spf}(A) \xhookrightarrow{j} \mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}}} = \mathrm{Spf}(B\{\mathbf{T}\}[[\mathbf{Z}]]) \xrightarrow{p} \mathfrak{Y} = \mathrm{Spf}(B)$$

with j a closed immersion given by an ideal $\mathcal{I} = I^\triangle \subset \mathcal{O}_{\mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}}}}$ where $I = \langle g_1, g_2, \ldots, g_k \rangle \subset B\{\mathbf{T}\}[[\mathbf{Z}]]$ and p the natural projection [2, Proposition 1.7]. The *Jacobian matrix of $\mathfrak{X}$ over $\mathfrak{Y}$ at x* [3, 5.12] is defined as

$$\mathrm{Jac}_{\mathfrak{X}/\mathfrak{Y}}(x) = \begin{pmatrix} \dfrac{\partial g_1}{\partial T_1}(x) & \cdots & \dfrac{\partial g_1}{\partial T_r}(x) & \dfrac{\partial g_1}{\partial Z_1}(x) & \cdots & \dfrac{\partial g_1}{\partial Z_s}(x) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial g_k}{\partial T_1}(x) & \cdots & \dfrac{\partial g_k}{\partial T_r}(x) & \dfrac{\partial g_k}{\partial Z_1}(x) & \cdots & \dfrac{\partial g_k}{\partial Z_s}(x) \end{pmatrix},$$

where for $u \in \{T_1, \ldots, T_r, Z_1, \ldots, Z_s\}$, $\frac{\partial g_i}{\partial u}(x)$ denotes the image of $\frac{\partial g_i}{\partial u} \in B\{\mathbf{T}\}[[\mathbf{Z}]]$ in $k(x)$, for all $i = 1, 2, \ldots, k$.

1.7. We will use the calculus of Čech cohomology, which forces to impose the separation hypothesis each time we need cohomology of degree greater than or equal to 2. Moreover, in that context the Čech cohomology agrees with the (usual) derived functor cohomology. This follows from [8, Ch. III, Exercise 4.11] in view of [1, Corollary 3.1.8].

2. Lifting of morphisms

2.1. Consider a commutative diagram of morphisms of pseudo-finite type in NFS

$$\begin{array}{ccc} \mathfrak{Z}_0 & \overset{i}{\hookrightarrow} & \mathfrak{Z} \\ {\scriptstyle u_0}\downarrow & \swarrow {\scriptstyle f} & \downarrow \\ \mathfrak{X} & \overset{}{\longrightarrow} & \mathfrak{Y} \end{array}$$

(2.1.1)

where $\mathfrak{Z}_0 \hookrightarrow \mathfrak{Z}$ is a closed formal subscheme given by a square zero ideal $\mathcal{I} \subset \mathcal{O}_{\mathfrak{Z}}$. A morphism $u : \mathfrak{Z} \to \mathfrak{X}$ is a *lifting of u_0 over $\mathfrak{Y}$* if it makes this diagram commutative. For instance, if f is étale, then for all such morphisms u_0, there always exists a unique lifting by [2, Corollary 2.5].

So the basic question is: When can we guarantee uniqueness and existence of a lifting for a $\mathfrak{Y}$-morphism $u_0 : \mathfrak{Z}_0 \to \mathfrak{X}$? In 2.2 it is shown that if $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^* \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}) = 0$, then the lifting is unique. Proposition 2.3 establishes that, whenever f is smooth, there exists an obstruction in $\mathrm{Ext}^1_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^* \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I})$ to the existence of such a lifting.

Observe that in the diagram above, $i : \mathfrak{Z}_0 \to \mathfrak{Z}$ is the identity as topological map and, therefore, we may identify $i_* \mathcal{O}_{\mathfrak{Z}_0} \equiv \mathcal{O}_{\mathfrak{Z}_0}$. Through this identification, we have that the ideal $\mathcal{I}$ is a $\mathcal{O}_{\mathfrak{Z}_0}$-module and $\mathcal{I} = i_* \mathcal{I}$.

2.2. Let us continue to consider the situation depicted in diagram (2.1.1). If there exists a lifting $u : \mathfrak{Z} \to \mathfrak{X}$ of u_0 over $\mathfrak{Y}$, then we claim that the set of liftings of u_0 over $\mathfrak{Y}$ is an affine space via

$$\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, u_{0*}\mathcal{I}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^* \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}).$$

Indeed, $u_{0*}\mathcal{I} = u_*\mathcal{I}$ and in view of this identification, from [5, (**0**, 20.1.1), (**0**, 20.3.1) and (**0**, 20.3.2)] we deduce that if $v : \mathfrak{Z} \to \mathfrak{X}$ is another lifting of u_0 over $\mathfrak{Y}$, the morphism $\mathcal{O}_{\mathfrak{X}} \xrightarrow{u^\natural - v^\natural} u_{0*}\mathcal{I}$ is a continuous $\mathfrak{Y}$-derivation. By [2, Lemma 3.6 and Theorem 3.5], there exists a unique morphism of $\mathcal{O}_{\mathfrak{X}}$-modules $\phi : \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to u_{0*}\mathcal{I}$ such that $\phi \circ \widehat{d}_{\mathfrak{X}/\mathfrak{Y}} = u^\natural - v^\natural$. On the other hand, given a morphism of $\mathcal{O}_{\mathfrak{X}}$-modules $\phi : \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to u_{0*}\mathcal{I}$, the map $v^\natural := u^\natural + \phi \circ \widehat{d}_{\mathfrak{X}/\mathfrak{Y}}$ defines another morphism $v : \mathfrak{Z} \to \mathfrak{X}$ that is a lifting of u_0.
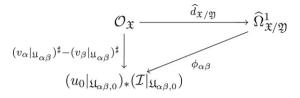
Moreover, given $r : \mathfrak{X} \to \mathfrak{X}'$ a $\mathfrak{Y}$-morphism of pseudo-finite type in NFS induces a morphism of $\mathcal{O}_{\mathfrak{X}}$-Modules $r^* \widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ which is compatible with the canonical derivation (*cf.* [2, Proposition 3.7]). Therefore, any lifting of u_0 over $\mathfrak{Y}$ leads to a lifting of $r \circ u_0$ over $\mathfrak{Y}$ preserving compatibility with the natural map $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^* \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^* r^* \widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}}, \mathcal{I})$.

Remark. Using the language of torsor theory, 2.2 says that the sheaf on $\mathfrak{Z}_0$ which associates to the open subset $\mathfrak{U}_0 \subset \mathfrak{Z}_0$ the set of liftings $\mathfrak{U} \to \mathfrak{X}$ of $u_0|_{\mathfrak{U}_0}$ over $\mathfrak{Y}$ – where $\mathfrak{U} \subset \mathfrak{Z}$ is the open subset corresponding to $\mathfrak{U}_0$ – is a pseudo torsor over $\mathcal{H}om_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^* \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I})$ which is functorial on $\mathfrak{X}$.

When can we guarantee for a diagram like (2.1.1) the existence of a lifting of u_0 over $\mathfrak{Y}$? In [2, Proposition 2.3], we have shown that if f is smooth and $\mathfrak{Z}$ is in NFS$_{\mathbf{af}}$, then there exists lifting of u_0 over $\mathfrak{Y}$. So, the issue amounts to patching local data to obtain global data.

Proposition 2.3. *Consider the commutative diagram (2.1.1) where $f : \mathfrak{X} \to \mathfrak{Y}$ is a smooth morphism. Then there exists an element (usually called the obstruction) $c_{u_0} \in \mathrm{Ext}^1_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^* \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I})$ such that: $c_{u_0} = 0$ if and only if there exists $u : \mathfrak{Z} \to \mathfrak{X}$ a lifting of u_0 over $\mathfrak{Y}$.*

Proof. Let $\{\mathfrak{U}_\alpha\}_{\alpha \in L}$ be an affine open covering of $\mathfrak{Z}$ and $\mathfrak{U} = \{\mathfrak{U}_{\alpha,0}\}_{\alpha \in L}$ the corresponding affine open covering of $\mathfrak{Z}_0$ such that, for all α, $\mathfrak{U}_{\alpha,0} \hookrightarrow \mathfrak{U}_\alpha$ is a closed immersion in $\mathsf{NFS_{af}}$ given by the square zero ideal $\mathcal{I}|_{\mathfrak{U}_\alpha}$. Since f is a smooth morphism, [2, Proposition 2.3] implies that for all α there exists a lifting $v_\alpha : \mathfrak{U}_\alpha \to \mathfrak{X}$ of $u_0|_{\mathfrak{U}_{\alpha,0}}$ over $\mathfrak{Y}$. For all couples of indexes α, β such that $\mathfrak{U}_{\alpha\beta} := \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta \neq \varnothing$, if we denote by $\mathfrak{U}_{\alpha\beta,0}$ the corresponding open formal subscheme of $\mathfrak{Z}_0$, from 2.2 we have that there exists a unique morphism of $\mathcal{O}_\mathfrak{X}$-modules $\phi_{\alpha\beta} : \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to (u_0|_{\mathfrak{U}_{\alpha\beta,0}})_*(\mathcal{I}|_{\mathfrak{U}_{\alpha\beta,0}})$ such that the following diagram

$$\begin{array}{ccc}
\mathcal{O}_\mathfrak{X} & \xrightarrow{\;\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}\;} & \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \\
{\scriptstyle (v_\alpha|_{\mathfrak{U}_{\alpha\beta}})^\sharp - (v_\beta|_{\mathfrak{U}_{\alpha\beta}})^\sharp}\Big\downarrow & \swarrow{\scriptstyle \phi_{\alpha\beta}} & \\
(u_0|_{\mathfrak{U}_{\alpha\beta,0}})_*(\mathcal{I}|_{\mathfrak{U}_{\alpha\beta,0}}) & &
\end{array}$$

commutes. Let $u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}|_{\mathfrak{U}_{\alpha\beta,0}} \to \mathcal{I}|_{\mathfrak{U}_{\alpha\beta,0}}$ be the morphism of $\mathcal{O}_{\mathfrak{U}_{\alpha\beta,0}}$-modules adjoint to $\phi_{\alpha\beta}$, which we continue to denote by $\phi_{\alpha\beta}$. The family of morphisms $\phi_{\mathfrak{U}_\bullet} := (\phi_{\alpha\beta})$ satisfies the cocycle condition; that is, for any α, β, γ such that $\mathfrak{U}_{\alpha\beta\gamma,0} := \mathfrak{U}_{\alpha,0} \cap \mathfrak{U}_{\beta,0} \cap \mathfrak{U}_{\gamma,0} \neq \varnothing$, we have that

$$\phi_{\alpha\beta}|_{\mathfrak{U}_{\alpha\beta\gamma,0}} - \phi_{\alpha\gamma}|_{\mathfrak{U}_{\alpha\beta\gamma,0}} + \phi_{\beta\gamma}|_{\mathfrak{U}_{\alpha\beta\gamma,0}} = 0 \tag{2.3.1}$$

so, $\phi_{\mathfrak{U}_\bullet} \in \check{Z}^1(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}))$. Moreover, its class

$$[\phi_{\mathfrak{U}_\bullet}] \in \check{H}^1(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}))$$

does not depend on the liftings $\{v_\alpha\}_{\alpha \in L}$. Indeed, for all arbitrary $\alpha \in L$, let $w_\alpha : \mathfrak{U}_\alpha \to \mathfrak{X}$ be a lifting of $u_0|_{\mathfrak{U}_{\alpha,0}}$ over $\mathfrak{Y}$ and let $\psi_{\mathfrak{U}_\bullet} := (\psi_{\alpha\beta}) \in \check{Z}^1(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}))$ be the corresponding cocycle defined as above. By 2.2, there exists a unique $\xi_\alpha \in \mathrm{Hom}_\mathfrak{X}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, (u_0|_{\mathfrak{U}_{\alpha,0}})_*(\mathcal{I}|_{\mathfrak{U}_{\alpha,0}}))$ such that $v_\alpha^\natural - w_\alpha^\natural = \xi_\alpha \circ \widehat{d}_{\mathfrak{X}/\mathfrak{Y}}$. Then for all couples of indexes α, β such that $\mathfrak{U}_{\alpha\beta} \neq \varnothing$ we have that $\psi_{\alpha\beta} = \phi_{\alpha\beta} + \xi_\beta|_{\mathfrak{U}_{\alpha\beta}} - \xi_\alpha|_{\mathfrak{U}_{\alpha\beta}}$. In other words, the cocycles $\psi_{\mathfrak{U}_\bullet}$ and $\phi_{\mathfrak{U}_\bullet}$ differ by a coboundary from which we conclude that $[\phi_{\mathfrak{U}_\bullet}] = [\psi_{\mathfrak{U}_\bullet}] \in \check{H}^1(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}))$. With an analogous argument, it is possible to prove that, given a refinement $\mathfrak{V}_\bullet$ of $\mathfrak{U}_\bullet$, we have $[\phi_{\mathfrak{U}_\bullet}] = [\phi_{\mathfrak{V}_\bullet}] \in \check{H}^1(\mathfrak{Z}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}))$.

We define [11, (5.4.15)]:

$$\begin{aligned}
c_{u_0} &:= [\phi_{\mathfrak{U}_\bullet}] \in \check{H}^1(\mathfrak{Z}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I})) \\
&= H^1(\mathfrak{Z}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I})).
\end{aligned}$$

Since f is smooth, [2, Proposition 4.8] implies that $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ is a locally free $\mathcal{O}_\mathfrak{X}$-module of finite rank, so,

$$c_{u_0} \in H^1(\mathfrak{Z}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I})) = \mathrm{Ext}^1(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}).$$

The element c_{u_0} is the obstruction to the existence of a lifting of u_0. If u_0 admits a lifting then it is clear that $c_{u_0} = 0$. Reciprocally, suppose that $c_{u_0} = 0$. From the family of morphisms $\{v_\alpha\}_{\alpha \in L}$ we are going to construct a collection of liftings $\{u_\alpha : \mathfrak{U}_\alpha \to \mathfrak{X}\}_{\alpha \in L}$ of $\{u_0|_{\mathfrak{U}_{\alpha,0}}\}_{\alpha \in L}$ over $\mathfrak{Y}$ that will patch into a morphism $u : \mathfrak{Z} \to \mathfrak{X}$. By hypothesis, there exists $\{\varphi_\alpha\}_{\alpha \in L} \in \check{C}^0(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}))$ such that for all couples of indexes α, β with $\mathfrak{U}_{\alpha\beta} \neq \varnothing$,

$$\varphi_\alpha|_{\mathfrak{U}_{\alpha\beta}} - \varphi_\beta|_{\mathfrak{U}_{\alpha\beta}} = \phi_{\alpha\beta}. \tag{2.3.2}$$

For all $\alpha \in L$, let $u_\alpha : \mathfrak{U}_\alpha \to \mathfrak{X}$ be the morphism that agrees with $u_0|_{\mathfrak{U}_{\alpha,0}}$ as a topological map and is given by

$$u_\alpha^\natural := v_\alpha^\natural - \varphi_\alpha \circ \widehat{d}_{\mathfrak{X}/\mathfrak{Y}}$$

as a map of topologically ringed spaces. By 2.2, we have that u_α is a lifting of $u_0|_{\mathfrak{U}_{\alpha,0}}$ over $\mathfrak{Y}$ for all α, and from (2.3.2) and (2.3.1) (for $\gamma = \beta$) we deduce that the morphisms $\{u_\alpha\}_{\alpha \in L}$ glue into a morphism $u : \mathfrak{Z} \to \mathfrak{X}$. □

2.4. Let $r : \mathfrak{X} \to \mathfrak{X}'$ be a $\mathfrak{Y}$-morphism of pseudo-finite type in NFS. From 2.2 and the last proof, it follows that the obstruction c_{u_0} leads to the obstruction $c_{r \circ u_0}$ through the natural map $\mathrm{Ext}^1_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{I}) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathfrak{Z}_0}}(u_0^*r^*\widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}}, \mathcal{I})$.

3. Lifting of smooth formal schemes: Uniqueness

Given a smooth morphism $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ and a closed immersion $\mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}$ defined by a square zero ideal $\mathcal{I}$, one can pose the following question: Suppose that there exists a smooth $\mathfrak{Y}$-formal scheme $\mathfrak{X}$ such that $\mathfrak{X} \times_\mathfrak{Y} \mathfrak{Y}_0 = \mathfrak{X}_0$. When is $\mathfrak{X}$ unique? We will answer this in the present section. It follows from Proposition 3.5 that if $\mathrm{Ext}^1(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}) = 0$, then $\mathfrak{X}$ is unique up to isomorphism.

3.1. Assume that $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ is a smooth morphism and $i\colon \mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}$ a closed immersion given by a square zero ideal $\mathcal{I} \subset \mathcal{O}_{\mathfrak{Y}}$, hence, $\mathfrak{Y}_0$ and $\mathfrak{Y}$ have the same underlying topological space. If there exists a smooth morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in NFS such that the diagram
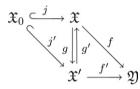
$$
\begin{array}{ccc}
\mathfrak{X}_0 & \xrightarrow{\;f_0\;} & \mathfrak{Y}_0 \\
{\scriptstyle j}\big\uparrow\rotatebox{90}{\subset} & & \big\uparrow \\
\mathfrak{X} & \xrightarrow{\;f\;} & \mathfrak{Y}
\end{array}
\tag{3.1.1}
$$

is cartesian we will say that $f : \mathfrak{X} \to \mathfrak{Y}$ is a *smooth lifting of $\mathfrak{X}_0$ over $\mathfrak{Y}$*.

Observe that, since f is flat [2, Proposition 4.8], then $j\colon \mathfrak{X}_0 \to \mathfrak{X}$ is a closed immersion given (up to isomorphism) by the square zero ideal $f^*\mathcal{I}$. The sheaf $\mathcal{I}$ is an $\mathcal{O}_{\mathfrak{Y}_0}$-module in a natural way, $f^*\mathcal{I}$ is a $\mathcal{O}_{\mathfrak{X}_0}$-module and it is clear that $f^*\mathcal{I}$ agrees with $f_0^*\mathcal{I}$ as an $\mathcal{O}_{\mathfrak{X}_0}$-module.

3.2. Denote by $\mathrm{Aut}_{\mathfrak{X}_0}(\mathfrak{X})$ the group of $\mathfrak{Y}$-automorphisms of $\mathfrak{X}$ that induce the identity on $\mathfrak{X}_0$. In particular, we have that $1_{\mathfrak{X}} \in \mathrm{Aut}_{\mathfrak{X}_0}(\mathfrak{X})$ and, therefore, by 2.2 there exists a bijection $\mathrm{Aut}_{\mathfrak{X}_0}(\mathfrak{X}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, j_* f_0^*\mathcal{I})$ defined using the map $g \in \mathrm{Aut}_{\mathfrak{X}_0}(\mathfrak{X}) \rightsquigarrow g^\natural - 1_{\mathfrak{X}}^\natural \in \mathrm{Dercont}_{\mathfrak{Y}}(\mathcal{O}_{\mathfrak{X}}, j_* f_0^*\mathcal{I})$.

3.3. If $\mathfrak{X}_0$ is in $\mathrm{NFS}_{\mathbf{af}}$ and $\mathfrak{X}_0 \overset{j'}{\hookrightarrow} \mathfrak{X}' \overset{f'}{\to} \mathfrak{Y}$ is another smooth lifting of $\mathfrak{X}_0$ over $\mathfrak{Y}$, then there exists a $\mathfrak{Y}$-isomorphism $g : \mathfrak{X} \xrightarrow{\sim} \mathfrak{X}'$ such that $g|_{\mathfrak{X}_0} = j'$. Indeed, by Proposition 2.3 and [1, Corollary 3.1.8] there are morphisms $g : \mathfrak{X} \to \mathfrak{X}'$, $g' : \mathfrak{X}' \to \mathfrak{X}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathfrak{X}_0 & \overset{j}{\hookrightarrow} & \mathfrak{X} \\
{\scriptstyle j'}\big\downarrow & {\scriptstyle g}\big\Updownarrow{\scriptstyle g'} & \big\downarrow{\scriptstyle f} \\
& \mathfrak{X}' \xrightarrow{\;f'\;} & \mathfrak{Y}
\end{array}
$$

From 2.2 and 3.2, it is easy to deduce that $g' \circ g \in \mathrm{Aut}_{\mathfrak{X}_0}(\mathfrak{X})$, $g \circ g' \in \mathrm{Aut}_{\mathfrak{X}_0}(\mathfrak{X}')$, therefore g is an isomorphism.
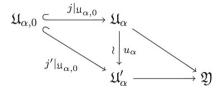
3.4. In the setting of 3.3, the set of $\mathfrak{Y}$-isomorphisms of $\mathfrak{X}$ onto $\mathfrak{X}'$ that make the diagram commutative is an affine space over $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I})$ (or, equivalently over $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}'}}(\widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}}, j'_* f_0^*\mathcal{I})$, by adjunction). Indeed, assume that $g : \mathfrak{X} \to \mathfrak{X}'$ and $h : \mathfrak{X} \to \mathfrak{X}'$ are two such $\mathfrak{Y}$-isomorphisms. From 2.2, there exists a unique homomorphism of $\mathcal{O}_{\mathfrak{X}'}$-modules $\phi : \widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}} \to j'_* f_0^*\mathcal{I}$ such that $g^\natural - h^\natural = \phi \circ \widehat{d}_{\mathfrak{X}'/\mathfrak{Y}}$. Reciprocally, if

$$
\phi \in \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}'}}(\widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}}, j'_* f_0^*\mathcal{I})
$$

and $g : \mathfrak{X} \to \mathfrak{X}'$ is a $\mathfrak{Y}$-isomorphism with $g|_{\mathfrak{X}_0} = j'$, the $\mathfrak{Y}$-morphism $h : \mathfrak{X} \to \mathfrak{X}'$ defined by $h^\natural = g^\natural + \phi \circ \widehat{d}_{\mathfrak{X}'/\mathfrak{Y}}$, which as topological space map is the identity, is an isomorphism. Indeed, using 2.2 and 3.2, it follows that $h \circ g^{-1} \in \mathrm{Aut}_{\mathfrak{X}_0}(\mathfrak{X}')$ and $g^{-1} \circ h \in \mathrm{Aut}_{\mathfrak{X}_0}(\mathfrak{X})$, therefore h is an isomorphism.

Proposition 3.5. *Let $\mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}$ be a closed immersion in* NFS *defined by a square zero ideal $\mathcal{I} \subset \mathcal{O}_{\mathfrak{Y}}$ and $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ a smooth morphism in* NFS *and suppose that there exists a smooth lifting of $\mathfrak{X}_0$ over $\mathfrak{Y}$. Then the set of isomorphism classes of smooth liftings of $\mathfrak{X}_0$ over $\mathfrak{Y}$ is an affine space over $\mathrm{Ext}^1(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I})$.*

Proof. Let $\mathfrak{X}_0 \overset{j}{\hookrightarrow} \mathfrak{X} \overset{f}{\to} \mathfrak{Y}$ and $\mathfrak{X}_0 \overset{j'}{\hookrightarrow} \mathfrak{X}' \overset{f'}{\to} \mathfrak{Y}$ be two smooth liftings over $\mathfrak{Y}$. Given an affine open covering $\mathfrak{U}_{\bullet} = \{\mathfrak{U}_{\alpha,0}\}_{\alpha \in L}$ of $\mathfrak{X}_0$, let $\{\mathfrak{U}_\alpha\}_{\alpha \in L}$ and $\{\mathfrak{U}'_\alpha\}_{\alpha \in L}$ be the corresponding affine open coverings of $\mathfrak{X}$ and $\mathfrak{X}'$, respectively. From 3.3, for each $\alpha \in L$, there exists an isomorphism of $\mathfrak{Y}$-formal schemes $u_\alpha : \mathfrak{U}_\alpha \xrightarrow{\sim} \mathfrak{U}'_\alpha$ such that the following diagram

$$
\begin{array}{ccc}
\mathfrak{U}_{\alpha,0} & \overset{j|_{\mathfrak{U}_{\alpha,0}}}{\hookrightarrow} & \mathfrak{U}_\alpha \\
{\scriptstyle j'|_{\mathfrak{U}_{\alpha,0}}}\big\downarrow & & \big\downarrow{\scriptstyle \wr}\,{\scriptstyle u_\alpha} \\
& \mathfrak{U}'_\alpha \xrightarrow{\qquad\qquad} & \mathfrak{Y}
\end{array}
$$

is commutative. By 3.4, for all couples of indexes α, β such that $\mathfrak{U}_{\alpha\beta,0} := \mathfrak{U}_{\alpha,0} \cap \mathfrak{U}_{\beta,0} \neq \varnothing$, if $\mathfrak{U}_{\alpha\beta} := \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta$, the difference between $u_\alpha^\natural|_{\mathfrak{U}_{\alpha\beta}}$ and $u_\beta^\natural|_{\mathfrak{U}_{\alpha\beta}}$ is measured by a homomorphism of $\mathcal{O}_{\mathfrak{U}_{\alpha\beta,0}}$-modules $\phi_{\alpha\beta} : \widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}|_{\mathfrak{U}_{\alpha\beta,0}} \to (f_0^*\mathcal{I})|_{\mathfrak{U}_{\alpha\beta,0}}$. Then

$\phi_{\mathfrak{U}_\bullet} := \{\phi_{\alpha\beta}\} \in \check{C}^1(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$ has the property that for all α, β, γ such that $\mathfrak{U}_{\alpha\beta\gamma,0} := \mathfrak{U}_{\alpha,0} \cap \mathfrak{U}_{\beta,0} \cap \mathfrak{U}_{\gamma,0} \neq \varnothing$, the cocycle condition

$$\phi_{\alpha\beta}|_{\mathfrak{U}_{\alpha\beta\gamma,0}} - \phi_{\alpha\gamma}|_{\mathfrak{U}_{\alpha\beta\gamma,0}} + \phi_{\beta\gamma}|_{\mathfrak{U}_{\alpha\beta\gamma,0}} = 0 \tag{3.5.1}$$

holds and, therefore, $\phi_{\mathfrak{U}_\bullet} \in \check{Z}^1(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$. The homology class of the element

$$c_{\mathfrak{U}_\bullet} := [\phi_{\mathfrak{U}_\bullet}] \in \check{H}^1(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$$

does not depend on the choice of the isomorphisms $\{u_\alpha\}_{\alpha\in L}$. Indeed, consider another collection of $\mathfrak{Y}$-isomorphisms $\{v_\alpha : \mathfrak{U}_\alpha \xrightarrow{\sim} \mathfrak{U}'_\alpha\}_{\alpha\in L}$ such that, for all $\alpha \in L$, $v_\alpha \circ j|_{\mathfrak{U}_{\alpha,0}} = j'|_{\mathfrak{U}_{\alpha,0}}$ and let $\psi_{\mathfrak{U}_\bullet} := \{\psi_{\alpha\beta}\}$ be the corresponding element in $\check{Z}^1(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$ defined in the same way as $\phi_{\mathfrak{U}_\bullet}$ from $\{u_\alpha\}_{\alpha\in L}$. Using 3.4, we obtain a collection $\{\xi_\alpha\}_{\alpha\in L} \in \check{C}^0(\mathfrak{U}_\bullet, \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$ satisfying that, for all $\alpha \in L$, their adjoints (for which we will use the same notation) are such that $u_\alpha - v_\alpha = \xi_\alpha \circ \widehat{d}_{\mathfrak{U}'_\alpha/\mathfrak{Y}}$, therefore, $[\phi_{\mathfrak{U}_\bullet}] = [\psi_{\mathfrak{U}_\bullet}] \in \check{H}^1(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$. If $\mathfrak{V}_\bullet$ is an affine open refinement of $\mathfrak{U}_\bullet$, by what we have already seen, we deduce that $c_{\mathfrak{U}_\bullet} = c_{\mathfrak{V}_\bullet}$. Let us define [11, (5.4.15)]

$$\begin{aligned} c := [\phi_{\mathfrak{U}_\bullet}] &\in \check{H}^1(\mathfrak{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I})) \\ &= H^1(\mathfrak{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I})) \\ &= \mathrm{Ext}^1(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}). \end{aligned}$$

Conversely, let $f: \mathfrak{X} \to \mathfrak{Y}$ be a smooth lifting of $\mathfrak{X}_0$ and consider $c \in \mathrm{Ext}^1(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I})$. Given $\mathfrak{U}_\bullet = \{\mathfrak{U}_{\alpha,0}\}_{\alpha\in L}$ an affine open covering of $\mathfrak{X}_0$, take $\{\mathfrak{U}_\alpha\}_{\alpha\in L}$ the corresponding affine open covering in $\mathfrak{X}$ and

$$\phi_{\mathfrak{U}_\bullet} = (\phi_{\alpha\beta}) \in \check{Z}^1(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$$

such that $c = [\phi_{\mathfrak{U}_\bullet}]$. For each couple of indexes α, β such that $\mathfrak{U}_{\alpha\beta} = \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta \neq \varnothing$, let us consider the morphism $u_{\alpha\beta} : \mathfrak{U}_{\alpha\beta} \to \mathfrak{U}_{\alpha\beta}$ which is the identity as topological map and is defined by $u_{\alpha\beta}^\natural := 1_{\mathfrak{U}_{\alpha\beta}}^\natural + \phi_{\alpha\beta} \circ \widehat{d}_{\mathfrak{U}_{\alpha\beta}/\mathfrak{Y}}$, as a map of topologically ringed spaces, where again $\phi_{\alpha\beta}$ also denotes its adjoint $\phi_{\alpha\beta}: \widehat{\Omega}^1_{\mathfrak{U}_{\alpha\beta}/\mathfrak{Y}} \to (j_*f_0^*\mathcal{I})|_{\mathfrak{U}_{\alpha\beta}}$, such that the following hold:

- $u_{\alpha\beta} \in \mathrm{Aut}_{\mathfrak{U}_{\alpha\beta,0}}(\mathfrak{U}_{\alpha\beta})$ (by 3.2);
- $u_{\alpha\beta}|_{\mathfrak{U}_{\alpha\beta\gamma}} \circ u_{\alpha\gamma}^{-1}|_{\mathfrak{U}_{\alpha\beta\gamma}} \circ u_{\beta\gamma}|_{\mathfrak{U}_{\alpha\beta\gamma}} = 1_{\mathfrak{U}_{\alpha\beta\gamma}}$, for any α, β, γ such that $\mathfrak{U}_{\alpha\beta\gamma} := \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta \cap \mathfrak{U}_\gamma \neq \varnothing$ (because $\{\phi_{\alpha\beta}\}$ satisfies the cocycle condition (3.5.1));
- $u_{\alpha\alpha} = 1_{\mathfrak{U}_\alpha}$ and $u_{\alpha\beta}^{-1} = u_{\beta\alpha}$.

Then the $\mathfrak{Y}$-formal schemes $\mathfrak{U}_\alpha$ glue into a smooth lifting $f' : \mathfrak{X} \to \mathfrak{Y}$ of $\mathfrak{X}_0$ through the morphisms $\{u_{\alpha\beta}\}$, since the morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is compatible with the family of isomorphisms $\{u_{\alpha\beta}\}$.

We leave the verification that these correspondences are mutually inverse to the reader. $\quad\square$

Remark. Proposition 3.5 can be rephrased in the language of torsor theory as follows: The sheaf on $\mathfrak{X}_0$ that associates to each open $\mathfrak{U}_0 \subset \mathfrak{X}_0$ the set of isomorphism classes of smooth liftings of $\mathfrak{U}_0$ over $\mathfrak{Y}$ is a pseudo torsor over $\mathrm{Ext}^1(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I})$.

Remark. With the hypothesis of Proposition 3.5, if $\mathfrak{X}_0$ is in NFS$_{\mathsf{af}}$, we have that $H^1(\mathfrak{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I})) = 0$ (cf. [1, Corollary 3.1.8]) and, therefore, there exists a unique isomorphism class of liftings of $\mathfrak{X}_0$ over $\mathfrak{Y}$.

4. Lifting of smooth formal schemes: Existence

We continue considering the set-up of the previous section, namely, a smooth morphism $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ and a closed immersion $\mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}$ defined by a square zero ideal $\mathcal{I}$. Let us pose the following question: Does there exist a smooth $\mathfrak{Y}$-formal scheme $\mathfrak{X}$ such that it holds that $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}_0 = \mathfrak{X}_0$? We will give the following local answer: for all $x \in \mathfrak{X}_0$, there exists an open $\mathfrak{U}_0 \subset \mathfrak{X}_0$ with $x \in \mathfrak{U}_0$ and a locally noetherian smooth formal scheme $\mathfrak{U}$ over $\mathfrak{Y}$ such that $\mathfrak{U}_0 = \mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{Y}_0$ (see Proposition 4.1). Globally, Theorem 4.2 provides an element in $\mathrm{Ext}^2(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I})$ whose vanishing is equivalent to the existence of such an $\mathfrak{X}$. In particular, whenever $\mathfrak{X}_0$ is in NFS$_{\mathsf{af}}$ Corollary 4.3 asserts the existence of $\mathfrak{X}$.

Proposition 4.1. *Let us consider in* NFS *a closed immersion* $\mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}$ *and a smooth morphism* $f' : \mathfrak{X} \to \mathfrak{Y}$. *For all points* $x \in \mathfrak{X}$, *there exists an open subset* $\mathfrak{U} \subset \mathfrak{X}$ *with* $x \in \mathfrak{U}$ *and a locally noetherian formal scheme* $\mathfrak{U}$ *smooth over* $\mathfrak{Y}$ *such that* $\mathfrak{U} = \mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{Y}_0$.
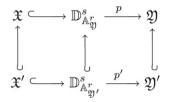
Proof. Since it is a local question, we may assume that the morphisms $\mathfrak{Y}' = \mathrm{Spf}(B') \hookrightarrow \mathfrak{Y} = \mathrm{Spf}(B)$ and $f' : \mathfrak{X}' = \mathrm{Spf}(A') \to \mathfrak{Y}' = \mathrm{Spf}(B')$ are in $\mathsf{NFS_{af}}$ and that there exist $r, s \in \mathbb{N}$ such that the morphism f' factors as

$$\mathfrak{X}' = \mathrm{Spf}(A') \hookrightarrow \mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}'}} = \mathrm{Spf}(B'\{\mathbf{T}\}[[\mathbf{Z}]]) \xrightarrow{p'} \mathfrak{Y}' = \mathrm{Spf}(B'),$$

where $\mathfrak{X}' \hookrightarrow \mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}'}}$ is a closed subscheme given by an ideal $\mathcal{I}' = (I')^{\triangle} \subset \mathcal{O}_{\mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}'}}}$, with $I' \subset B'\{\mathbf{T}\}[[\mathbf{Z}]]$ an ideal, and p' is the canonical projection (see 1.6). Fix $x \in \mathfrak{X}'$. As f' is smooth, by the matrix Jacobian criterion for the affine formal space and the affine formal disc ([3, Corollary 5.13]), we have that there exists $\{g'_1, g'_2, \ldots, g'_l\} \subset I'$ such that:

$$\langle g'_1, g'_2, \ldots, g'_l \rangle \mathcal{O}_{\mathfrak{X}',x} = I'_x \quad \text{and} \quad \mathrm{rg}(\mathrm{Jac}_{\mathfrak{X}'/\mathfrak{Y}'}(x)) = l. \tag{4.1.1}$$
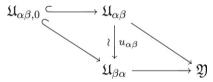
Replacing, if necessary, $\mathfrak{X}'$ by a smaller affine open neighborhood of x, we may assume that $I' = \langle g'_1, g'_2, \ldots, g'_l \rangle$. Let $\{g_1, g_2, \ldots, g_l\} \subset B\{\mathbf{T}\}[[\mathbf{Z}]]$ be such that $g_i \in B\{\mathbf{T}\}[[\mathbf{Z}]] \rightsquigarrow g'_i \in B'\{\mathbf{T}\}[[\mathbf{Z}]]$ through the continuous homomorphism of rings $B\{\mathbf{T}\}[[\mathbf{Z}]] \twoheadrightarrow B'\{\mathbf{T}\}[[\mathbf{Z}]]$ induced by $B \twoheadrightarrow B'$. Put $I := \langle g_1, g_2, \ldots, g_l \rangle \subset B\{\mathbf{T}\}[[\mathbf{Z}]]$ and $\mathfrak{X} := \mathrm{Spf}(B\{\mathbf{T}\}[[\mathbf{Z}]]/I)$. It holds that $\mathfrak{X}' \subset \mathfrak{X}$ is a closed subscheme and that in the diagram

the squares are cartesian. From (4.1.1), we deduce that $\mathrm{rg}(\mathrm{Jac}_{\mathfrak{X}/\mathfrak{Y}}(x)) = l$ and, applying the Jacobian criterion for the affine formal space and the affine formal disc, it follows that $\mathfrak{X} \to \mathfrak{Y}$ is smooth at $x \in \mathfrak{X}$ To finish the proof, it suffices to take $\mathfrak{U} \subset \mathfrak{X}$, an open neighborhood of $x \in \mathfrak{X}$ such that the morphism $\mathfrak{U} \to \mathfrak{Y}$ is smooth, and $\mathfrak{U}$ the corresponding open set in $\mathfrak{X}'$. $\square$

Theorem 4.2. *Let us consider in* NFS *a closed immersion* $\mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}$ *given by a square zero ideal* $\mathcal{I} \subset \mathcal{O}_{\mathfrak{Y}}$ *and* $f_0 : \mathfrak{X}_0 \to \mathfrak{Y}_0$ *a smooth morphism with* $\mathfrak{X}_0$ *a separated formal scheme. Then there is an element* $c_{f_0} \in \mathrm{Ext}^2(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I})$ *such that:* c_{f_0} *vanishes if and only if there exists a smooth lifting* $\mathfrak{X}$ *of* $\mathfrak{X}_0$ *over* $\mathfrak{Y}$.

Proof. From Proposition 4.1, there exists an affine open covering $\mathfrak{U}_\bullet = \{\mathfrak{U}_{\alpha,0}\}_{\alpha \in L}$ of $\mathfrak{X}_0$, such that, for all $\alpha \in L$, there exists a smooth lifting $\mathfrak{U}_\alpha$ of $\mathfrak{U}_{\alpha,0}$ over $\mathfrak{Y}$. As $\mathfrak{X}_0$ is a separated formal scheme $\mathfrak{U}_{\alpha\beta,0} := \mathfrak{U}_{\alpha,0} \cap \mathfrak{U}_{\beta,0}$ is an affine open set for any α, β and, if we call $\mathfrak{U}_{\alpha\beta} \subset \mathfrak{U}_\alpha$ and $\mathfrak{U}_{\beta\alpha} \subset \mathfrak{U}_\beta$ the corresponding open subsets, from 3.3, there exists an isomorphism $u_{\alpha\beta} : \mathfrak{U}_{\alpha\beta} \xrightarrow{\sim} \mathfrak{U}_{\beta\alpha}$ such that the following diagram

commutes. For any α, β, γ such that $\mathfrak{U}_{\alpha\beta\gamma,0} := \mathfrak{U}_{\alpha,0} \cap \mathfrak{U}_{\beta,0} \cap \mathfrak{U}_{\gamma,0} \neq \varnothing$, let us write $\mathfrak{U}_{\alpha\beta\gamma} := \mathfrak{U}_{\alpha\beta} \times_{\mathfrak{U}_\alpha} \mathfrak{U}_{\alpha\gamma}$. It holds that

$$u_{\alpha\beta\gamma} := u_{\alpha\gamma}^{-1}|_{\mathfrak{U}_{\gamma\beta} \cap \mathfrak{U}_{\gamma\alpha}} \circ u_{\beta\gamma}|_{\mathfrak{U}_{\beta\alpha} \cap \mathfrak{U}_{\beta\gamma}} \circ u_{\alpha\beta}|_{\mathfrak{U}_{\alpha\beta} \cap \mathfrak{U}_{\alpha\gamma}} \in \mathrm{Aut}_{\mathfrak{U}_{\alpha\beta\gamma,0}}(\mathfrak{U}_{\alpha\beta\gamma}).$$

Applying 3.2, we get a unique $\phi_{\alpha\beta\gamma} \in \Gamma(\mathfrak{U}_{\alpha\beta\gamma,0}, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$ whose adjoint satisfies the relation $u_{\alpha\beta\gamma}^\natural - 1^\natural_{\mathfrak{U}_{\alpha\beta\gamma}} = \phi_{\alpha\beta\gamma} \circ \widehat{d}_{\mathfrak{U}_{\alpha\beta\gamma}/\mathfrak{Y}}$. Let $\mathfrak{U}_{\alpha\beta\gamma\delta,0} := \mathfrak{U}_{\alpha,0} \cap \mathfrak{U}_{\beta,0} \cap \mathfrak{U}_{\gamma,0} \cap \mathfrak{U}_{\delta,0}$. By the previous discussion, the cochain $\phi_{\mathfrak{U}_\bullet} := (\phi_{\alpha\beta\gamma}) \in \check{C}^2(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$ satisfies the cocycle condition

$$\phi_{\alpha\beta\gamma}|_{\mathfrak{U}_{\alpha\beta\gamma\delta,0}} - \phi_{\alpha\gamma\delta}|_{\mathfrak{U}_{\alpha\beta\gamma\delta,0}} + \phi_{\beta\gamma\delta}|_{\mathfrak{U}_{\alpha\beta\gamma\delta,0}} - \phi_{\beta\delta\alpha}|_{\mathfrak{U}_{\alpha\beta\gamma\delta,0}} = 0 \tag{4.2.1}$$

for any $\alpha, \beta, \gamma, \delta$ such that $\mathfrak{U}_{\alpha\beta\gamma\delta,0} \neq \varnothing$ and, therefore,

$$\phi_{\mathfrak{U}_\bullet} \in \check{Z}^2(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I})).$$

Using 3.4 and reasoning in an analogous way as in the proof of Proposition 3.5, it is easily seen that the definition of

$$c_{\mathfrak{U}_\bullet} := [\phi_{\mathfrak{U}_\bullet}] \in \check{H}^2(\mathfrak{U}_\bullet, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$$

does not depend on the choice of the family of isomorphisms $\{u_{\alpha\beta}\}$. Furthermore, if $\mathfrak{V}_\bullet$ is an affine open refinement of $\mathfrak{U}_\bullet$, then $c_{\mathfrak{U}_\bullet} = c_{\mathfrak{V}_\bullet} \in \check{H}^2(\mathfrak{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^*\mathcal{I}))$. By [2, Proposition 4.8], $\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}$ is a locally free $\mathcal{O}_{\mathfrak{X}_0}$-module. Since

$\mathfrak{X}_0$ is separated, using [1, Corollary 3.1.8] and [8, Ch. III, Exercise 4.11], we have that $\check{\mathrm{H}}^2(\mathfrak{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^* \mathscr{I})) = \mathrm{H}^2(\mathfrak{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^* \mathscr{I}))$. We set

$$c_{f_0} := [\phi_{\mathfrak{U}_\bullet}] \in \mathrm{H}^2(\mathfrak{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^* \mathscr{I})) = \mathrm{Ext}^2(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^* \mathscr{I}).$$

Let us show that c_{f_0} is the obstruction to the existence of a smooth lifting of $\mathfrak{X}_0$ over $\mathfrak{Y}$. If there exists a smooth lifting $\mathfrak{X}$ of $\mathfrak{X}_0$ over $\mathfrak{Y}$, one could take the isomorphisms $\{u_{\alpha\beta}\}$ above as the identities, then $c_{f_0} = 0$, trivially. Reciprocally, let $\mathfrak{U} = \{\mathfrak{U}_{\alpha,0}\}_{\alpha \in L}$ be an affine open covering of $\mathfrak{X}_0$ and, for each α, $\mathfrak{U}_\alpha$ a smooth lifting of $\mathfrak{U}_{\alpha,0}$ over $\mathfrak{Y}$ such that, with the notations established at the beginning of the proof, $c_{f_0} = [\phi_{\mathfrak{U}_\bullet}]$ with

$$\phi_{\mathfrak{U}_\bullet} = (\phi_{\alpha\beta\gamma}) \in \check{\mathrm{Z}}^2(\mathfrak{U}, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^* \mathscr{I})).$$

In view of $c_{f_0} = 0$, we are going to glue the $\mathfrak{Y}$-formal schemes $\{\mathfrak{U}_\alpha\}_{\alpha \in L}$ into a smooth lifting of $\mathfrak{X}_0$ over $\mathfrak{Y}$. By hypothesis, we have that $\phi_{\mathfrak{U}_\bullet}$ is a coboundary and therefore, there exists $(\phi_{\alpha\beta}) \in \check{\mathrm{C}}^1(\mathfrak{U}, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^* \mathscr{I}))$ such that, for any α, β, γ with $\mathfrak{U}_{\alpha\beta\gamma,0} \neq \varnothing$,

$$\phi_{\alpha\beta}|_{\mathfrak{U}_{\alpha\beta\gamma,0}} - \phi_{\alpha\gamma}|_{\mathfrak{U}_{\alpha\beta\gamma,0}} + \phi_{\beta\gamma}|_{\mathfrak{U}_{\alpha\beta\gamma,0}} = \phi_{\alpha\beta\gamma}. \tag{4.2.2}$$

For each couple of indexes α, β such that $\mathfrak{U}_{\alpha\beta,0} \neq \varnothing$, let $v_{\alpha\beta} : \mathfrak{U}_{\alpha\beta} \to \mathfrak{U}_{\beta\alpha}$ be the morphism which is the identity as topological map, and that, as the topologically ringed spaces map is given by $v_{\alpha\beta}^\natural := u_{\alpha\beta}^\natural - \phi_{\alpha\beta} \circ \widehat{d}_{\mathfrak{X}/\mathfrak{Y}}|_{\mathfrak{U}_{\alpha\beta}}$. The family $\{v_{\alpha\beta}\}$ satisfies:

- Each map $v_{\alpha\beta}$ is an isomorphism of $\mathfrak{Y}$-formal schemes (by 3.4).
- For any α, β, γ such that $\mathfrak{U}_{\alpha\beta\gamma,0} \neq \varnothing$,

$$v_{\alpha\gamma}^{-1}|_{\mathfrak{U}_{\gamma\beta} \cap \mathfrak{U}_{\gamma\alpha}} \circ v_{\beta\gamma}|_{\mathfrak{U}_{\beta\alpha} \cap \mathfrak{U}_{\beta\gamma}} \circ v_{\alpha\beta}|_{\mathfrak{U}_{\alpha\beta} \cap \mathfrak{U}_{\alpha\gamma}} = 1_{\mathfrak{U}_{\alpha\beta} \cap \mathfrak{U}_{\alpha\gamma}}$$

  by (4.2.1) and (4.2.2).
- For any α, β, $v_{\alpha\alpha} = 1_{\mathfrak{U}_\alpha}$ and $v_{\alpha\beta}^{-1} = v_{\beta\alpha}$.

  Thus, the $\mathfrak{Y}$-formal schemes $\{\mathfrak{U}_\alpha\}$ glue into a smooth lifting $f : \mathfrak{X} \to \mathfrak{Y}$ of $\mathfrak{X}_0$ over $\mathfrak{Y}$ through the gluing morphisms $\{v_{\alpha\beta}\}$. $\square$

Corollary 4.3. *With the hypothesis of Theorem 4.2, if $\mathfrak{X}_0$ is affine, there exists a lifting of $\mathfrak{X}_0$ over $\mathfrak{Y}$.*

Proof. From [1, Corollary 3.1.8], we have that $\mathrm{H}^2(\mathfrak{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}_0}}(\widehat{\Omega}^1_{\mathfrak{X}_0/\mathfrak{Y}_0}, f_0^* \mathscr{I})) = 0$ and the result follows from the last proposition. $\square$

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