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A note on an example by van Mill

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Abstract

Adapting an earlier example by J. van Mill, we prove that there exists a zero-dimensional compact space of countable π-weight and uncountable character that is homogeneous under $\mathsf{MA} + \neg\mathsf{CH}$, but not under CH.

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1. Introduction

In [4] van Mill constructed a compact Hausdorff space of countable π-weight and character $\aleph_1$ with the curious property that $\mathsf{MA} + \neg\mathsf{CH}$ implies it is homogeneous whereas CH implies that it is *not*.

The space was obtained as a resolution of the Cantor set, where every point was replaced by the uncountable torus T^{ω_1}. Close inspection of the proof of homogeneity from $\mathsf{MA} +$

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¬CH reveals that one needs any possible replacement, Y, of T^{ω_1} to have the following properties:

(1) Y is compact Hausdorff and homogeneous;
(2) the weight and character of Y are $\aleph_1$;
(3) there are an autohomeomorphism η of Y and a point d such that the positive orbit $\{\eta^n(d)\colon n \in \omega\}$ is dense; and
(4) Y is a retract of $\gamma\mathbb{N}$, whenever $\gamma\mathbb{N}$ is a compactification of $\mathbb{N}$ with Y as its remainder.

In Section 2 of this note we shall show that these four properties do indeed suffice for a proof of homogeneity from MA + ¬CH and in Section 3 we exhibit a zero-dimensional space with these properties. This then will establish that there is a zero-dimensional compact Hausdorff space of countable π-weight and character $\aleph_1$ with the curious property that MA + ¬CH implies it is homogeneous whereas CH implies that it is *not*. Indeed, the power $2^{\omega_1\times\mathbb{Z}}$ with the mapping η, defined by $\eta(x)(\alpha, n) = x(\alpha, n+1)$, is as required.

2. The construction

To keep our presentation self-contained we give an alternative description of van Mill's construction. We fix a compact space Y as in the introduction, along with the map η and the point d; for each $n \in \omega$ we write $d_n = \eta^n(d)$.

The underlying set of X is the product $\mathsf{C} \times Y$, where C is the Cantor set 2^ω. Before we define the topology on X we fix some notation.

Given $s \in 2^{<\omega}$, so s is a finite sequence of zeros and ones, we put

$$[s] = \{x \in \mathsf{C}\colon s \subseteq x\}.$$

The family $\{[s]\colon s \in 2^{<\omega}\}$ is the canonical base for the topology of C. If $s \in 2^{<\omega}$ and $x \in \mathsf{C}$ then $s * x$ denotes the concatenation of s and x.

Given $x \in \mathsf{C}$ and $n \in \omega$ we put $U_{x,n} = [x \restriction n]$, the nth basic neighbourhood of x, and $C_{x,n} = U_{x,n} \setminus U_{x,n+1}$. Note that $C_{x,n}$ is of the form $U_{y,n+1}$ for some suitably chosen $y \in \mathsf{C}$.

Let $\langle x, y\rangle$ be a point of X. The basic neighbourhoods of $\langle x, y\rangle$ will be those of the form

$$U_{x,n} \otimes W = \bigl(\{x\} \times W\bigr) \cup \bigcup\{C_{x,m} \times Y\colon m \geqslant n,\ d_m \in W\},$$

with $n \in \omega$ and where W runs through the neighbourhoods of y in Y.

We will use this description of X but we invite the interested reader to verify that X is indeed a resolution of C, where each x is resolved into Y via the map $f_x : \mathsf{C} \setminus \{x\} \to Y$ defined by $f_x(y) = d_n$ iff $y \in C_{x,n}$, see [2,5] for details on resolutions.

The following lemmas are easily verified and left to the reader.

Lemma 2.1. *If $U_{x,n}$ and W are clopen then so is $U_{x,n} \otimes W$, hence X is zero-dimensional if Y is.*

Lemma 2.2. *The character of $\langle x, y\rangle$ in X is the same as the character of y in Y, hence $\chi(X) = \aleph_1$.*

Lemma 2.3. *The family* $\{[s] \times Y\colon\ s \in 2^{<\omega}\}$ *is a* π*-base for* X*, hence* $\pi(X) = \aleph_0$.

The next lemma should be compared with [5, Theorem 3.1.33].

Lemma 2.4. *The space* X *is compact Hausdorff.*

Proof. Let $\mathcal{U}$ be a basic open cover of X. Much like in a standard proof that the product of two compact spaces is compact one proves that around every vertical line one can put an open strip that is covered by finitely many elements of $\mathcal{U}$. Let $z \in C$. In the case that there is a set $U_{x,n} \otimes W \in \mathcal{U}$ such that $z \in C_{x,m}$ for some m we are done. In the other case for every $y \in Y$ there are W_y and n_y such that $y \in W_y$ and $U_{z,n_y} \otimes W_y \in \mathcal{U}$; finitely many of these, indexed by the finite set F say, will cover $\{z\} \times Y$. Let $n = \max\{n_y\colon\ y \in F\}$, then these sets cover the strip $U_{z,n} \times Y$ as well. $\square$

At this point we know that CH implies that X is not homogeneous, because under CH we have $\chi(X) = 2^{\pi(X)}$. Corollary 1.2 in [4] forbids this for homogeneous compacta.

We now begin to work toward a proof that X is homogeneous if $\mathsf{MA} + \neg\mathsf{CH}$ is assumed. As a first step we show that points with the same second coordinate are similar.

Lemma 2.5. *For each* $a \in C$ *the map* T_a *defined by* $T_a(x, y) = \langle x + a, y\rangle$ *is an autohomeomorphism of* X.

Proof. One easily verifies that $T_a[U_{x,n} \otimes W] = U_{x+a,n} \otimes W$, and this suffices. $\square$

The hard work will be in establishing that points with the same first coordinate are similar. We begin by showing that the special clopen sets $[s] \times Y$ are all homeomorphic and we give canonical homeomorphisms between them.

Lemma 2.6. *Let* $s, t \in 2^{<\omega}$*, put* $k = |t| - |s|$ *and define* $\xi_{s,t} : [s] \times Y \to [t] \times Y$ *by* $\xi_{s,t}(s * x, y) = \langle t * x, \eta^k(y)\rangle$. *Then* $\xi_{s,t}$ *is a homeomorphism.*

Proof. It is not hard to show that $\xi_{s,t}[U_{s*x,n} \otimes W] = U_{t*x,n+k} \otimes \eta^k[W]$, which suffices. $\square$

In this lemma k may be positive or negative and we need both possibilities, see the last three lines of this section.

For ease of notation we let e be the point of C with all coordinates zero and we abbreviate $U_{e,n}$ and $C_{e,n}$ by U_n and C_n respectively. We shall prove, assuming $\mathsf{MA} + \neg\mathsf{CH}$, that for every homeomorphism $f : Y \to Y$ there is a homeomorphism $\bar{f} : X \to X$ such that $\bar{f}(e, y) = \langle e, f(y)\rangle$ for all $y \in Y$. This will complete the proof that X is homogeneous under this assumption because it shows that all points of the form $\langle e, y\rangle$ are similar.

For $n \in \omega$ let $x_n \in$ C be the point in C_n with all coordinates zero except for the nth. Let $E = \{\langle x_n, d_n\rangle\colon\ n \in \omega\}$; note that E is discrete. The proof of the following lemma is that of [4, Lemma 5.1].

Lemma 2.7. $\overline{E} = E \cup (\{e\} \times Y)$.

Thus, $\overline{E}$ is a compactification of $\mathbb{N}$, whose remainder is Y.

Let f be an autohomeomorphism of Y, considered to be acting on $\{e\} \times Y$. Theorem 4.2 from [4], an extension of a result due to Matveev [3], now implies that there is an autohomeomorphism F of $\overline{E}$ that extends f. This requires assumptions (1) and (4) on Y. It suffices to know that the weight of Y is less than the cardinal $\mathfrak{p}$; the inequality $\aleph_1 < \mathfrak{p}$ follows from $\mathsf{MA} + \neg\mathsf{CH}$.

Note that the action of F on E is given by a permutation τ of ω.

We use τ to define our extension $\bar{f}$: use the maps $\xi_{s,t}$ from Lemma 2.6 to map $C_n \times Y$ onto $C_{\tau(n)} \times Y$ for all n. The verification that $\bar{f}$ is a homeomorphism is as in [4].

3. The input

We now show that the space $Y = 2^{\omega_1 \times \mathbb{Z}}$ provides suitable input for the construction in the previous section.

It is clear that Y is zero-dimensional, compact and Hausdorff and that its weight and character are $\aleph_1$. Since Y is a product of second countable compacta, condition (4) is also satisfied, by [4, Corollary 4.5].

We need to find an autohomeomorphism η and a point d such that $\{\eta^n(d)\colon n \in \omega\}$ is dense in Y. The map $\eta : Y \to Y$ is defined co-ordinatewise as follows (with $\alpha \in \omega_1$ and $i \in \mathbb{Z}$),

$$\eta(y)(\alpha, i) = y(\alpha, i + 1).$$

We may think of points of Y as ω_1 by $\mathbb{Z}$ matrices. The action of η on such a matrix consists of shifting every row one step downwards. For $n < \omega$, $[-n, n]$ is the set $\{-n, -n + 1, \ldots, n - 1, n\}$.

To make the point d we take a countable dense subset Q of $2^{\omega_1 \times \mathbb{Z}}$ (cf. [1, Theorem 2.3.15]) and we enumerate $Q \times \omega$ as $\{\langle q_k, n_k\rangle\colon k < \omega\}$. We define the point d by concatenating the restrictions $q_k \restriction \omega_1 \times [-n_k, n_k]$. First write $N_k = \sum_{j<k}(2 \cdot n_j + 1)$ for all k and then define, for each α and each n

$$d(\alpha, n) = \begin{cases} 0 & n < 0, \\ q_k(\alpha, n - N_k - n_k) & N_k \leqslant n < N_{k+1}. \end{cases}$$

Next we verify that the point d has a dense positive orbit under η. Observe that it follows by construction that for all $k < \omega$ we have

$$\eta^{N_k + n_k}(d) \restriction \bigl(\omega_1 \times [-n_k, n_k]\bigr) = q_k \restriction \bigl(\omega_1 \times [-n_k, n_k]\bigr).$$

Now let an arbitrary basic open subset U of $2^{\omega_1 \times \mathbb{Z}}$ be given by a function $s : F \to 2$, where $F \subseteq \omega_1 \times \mathbb{Z}$ is finite. Thus U is given by

$$U = \bigl\{y \in 2^{\omega_1 \times \mathbb{Z}}\colon s \subseteq y\bigr\}.$$

Since F is finite, we may find n such that $F \subseteq \omega_1 \times [-n, n]$. The set Q was chosen dense in $2^{\omega_1 \times \mathbb{Z}}$, so there is a $k < \omega$ with $q_k \in U$ and $n_k = n$. It follows that

$$q_k \restriction \bigl(\omega_1 \times [-n_k, n_k]\bigr) \supseteq s$$

from which it follows that

$$\eta^{N_k+n_k}(d) \restriction \bigl(\omega_1 \times [-n_k, n_k]\bigr) \supseteq s.$$

This implies that $\eta^{N_k+n_k}(d) \in U$.

We find that the set $\{\eta^n(d)\colon n < \omega\}$ is dense in $2^{\omega_1 \times \mathbb{Z}}$, which means that we are done.

Remark 3.1. This argument shows that also $2^{\mathfrak{c} \times \mathbb{Z}}$ has a point with a dense orbit under the shift, that is, not only is the Cantor cube $2^{\mathfrak{c}}$ separable, it even has an autohomeomorphism η and a point d whose positive orbit is dense.

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