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Rational curves on minuscule Schubert varieties

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Introduction

Let us denote by $\mathfrak{C}$ the variety of lines in $\mathbb{P}^3$ meeting a fixed line, it is a grassmannian (and hence minuscule) Schubert variety. In [16] we described the irreducible components of the scheme of morphisms from $\mathbb{P}^1$ to $\mathfrak{C}$ and the general morphism in each irreducible component.

In this text we study the scheme of morphisms from $\mathbb{P}^1$ to any minuscule Schubert variety X. Let us recall that we studied in [15] the scheme of morphisms from $\mathbb{P}^1$ to any homogeneous variety. The main idea, in the case of a minuscule Schubert variety X, is to restrict ourselves to the dense orbit under the stabilizer $\mathrm{Stab}(X)$ of X and apply the results of [15].

More precisely, let U be the dense orbit under $\mathrm{Stab}(X)$ in X and let Y be the complement. Because X is a minuscule Schubert variety the closed subset Y of X is of codimension at least 2 (see Section 2.2). This fact and the stratification of X by Schubert subvarieties gives us a surjective morphism (see Section 1):

$$s : \mathrm{Pic}(U)^\vee \to A_1(X).$$

For any class $\alpha \in A_1(X)$, we can consider a certain morphism:

$$j : \coprod_{s(\beta)=\alpha} \mathbf{Hom}_\beta\big(\mathbb{P}^1, U\big) \to \mathbf{Hom}_\alpha\big(\mathbb{P}^1, X\big)$$

where $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ is the scheme of morphisms $f : \mathbb{P}^1 \to X$ with $f_*[\mathbb{P}^1] = \alpha$ and $\mathbf{Hom}_\beta(\mathbb{P}^1, U)$ is the scheme of morphisms $g : \mathbb{P}^1 \to U$ such that $[g] = \beta$ where $[g]$ is

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the linear function $L \mapsto \deg(g^*L)$ on $\mathrm{Pic}(U)$. As $Y = X \setminus U$ lies in codimension 2, we expect the image of this morphism to be dense (this is the crucial point of the proof). This condition means that any morphism $\mathbb{P}^1 \to X$ can be deformed such that the image of this deformation does not meet Y. If the morphism j defined above is dominant, we may apply the results of [15] to prove that $\mathbf{Hom}_\beta(\mathbb{P}^1, U)$ is irreducible as soon as it is non-empty and the images of these irreducible $\mathbf{Hom}_\beta(\mathbb{P}^1, U)$ will give the irreducible components of $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$.

Let us denote by $\mathfrak{ne}(\alpha)$ the subset of $\mathrm{Pic}(U)^\vee$ given elements β such that $s(\beta) = \alpha$ and $\mathbf{Hom}_\beta(\mathbb{P}^1, U)$ is non-empty (see Section 1 for a more precise definition in terms of roots). We prove

Theorem 0.1. *The irreducible components of the scheme of morphisms* $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ *are indexed by* $\mathfrak{ne}(\alpha)$.

Here is an outline of the paper. In the first section we define the surjective map s of the introduction and the set $\mathfrak{ne}(\alpha)$ for X any Schubert variety and $\alpha \in A_1(X)$. In the second section we recall the definition of a minuscule Schubert variety and its properties. We also prove a positivity result on roots we will need later. In the third section we recall the construction of the Bott–Samelson resolution $\pi : \widetilde{X} \to X$ of a Schubert variety X and describe some cycles on $\widetilde{X}$. In the fourth section, we construct some big families of curves on $\widetilde{X}$ contracted by π. In the fifth section we study the scheme of morphisms $\mathbf{Hom}_{\tilde{\alpha}}(\mathbb{P}^1, \widetilde{X})$ and prove some smoothing results with the curves contructed in the fourth section. In the last section we prove our main result.

The key point as indicated above is to prove that the map j is dominant that is to say that any morphism $f : \mathbb{P}^1 \to X$ can be factorised in U (modulo deformation). We prove this by lifting f in $\tilde{f}$ on $\widetilde{X}$. It is now sufficient to prove that the lifted curve $\tilde{f}$ of a general curve f does not meet the divisors contracted by π. If $\tilde{f}$ does meet a contracted divisor D then we add a "line" $L \subset D$ with $L \cdot D = -1$ constructed in the fourth section and smooth the union $\tilde{f}(\mathbb{P}^1) \cup L$. The intersection with D is lowered by one in the operation. We conclude by induction on the number of intersection of $\tilde{f}$ with the contracted divisors.

Remark 0.2. (i) The variety $\mathfrak{C}$ can also be seen as a cone over a smooth 2-dimensional quadric embedded in $\mathbb{P}^3$. We treat more generally the case of a cone X over an homogeneous variety in the forthcoming paper [17]. In this situation we can also define for $\alpha \in A_1(X)$ a class $\mathfrak{ne}(\alpha)$ as previously but the irreducible components of $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ are not always indexed by $\mathfrak{ne}(\alpha)$. It is the case if and only if the projectivised tangent cone of the singularity (here the embedded homogeneous variety) contains lines.

(ii) This condition on the existence of lines in the projectivised tangent cone of the singularity also appears for more general Schubert varieties.

(iii) In [3], M. Brion and P. Polo proved that the singularities of minuscule Schubert varieties are locally isomorphic to cones over homogeneous varieties. With the results of [17] this implies that the key problem of factorising morphisms through U is locally true. Unfortunately it is not obvious to prove the global results thanks to this local property. It is nevertheless a good guide for intuition and we solve here the global problem using Bott–Samelson resolutions.

1. Preliminary

In this section we explain the results on cycles used in the introduction. We describe the surjective morphism $s:\mathrm{Pic}(U)^\vee \to A_1(X)$ and define the set of classes $\mathfrak{ne}(\alpha)$ for $\alpha \in A_1(X)$.

Let X be a scheme of dimension n. Denote by $Z_k(X)$ the group of k-cycles on X and by $Z_k^{\equiv}(X)$ and $Z_k^r(X)$ the subgroups of cycles trivial for the numerical and rational equivalence. Let us denote by $N_k(X)$ and $A_k(X)$ the corresponding quotients. The Picard group is the image in $A_{n-1}(X)$ of the subgroup of Cartier divisors in $Z_{n-1}(X)$ and we denote by $N^1(X)$ the quotient of $\mathrm{Pic}(X)$ by numerical equivalence.

Lemma 1.1. *Let $X \subset G/P$ be a Schubert variety (G a Lie group and P a parabolic subgroup of G). Then one has*

(i) $\mathrm{Pic}(X) \simeq N^1(X)$,
(ii) $A_1(X) \simeq N_1(X)$.

In particular we have $A_1(X) \simeq \mathrm{Pic}(X)^\vee$.

Proof. (i) Thanks to the results of [6] the groups $A_*(X)$ are free generated by Schubert subvarieties and, furthermore, rational and algebraic equivalence are the same. So on the one hand, the Picard group is contained in $A_{n-1}(X)$ and is in particular free.

On the other hand, thanks to [5, Example 19.3.3], we know that a Cartier divisor D is numerically trivial if for some $m \in \mathbb{N}$ we have mD is algebraically trivial. This implies for Schubert varieties that mD is rationally trivial and because $\mathrm{Pic}(X)$ is torsion free D is trivial in $\mathrm{Pic}(X)$. This implies that $\mathrm{Pic}(X) \simeq N^1(X)$.

(ii) The results of [6] also imply that $A_1(X)$ is generated by the one-dimensional Schubert varieties in X. But on G/P there is a duality between the Picard group and one-dimensional Schubert varieties. In particular for any one-dimensional Schubert variety Z there is a line bundle L_Z such that $L_Z \cdot Z = 1$ and L_Z is trivial on any other one-dimensional Schubert variety. If the $Z_i \subset X$ are the one-dimensional Schubert varieties in X then the restrictions of the L_{Z_i} to X form a dual family to the Z_i. In particular, the Z_i are numerically independent. As they form a basis of $A_1(X)$ we have $A_1(X) \simeq N_1(X)$.

The duality comes from general duality between $N_1(X)$ and $N^1(X)$. □

Let U be the smooth locus of X. If X is minuscule (see definition in Section 2) this smooth locus U is the dense orbit under $\mathrm{Stab}(X)$ in X (see [3][1]). Let Y be the complement of U in X. Because X is a normal variety the closed subset Y is of codimension at least 2, this in particular implies that $\mathrm{Pic}(U) = A_{n-1}(U) \simeq A_{n-1}(X)$. We now have the following inclusion:

$$\mathrm{Pic}(X) \subset A_{n-1}(X) \simeq \mathrm{Pic}(U)$$

[1] We do not need the results of [3] to define $\mathfrak{ne}(\alpha)$, see Theorem 6.7, but it is more simple with this fact on the singular locus.

giving the surjection

$$s : \mathrm{Pic}(U)^\vee \to A_1(X).$$

With these notations we make the following:

Definition 1.2. Let X be any Schubert variety and let $\alpha \in A_1(X)$. We define the set $\mathfrak{ne}(\alpha) \subset A_{n-1}(X)^\vee$.

Let us make the identification $A_{n-1}(X) \simeq \mathrm{Pic}(U)$. The elements of $\mathfrak{ne}(\alpha)$ are the elements $\beta \in \mathrm{Pic}(U)^\vee$ such that $s(\beta) = \alpha$ and there exists a curve $C \subset U$ with $[C] = \beta$ as a linear form on $\mathrm{Pic}(U)$ (β is effective).

In the case of minuscule Schubert variety $X \subset G/P$ we describe $\mathfrak{ne}(\alpha)$ more precisely: the smooth part U is the dense orbit under $\mathrm{Stab}\, X$. Let R be the Levi subgroup of $\mathrm{Stab}(X)$, the orbit U is of the form $QP/P \simeq Q/Q \cap P$ where $Q = \mathrm{Stab}(X)$ is a parabolic subgroup of G. We proved in [15, Proposition 5] that this orbit is a tower of affine bundles over the homogeneous variety $R/R \cap P$. In particular $\mathrm{Pic}(U) \simeq \mathrm{Pic}(R/R \cap P)$ is given in terms of weights with a particular weight given by the generator of $\mathrm{Pic}(X)$. Furthermore, we proved in [15] that the elements $\beta \in \mathrm{Pic}(R/R \cap P)^\vee$ are effective if they are in the dual cone of the cone of effective divisor, in other words they lie in the cone spanned by the positive roots.

Example 1.3. If X is a grassmannian Schubert variety given by a partition λ, consider the associated Young diagram (see for example [13]). Then the Picard group $\mathrm{Pic}(U)$ is free and has as many generators $(L_i)_{i \in [1,r]}$ as the numbers of outer corner of the Young diagram, which is the number of strict inequalities in the partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0)$. The generator L of $\mathrm{Pic}(X)$ is given by

$$L = \sum_{i \in [1,r]} L_i.$$

If $\alpha \in A_1(X)$ is such that $\alpha \cdot L = d$ then $\mathfrak{ne}(\alpha)$ is given by the r-tuples $(b_i)_{i \in [1,r]}$ of non-negative integers such that

$$\sum_{i \in [1,r]} b_i = d.$$

The number of irreducible components is $\binom{d+r-1}{d}$.

Remark 1.4. The scheme $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ is the scheme of morphisms from $\mathbb{P}^1$ to X of class α (for more details see [8,14]).

In general, this will just mean that $\alpha \in A_1(X)$ and that $f_*[\mathbb{P}^1] = \alpha$ but sometimes (in particular in the introduction for the open part U) we consider $\alpha \in \mathrm{Pic}(X)^\vee$ and the class of a morphism $f : \mathbb{P}^1 \to X$ will be the linear form $\mathrm{Pic}(X) \to \mathbb{Z}$ given by $L \mapsto \deg(f^* L)$.

In the case of a minuscule Schubert variety X the two notion coincide because of the previous lemma.

In the case of the open part U of a minuscule Schubert variety X, these scheme are connected components of the scheme of morphisms with a fixed 1-cycle class.

2. Minuscule Schubert varieties

2.1. Definitions

In this section we recall the notion of minuscule weight and study the related homogeneous and Schubert varieties. Our basic reference will be [11].

Let G be a semi-simple algebraic group, fix T a maximal torus and B a Borel subgroup containing T. Let us denote by Δ the set of all roots, by Δ^+ (respectively Δ^-) the set of positive (respectively negative) roots, by S the set of simple roots associated to the data (G, T, B) and by W the associated Weyl group. If P is a parabolic subgroup containing B we note W_P the subgroup of W corresponding to P. Let us finally denote by $\widetilde{B}$ the opposite Borel subgroup (corresponding to the negative roots) and by i the Weyl involution on simple roots. This involution sends a simple root β on $-w_0(\beta)$ and is also defined on fundamental weights.

Definition 2.1. Let ϖ be a fundamental weight,

(i) we say that ϖ is minuscule if we have $\langle\alpha^\vee, \varpi\rangle \leqslant 1$ for all positive roots $\alpha \in \Delta^+$;
(ii) we say that ϖ is cominuscule if $\langle\alpha_0^\vee, \varpi\rangle = 1$ where α_0 is the longest root.

With the notation of Bourbaki [2], the minuscule and cominuscule weights are:

Type	Minuscule	Cominuscule
A_n	$\varpi_1 \cdots \varpi_n$	same weights
B_n	ϖ_n	ϖ_1
C_n	ϖ_1	ϖ_n
D_n	ϖ_1, ϖ_{n-1} and ϖ_n	same weights
E_6	ϖ_1 and ϖ_6	same weights
E_7	ϖ_7	same weight
E_8	none	none
F_4	none	none
G_2	none	none

Remark 2.2. The Weyl involution i acts on minuscule and on cominuscule weights.

Definition 2.3. Let ϖ be a minuscule weight and let P_ϖ be the associated parabolic subgroup. The homogeneous variety G/P_ϖ is then said to be minuscule. The Schubert varieties of a minuscule homogeneous variety are called minuscule Schubert varieties.

Remark 2.4. To study minuscule homogeneous varieties and their Schubert varieties, it is sufficient to restrict ourselves to simply-laced groups.

In fact the variety G/P_{ϖ_n} with $G = \mathrm{Spin}_{2n+1}$ is isomorphic to the variety $G'/P'_{\varpi_{n+1}}$ with $G' = \mathrm{Spin}_{2n+2}$ and there is a one to one correspondence between Schubert varieties thanks to this isomorphism. The same situation occurs with G/P_{ϖ_1}, $G = \mathrm{Sp}_{2n}$ and G'/P'_{ϖ_1}, $G' = \mathrm{SL}_{2n}$.

2.2. Divisors on minuscule Schubert varieties

In this section we describe the divisors on minuscule Schubert varieties. For proofs and more details see [11].

Definition 2.5. Let $\bar{\phi} \in W/W_{P_\varpi}$ and let $X(\bar{\phi})$ the associated Schubert variety. A Schubert divisor $X(s_\beta\bar{\phi})$ in $X(\bar{\phi})$ defined by a *simple* root β is called a moving divisor. All other Schubert divisor are said to be stationary.

Remark 2.6. The term "moving divisor" comes from the fact that the Schubert variety $X(\bar{\phi})$ is stable under the action of $U_{-\beta}$ whereas $X(s_\beta\bar{\phi})$ is moved by $U_{-\beta}$ in $X(\bar{\phi})$ (see [11]).

We have the following proposition [12, Lemma 1.14]:

Proposition 2.7. *With the notation of the Definition* 2.5 *then $X(s_\beta\bar{\phi})$ is a moving divisor in $X(\bar{\phi})$ if and only if $\bar{\phi}$ has a reduced expression starting with s_β.*

We now have the following theorem ([10, Theorem 1] or [11, Theorem 3.10]) which describes the divisors of a minuscule Schubert variety:

Theorem 2.8. *Let X be a minuscule Schubert variety, then every Schubert divisor in X is a moving divisor.*

Remark 2.9. (i) This theorem is equivalent to the fact that weak and strong Bruhat orders coincide on minuscule Schubert varieties.

(ii) Let U be the dense orbit in X under the action of stabilizer $\mathrm{Stab}(X) \subset G$. Let Y be the complement of U in X. A consequence of this theorem is that Y is in codimension at least 2.

2.3. A positivity result

Let $(\gamma_i)_{i\in[1,n]}$ be a sequence of simple roots and define $\phi = s_{\gamma_1}\cdots s_{\gamma_n}$. We suppose in addition that $l(\phi) = n$. Set $\beta_i = i(\gamma_i)$ and let us define a sequence of roots $(\alpha_i)_{i\in[1,n]}$ by

$$\alpha_1 = \beta_1, \quad \alpha_2 = s_{\beta_1}(\beta_2), \quad \dots, \quad \alpha_n = s_{\beta_1}\cdots s_{\beta_{n-1}}(\beta_n).$$

Remark that this construction is involutive in the sense that if the $(\alpha_i)_{i\in[1,n]}$ are given we can recover the $(\beta_i)_{i\in[1,n]}$ by the formulae

$$\beta_1 = \alpha_1, \quad \beta_2 = s_{\alpha_1}(\alpha_2), \quad \dots, \quad \beta_n = s_{\alpha_1}\cdots s_{\alpha_{n-1}}(\alpha_n).$$

Remark 2.10. We use these notations to fit with those of the Bott–Samelson resolution.

Proposition 2.11. *Let ϖ be a minuscule weight. Suppose that ϕ is the smallest element in the class $\bar{\phi} \in W/W_{P_\varpi}$. Then for all $(i, j) \in [1, n]$ we have*

$$\langle \alpha_i^\vee, \alpha_j \rangle \geqslant 0.$$

Proof. Let us define the sequence $(\tilde{\beta}_i)_{i\in[1,n]}$ of simple roots as being the sequence $(\beta_i)_{i\in[1,n]}$ with reversed order, that is to say $\tilde{\beta}_i = \beta_{n+1-i}$. With this sequence we can construct a sequence $(\tilde{\alpha}_i)_{i\in[1,n]}$ by

$$\tilde{\alpha}_1 = \tilde{\beta}_1, \quad \tilde{\alpha}_2 = s_{\tilde{\beta}_1}(\tilde{\beta}_2), \quad \ldots, \quad \tilde{\alpha}_n = s_{\tilde{\beta}_1} \cdots s_{\tilde{\beta}_{n-1}}(\tilde{\beta}_n). \qquad \square$$

Lemma 2.12. *For all $i \in [1, n]$, we have*

$$\langle \alpha_i^\vee, \alpha_j \rangle = \langle \tilde{\alpha}_{n+1-i}^\vee, \tilde{\alpha}_{n+1-j} \rangle.$$

Proof. We have

$$\begin{aligned}\langle \tilde{\alpha}_{n+1-i}^\vee, \tilde{\alpha}_{n+1-j} \rangle &= \langle s_{\tilde{\beta}_1} \cdots s_{\tilde{\beta}_{n-i}}(\tilde{\beta}_{n+1-i})^\vee, s_{\tilde{\beta}_1} \cdots s_{\tilde{\beta}_{n-j}}(\tilde{\beta}_{n+1-j}) \rangle \\ &= \langle s_{\beta_n} \cdots s_{\beta_{i+1}}(\beta_i)^\vee, s_{\beta_n} \cdots s_{\beta_{j+1}}(\beta_j) \rangle = \langle s_{\beta_1} \cdots s_{\beta_i}(\beta_i)^\vee, s_{\beta_1} \cdots s_{\beta_j}(\beta_j) \rangle\end{aligned}$$

where we applied $s_{\beta_1} \cdots s_{\beta_n}$ to get the last equality. But we have

$$\begin{aligned}\langle s_{\beta_1} \cdots s_{\beta_i}(\beta_i)^\vee, s_{\beta_1} \cdots s_{\beta_j}(\beta_j) \rangle &= \langle s_{\beta_1} \cdots s_{\beta_{i-1}}(-\beta_i)^\vee, s_{\beta_1} \cdots s_{\beta_{j-1}}(-\beta_j) \rangle \\ &= \langle \alpha_i^\vee, \alpha_j \rangle. \qquad \square\end{aligned}$$

It is thus enough to prove the result on the sequence $(\tilde{\alpha}_i)_{i\in[1,n]}$. As ϕ is the smallest element in $\bar{\phi}$, the reduced expression $\phi = s_{\gamma_1} \cdots s_{\gamma_n} = s_{i(\tilde{\beta}_n)} \cdots s_{i(\tilde{\beta}_1)}$ in W is still reduced in W/W_{P_ϖ}. Let us prove the following lemma:

Lemma 2.13. *Let β the only simple root such that $\langle \beta^\vee, i(\varpi) \rangle = 1$. For all $i \in [1, n]$, the roots $\tilde{\alpha}_i$ are such that*

$$\tilde{\alpha}_i \geqslant \beta$$

that is to say, for all fundamental weights ϖ_k we have $\langle \varpi_k^\vee, \alpha_i - \beta \rangle \geqslant 0$.

Proof. Because the expression $\bar{\phi} = s_{i(\tilde{\beta}_n)} \cdots s_{i(\tilde{\beta}_1)}$ is reduced in W/W_{P_ϖ}, we have for all $i \in [1, n]$

$$\langle i(\tilde{\beta}_{i+1})^\vee, s_{i(\tilde{\beta}_i)} \cdots s_{i(\tilde{\beta}_1)}(-\varpi) \rangle < 0.$$

Remark that this (and in fact the whole lemma) is valid for any fundamental weight ϖ (the minuscule hypothesis implies more precisely that this bracket has to be -1). Let us calculate

$$\begin{aligned}\left\langle\tilde{\alpha}_{i+1}^\vee,-i(\varpi)\right\rangle &= \left\langle s_{\tilde{\beta}_1}\cdots s_{\tilde{\beta}_i}(\tilde{\beta}_{i+1})^\vee,-i(\varpi)\right\rangle\left\langle\tilde{\beta}_{i+1}^\vee,s_{\tilde{\beta}_i}\cdots s_{\tilde{\beta}_1}\left(-i(\varpi)\right)\right\rangle \\ &= \left\langle i(\tilde{\beta}_{i+1})^\vee,s_{i(\tilde{\beta}_i)}\cdots s_{i(\tilde{\beta}_1)}(-\varpi)\right\rangle<0,\end{aligned}$$

that is to say $\langle\tilde{\alpha}_{i+1}^\vee,i(\varpi)\rangle>0$. Writing $\tilde{\alpha}_{i+1}$ in terms of simple roots, we see that the coefficient of β has to be strictly positive (in fact it has to be one because ϖ is cominuscule). This exactly means that $\tilde{\alpha}_{i+1}\geqslant\beta$. □

It is now an easy check on the tables of [2] to see that for these roots and a minuscule weight ϖ we always have

$$\left\langle\tilde{\alpha}_i^\vee,\tilde{\alpha}_j\right\rangle\geqslant 0.$$

Remark 2.14. Here is another proof (due to the referee) of this proposition. Let us define $\mathrm{Inv}(w)$ to be the set of roots $\{\alpha_1,\dots,\alpha_n\}$ already defined.

J.R. Stembridge [18,19] defines an element of W to be fully commutative if its reduced decomposition is unique up to commuting factors $s_is_j=s_js_i$. He also proves that in the (co)minuscule case, every $w\in W/W_{P_\varpi}$ of minimal length in its class is fully commutative.

Then S. Billey and A. Postnikov [1] prove that for all non-orthogonal roots α and β in $\mathrm{Inv}(w)$ then the vectors $\mathrm{Inv}(w)\cap\mathrm{Span}_{\mathbb{R}}(\alpha,\beta)$ are isomorphic to a proper subset of the positive roots of type A_2, B_2 or G_2. The last case being here impossible we get the positivity result.

Corollary 2.15. *With the above notations and the remark of Lemma* 2.13*, the fact that the expression $\phi=s_{i(\beta_1)}\cdots s_{i(\beta_n)}$ is reduced implies that $\langle i(\beta_n),-\varpi\rangle<0$ or equivalently $\langle\beta_n,i(\varpi)\rangle>0$. This is possible if and only if $\beta_n=\beta$.*

Let $k\in[1,n]$, if there exists an $i<k$ such that $\beta_i=\beta_k$ (respectively if there exists an $i>k$ such that $\beta_i=\beta_k$) we will denote by $p(k)$ (respectively $n(k)$) the biggest (respectively smallest) integer $i\in[1,k-1]$ (respectively $i\in[k+1,n]$) such that $\beta_i=\beta_k$.

Corollary 2.16. *Let j such that $\beta_j=\beta$.*

(i) *We have $\langle\alpha_i^\vee,\alpha_j\rangle=0$ if for all $k\in[i+1,j]$, $\langle\beta_i^\vee,\beta_k\rangle=0$.*
(ii) *Otherwise we have $\langle\alpha_i^\vee,\alpha_j\rangle=1$ if $i>p(j)$ or if $i<p(j)$ and for all $k\in[i+1,p(j)]$, $\langle\beta_i^\vee,\beta_k\rangle=0$. In all other cases we have $\langle\alpha_i^\vee,\alpha_j\rangle=0$.*

Proof. We have seen that $\langle\alpha_i^\vee,\alpha_j\rangle=\langle\tilde{\alpha}_{n+1-i}^\vee,\tilde{\alpha}_{n+1-j}\rangle$ and composing with $s_{\tilde{\beta}_1}\cdots s_{\tilde{\beta}_{n-j}}$ we can assume that $n+1-j=1$ (i.e., $j=n$). We thus have to calculate

$$\left\langle\tilde{\alpha}_{n+1-i}^\vee,\tilde{\alpha}_1\right\rangle=\left\langle\tilde{\beta}_1^\vee,\tilde{\alpha}_{n+1-i}\right\rangle=\left\langle\beta^\vee,\tilde{\alpha}_{n+1-i}\right\rangle$$

(we use here the fact that $R = R^\vee$ and the fact that $\beta_j = \beta$). We first have to prove that $\langle \beta^\vee, \tilde{\alpha}_{n+1-i} \rangle = 0$ if for all $k \in [1, n+1-i]$, $\langle \tilde{\beta}^\vee_{n+1-i}, \tilde{\beta}_k \rangle = 0$.

And otherwise we have to prove that $\langle \beta^\vee, \tilde{\alpha}_{n+1-i} \rangle = 1$ if $n+1-i < n(1)$ or if $n+1-i > n(1)$ and for all $k \in [n(1), n+1-i]$, $\langle \tilde{\beta}^\vee_{n+1-i}, \tilde{\beta}_k \rangle = 0$ and that in all other cases we have $\langle \beta^\vee, \tilde{\alpha}_{n+1-i} \rangle = 0$.

(i) In this case, it is easy to see that $\tilde{\alpha}_{n+1-i} = \tilde{\beta}_{n+1-i}$ and we have the vanishing.

(ii) Let us define $\alpha = s_{\tilde{\beta}_2} \cdots s_{\tilde{\beta}_{n-i}}(\tilde{\beta}_{n+1-i})$. We have $\tilde{\alpha}_{n+1-i} = s_{\tilde{\beta}_1}(\alpha) = s_\beta(\alpha)$. And recall that the simple root β always appears in $\tilde{\alpha}_{n+1-i}$ (Lemma 2.13) with multiplicity 1 (because ϖ is a cominuscule weight).

In the first case, we see that the simple root β does not appear in α. But we have $\tilde{\alpha}_{n+1-i} = s_\beta(\alpha) = \alpha - \langle \beta^\vee, \alpha \rangle \beta$ thus $\langle \beta^\vee, \alpha \rangle = -1$.

In the second case, applying Lemma 2.13 to the sequence $n(1), \ldots, n+1-i$ we see that the simple root β appears in $s_{\tilde{\beta}_{n(1)}} \cdots s_{\tilde{\beta}_{n-i}}(\tilde{\beta}_{n+1-i})$ with multiplicity 1. As β does not appear in $\tilde{\beta}_2, \ldots, \tilde{\beta}_{n(1)-1}$, we see that β appears in α with multiplicity 1. But we have $\tilde{\alpha}_{n+1-i} = s_\beta(\alpha) = \alpha - \langle \beta^\vee, \alpha \rangle \beta$ thus $\langle \beta^\vee, \alpha \rangle = 0$.

We conclude because $\langle \beta^\vee, \tilde{\alpha}_{n+1-i} \rangle \langle \beta^\vee, s_\beta(\alpha) \rangle = -\langle \beta^\vee, \alpha \rangle$. □

Remark 2.17. The formula of Corollary 2.16 is more simple if we use commutation relation between the simple root β_k: let j such that $\beta_j = \beta$, then we have $\langle \alpha_i^\vee, \alpha_j \rangle = 0$ if modulo commutation we can exchange s_{β_i} and s_{β_j}. If not we also have $\langle \alpha_i^\vee, \alpha_j \rangle = 0$ if $i < p(j)$ and we cannot commute s_{β_i} and $s_{\beta_{p(j)}}$.

Let us prove the following:

Corollary 2.18. *We have the formula*

$$\sum_{k=i+1,\ \beta_k=\beta}^{n} \langle \alpha_i^\vee, \alpha_k \rangle = \begin{cases} 1, & \text{if } \beta_i \neq \beta, \\ 0, & \text{if } \beta_i = \beta. \end{cases}$$

Proof. We apply the previous corollary. We know that $\beta_n = \beta$ and we cannot commute s_{β_i} and s_{β_n} (otherwise the expression would not be reduced). Let j be the smallest integer $k \in [i+1, n]$ such that $\beta_k = \beta$ and we can not commute s_{β_i} and s_{β_k}.

We have $\langle \alpha_i^\vee, \alpha_k \rangle = 0$ for all $k \in [i+1, n]$ with $\beta_k = \beta$ and $k \neq j$. For $k = j$, we have

$$\langle \alpha_i^\vee, \alpha_k \rangle = \begin{cases} 1, & \text{if } \beta_i \neq \beta, \\ 0, & \text{if } \beta_i = \beta. \end{cases} \qquad \square$$

As is Proposition 2.11, it is easy to check on the tables of [2] the following

Fact 2.19. *If ϖ is minuscule and $(\alpha_i)_{i \in [1,n]}$ as above, then for all i and j in $[1, n]$, one has $\langle \alpha_i^\vee, \alpha_j \rangle \leqslant 2$ with equality if and only if $\alpha_i = \alpha_j$.*

Let us prove the following corollary that we will need later:

Corollary 2.20. *Let i, x and j in $[1,n]$. If $\langle\alpha_i^\vee,\alpha_x\rangle = 1$ then for all $j\in[1,n]$, one has*

$$\bigl\langle\alpha_i^\vee, s_{\alpha_x}(\alpha_j)\bigr\rangle \geqslant -1.$$

Proof. We have

$$\bigl\langle\alpha_i^\vee, s_{\alpha_x}(\alpha_j)\bigr\rangle = \bigl\langle\alpha_i^\vee,\alpha_j\bigr\rangle - \bigl\langle\alpha_i^\vee,\alpha_x\bigr\rangle\bigl\langle\alpha_x^\vee,\alpha_j\bigr\rangle = \bigl\langle\alpha_i^\vee,\alpha_j\bigr\rangle - \bigl\langle\alpha_x^\vee,\alpha_j\bigr\rangle.$$

The preceding fact tells us that $\langle\alpha_x^\vee,\alpha_j\rangle \leqslant 2$ with equality only if $\alpha_x = \alpha_j$. In case of equality we have $\langle\alpha_i^\vee,\alpha_j\rangle = \langle\alpha_i^\vee,\alpha_x\rangle = 1$ thus $\langle\alpha_i^\vee, s_{\alpha_x}(\alpha_j)\rangle = -1$.

If $\alpha_x \neq \alpha_j$, then Proposition 2.11 tells us that $\langle\alpha_i^\vee,\alpha_j\rangle \geqslant 0$ and we have $\langle\alpha_x^\vee,\alpha_j\rangle \leqslant 1$ thus $\langle\alpha_i^\vee, s_{\alpha_x}(\alpha_j)\rangle \geqslant -1$. □

3. The Bott–Samelson resolutions

In this section we briefly describe the Bott–Samelson construction which gives a resolution of any Schubert variety in G/B and in G/P for any parabolic subgroup P. We describe this construction as M. Demazure did in [4] we refer to this article for more details.

3.1. Construction

Let $\phi\in W$ with $l(\phi) = n$. We recall in this section M. Demazure's construction [4] of a resolution of the dimension n Schubert variety $X(\phi) = \overline{B\phi B/B} \subset G/B$ associated to a reduced decomposition $\phi = s_{\gamma_1}\cdots s_{\gamma_n}$ with $\gamma_i\in S$.

Let w_0 be the longest element of W and define the element $w = w_0\phi^{-1}w_0$. The preceding reduced expression leads to the reduced expression

$$w = s_{i(\gamma_n)}\cdots s_{i(\gamma_1)}.$$

If we choose any reduced expression

$$ww_0 = s_{i(\gamma_{n+1})}\cdots s_{i(\gamma_N)}$$

with $\gamma_i\in S$ and $N = l(w_0)$, then $w_0 = s_{i(\gamma_1)}\cdots s_{i(\gamma_N)}$ is a reduced expression of w_0. To keep the same notation with [4], let us note $\beta_i = i(\gamma_i)$, we have:

$$w_0 = s_{\beta_1}\cdots s_{\beta_N},\qquad w = s_{\beta_n}\cdots s_{\beta_1}\quad\text{and}\quad ww_0 = s_{\beta_{n+1}}\cdots s_{\beta_N}.$$

With the sequence $(\beta_i)_{i\in[1,N]}$, we define the following sequence $(\alpha_i)_{i\in[1,N]}$ of roots by:

$$\alpha_1 = \beta_1,\quad \alpha_2 = s_{\beta_1}(\beta_2),\quad \ldots,\quad \alpha_N = s_{\beta_1}\cdots s_{\beta_{N-1}}(\beta_N).$$

The α_i are distinct and $\Delta^+ = \{\alpha_i / i\in[1,N]\}$. Define $w_i = s_{\alpha_i}\in W$ (we will also for simplicity of notations sometimes consider w_i as an element of G). We have

$$w_i = s_{\beta_1} \cdots s_{\beta_{i-1}} s_{\beta_i} s_{\beta_{i-1}} \cdots s_{\beta_1}, \qquad w = w_1 \cdots w_n,$$
$$w_0 = w_1 \cdots w_N \quad \text{and} \quad w_0^{-1}\phi = w_{N-n+1} \cdots w_N.$$

We define a sequence $(B_i)_{i\in[0,N]}$ of Borel subgroups containing T by induction:

$$B_0 = \widetilde{B} \quad \text{and} \quad B_{i+1} = w_{i+1} B_i w_{i+1}^{-1}.$$

Denote by P_i the parabolic subgroup generated by B_{i-1} and B_i we get a sequence of codimension one inclusions:

$$B_0 \subset P_1 \supset B_1 \subset \cdots \supset B_{n-1} \subset P_N \supset B_N.$$

Finally we construct a sequence of varieties $(X_i)_{i\in[0,N]}$ endowed with a right action of B_i by induction:

$$X_0 = B_0 \quad \text{and} \quad X_{i+1} = X_i \times^{B_i} P_{i+1}$$

where the second term is the contracted product of X_i and P_i over B_i (see [4, Par. 2.3]). The quotient X_i/B_i is well defined and we get a sequence of $\mathbb{P}^1$-bundles f_i with canonical sections σ_i:

$$X_0/B_0 \xleftarrow{f_1} X_1/B_1 \leftarrow \cdots \leftarrow X_{N-1}/B_{N-1} \xleftarrow{f_N} X_N/B_N.$$

The scheme X_i/B_i is the quotient of $P_1 \times \cdots \times P_i$ by the right action of $B_1 \times \cdots \times B_i$ given by

$$(p_1, \ldots, p_i) \cdot (b_1, \ldots, b_i) = \big(p_1 b_1, \ldots, b_{i-1}^{-1} p_i b_i\big).$$

The projection f_i sends the class of $(p_1, \ldots, p_i)$ to the class of $(p_1, \ldots, p_{i-1})$ whereas the section σ_i sends the class of $(p_1, \ldots, p_{i-1})$ to the class of $(p_1, \ldots, p_{i-1}, w_i)$.

The multiplication morphism $P_1 \times \cdots \times P_N \to G$ factorises through $X_N \to G$ which is P_N equivariant and in particular B_N equivariant. We thus get a morphism

$$X_N/B_N \to G/B_N = G/B$$

which is birational and such that the restriction to $\sigma_N \cdots \sigma_{n+1}(X_n/B_n)$ is birational on the Schubert variety

$$\overline{\widetilde{B} w_0^{-1} \phi B/B} \simeq X(\phi).$$

This construction gives us the resolution

$$\pi : X_n/B_n \to X(\phi).$$

Let P be a parabolic subgroup containing P and let $\bar{\phi} \in W/W_P$. We want to construct a resolution of the Schubert variety

$$X(\bar{\phi}) = \overline{B\bar{\phi}P/P} \subset G/P.$$

For this choose ϕ the smallest element in the class $\bar{\phi}$. The morphism $X(\phi) \to X(\bar{\phi})$ induced by the projection $G/B \to G/P$ is birational. So the morphism

$$\pi : X_n/B_n \to X(\bar{\phi})$$

is a resolution. We will denote by $\widetilde{X}(\bar{\phi})$ the scheme X_n/B_n.

Remark 3.1. If we have $\langle \beta_i^\vee, \beta_{i+1} \rangle = 0$ for some i, then the Bott–Samelson resolution associated to the sequence $(\beta_k)_{k\in[1,n]}$ is the same as the Bott–Samelson resolution associated to the sequence $(\beta'_k)_{k\in[1,n]}$ where $\beta'_k = \beta_k$ for $k \notin \{i, i+1\}$, $\beta'_i = \beta_{i+1}$ and $\beta'_{i+1} = \beta_i$.

3.2. Curves and divisors on the Bott–Samelson resolution

In his paper [4], M. Demazure studies some special cycles on the varieties X_N/B_N. Denote $Z_i = f_N^{-1} \cdots f_{i+1}^{-1}(\mathrm{Im}(\sigma_i))$. It is a divisor in X_N/B_N. For any $K \subset [1, N]$ denote by

$$Z_K = \bigcap_{i\in K} Z_i$$

which is a codimension $|K|$ subvariety of X_N/B_N. The classes of the Z_K form a basis of the Chow group of X_N/B_N (cf. [4, Par. 4, Proposition 1]). Remark that for any $k \in [1, N]$, we have $X_k/B_k = Z_{[k+1,N]}$. We can in this way define subvarieties of $\widetilde{X}(\bar{\phi})$:

- Denote by $D_i = Z_{\{i\}\cup[n+1,N]}$. This is a divisor on $\widetilde{X}(\bar{\phi})$ and these divisors form a basis of the Picard group of $\widetilde{X}(\bar{\phi})$.
- Define the curve $C_i = Z_{[1,N]-\{i\}}$. These curves for $i \in [1, n]$ form a basis of $A_1(\widetilde{X}(\bar{\phi}))$.

Denote by ξ_i the class of Z_i in the Chow group of X_N/B_N. M. Demazure describes completely the Chow group of X_N/B_N in the following

Theorem 3.2 (Demazure [4, Par. 4, Proposition 1]). *The Chow group of X_N/B_N is generated over $\mathbb{Z}$ by the $(\xi_i)_{i\in[1,N]}$ with the relations*:

$$\xi_i \cdot \left(\xi_i + \sum_{j=1}^{i-1} \langle \alpha_j^\vee, \alpha_i \rangle \xi_j\right) = 0 \quad \textit{for all } i \in [1, N].$$

With the above notation we have $[C_i] = \prod_{j\neq i} \xi_j$ and we can use the previous theorem to prove

Proposition 3.3. *We have*

$$[C_i]\cdot\xi_j=\begin{cases}0, & \text{for } i>j,\\ 1, & \text{for } i=j,\\ \langle\beta_i^\vee,\beta_j\rangle, & \text{for } i<j.\end{cases}$$

Proof. The preceding theorem leads by an easy induction to

Fact 3.4. *We have the following formula in* $A(X_N/B_N)$:

$$[C_i]\cdot\xi_j=\begin{cases}0, & \text{for } i>j,\\ 1, & \text{for } i=j,\\ \sum_{k=1}^{j-i}(-1)^k\sum_{i=i_0<\cdots<i_k=j}\prod_{x=0}^{k-1}\langle\alpha_x^\vee,\alpha_{x+1}\rangle, & \text{for } i<j.\end{cases}$$

We prove the following lemma to conclude the proof:

Lemma 3.5. *For* $i<j$, *we have the following formula*:

$$\sum_{k=1}^{j-i}(-1)^k\sum_{i=i_0<\cdots<i_k=j}\prod_{x=0}^{k-1}\langle\alpha_x^\vee,\alpha_{x+1}\rangle=\langle\beta_i^\vee,\beta_j\rangle.$$

Proof. Let us first remark that the β_i can be constructed thanks to the α_i in the following way:

$$\beta_1=\alpha_1,\quad \beta_2=s_{\alpha_1}(\alpha_2),\quad \ldots,\quad \beta_N=s_{\alpha_1}\cdots s_{\alpha_{N-1}}(\alpha_N).$$

Calculating

$$\begin{aligned}\langle\beta_i^\vee,\beta_j\rangle&=\langle s_{\alpha_1}\cdots s_{\alpha_{i-1}}(\alpha_i)^\vee,s_{\alpha_1}\cdots s_{\alpha_{j-1}}(\alpha_j)\rangle=\langle\alpha_i^\vee,s_{\alpha_i}\cdots s_{\alpha_{j-1}}(\alpha_j)\rangle\\&=-\langle\alpha_i s_{\alpha_{i+1}}\cdots s_{\alpha_{j-1}}(\alpha_j)\rangle.\end{aligned}$$

Furthermore, we can write

$$s_{\alpha_i}\cdots s_{\alpha_{j-1}}(\alpha_j)=\sum_{k=i}^{j}x_{k,j}\alpha_k$$

with $x_{k,j}\in\mathbb{Z}$ not depending on i. On the one hand, we get by an easy induction the equality:

$$x_{i,j}=\sum_{k=1}^{j-i}(-1)^k\sum_{i=i_0<\cdots<i_k=j}\prod_{x=0}^{k-1}\langle\alpha_x^\vee,\alpha_{x+1}\rangle.$$

On the other hand, we have

$$\langle \alpha_i^\vee, s_{\alpha_i} \cdots s_{\alpha_{j-1}}(\alpha_j) \rangle = \sum_{k=i}^{j} x_{k,j} \langle \alpha_i^\vee, \alpha_k \rangle$$

and

$$-\langle \alpha_i^\vee, s_{\alpha_{i+1}} \cdots s_{\alpha_{j-1}}(\alpha_j) \rangle = -\sum_{k=i+1}^{j} x_{k,j} \langle \alpha_i^\vee, \alpha_k \rangle$$

summing the two equalities we get

$$2\langle \alpha_i^\vee, s_{\alpha_i} \cdots s_{\alpha_{j-1}}(\alpha_j) \rangle = \langle \alpha_i^\vee, s_{\alpha_i} \cdots s_{\alpha_{j-1}}(\alpha_j) \rangle - \langle \alpha_i^\vee, s_{\alpha_{i+1}} \cdots s_{\alpha_{j-1}}(\alpha_j) \rangle = x_{i,j} \langle \alpha_i^\vee, \alpha_i \rangle$$

concluding the proof of the lemma. □

The proposition follows from Fact 3.4 and Lemma 3.5. □

Remark 3.6. The formulae of Proposition 3.3 are still valid on $\widetilde{X}(\bar{\phi})$.

Let us introduce some notations (see also [4]). If λ is a character of the torus T let us denote by $Ł_i(\lambda)$ the associated line bundle on X_i/B_i (recall that $T \subset B_i$). Let us now denote by T_i the relative tangent sheaf of the $\mathbb{P}^1$-fibration $f_i : X_i/B_i \to X_{i-1}/B_{i-1}$. Thanks to [4, Par. 2, Proposition 1] and an easy induction on i we get

Fact 3.7. *Let us still denote $Ł_i(\lambda)$ the corresponding class in $A^*(X_i/B_i)$ then we have the formula*:

$$Ł_i(\lambda) = \sum_{k=1}^{i} \langle \alpha_k^\vee, \lambda \rangle \cdot \xi_k.$$

Furthermore, M. Demazure remarks [4, Par. 2, remark following Proposition 1] that we have $T_i = Ł_i(\alpha_i)$ so that we get the following

Corollary 3.8. *Let us still denote T_i the corresponding class in $A^*(X_i/B_i)$ then we have the formula*:

$$T_i = \sum_{k=1}^{i} \langle \alpha_k^\vee, \alpha_i \rangle \cdot \xi_k.$$

Remark that the factor of ξ_i in T_i is 2. We get the

Corollary 3.9. *Let C be a curve on X_i/B_i. Suppose that for all $k \in [1,i]$ we have $[C] \cdot \xi_k \geqslant 0$ and $\langle \alpha_k^\vee, \alpha_i \rangle \geqslant 0$ then for all k we have*

$$[C] \cdot (T_k - \xi_k) \geqslant 0 \quad \textit{and in particular} \quad [C] \cdot T_k \geqslant 0$$

where we still denote by T_k the pull-back of T_k on X_i/B_i.

Finally if ϕ is the smallest element in the class $\bar{\phi} \in W/W_{P_\varpi}$ with ϖ a minuscule weight, the results of Proposition 2.11 gives us

Corollary 3.10. *Let C be a curve on $\widetilde{X}(\bar{\phi})$ the resolution of $X(\bar{\phi})$. Suppose that for all $k \in [1,n]$ we have $[C] \cdot \xi_k \geqslant 0$ then for all k we have*

$$[C] \cdot (T_k - \xi_k) \geqslant 0 \quad \textit{and in particular} \quad [C] \cdot T_k \geqslant 0$$

where we still denote by T_k the pull-back of T_k on $\widetilde{X}(\bar{\phi})$.

Proposition 3.11. *We have*

$$[C_i] \cdot T_j = \begin{cases} 0, & \textit{for } i > j, \\ \langle \beta_i^\vee, \beta_j \rangle, & \textit{for } i \leqslant j. \end{cases}$$

Proof. Thanks to Corollary 3.8 the result is clear for $i > j$. Let $i \leqslant j$ and let us use Corollary 3.8 and Proposition 3.3 to get

$$[C_i] \cdot T_j = \sum_{k=1}^{j} \langle \alpha_k^\vee, \alpha_j \rangle [C_i] \cdot \xi_k = \sum_{k=i+1}^{j-1} \langle \alpha_k^\vee, \alpha_j \rangle \cdot \langle \beta_i^\vee, \beta_k \rangle + \langle \alpha_i^\vee, \alpha_j \rangle + 2\langle \beta_i^\vee, \beta_j \rangle.$$

Lemma 3.12. *We have the formula*

$$\sum_{k=i+1}^{j-1} \langle \alpha_k^\vee, \alpha_j \rangle \cdot \langle \beta_i^\vee, \beta_k \rangle = -\langle \alpha_i^\vee, \alpha_j \rangle - \langle \beta_i^\vee, \beta_j \rangle.$$

Proof. Because the construction of α_i in terms of β_i is symmetric to the construction of β_i in terms of α_i the formula of Lemma 3.5 is valid when we exchange the roles of the α_i and of the β_i so we get for $k < j$:

$$\langle \alpha_k^\vee, \alpha_j \rangle = \sum_{u=1}^{j-k} (-1)^u \sum_{k=i_0 < \cdots < i_u = j} \prod_{x=0}^{u-1} \langle \beta_{i_x}^\vee, \beta_{i_{x+1}} \rangle.$$

We thus obtain

$$\sum_{k=i+1}^{j-1} \langle \alpha_k^\vee, \alpha_j \rangle \cdot \langle \beta_i^\vee, \beta_k \rangle = \sum_{k=i+1}^{j} \sum_{u=1}^{j-k} (-1)^u \sum_{k=i_0 < \cdots < i_u = j} \prod_{x=0}^{u-1} \langle \beta_{i_x}^\vee, \beta_{i_{x+1}} \rangle \cdot \langle \beta_i^\vee, \beta_k \rangle.$$

If we set $i_{-1}=i$ we get

$$\begin{aligned}\sum_{k=i+1}^{j-1}\langle\alpha_k^\vee,\alpha_j\rangle\cdot\langle\beta_i^\vee,\beta_k\rangle &= \sum_{u=1}^{j-i-1}\sum_{k=i+1}^{j-u}(-1)^u\sum_{i=i_{-1}<k=i_0<\cdots<i_u=j}\prod_{x=-1}^{u-1}\langle\beta_{i_x}^\vee,\beta_{i_{x+1}}\rangle\\ &= \sum_{u=1}^{j-i-1}(-1)^u\sum_{i=i_{-1}<i_0<\cdots<i_u=j}\prod_{x=-1}^{u-1}\langle\beta_{i_x}^\vee,\beta_{i_{x+1}}\rangle\\ &= \sum_{u=2}^{j-i}(-1)^{u+1}\sum_{i=i_0<\cdots<i_u=j}\prod_{x=0}^{u}\langle\beta_{i_x}^\vee,\beta_{i_{x+1}}\rangle\\ &= -\langle\beta_i^\vee,\beta_j\rangle+\sum_{u=1}^{j-i}(-1)^{u+1}\sum_{i=i_0<\cdots<i_u=j}\prod_{x=0}^{u}\langle\beta_{i_x}^\vee,\beta_{i_{x+1}}\rangle\\ &\quad-\langle\beta_i^\vee,\beta_j\rangle-\langle\alpha_i^\vee,\alpha_j\rangle. \qquad\square\end{aligned}$$

This lemma with the preceding formula ends the proof. □

4. Some more curves on $\widetilde{X}(\bar{\phi})$

4.1. Effective and contracted curves

In this section, we study some more curves on $\widetilde{X}(\bar{\phi})$. In particular those which are contracted by the projection $\pi:\widetilde{X}(\bar{\phi})\to X(\bar{\phi})$.

Let us look at the restriction of π on the curve C_j. M. Demazure [4, Par. 3, Theorem 1] proves that the curve is contracted if and only if $l(w_1\cdots w_{j-1}w_{j+1}\cdots w_n)>n$. But a simple calculation gives

$$w_1\cdots w_{j-1}w_{j+1}\cdots w_n=s_{\beta_j}w$$

so the curve is not contracted in G/B if and only if $l(s_{\beta_j}w)=l(w)-1$ in W and is not contracted in G/P if this equality is true for the minimal representatives in W of $\overline{s_{\beta_j}w}$ and $\overline{w}$ in W/W_P. This means that there exists a minimal reduced expression of $\overline{w}$ beginning with s_{β_j}. But for any reduced expression $\overline{w}=s_{\beta_n}\cdots s_{\beta_1}$ we have seen in Corollary 2.15 that we must have $\beta_n=\beta$ (where β is the unique simple root such that $\langle\beta^\vee,i(\varpi)\rangle=1$). This would imply that $\beta_j=\beta=\beta_n$. In the other cases the curve C_j is contracted (in general, i.e., when ϖ is not minuscule and even not fundamental, the curve C_j is contracted if and only if $\langle\beta_j^\vee,i(\varpi)\rangle>0$).

For any integer $j\in[1,n]$ let us define $n(j)=\min\{k>j/b_k=b_j\}$. If $n(j)$ does not exist then $j=m(i)=\max\{k/\beta_k=\beta_i\}$ for some $i\in[1,n]$. Let us consider the case where $n(j)$ exists. A point t in C_j (respectively $C_{n(j)}$) is the image in $\widetilde{X}(\bar{\phi})$ of a n-uple $(w_1,\dots,w_{j-1},x(t),w_{j+1},\dots,w_n)\in P_1\times\cdots\times P_n$ (respectively $(w_1,\dots,w_{n(j)-1},y(t),$

$w_{n(j)+1}, \dots, w_n)$). Because the curves C_j and $C_{n(j)}$ have the same image $X(s_{b_j} w)$ under π, we thus have the equation

$$w_1 \cdots w_{j-1} \cdot x(t) \cdot w_{j+1} \cdots w_n = w_1 \cdots w_{n(j)-1} \cdot y(t) \cdot w_{n(j)+1} \cdots w_n.$$

If we consider the curve $\widetilde{C}_j$ parametrized by t defined by the images of

$$\left(w_1, \dots, w_{j-1}, w_j^{-1} x(t), w_{j+1}, \dots, w_{n(j)-1}, w_{n(j)} y(t)^{-1}, w_{n(j)+1}, \dots, w_n\right)$$

in $\widetilde{X}(\bar{\phi})$ we see that its image by π is $w_1 \cdots w_{j-1} w_{j+1} \cdots w_{n(j)-1} w_{n(j)+1} \cdots w_{n-1}$ a constant. The curve $\widetilde{C}_j$ is contracted by π.

If $n(j)$ does not exists, then we define $\widetilde{C}_j = C_j$.

Lemma 4.1. *We have* $[\widetilde{C}_j] = [C_j] - [C_{n(j)}]$.

Proof. The projection of $\widetilde{C}_j$ and C_j on $X_{n(j)-1}/B_{n(j)-1}$ are the same. This implies that $[C_j] - [\widetilde{C}_j] = a[C_{n(j)}]$ with $a \in \mathbb{Z}$. Apply π_* to this equation to get $\pi_*[C_j] - \pi_*[\widetilde{C}_j] = a\pi_*[C_{n(j)}]$. But we have $\pi_*[C_j] = \pi_*[C_{n(j)}]$ and $\pi_*[\widetilde{C}_j] = 0$ thus $a = 1$. □

Proposition 4.2. *The classes* $[\widetilde{C}_j]$ *generate* $A_1(\widetilde{X}(\bar{\phi}))$ *over* $\mathbb{Z}$. *Furthermore, they generate the cone of effective curves, i.e., they generate the extremal rays.*

Proof. The first assertion is trivial because the classes $[C_j]$ generate $A_1(\widetilde{X}(\bar{\phi}))$ over $\mathbb{Z}$.

For the second, we proceed by induction on j: we prove that the classes $[\widetilde{C}_k]$ for $k \leqslant j$ generated the effective cone of X_j/B_j (by abuse of notation we still denote by $[\widetilde{C}_k]$ the image of the class $[\widetilde{C}_k]$ in X_j/B_j). It is true for $j = 1$ assume it is true for $j - 1$ and let

$$[C] = \sum_{k=1}^{j} a_k [\widetilde{C}_k]$$

the class of an effective curve. By projection on X_{j-1}/B_{j-1} we obtain the class

$$f_{j_*}[C] = \sum_{k=1}^{j-1} a_k [\widetilde{C}_k]$$

which has to be effective so by induction we have $a_k \geqslant 0$ for $k < j$. Now by projection on G/P_j we get

$$\pi_*[C] = \sum_{k=1}^{j} a_k \pi_*[\widetilde{C}_k].$$

The class $[\widetilde{C}_j]$ is not contracted. The only classes $[\widetilde{C}_k]$ that are not contracted by π are such that $[\widetilde{C}_k] = [C_k]$ and $l(s_{\beta_k} w) = l(w) - 1$. The image is then the Schubert variety associated to s_{β_k}. The first condition implies that for these not contracted curves, all the β_k are distinct. But the associated Schubert varieties are independent in $A_1(G/B)$ and because the image is effective we have $a_k \geqslant 0$ for all those k and in particular $a_j \geqslant 0$. □

4.2. Curves on contracted divisors

Let $x \in [1, n]$ such that the divisor D_x is contracted by π. We are going to construct special curves on D_x (recall that $[D_x] = \xi_x$).

Lemma 4.3. *There exists $i \in [1, n]$ such that $[C_i] \cdot \xi_x = -1$.*

Proof. Recall that we have (Proposition 3.3)

$$[C_i] \cdot \xi_x = \begin{cases} 0, & \text{for } i > x, \\ 1, & \text{for } i = x, \\ \langle \beta_i^\vee, \beta_x \rangle, & \text{for } i < x. \end{cases}$$

We have to choose $i < x$ and for such an i, as the group is simply laced we have $[C_i] \cdot \xi_x = -1$, 0 or 2. If for all $i < x$ this intersection is zero then for all $i < x$ the symmetry s_{β_i} commutes with s_{β_x} so that the reduced expression $w = s_{\beta_n} \cdots s_{\beta_1}$ can be written $w = s_{\beta_n} \cdots s_{\beta_{x+1}} s_{\beta_{x-1}} \cdots s_{\beta_1} s_{\beta_x}$. We have a reduced expression

$$\bar{\phi} = s_{\gamma_x} s_{\gamma_1} \cdots s_{\gamma_{x-1}} s_{\gamma_{x+1}} \cdots s_{\gamma_n}$$

meaning that the image of D_x in $X(\bar{\phi})$ is a moving divisor. This is impossible because D_x is contracted. Let i be the biggest $i < x$ such that $[C_i] \cdot \xi_x \neq 0$. If the intersection is 2 this means that $\beta_i = \beta_x$. But because for all $k \in [i+1, x-1]$, we have $\langle \beta_k^\vee, \beta_x \rangle = 0$, we see that s_{β_x} commutes with all s_{β_k} with $k \in [i+1, x-1]$. We have:

$$\begin{aligned} \bar{\phi} &= s_{\gamma_1} \cdots s_{\gamma_{i-1}} s_{\gamma_i} s_{\gamma_{i+1}} \cdots s_{\gamma_{x-1}} s_{\gamma_x} s_{\gamma_{x+1}} \cdots s_{\gamma_n} = s_{\gamma_1} \cdots s_{\gamma_{i-1}} s_{\gamma_i} s_{\gamma_x} s_{\gamma_{i+1}} \cdots s_{\gamma_{x-1}} s_{\gamma_{x+1}} \cdots s_{\gamma_n} \\ &= s_{\gamma_1} \cdots s_{\gamma_{i-1}} s_{\gamma_{i+1}} \cdots s_{\gamma_{x-1}} s_{\gamma_{x+1}} \cdots s_{\gamma_n} \end{aligned}$$

that is to say the expression $\bar{\phi} = s_{\gamma_1} \cdots s_{\gamma_n}$ was not reduced, a contradiction. □

Remark 4.4. In particular there exists an $i \in [1, n]$ such that $\langle \alpha_i^\vee, \alpha_x \rangle = 1$ (choose the i of the preceding proof and we have $\langle \alpha_i^\vee, \alpha_x \rangle = -\langle \beta_i^\vee, \beta_x \rangle = 1$).

Let $i \in [1, n]$ and let us define the following classes of curves:

$$[\widehat{C}_i] = [C_i] + \sum_{k=i+1}^{n} \langle \alpha_i^\vee, \alpha_k \rangle [C_k].$$

Lemma 4.5. *We have the formulae*

$$[\widehat{C}_i]\cdot\xi_j=\delta_{i,j}\quad\textit{and}\quad[\widehat{C}_i]\cdot T_j=\begin{cases}0, & \textit{for } i>j,\\ \langle\alpha_i^\vee,\alpha_j\rangle, & \textit{for } i\leqslant j.\end{cases}$$

Proof. We use Proposition 3.3 and Lemma 3.12 to get

$$\begin{aligned}[\widehat{C}_i]\cdot\xi_j&=\left([C_i]+\sum_{k=i+1}^{n}\langle\alpha_i^\vee,\alpha_k\rangle[C_k]\right)\cdot\xi_j\\ &=\begin{cases}0, & \text{for } i>j,\\ 1, & \text{for } i=j,\\ \langle\beta_i^\vee,\beta_j\rangle+\sum_{k=i+1}^{j-1}\langle\alpha_i^\vee,\alpha_k\rangle\langle\beta_k^\vee,\beta_j\rangle+\langle\alpha_i^\vee,\alpha_j\rangle, & \text{for } i<j,\end{cases}\\ &=\begin{cases}0, & \text{for } i>j,\\ 1, & \text{for } i=j,\\ \langle\beta_i^\vee,\beta_j\rangle-\langle\beta_i^\vee,\beta_j\rangle-\langle\alpha_i^\vee,\alpha_j\rangle+\langle\alpha_i^\vee,\alpha_j\rangle, & \text{for } i<j,\end{cases}\end{aligned}$$

proving the first formula. For the second one we use Proposition 3.11 and Lemma 3.12 to get

$$\begin{aligned}[\widehat{C}_i]\cdot T_j&=\left([C_i]+\sum_{k=i+1}^{n}\langle\alpha_i^\vee,\alpha_k\rangle[C_k]\right)\cdot T_j\\ &=\begin{cases}0, & \text{for } i>j,\\ \langle\beta_i^\vee,\beta_j\rangle+\sum_{k=i+1}^{j}\langle\alpha_i^\vee,\alpha_k\rangle\langle\beta_k^\vee,\beta_j\rangle, & \text{for } i\leqslant j,\end{cases}\\ &=\begin{cases}0, & \text{for } i>j,\\ 1, & \text{for } i=j,\\ \langle\beta_i^\vee,\beta_j\rangle-\langle\beta_i^\vee,\beta_j\rangle-\langle\alpha_i^\vee,\alpha_j\rangle+2\langle\alpha_i^\vee,\alpha_j\rangle, & \text{for } i<j,\end{cases}\end{aligned}$$

concluding the proof. □

Now let $i\in[1,n]$ such that $\langle\alpha_i^\vee,\alpha_x\rangle=1$ (there exists such an i thanks to Remark 4.4). We define the class:

$$[\Gamma_{x,i}]=[\widehat{C}_i]-\langle\alpha_i^\vee,\alpha_x\rangle[\widehat{C}_x]=[\widehat{C}_i]-[\widehat{C}_x]$$

and prove the following:

Proposition 4.6. *We have*:

(i) $[\Gamma_{x,i}]\cdot\xi_x=-1$ *so all curves* $C\in[\Gamma_{x,i}]$ *are contained in* D_x.
(ii) *The scheme* $\mathbf{Hom}_{[\Gamma_{x,i}]}(\mathbb{P},\widetilde{X}(\bar{\phi}))$ *is irreducible and smooth* (*in particular non-empty*).

(iii) *The open part* $D_x - \bigcup_{k\neq x}(D_x \cap D_k)$ *of the divisor* D_x *is covered by curves* $C \in [\Gamma_{x,i}]$.
(iv) *All curves* $C \in [\Gamma_{x,i}]$ *are contracted by* π.

Proof. (i) This is a simple application of Lemma 4.5.

(ii) Recall that $\widetilde{X}(\bar{\phi})$ is a sequence of $\mathbb{P}^1$-bundles. We proceed by induction on the X_j/B_j (by abuse of notation, we still denote by $[\Gamma_{x,i}]$ the push-forward of $[\Gamma_{x,i}]$ in $A_1(X_j/B_j)$). Let us denote by $\varphi : X \to Y$ the morphism $f_j : X_j/B_j \to X_{j-1}/B_{j-1}$ and by T the relative tangent sheaf. We have a section $\sigma = \sigma_j$ of φ and we denote by $\xi = \xi_j$ the divisor image of the section. We have:

$$\sigma_*\varphi_*[\Gamma_{x,i}] = \begin{cases} 0, & \text{for } j \leqslant i, \\ [\Gamma_{x,i}] - \langle \alpha_i^\vee, \alpha_j\rangle \cdot [C_j], & \text{for } i < j < x, \\ [\Gamma_{x,i}], & \text{for } j = x, \\ [\Gamma_{x,i}] - \langle \alpha_i^\vee, s_{\alpha_x}(\alpha_j)\rangle \cdot [C_j], & \text{for } j > x. \end{cases}$$

Proposition 3.3 and Lemma 4.5 give us

$$[\Gamma_{x,i}] \cdot \xi = \begin{cases} 1, & \text{for } j = i, \\ -1, & \text{for } j = x, \\ 0, & \text{otherwise,} \end{cases} \qquad \sigma_*\varphi_*[\Gamma_{x,i}] \cdot \xi = \begin{cases} 0, & \text{for } j \leqslant i, \\ -\langle \alpha_i^\vee, \alpha_j\rangle, & \text{for } i < j \leqslant x, \\ -\langle \alpha_i^\vee, s_{\alpha_x}(\alpha_j)\rangle, & \text{for } j > x, \end{cases}$$

and

$$[\Gamma_{x,i}] \cdot T = \begin{cases} 0, & \text{for } j < i, \\ \langle \alpha_i^\vee, \alpha_j\rangle, & \text{for } i \leqslant j < x, \\ \langle \alpha_i^\vee, s_{\alpha_x}(\alpha_j)\rangle, & \text{for } j \geqslant x. \end{cases}$$

Let us denote by $[\Gamma]$ the class of $[\Gamma_{x,i}]$ in $X = X_j/B_j$ and let $f \in \mathbf{Hom}_{\varphi_*[\Gamma]}(\mathbb{P}^1, Y)$. We want to study the fiber over f of the morphism

$$\mathbf{Hom}_{[\Gamma]}\big(\mathbb{P}^1, X\big) \to \mathbf{Hom}_{\varphi_*[\Gamma]}(\mathbb{P}, Y)$$

that is to say the morphisms $f' \in \mathbf{Hom}_{[\Gamma]}(\mathbb{P}^1, X)$ such that $f = \varphi \circ f'$. We look for a section of the $\mathbb{P}^1$-bundle φ pulled-back by f. Let E be the rank two vector bundle defining the $\mathbb{P}^1$-bundle. We can choose E such that $f^*E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ with $a \geqslant 0$.

The section $f \circ \sigma$ is given by a surjection $f^*E \to \mathcal{O}_{\mathbb{P}^1}(z)$ with $2z - a = \sigma_*\varphi_*[\Gamma] \cdot \xi$. A morphism f' is simply given by a surjection $f^*E \to \mathcal{O}_{\mathbb{P}^1}(y)$ such that $y + z - a = [\Gamma] \cdot \xi$ and $2y - a = [\Gamma] \cdot T$.

Remark 4.7. The section $f \circ \sigma$ always exists. We must thus have $z = 0$ or $z \geqslant a$. This implies that

- if $\sigma_*\varphi_*[\Gamma] \cdot \xi = 0$ then $a = z = 0$;
- if $\sigma_*\varphi_*[\Gamma] \cdot \xi < 0$ then $z = 0$ and $a = -\sigma_*\varphi_*[\Gamma] \cdot \xi$;

- if $\sigma_*\varphi_*[\Gamma]\cdot\xi > 0$ then $\langle\alpha_i^\vee, s_{\alpha_x}(\alpha_j)\rangle = -\sigma_*\varphi_*[\Gamma]\cdot\xi < 0$ and in fact $\langle\alpha_i^\vee, s_{\alpha_x}(\alpha_j)\rangle = -1$ (Corollary 2.20). In this case we have $2z - a = 1$ and this implies $z = a = 1$.

The section f' will exist if there exists an integer y such that $y = 0$ or $y \geqslant a$. In the case $j = i$, we have $z = a = 0$, $y = 1$ and f' exists. In the case $j = x$ we have $y = z = 0$, $a = 1$ and f' exists. In the other cases we always have $y + z - a = 0$. This implies that

- if $\sigma_*\varphi_*[\Gamma]\cdot\xi = 0$ then $y = 0$;
- if $\sigma_*\varphi_*[\Gamma]\cdot\xi < 0$ then $y = a$;
- if $\sigma_*\varphi_*[\Gamma]\cdot\xi > 0$ then $y = 0$.

In conclusion there always exists a section f' of f with the required invariants.

We will use the following proposition (see [15, Proposition 4]):

Proposition 4.8. *Let $\varphi: X \to Y$ a $\mathbb{P}^1$-bundle with relative tangent sheaf T and let $[\Gamma] \in A_1(X)$ such that $[\Gamma]\cdot T \geqslant 0$, then $\mathbf{Hom}_{[\Gamma]}(\mathbb{P}^1, X)$ is an open subset of a projective bundle over $\mathbf{Hom}_{\varphi_*[\Gamma]}(\mathbb{P}^1, Y)$. In particular, if $\mathbf{Hom}_{\varphi_*[\Gamma]}(\mathbb{P}^1, Y)$ is irreducible, the same is true for $\mathbf{Hom}_{[\Gamma]}(\mathbb{P}^1, X)$ as soon as it is non-empty.*

This proposition with be useful for the fibration f_j if we have $[\Gamma_{x,i}]\cdot T_j \geqslant 0$. The only cases where the previous proposition does not apply is when $\langle\alpha_i^\vee, s_{\alpha_x}(\alpha_j)\rangle < 0$ and in fact $\langle\alpha_i^\vee, s_{\alpha_x}(\alpha_j)\rangle = -1$ (Lemma 2.20). There are two distinct cases where this may occur. If $j = x$ then $[\Gamma_{x,i}]\cdot\xi_j = -1$ and $[\Gamma_{x,i}]\cdot T_j = -1$. If $j > x$ and $\langle\alpha_i^\vee, s_{\alpha_x}(\alpha_j)\rangle = -1$ then $[\Gamma_{x,i}]\cdot\xi_j = 0$ and $[\Gamma_{x,i}]\cdot T_j = -1$.

The first case $j = x$ is treated thanks to

Lemma 4.9. *Let $\varphi: X \to Y$ a $\mathbb{P}^1$-bundle with relative tangent sheaf T and with a section σ. Denote ξ the divisor $\sigma(Y)$ and let $[\Gamma] \in A_1(X)$ such that $[\Gamma]\cdot\xi = -1$, $[\Gamma]\cdot T = -1$ and $\sigma_*\varphi_*[\Gamma]\cdot\xi = -1$. Suppose that $\mathbf{Hom}_{\varphi_*[\Gamma]}(\mathbb{P}^1, Y)$ is normal then we have*

$$\mathbf{Hom}_{[\Gamma]}\big(\mathbb{P}^1, X\big) \simeq \mathbf{Hom}_{\varphi_*[\Gamma]}\big(\mathbb{P}^1, Y\big).$$

Proof. Let $f \in \mathbf{Hom}_{\varphi_*[\Gamma]}(\mathbb{P}^1, Y)$, we have to prove (by Zariski Main theorem) that there is exactly one morphism $f' \in \mathbf{Hom}_{[\Gamma]}(\mathbb{P}^1, X)$ such that $f = \varphi\circ f'$. But with the above notation and thanks to Remark 4.7 we have $y = z = 0$ and $a = 1$. The morphism f' has to be $\sigma\circ f$. □

The second case $j > x$ is treated thanks to

Lemma 4.10. *Let $\varphi: X \to Y$ a $\mathbb{P}^1$-bundle with relative tangent sheaf T and with a section σ. Denote ξ the divisor $\sigma(Y)$ and let $[\Gamma] \in A_1(X)$ such that $[\Gamma]\cdot\xi = 0$, $[\Gamma]\cdot T = -1$ and $\sigma_*\varphi_*[\Gamma]\cdot\xi = 1$. Suppose that $\mathbf{Hom}_{\varphi_*[\Gamma]}(\mathbb{P}^1, Y)$ is normal then we have*

$$\mathbf{Hom}_{[\Gamma]}\big(\mathbb{P}^1, X\big) \simeq \mathbf{Hom}_{\varphi_*[\Gamma]}\big(\mathbb{P}^1, Y\big).$$

Proof. Let $f \in \mathbf{Hom}_{\varphi_*[\Gamma]}(\mathbb{P}^1, Y)$, we have to prove (by Zariski Main theorem) that there is exactly one morphism $f' \in \mathbf{Hom}_{[\Gamma]}(\mathbb{P}^1, X)$ such that $f = \varphi \circ f'$. But with the above notation and thanks to Remark 4.7 we have $y = 0$ and $z = a = 1$. The morphism f' is given by the unique self-negative section of $\mathbb{P}_{\mathbb{P}^1}(f^*E)$. □

(iii) Let us note that thanks to Remark 4.7, Lemmas 4.9 and 4.10 there are curves $C \in [\Gamma_{x,i}]$ such that C is not contained in any intersection $D_x \cap D_j$ (we always have $C \subset D_x$) and thus always meet the open part $D_x - \bigcup_{k \neq x}(D_x \cap D_k)$ of the divisor D_x.

But the orbit of the unipotent part U of B acting on D_x is exactly $D_x - \bigcup_{k \neq x}(D_x \cap D_k)$. Translating C thanks to the action of U we see that the curves $C \in [\Gamma_{x,i}]$ cover $D_x - \bigcup_{k \neq x}(D_x \cap D_k)$.

(iv) We have seen that all the curves $[\widetilde{C}_k]$ are contracted by π except $[\widetilde{C}_n]$. We just have to prove that the coefficient a_n of $[\widetilde{C}_n]$ in $[\Gamma_{x,i}]$ is zero. Let us set

$$A = \sum_{k=i+1,\ \beta_k = \beta}^{n} \langle \alpha_i^\vee, \alpha_k \rangle - \sum_{k=x+1,\ \beta_k = \beta}^{n} \langle \alpha_x^\vee, \alpha_k \rangle.$$

We have

$$a_n = \begin{cases} A, & \text{if } \beta_i \neq \beta \text{ and } \beta_x \neq \beta, \\ A+1, & \text{if } \beta_i = \beta \text{ and } \beta_x \neq \beta, \\ A-1, & \text{if } \beta_i \neq \beta \text{ and } \beta_x = \beta, \\ A, & \text{if } \beta_i = \beta \text{ and } \beta_x = \beta. \end{cases}$$

We now apply Corollary 2.18 to see that $a_n = 0$ in all cases. □

Remark 4.11. If the fiber of the projection $\pi : D_x \to \pi(D_x)$ is a curve then its class has to be $[\Gamma_{x,i}]$. In general, the generic fiber is covered by curves in the class $[\Gamma_{x,i}]$. For more details on the fiber of the Bott–Samelson resolution see [7].

5. The scheme of morphisms for $\widetilde{X}(\bar{\phi})$

5.1. Irreducibility

We will prove in this section that for some classes $\alpha \in A_1(\widetilde{X}(\bar{\phi}))$ the scheme of morphisms $\mathbf{Hom}_\alpha(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ is irreducible and smooth. We will essentially need Proposition 4.8 (see [15, Proposition 4]).

Let us now consider a class $\alpha \in A_1(\widetilde{X}(\bar{\phi}))$ such that $\alpha \cdot \xi_i \geqslant 0$ for all $i \in [1, n]$. Thanks to Corollary 3.10 we know that $\alpha \cdot T_i \geqslant 0$ and $\alpha \cdot (T_i - \xi_i) \geqslant 0$.

Proposition 5.1.

(i) *The scheme of morphisms* $\mathbf{Hom}_\alpha(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ *is irreducible and smooth of dimension*

$$\int_\alpha c_1(T_{\widetilde{X}(\bar{\phi})}) + \dim\bigl(\widetilde{X}(\bar{\phi})\bigr).$$

(ii) *If the class* α *is such that* $\alpha \cdot \xi_x = 0$ *for all* $x \in [1, n]$ *with* D_x *a contracted divisor, then a general element* $f \in \mathbf{Hom}_\alpha\bigl(\mathbb{P}^1, \widetilde{X}(\bar{\phi})\bigr)$ *is contained in the regular locus of* π.

Proof. (i) We proceed by induction, for the first step, we have to study the scheme of morphisms from $\mathbb{P}^1$ to $\mathbb{P}^1$. This scheme is irreducible and smooth. We go by induction thanks to Proposition 4.8. We only have to prove that the scheme is non-empty. However, with the notations of the preceding section for $\mathbb{P}^1$-fibrations, we have $f^*E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ with section σ given by a surjection $f^*E \to \mathcal{O}_{\mathbb{P}^1}(z)$ and we look for a section $f^*E \to \mathcal{O}_{\mathbb{P}^1}(y)$. Because of the relations $\alpha \cdot T_i \geqslant 0$ and $\alpha \cdot (T_i - \xi_i) \geqslant 0$ we see that $y \geqslant z$ and $y \geqslant a - z$. This implies that $y \geqslant a$ proving the existence of a surjection $f^*E \to \mathcal{O}_{\mathbb{P}^1}(y)$.

(ii) Let f a general element. Thanks to the discussion above, we may assume that this element will meet the non-contracted divisors D_i in distinct points and will not meet the contracted divisors. In particular f will never meet intersections $D_i \cap D_j$ with $i \neq j$. In particular, the only B-orbits of the Bott–Samelson resolution that f will meet are the dense orbit and the orbits dense in D_i for a non-contracted divisor. These orbits are contained in the regular locus so this in particular proves that f is contained in the regular locus. □

5.2. Smoothing curves on $\widetilde{X}(\bar{\phi})$

Let $\alpha \in A_1(\widetilde{X}(\bar{\phi}))$ as above.

Lemma 5.2. *There exists* $\tilde{f} \in \mathbf{Hom}_\alpha(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ *such that* $\tilde{f}(\mathbb{P}^1)$ *is not contained in any* D_i *and does not meet any intersection* $D_i \cap D_j$.

Proof. Because the scheme $\mathbf{Hom}_\alpha(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ is irreducible, if it exists, a general morphism will have the required property.

Let $i < j$, we construct this curve $\tilde{f}$ by induction on the $\mathbb{P}^1$-fibrations. For all fibrations except for the fibrations f_i and f_j, we take any section.

For the fibration f_i we have by induction a morphism $\tilde{f}_{i-1} : \mathbb{P}^1 \to X_{i-1}/B_{i-1}$ and with the notations of the proof of the previous proposition: a rank 2 vector bundle $\tilde{f}_{i-1}^*E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ with $a \geqslant 0$; a surjection $\tilde{f}_{i-1}^*E \to \mathcal{O}_{\mathbb{P}^1}(z)$ (corresponding to the divisor D_i) and we look for a surjection $\tilde{f}_{i-1}^*E \to \mathcal{O}_{\mathbb{P}^1}(y)$. With our hypothesis on α we have $y \geqslant z$ and $y \geqslant a - z$ (cf. proof of the preceding proposition) so there always exists a section and we can choose it such that the image is not contained in D_i (because $y \geqslant a - z$).

For the fibration f_j we have by induction a morphism $\tilde{f}_{j-1} : \mathbb{P}^1 \to X_{j-1}/B_{j-1}$ and with the notations of the proof of the previous proposition: a rank 2 vector bundle $\tilde{f}_{j-1}^*E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ with $a \geqslant 0$; a surjection $\tilde{f}_{j-1}^*E \to \mathcal{O}_{\mathbb{P}^1}(z)$ (corresponding to the divisor D_j) and we look for a surjection $\tilde{f}_{j-1}^* E \to \mathcal{O}_{\mathbb{P}^1}(y)$. We know that $\tilde{f}_{j-1}(\mathbb{P}^1)$ is not contained in D_i. There are a finite number of points in $\mathbb{P}^1$, say $x_1, \ldots, x_k$ such that $\tilde{f}_{j-1}(x_l) \in D_i$. With our hypothesis on α we have $y \geqslant z$ and $y \geqslant a - z$ (cf. proof of the preceding proposition) so there always exists a section and we can choose it such that the composition $\mathcal{O}_{\mathbb{P}^1}(a-z) \to f^*E \to \mathcal{O}_{\mathbb{P}^1}(y)$ is non-zero for $x_1, \ldots, x_k$. Then the new curve does not meet $D_i \cap D_j$.

Because the condition is open we can find a curve for which it is true for all i and j. □

Corollary 5.3. *Let $\alpha \in A_1(\widetilde{X}(\bar{\phi}))$ such that $\alpha \cdot \xi_k \geqslant 0$ for all $k \in [1, n]$ and $\alpha \cdot \xi_x > 0$ for some $x \in [1, n]$. Then there exists $\tilde{f} \in \mathbf{Hom}_\alpha(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ such that $\tilde{f}(\mathbb{P}^1)$ meets D_x in $D_x - \bigcup_{k \neq x}(D_x \cap D_k)$.*

Proof. Let $\tilde{f}$ as in the preceding lemma. We know that $\tilde{f}(\mathbb{P}^1)$ is not contained in D_x but has to meet D_x (because of the intersection number). The curve $\tilde{f}(\mathbb{P}^1)$ does not meet any intersection $D_i \cap D_j$, in particular it does not meet the intersection $D_x \cap D_k$ for all k. □

Let us now suppose that $\dim(X(\bar{\phi})) \geqslant 3$. When $\dim(X(\bar{\phi})) \leqslant 2$ then $X(\bar{\phi})$ is $\mathbb{P}^1$ or $\mathbb{P}^2$ for which the scheme of morphisms is well known. Let D_x be a contracted divisor, $\alpha \in A_1(\widetilde{X}(\bar{\phi}))$ and $\tilde{f} \in \mathbf{Hom}_\alpha(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ as in the preceding corollary. There exists $x_0 \in \mathbb{P}^1$ such that

$$\tilde{f}(x_0) \in D_x - \bigcup_{k \neq x}(D_x \cap D_k)$$

and thanks to Proposition 4.6, for any integer $i < x$ with $\langle \alpha_i^\vee, \alpha_x \rangle = 1$ there exists a curve $C \in [\Gamma_{x,i}]$ such that $\tilde{f}(x_0) \in C$.

Proposition 5.4. *Then there exists a deformation $\tilde{f}'$ of $\tilde{f}$ in $\mathbf{Hom}_\alpha(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ and an integer i with $\langle \alpha_i^\vee, \alpha_x \rangle = 1$ such that $\tilde{f}'(\mathbb{P}^1)$ and C meet exactly in $\tilde{f}(x_0)$ and transversally.*

Proof. Let us first assume that $x < n$.

Lemma 5.5. *There exists $j > x$ such that $\langle \alpha_x^\vee, \alpha_j \rangle = 1$.*

Proof. It is enough to prove that there exists $j > x$ such that $\langle \beta_x^\vee, \beta_j \rangle \neq 0$. Indeed taking the smallest such j we must have $\langle \beta_x^\vee, \beta_j \rangle = -1$ because otherwise we would have $\langle \beta_x^\vee, \beta_j \rangle = 2$ that is to say $\beta_x = \beta_j$. But for $k \in [x+1, j-1]$ we have $\langle \beta_x^\vee, \beta_k \rangle = 0$ so in this case we have

$$\begin{aligned}
\bar{\phi} &= s_{i(\beta_1)} \cdots s_{i(\beta_{x-1})} s_{i(\beta_x)} s_{i(\beta_{x+1})} \cdots s_{i(\beta_{j-1})} s_{i(\beta_j)} s_{i(\beta_{j+1})} \cdots s_{i(\beta_n)} \\
&= s_{i(\beta_1)} \cdots s_{i(\beta_{x-1})} s_{i(\beta_x)} s_{i(\beta_j)} s_{i(\beta_{x+1})} \cdots s_{i(\beta_{j-1})} s_{i(\beta_{j+1})} \cdots s_{i(\beta_n)} \\
&= s_{i(\beta_1)} \cdots s_{i(\beta_{x-1})} s_{i(\beta_{x+1})} \cdots s_{i(\beta_{j-1})} s_{i(\beta_{j+1})} \cdots s_{i(\beta_n)}
\end{aligned}$$

that is to say the expression $\bar{\phi} = s_{i(\beta_1)} \cdots s_{i(\beta_n)}$ was not reduced, a contradiction. Thus we have $\langle \beta_x^\vee, \beta_j \rangle = -1$. For such a j we have $\langle \beta_x^\vee, \beta_k \rangle = 0$ for $k \in [x+1, j-1]$ and thus

$$\langle \alpha_x^\vee, \alpha_j \rangle = -\langle \beta_x^\vee, \beta_j \rangle = 1.$$

We have to prove that there exists $j > x$ such that $\langle \beta_x^\vee, \beta_j \rangle \neq 0$. If not we would have:

$$\bar{\phi} = s_{i(\beta_1)} \cdots s_{i(\beta_{x-1})} s_{i(\beta_x)} s_{i(\beta_{x+1})} \cdots s_{i(\beta_n)} = s_{i(\beta_1)} \cdots s_{i(\beta_{x-1})} s_{i(\beta_{x+1})} \cdots s_{i(\beta_n)} s_{i(\beta_x)}$$

and we would have $\beta_x = \beta_n$ (Remark 2.15) thus $\langle \beta_x^\vee, \beta_n \rangle = 2 \neq 0$, a contradiction. □

In the case $x < n$ let j be as in the lemma and consider the line bundles T_x and T_j. We have the formula (Corollary 3.8):

$$T_i = \sum_{k=1}^{i} \langle \alpha_k^\vee, \alpha_i \rangle \cdot \xi_k.$$

But $\langle \alpha_k^\vee, \alpha_i \rangle \geqslant 0$ for all i and k (Proposition 2.11) and $\alpha \cdot \xi_k \geqslant 0$ for all k by assumption, therefore

$$\alpha \cdot T_x \geqslant \langle \alpha_x^\vee, \alpha_x \rangle \alpha \cdot \xi_x = 2\alpha \cdot \xi_x > 0 \quad \text{and} \quad \alpha \cdot T_j \geqslant \langle \alpha_x^\vee, \alpha_j \rangle \alpha \cdot \xi_x = \alpha \cdot \xi_x > 0.$$

We construct the required $\tilde{f}'$ by induction on the fibrations. Let us denote by $g : \mathbb{P}^1 \to \widetilde{X}(\bar{\phi})$ the morphism whose image is C (cf. Proposition 4.6) and define $P = \tilde{f}(x_0)$ and P_k the image of P in X_k/B_k. Let us denote by $\tilde{f}_k$ (respectively g_k) the morphism from $\mathbb{P}^1$ to X_k/B_k induced by $\tilde{f}$ (respectively by g). We construct $\tilde{f}'$ by induction on the $\mathbb{P}^1$-fibration beginning with $\tilde{f}_{x-1}$.

Lemma 5.6. *Let $\varphi : X \to Y$ a $\mathbb{P}^1$-bundle and $\alpha \in A_1(X)$ such that $\alpha \cdot T > 0$ (T is the relative tangent sheaf). Let $f \in \mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ and $g : \mathbb{P}^1 \to X$ such that there exists $x_0 \in \mathbb{P}^1$ with $f(x_0) = g(x_0)$ and such that the images of $\varphi \circ f$ and $\varphi \circ g$ are distinct.*

Then there exists a deformation f' of f meeting g exactly in $f(x_0)$ and transversally.

Proof. Because the images are distinct the curves $\varphi \circ f(\mathbb{P}^1)$ and $\varphi \circ g(\mathbb{P}^1)$ meet each other in a finite number of points, say x_0 and $x_1, \ldots, x_k$. Let E a rank 2 vector bundle defining the fibration, we can choose E such that $(\varphi \circ f)^* E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ with $a \geqslant 0$. The morphism f is given by a surjection $s : (\varphi \circ f)^* E \to \mathcal{O}_{\mathbb{P}^1}(y)$ with $\alpha \cdot T = 2y - a > 0$ (this implies $y > 0$). A general surjection $s' : (\varphi \circ f)^* E \to \mathcal{O}_{\mathbb{P}^1}(y)$ will give a deformation of f. Because $y > 0$, we can take such a surjection such that $s'(x_i) \neq s(x_i)$ for $i \in [1, k]$, $s'(x_0) = s(x_0)$ but are not equal at order 2 in x_0. This gives us a morphism f' whose image meets the image of g only in $f(x_0)$ and transversally. □

If $\tilde{f}_{x-1}(\mathbb{P}^1) \neq g_{x-1}(\mathbb{P}^1)$ then thanks to the lemma we can construct $\tilde{f}'_x$ a deformation of $\tilde{f}_x$ meeting g_x only in P_x. Taking by induction any section of $\tilde{f}'_x$ passing through the points P_k for $k > x$ (this is possible because $\alpha \cdot T_k \geqslant 0$ for all k) we get the required deformation.

On the contrary if $\tilde{f}_{x-1}(\mathbb{P}^1) = g_{x-1}(\mathbb{P}^1)$ we use the following

Lemma 5.7. *Let $\varphi : X \to Y$ a $\mathbb{P}^1$-bundle and $\alpha \in A_1(X)$ such that $\alpha \cdot T > 0$ (T is the relative tangent sheaf). Let $f \in \mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ and $x_0 \in \mathbb{P}^1$.*

There exists a deformation f' of f such that f' and f have distinct images still meeting in $f(x_0)$.

Proof. Let E a rank 2 vector bundle defining the fibration, we can choose E such that $(\varphi \circ f)^* E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ with $a \geqslant 0$. The morphism f is given by a surjection $s : (\varphi \circ f)^* E \to \mathcal{O}_{\mathbb{P}^1}(y)$ with $\alpha \cdot T = 2y - a > 0$ (this implies $y > 0$). A general surjection $s' : (\varphi \circ f)^* E \to \mathcal{O}_{\mathbb{P}^1}(y)$ will give a deformation of f. Because $y > 0$, we can take such a surjection such that $s' \neq s$ and $s'(x_0) = s(x_0)$. This gives us the deformation f'. □

If $\tilde{f}_x = g_x$ then, thanks to the lemma we can construct $\tilde{f}'_x$ a deformation of $\tilde{f}_x$ meeting g_x in P_x and a finite number of points. If $\tilde{f}_x \neq g_x$ we can take $\tilde{f}'_x = \tilde{f}_x$. Taking by induction any section of $\tilde{f}'_x$ passing through the points P_k for $x < k < j$ (this is possible because $\alpha \cdot T_k \geqslant 0$ for all k) we get a deformation $\tilde{f}'_{j-1}$ of $\tilde{f}_{j-1}$ meeting g_{j-1} in P_{j-1} and a finite number of points. Because we have $\alpha \cdot T_j > 0$ we can use Lemma 5.6 to contruct a deformation $\tilde{f}'_j$ of $\tilde{f}_j$ meeting g_j exactly in P_j and transversally. Taking by induction any section of $\tilde{f}'_j$ passing through the points P_k for $k > j$ (this is possible because $\alpha \cdot T_k \geqslant 0$ for all k) we get the required deformation.

The only case left is the case $x = n$. In this case, because $n \geqslant 3$, we can consider β_{n-1} and β_{n-2}. Let us prove that $\langle \alpha_{n-2}^\vee, \alpha_n \rangle = \langle \alpha_{n-1}^\vee, \alpha_n \rangle = 1$. For $\langle \alpha_{n-1}^\vee, \alpha_n \rangle = -\langle \beta_{n-1}^\vee, \beta_n \rangle = 1$ it is just Corollary 2.16. For $\langle \alpha_{n-2}^\vee, \alpha_n \rangle$ we can apply Corollary 2.16 and it will be true except if $\beta_{n-2} = \beta_n = \beta$. But in this case we have

$$0 \leqslant \langle \alpha_{n-2}^\vee, \alpha_n \rangle = \langle \beta^\vee, s_\beta s_{\beta_{n-1}}(\beta_n) \rangle \langle \beta^\vee, s_\beta(\beta_{n-1} + \beta_n) \rangle = \langle \beta^\vee, \beta_{n-1} \rangle = -1.$$

This is impossible and we must have $\langle \alpha_{n-2}^\vee, \alpha_n \rangle = \langle \alpha_{n-1}^\vee, \alpha_n \rangle = 1$. This in particular implies that there are at least two $i < x = n$ such that $\langle \alpha_i^\vee, \alpha_x \rangle = 1$, namely $i = n-1$ and $i = n-2$.

Let us now consider the morphism $\tilde{f}_{n-1} : \mathbb{P}^1 \to X_{n-1}/B_{n-1}$ induced by $\tilde{f}$ and two morphisms $g : \mathbb{P}^1 \to \widetilde{X}(\bar{\phi})$ and $h : \mathbb{P}^1 \to \widetilde{X}(\bar{\phi})$ such that $g_*[\mathbb{P}^1] = [\Gamma_{n,n-1}]$ and $h_*[\mathbb{P}^1] = [\Gamma_{n,n-2}]$. Because the classes $[\Gamma_{n,n-1}]$ and $[\Gamma_{n,n-2}]$ are distinct, the morphism $\tilde{f}_{n-1}$ has to be distinct from one of the morphisms $g_{n-1} : \mathbb{P}^1 \to X_{n-1}/B_{n-1}$ and $h_{n-1} : \mathbb{P}^1 \to X_{n-1}/B_{n-1}$ deduced from g and h. Let us say that $\tilde{f}_{n-1} \neq g_{n-1}$ then applying Lemma 5.6 we get a deformation $\tilde{f}'$ of $\tilde{f}$ meeting g only in $\tilde{f}(x_0)$ and transversally. □

Proposition 5.8. *Let C and $\tilde{f}'$ as in the preceding proposition, the curve $\tilde{f}'(\mathbb{P}^1) \cup C$ can be smoothed that is to say deformed to a smooth curve. The smoothing is the image of a morphism $\hat{f} : \mathbb{P}^1 \to \widetilde{X}(\bar{\phi})$ and we have*

$$[\hat{f}(\mathbb{P}^1)] \cdot \xi_x = [\tilde{f}'(\mathbb{P}^1)] \cdot \xi_x - 1 = [\tilde{f}(\mathbb{P}^1)] \cdot \xi_x - 1.$$

Proof. We will use the following proposition proved in [9, Corollary 1.2] for $\mathbb{P}^3$ but valid for any smooth variety X:

Proposition 5.9. *Let D be a nodal curve in a smooth variety X and assume that $H^1(T_X|_D) = 0$ then there exists a smooth deformation of D.*

In order to prove Proposition 5.8 it suffices to prove that $H^1(T_X|_D) = 0$ where $X = \widetilde{X}(\bar{\phi})$ and $D = \tilde{f}'(\mathbb{P}^1) \cup C$ (which is a nodal curve). Let P be the intersection point, we have the exact sequence

$$0 \to \mathcal{O}_{\tilde{f}'(\mathbb{P}^1)}(-P) \to \mathcal{O}_D \to \mathcal{O}_C \to 0$$

and it is enough to prove that $H^1(T_X|_C) = 0$ and $H^1(T_X|_{\tilde{f}'(\mathbb{P}^1)}(-P)) = 0$.

One more time we do it by induction on the fibrations. Denote by $\tilde{f}'_k : \mathbb{P}^1 \to X_k/B_k$ the morphism induced by $\tilde{f}'$ and C_k the image of C in X_k/B_k. We assume that

$$H^1(T_{X_{j-1}/B_{j-1}}|_{C_{j-1}}) = 0 \quad \text{and} \quad H^1\big(T_{X_{j-1}/B_{j-1}}|_{\tilde{f}'_{j-1}(\mathbb{P}^1)}(-P)\big) = 0.$$

We are going to prove that $H^1(T_{X_j/B_j}|_{C_j}) = 0$ and $H^1(T_{X_j/B_j}|_{\tilde{f}'_j(\mathbb{P}^1)}(-P)) = 0$.

We have an exact sequence

$$0 \to T_j \to T_{X_j/B_j} \to T_{X_{j-1}/B_{j-1}} \to 0$$

so it suffices to prove that $H^1(T_j|_{C_j}) = 0$ and $H^1(T_j|_{\tilde{f}'_j(\mathbb{P}^1)}(-P)) = 0$. But we have seen in the proof of Proposition 4.6 that $[\Gamma_{x,i}] \cdot T_j \geqslant -1$ thus the restriction of T_j on C_j is $\mathcal{O}_{\mathbb{P}^1}(u)$ with $u \geqslant -1$ and we have the first vanishing. In the same way, we have $\alpha \cdot T_j \geqslant 0$ thus the restriction of T_j on $\tilde{f}'_j(\mathbb{P}^1)$ is $\mathcal{O}_{\mathbb{P}^1}(v)$ with $v \geqslant 0$ and we have the second vanishing. □

6. Curves on minuscule Schubert varieties

In this section, we prove our main theorem on the irreducible components of the scheme of morphisms from $\mathbb{P}^1$ to $X(\bar{\phi})$ a minuscule Schubert variety.

6.1. Moving out Schubert subvarieties

We begin to prove that a general curve in $X(\bar{\phi})$ is not contained in a Schubert subvariety:

Proposition 6.1. *Consider a morphism $f : \mathbb{P}^1 \to X(\bar{\phi})$ such that f factors through a Schubert variety $X(\bar{\phi}') \subset X(\bar{\phi})$ (with $\bar{\phi}' < \bar{\phi}$) then there exists a deformation $f' : \mathbb{P}^1 \to X(\bar{\phi})$ of f such that f' does not factor through $X(\bar{\phi}')$.*

Proof. Restricting ourselves to a smaller Schubert variety, we may assume that the Schubert variety $X(\bar{\phi}')$ is a Schubert divisor $X(\bar{s}_\beta\phi)$ of $X(\bar{\phi})$. But (Theorem 2.8) this divisor has to be a moving divisor so β is simple and (Proposition 2.7) there exists a reduced expression

$$\bar{\phi} = s_{\gamma_1}\cdots s_{\gamma_n}$$

of $\bar{\phi}$ where $\gamma_1 = \beta$ and $\bar{\phi}' = s_\beta\bar{\phi}$. Consider the Bott–Samelson resolution $\widetilde{X}(\bar{\phi}')$. If we denote with a prime the corresponding elements in the Bott–Samelson construction we have

$$B_{i+1} = s_{i(\beta)}(B_i'), \qquad P_{i+1} = s_{i(\beta)}(P_i')$$

thus

$$\widetilde{X}(\bar{\phi}') = X_{n-1}'/B_{n-1}' = s_{i(\beta)}\big(f_n^{-1}\cdots f_2^{-1}\sigma_1(X_0/B_0)\big).$$

This shows that we can identify $\widetilde{X}(\bar{\phi}')$ with the subscheme $f_n^{-1}\cdots f_2^{-1}\sigma_1(X_0/B_0)$ of $\widetilde{X}(\bar{\phi})$ which is a fiber of the projection of $\widetilde{X}(\bar{\phi})$ on X_1/B_1. We have the commutative diagram:

$$\begin{array}{ccc} \widetilde{X}(\bar{\phi}') & \hookrightarrow & \widetilde{X}(\bar{\phi}) \\ \downarrow & & \downarrow \\ X(\bar{\phi}') & \hookrightarrow & X(\bar{\phi}). \end{array}$$

The unipotent group $U_{-i(\beta)}$ acts equivariantly on the second vertical map and moves the first one. We may assume by induction that f does not factor through any Schubert subvariety of $X(\bar{\phi}')$ so we can find a section $g:\mathbb{P}^1 \to \widetilde{X}(\bar{\phi}') \subset \widetilde{X}(\bar{\phi})$ of f. We can deform g in $X(\bar{\phi})$ thanks to the action of $U_{-i(\beta)}$ and we obtain a morphism $g:\mathbb{P}^1 \to \widetilde{X}(\bar{\phi})$ not contained in $\widetilde{X}(\bar{\phi}')$. Projecting on $X(\bar{\phi})$ gives a deformation f' of f not contained in $X(\bar{\phi}')$. □

Corollary 6.2. *For any morphism $f:\mathbb{P}^1 \to X(\bar{\phi})$ there exists a deformation f' of f such that f' does not factor through any $X(\bar{\psi}) \subset X(\bar{\phi})$ (with $\bar{\psi} < \bar{\phi}$).*

Proof. Remark that if f does not factor through a subvariety of $X(\bar{\phi})$ then it is also the case of any deformation. As there is a finite number of Schubert varieties contained in $X(\bar{\phi})$, we apply the preceding proposition for each subvariety containing the image of f. □

Let $\pi:\widetilde{X}(\bar{\phi}) \to X(\bar{\phi})$ a Bott–Samelson resolution. The preceding result implies that for any morphism $f:\mathbb{P}^1 \to X(\bar{\phi})$ there exist a deformation $f':\mathbb{P}^1 \to X(\bar{\phi})$ of f (the deformation of the previous corollary) such that $f'(\mathbb{P}^1)$ meets the regular locus of π. We can thus consider a section $\tilde{f}$ of f':

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\tilde{f}} & \widetilde{X}(\bar{\phi}) \\ & \searrow^{f'} & \downarrow \pi \\ & & X(\bar{\phi}). \end{array}$$

Remark 6.3. The image $\tilde{f}(\mathbb{P}^1)$ is not contained in any divisor D_i on $\widetilde{X}(\bar{\phi})$. In fact, if it was the case it would means that $f'(\mathbb{P}^1)$ is contained in $\pi(D_i)$ which is a strict Schubert subvariety of $X(\bar{\phi})$. This is impossible.

Corollary 6.4. *The morphism $\tilde{f}$ constructed from f thanks to Corollary* 6.2 *is such that*

$$\big[\tilde{f}\big(\mathbb{P}^1\big)\big] \cdot \xi_i \geqslant 0 \quad \textit{for all } i \in [1, n].$$

Proposition 6.5. *For any morphism $f : \mathbb{P}^1 \to X(\bar{\phi})$ there exist a deformation f' of f such that f' does not meet the image $\pi(D_x)$ of any contracted divisor D_x.*

Proof. We can replace f by the deformation f' of Corollary 6.2. We can thus assume that $f(\mathbb{P}^1)$ is not contained in any $\pi(D_i)$ for $i \in [1, n]$. We then have a section $\tilde{f} : \mathbb{P}^1 \to \widetilde{X}(\bar{\phi})$ of f. Let us denote $\alpha = \tilde{f}_*[(\mathbb{P}^1)]$, we have $\alpha \cdot \xi_i \geqslant 0$ for all i. Define the subset $A \subset [1, n]$ of all integers k such that D_k is a contracted divisor and set

$$l(\alpha) = \sum_{k \in A} \alpha \cdot \xi_k.$$

We prove the result by induction on $l(\alpha)$. If $l(\alpha) = 0$ then $\tilde{f}$ does not meet any contracted divisor so f does not meet the image $\pi(D_x)$ of any contracted divisor D_x. Let x be the smallest element in A such that there exists a morphism $g : \mathbb{P}^1 \to X(\bar{\phi})$ not contained in any Schubert subvariety with a section $\tilde{g} : \mathbb{P}^1 \to \widetilde{X}(\bar{\phi})$ such that $\beta = \tilde{g}_*[(\mathbb{P}^1)]$ with $l(\beta) = l(\alpha)$, $\beta \cdot \xi_x > 0$ and for which we have not constructed the required deformation yet. Thanks to Propositions 5.4 and 5.8, for such a g and $\tilde{g}$ a section in $\widetilde{X}(\bar{\phi})$ there exists a deformation $\tilde{g}'$ of $\tilde{g}$, an integer $i < x$ with $\langle \alpha_i^\vee, \alpha_x \rangle = 1$ and a curve $C \in [\Gamma_{x,i}]$ such that $\tilde{g}'(\mathbb{P}^1) \cup C$ can be smoothed in $\hat{g}(\mathbb{P}^1)$. The morphism $\pi \circ \hat{g}$ deforms to g and we have $\hat{\beta} = \hat{g}_*[\mathbb{P}^1] = \beta + [\Gamma_{x,i}]$. We thus have

$$\hat{\beta} \cdot \xi_k \begin{cases} \beta \cdot \xi_k + 1, & \text{for } k = i, \\ \beta \cdot \xi_k - 1, & \text{for } k = x, \\ \beta \cdot \xi_k, & \text{otherwise.} \end{cases}$$

If $i \in A$ then because of our minimality assumption on x we know that there exists a required deformation for $\pi \circ \hat{g}$ and we can conclude because g is a deformation of this deformation. If on the contrary $i \notin A$ then we have

$$l(\hat{\beta}) = \sum_{k \in A} \hat{\beta} \cdot \xi_k = l(\beta) - 1 = l(\alpha) - 1$$

and by induction there exists a required deformation for $\pi \circ \hat{g}$ and we can conclude as above. □

Corollary 6.6. *There exists a dense open subset of* $\mathbf{Hom}(\mathbb{P}^1, X(\bar{\phi}))$ *whose elements do not meet the image* $\pi(D_x)$ *of any contracted divisor* D_x *and are not contained in any* $\pi(D_k)$ *for* $k \in [1, n]$.

Theorem 6.7. *Let* $\alpha \in A_1(X(\bar{\phi}))$*, the irreducible components of* $\mathbf{Hom}_\alpha(\mathbb{P}^1, X(\bar{\phi}))$ *are indexed by* $\mathfrak{ne}(\alpha)$.

Proof. We have a surjective morphism $\pi_* : A_1(\widetilde{X}(\bar{\phi})) \to A_1(X(\bar{\phi}))$ and a natural morphism

$$\coprod_{\pi_*(\tilde{\alpha}) = \alpha} \mathbf{Hom}_{\tilde{\alpha}}\big(\mathbb{P}^1, \widetilde{X}(\bar{\phi})\big) \to \mathbf{Hom}_\alpha\big(\mathbb{P}^1, X(\bar{\phi})\big).$$

Let us prove that the irreducible components of $\mathbf{Hom}_\alpha(\mathbb{P}^1, X(\bar{\phi}))$ are indexed by the set $C(\alpha)$ of classes $\tilde{\alpha} \in A_1(\widetilde{X}(\bar{\phi}))$ such that $\pi_*(\tilde{\alpha}) = \alpha$, $\tilde{\alpha} \cdot \xi_k \geqslant 0$ for all $k \in [1, n]$ and $\tilde{\alpha} \cdot \xi_x = 0$ for all x such that D_x is a contracted divisor.

Because of Corollary 6.6 we know that a general morphism $f \in \mathbf{Hom}_\alpha(\mathbb{P}^1, X(\bar{\phi}))$ can be lifted into $\tilde{f} \in \mathbf{Hom}_{\tilde{\alpha}}(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ such that $\tilde{\alpha} \in C(\alpha)$. We thus have a dominant morphism

$$\coprod_{\tilde{\alpha} \in C(\alpha)} \mathbf{Hom}_{\tilde{\alpha}}\big(\mathbb{P}^1, \widetilde{X}(\bar{\phi})\big) \to \mathbf{Hom}_\alpha\big(\mathbb{P}^1, X(\bar{\phi})\big).$$

Let $\tilde{\alpha} \in C(\alpha)$ and $\tilde{f}$ a general element in $\mathbf{Hom}_{\tilde{\alpha}}(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ (this scheme is irreducible thanks to Proposition 5.1). We know (Corollary 6.6 and Proposition 5.1) that its image is contained is the regular locus of π. If the morphism $\pi \circ \tilde{f}$ was in the image of $\mathbf{Hom}_{\tilde{\alpha}'}(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ then we would have a morphism $\tilde{f}'$ of class $\tilde{\alpha}'$ such that $\pi \circ \tilde{f}' = \pi \circ \tilde{f}$. But because these curves are contained in the regular locus of π this implies that $\tilde{f} = \tilde{f}'$ and $\tilde{\alpha}' = \tilde{\alpha}$. The images of the $\mathbf{Hom}_{\tilde{\alpha}}(\mathbb{P}^1, \widetilde{X}(\bar{\phi}))$ for $\tilde{\alpha} \in C(\alpha)$ are the irreducible components of $\mathbf{Hom}_\alpha(\mathbb{P}^1, X(\bar{\phi}))$.

To conclude the proof we have to show that $C(\alpha) = \mathfrak{ne}(\alpha)$. We begin with the following

Lemma 6.8. *The kernel* K *of the map* $\pi_* : A_{n-1}(\widetilde{X}(\bar{\phi})) \to A_{n-1}(X(\bar{\phi}))$ *is generated by the classes* ξ_x *of the contracted divisors* D_x.

Proof. M. Demazure proved in [4] that the morphism π is an isomorphism on the big cell of the Schubert variety $X(\bar{\phi})$. This in particular implies that the locus $\widetilde{D}$ in $\widetilde{X}(\bar{\phi})$ where π

is not an isomorphism is contained in $\bigcup_i D_i$. Moreover, if the divisor D_i is not contracted the open part $D_i - \bigcup_{j\neq i}(D_i \cap D_j)$ is not contained in $\widetilde{D}$ so that the codimension one part (in $\widetilde{X}(\bar{\phi})$) of $\widetilde{D}$ is the union of the contracted divisors D_x.

Let us denote by $\widetilde{U}$ the open part in $\widetilde{X}(\bar{\phi})$ where π is an isomorphism and U its image in $X(\bar{\phi})$. On the one hand, the kernel of the surjective map $A_{n-1}(\widetilde{X}(\bar{\phi})) \to A_{n-1}(\widetilde{U})$ is generated by the contracted divisors D_x. On the other hand, we have $A_{n-1}(\widetilde{U}) = A_{n-1}(U)$ and because the complement of U in $X(\bar{\phi})$ is in codimension at least 2 (it is the image of $\widetilde{D}$ with fibers of dimension at least 1 because Schubert varieties are normal), we have $A_{n-1}(U) = A_{n-1}(X(\bar{\phi}))$. $\square$

As $\widetilde{X}(\bar{\phi}))$ is smooth and projective we can identify $A_{n-1}(\widetilde{X}(\bar{\phi}))^\vee$ with $A_1(\widetilde{X}(\bar{\phi}))$ and π_* gives us a morphism $A_{n-1}(\widetilde{X}(\bar{\phi}))^\vee \to A_1(X(\bar{\phi}))$. The lemma leads to the following diagram whose first line is exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{n-1}\big(X(\bar{\phi})\big)^\vee & \longrightarrow & A_{n-1}\big(\widetilde{X}(\bar{\phi})\big)^\vee & \longrightarrow & K^\vee & \longrightarrow & 0 \\ & & & \searrow^{s} & \downarrow{\scriptstyle \pi} & & & & \\ & & & & A_1\big(X(\bar{\phi})\big). & & & & \end{array}$$

Now we can translate the definition of $C(\alpha)$ in terms of $A_{n-1}(X(\bar{\phi}))^\vee$. Indeed, because of the vanishing condition on contracted divisor, all the elements of $C(\alpha)$ are in $A_{n-1}(X(\bar{\phi}))^\vee$ and go on α by s. What is left to prove is the following

Lemma 6.9. *An element $\tilde{\alpha} \in A_{n-1}(X(\bar{\phi}))^\vee$ seen as an element in $A_1(\widetilde{X}(\bar{\phi}))$ is effective if and only if $\tilde{\alpha} \cdot \xi_i \geqslant 0$ for all $i \in [1, n]$.*

Proof. We have seen Proposition 5.1 that if all the intersection $\tilde{\alpha} \cdot \xi_i$ are non-negative then the class is effective.

Let $\tilde{\alpha} \in A_{n-1}(X(\bar{\phi}))^\vee$ an effective class. Because $\tilde{\alpha}$ is in $A_{n-1}(X(\bar{\phi}))^\vee$ we know that its intersection with all contracted D_x are 0. Let D_i a not contracted divisor, then its image in $X(\bar{\phi})$ is a moving divisor (Theorem 2.8). Let C a curve of class $\tilde{\alpha}$, if C is not contained in D_i then $C \cdot \xi_i \geqslant 0$. If C is contained in D_i then as in the proof of Proposition 6.1 we can deform this curve in the class $\tilde{\alpha}$ so that it is not contained in D_i and we have $C \cdot \xi_i \geqslant 0$. $\square$

This proves that $C(\alpha) = \mathfrak{ne}(\alpha)$ and the theorem follows. Indeed, $\mathfrak{ne}(\alpha)$ is given (cf. Section 1) by the elements $\beta \in \mathrm{Pic}(U)^\vee$ in the dual of the cone of effective divisors (U is the dense orbit under $\mathrm{Stab}(X(\bar{\phi}))$). But $\mathrm{Pic}(U) = A_{n-1}(U) = A_{n-1}(X(\bar{\phi}))$ and the effective cone is generated by the $\pi_*\xi_i$ with D_i not contracted. $\square$

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