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Degeneration of the Solutions of Certain Well-Posed Systems of Partial Differential Equations Depending on a Small Parameter

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1. INTRODUCTION, NOTATION

In 1950 N. Levinson [1] considered the Dirichlet problem for the system

$$\epsilon \Delta u + A(x, y)\, u_x + B(x, y)\, u_y + C(x, y)\, u = D(x, y)$$

$$u \big|_{\partial R} = \phi$$

in a region $R \subset E_2$ with a suitable boundary ∂R, where ϵ is a small positive parameter. The objective was to determine whether or not the solution $u = u_\epsilon$ of this problem tended to some solution U of the reduced equation

$$A(x, y)\, U_x + B(x, y)\, U_y + C(x, y)\, U = D(x, y)$$

obtained by setting $\epsilon = 0$ in the original equation. Clearly, it is not in general possible for U to satisfy all the boundary conditions which u does; hence u cannot tend to U as $\epsilon \to 0$ in the whole region $R + \partial R$. Under suitable hypotheses Levinson found that if U satisfies the same boundary values as u on a certain subset S of ∂R, the following asymptotic formula holds in a certain subset $\bar{R}'$ of R:

$$u_\epsilon(x, y) = U(x, y) + h(x, y)\, e^{-g(x,y)/\epsilon} + 0(\epsilon^{1/2}).$$

Here $g(x, y)$ is a function which is positive outside $(\partial R - S) \cap \bar{R}'$ ($\bar{R}'$ = closure of R') and equals zero on $(\partial R - S) \cap \bar{R}'$. Thus it follows that $u_\epsilon \to U$ uniformly in every closed subregion of $\bar{R}'$ not containing any part of $(\partial R - S)$. The term $h(x, y)\, e^{-g(x,y)/\epsilon}$ compensates for the fact that $U(x, y)$ cannot in general equal the boundary values of $u(x, y)$ on $(\partial R - S) \cap \bar{R}'$ because it is the solution of a lower-order equation.

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The work of Levinson has been greatly expanded by Višik and Lyusternik [2], [3], who consider the degeneration of higher-order elliptic equations and parabolic equations as well. These two papers, especially the former, have become standard references on regular degeneration problems.

Already in his *Lectures on Cauchy's Problem*, J. Hadamard had considered the heat equation

$$u_{xx} - u_t = 0$$

as a limiting case ($\epsilon = 0$) of the hyperbolic equation

$$u_{xx} + \epsilon u_{xt} - u_t = 0$$

by showing that the Riemann function for this latter equation (which is equivalent to the telegraphist's equation) tends, as $\epsilon \to 0$, to the fundamental solution of the heat equation. The method used is the direct one of asymptotically expanding the Riemann function.

In several papers, [4], [5], [6], [7], M. Zlamal has considered both the Cauchy problem and the mixed problem for the telegraphist's equation

$$\epsilon \frac{\partial^2 u_\epsilon}{\partial t^2} + \beta \frac{\partial u_\epsilon}{\partial t} - \Delta^2 u_\epsilon = 0$$

with a small parameter ϵ multiplying the highest-order time derivative. Here the order of the equation does not drop when the parameter is set equal to zero, but rather the equation changes from hyperbolic to parabolic type, and for this reason the solution U of the degenerate equation

$$\beta \frac{\partial U}{\partial t} - \Delta^2 U = 0$$

cannot satisfy all the initial conditions which u_ϵ can. This problem may be considered the prototype of the systems considered herein; a treatment of the pure Cauchy problem for the telegraphist's equation is given in Section 6 as an example of the more general results obtained.

The following notation will be used throughout. All problems will be considered for $n + 1$ independent variables, of which one, the time t, will be singled out, the remainder being grouped into the n-vector of space variables $\mathbf{x}$. ζ will denote the space coordinates in the Fourier-transform space, α an n-vector whose components are nonnegative integers. For any n-vector $y = (y_1, \ldots, y_n)$ we define $|y| = \sum_{i=1}^{n} |y_i|$; this introduces a norm on E_n which is equivalent to the Euclidean norm. We shall write $f(\mathbf{x})$ for an N-column each of whose entries is a function of $\mathbf{x}$; the Fourier transform of this column will be denoted $\hat{f}(\zeta)$, provided it exists:

$$\hat{f}(\zeta) = (2\pi)^{-n/2} \int_{E_n} e^{i(\mathbf{x},\zeta)} f(\mathbf{x})\, d\mathbf{x}.$$

Here $(\mathbf{x}, \zeta)$ is the ordinary inner product of $\mathbf{x}$ and ζ, and the integral is taken over the entire $\mathbf{x}$-space E_n. $\mathbf{x}^\alpha$ will mean $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$; writing $D = (D_1, D_2, \ldots, D_n)$, where $D_i = (\partial/\partial x_i)$, D^α will denote $D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$. We will consistently use $A(\epsilon)$ to denote a diagonal $N \times N$ matrix which has $N - m$ entries ϵ followed by m entries 1; ϵ will always be a small non-negative parameter. If C is a matrix (or a vector), we denote by $| C |$ any suitable norm equivalent to the Euclidean one $(\sum_{i,j} | c_{ij} |^2)^{1/2}$.

Our concern will be the following system of N partial differential equations of the first order in t:

$$L_\epsilon[V] \equiv A(\epsilon)\, V_t + \sum_{|\alpha| \leqslant p} B_\alpha D^\alpha V = 0, \tag{1}$$

where p is a positive integer and the B_α are $N \times N$ matrices of constants. The vector V is to satisfy

$$V(0, \mathbf{x}) = f(\mathbf{x}). \tag{2}$$

The solution of course depends on ϵ and may be denoted V_ϵ when it is desirable to display this dependence. The problem to be considered here is the determination of conditions under which a limit problem, the ***degenerate problem***, associated with the degenerate operator L_0 is meaningful, and then to study the degeneration of the solution V_ϵ of (1), (2) "to" the solution V_0 of this degenerate problem. As could be expected, boundary layer phenomena like those considered by Levinson and Višik and Lyusternik can appear in this degeneration. For our purpose a boundary layer term may be defined as a term in the solution which goes to zero with ϵ uniformly for $t \in [\delta, a]$, $0 < \delta < a \leqslant \infty$, and all x, but which does not go to zero for $t = 0$; for example, $e^{-t/\epsilon}$ on $[0, \infty)$ displays this behavior. Such terms will be found to enter because V_ϵ satisfies boundary conditions that, in general, V_0 cannot; thus one might expect an expansion of V_ϵ along the lines of

$$V_\epsilon = V_0 + \text{boundary layer terms} + \text{quantities which go to zero with } \epsilon.$$

It is shown here that, under sufficiently strong conditions on the problem (1), (2) and the degenerate problem considered, this will indeed be the case, and, moreover, boundary layer terms will not actually enter into certain elements of V_ϵ. For such an element, say V_ϵ^i, it will then follow that $V_\epsilon^i \to V_0^i$ as $\epsilon \to 0$ uniformly in $\mathbf{x} \in E_n$ and $t \in [0, a]$.

A similar result will be established for the inhomogeneous problem.

We proceed as follows. The problem (1), (2) is well known [1] when the summation is restricted to terms such that $|\alpha| = 1$, and only a slight generalization is introduced in solving the present problem. However, the method and some of the estimates, as well as the result, will be needed later, so the matter will be fully treated here in Section 2. Next we determine conditions under which the degenerate problem is well posed and obtain a solution, explicit up to inverse Fourier transformation. The conditions imposed are such that the degenerate problem can be reduced to a system for the last m components of V_0; the solution of this system for these components then determines V_0 completely. In Section 4 we investigate the nature of the degeneration in the Fourier-transformed space; estimates made in this space are then Fourier inverted to yield estimates in the original $\mathbf{x}$-space. Conditions somewhat stronger than those needed in Section 2 and 3 are imposed in order to guarantee regular degeneration. In Section 5 we treat the problem with an inhomogeneous equation by reducing it to two problems with homogeneous equations and inhomogeneous data (these two problems could be combined by imposing stronger differentiability on the data) and a problem with an inhomogeneous equation and homogeneous data. It is only this last problem which is actually discussed, as the other two are amenable to the treatment in Section 4. Moreover, this problem with an inhomogeneous equation is of a special form such that no boundary layer terms are encountered in the degeneration of its solution. Finally, in Section 6 we conclude with a few examples.

The main results are Theorem 4 of Section 4 and Theorem 8 of Section 5.

In the following, expressions such as "goes as" and "behaves like" will mean "is asymptotic to" (as $\epsilon \to 0$).

I wish to thank Professor Reuben Hersh for his advice and criticism throughout the course of this work.

2. The Nondegenerate Problem

We assume that the initial data f are in $L_1(E^n)$, multiply (1) and (2) by $(2\pi)^{-n/2} e^{i(\mathbf{x},\boldsymbol{\zeta})}$, and integrate over E^n, obtaining, by elementary properties of the Fourier transform,

$$\hat{L}_\epsilon[v] \equiv A(\epsilon)\, v_t + \left[\sum_{|\alpha| \leqslant p} (-i\boldsymbol{\zeta})^\alpha B_\alpha \right] v = 0,$$

$$v(0, \boldsymbol{\zeta}) = \hat{f}(\boldsymbol{\zeta});$$

here $v(t, \boldsymbol{\zeta})$ is the Fourier transform of $V(t, \mathbf{x})$.

For convenience we set

$$M(\zeta) = \sum_{|\alpha| \leqslant p} (-i\zeta)^{\alpha} B_{\alpha};$$

the transformed problem then can be written

$$A(\epsilon) v_t + Mv = 0 \tag{3}$$

$$v(0, \zeta) = \hat{f}(\zeta). \tag{4}$$

Throughout this section ϵ will be regarded as a fixed parameter. We define the characteristic form $Q_\epsilon(\eta, \zeta)$ associated with Eq. (3) by

$$Q_\epsilon(\eta, \zeta) = \det (i\eta A(\epsilon) + M(\zeta)).$$

If $Z_\epsilon(t, \zeta)$ is the solution matrix for the system of ordinary differential equations (3) satisfying $Z_\epsilon(0, \zeta) = I$, and if the integral

$$V_\epsilon(t, \mathbf{x}) \equiv (2\pi)^{-n/2} \int_{E^n} e^{-i(\mathbf{x},\zeta)} Z_\epsilon(t, \zeta) \hat{f}(\zeta)\, d\zeta \tag{5}$$

and its various formal derivatives in t and $\mathbf{x}$ converge uniformly to functions continuous on $[0, \infty) \times E^n$, the integral (5) represents the classical solution of the problem (1), (2). From the theory of ordinary differential equations $Z_\epsilon(t, \zeta)$ has the representation

$$Z_\epsilon(t, \zeta) = \frac{1}{2\pi} \oint_{\Gamma} e^{i\eta t} (i\eta A(\epsilon) + M(\zeta))^{-1} A(\epsilon)\, d\eta \tag{6}$$

where for each ζ the path Γ in the complex η-plane encircles all roots of $Q_\epsilon(\eta, \zeta) = 0$.

To show that (5) and its various formal derivatives appearing in (1) actually do define functions continuous on $[0, \infty) \times E^n$ it is necessary to investigate the structure of $Z_\epsilon(t, \zeta)$ and to make assumptions on the problem (1), (2). To this end we first select a particular path Γ for each ζ. *We assume that the (not necessarily distinct) roots of $Q_\epsilon(\eta, \zeta)$, $\eta_1, \ldots, \eta_N$, satisfy the conditions* $\operatorname{Im}(\eta_i) \geqslant c_\epsilon$ *for some (possibly negative) constant c_ϵ for all real ζ.* This is simply the condition that the system (1) is well posed in the sense of Petrovsky. For each η_k consider the rectangle

$$\{\eta : \operatorname{Re}(\eta_k) - 1 \leqslant \operatorname{Re}(\eta) \leqslant \operatorname{Re}(\eta_k) + 1, \operatorname{Im}(\eta_k) - 1 \leqslant \operatorname{Im}(\eta) \leqslant \operatorname{Im}(\eta_k) + 1\};$$

let Γ be the positively oriented boundary of the union of these rectangles.

Note that on Γ, $\operatorname{Im}(\eta) \geqslant c_\epsilon - 1$. Γ is clearly a path enclosing all the roots η_k; moreover, for η on Γ, $|\eta - \eta_k| \geqslant 1$, so that

$$|Q_\epsilon(\eta, \zeta)| = \prod_{k=1}^{N} |\eta - \eta_k| \geqslant 1$$

on $\Gamma = \Gamma(\zeta)$. It is possible, however, that $|Q_\epsilon(\eta, \zeta)|$ may go as a power of $|\zeta|$ as $|\zeta| \to \infty$ due to the fact that the various distinct closed curves of which Γ may be composed for large $|\zeta|$ tend apart in the η plane like some power of $|\zeta|$. For example, if in one space dimension $Q_\epsilon(\eta, \zeta) = \eta^2 - \zeta^2$, then $|Q_\epsilon(\eta, \zeta)| \sim 2|\zeta|$ for $|\zeta|$ large and η on $\Gamma(\zeta)$. As we shall see, the faster Q_ϵ grows at ∞, the better our estimate for Z_ϵ. In order to derive as good an estimate as the method employed will allow, *we assume therefore that*

$$|Q_\epsilon(\eta, \zeta)| \sim |\zeta|^l \qquad \text{as} \qquad |\zeta| \to \infty$$

for some $l \geqslant 0$; in any case one can take $l = 0$ and use the estimate $|Q_\epsilon(\eta, \zeta)| \geqslant 1$ in the analysis to follow.

The length of Γ clearly cannot exceed $8N$. Observe that the η_k are roots of an equation of degree N in η such that the coefficient of η^j is a polynomial in $\zeta_1, \ldots, \zeta_n$ of degree $(N - j)p$ at most; hence the roots η can become infinite at worst like $|\zeta|^p$; i.e., for some constant K and $|\zeta|$ sufficiently large

$$|\eta_k| \leqslant K|\zeta|^p, \qquad k = 1, \ldots, N.$$

Then for η on Γ and $|\zeta|$ large it is clear that

$$|\eta| \leqslant K|\zeta|^p + \sqrt{2}.$$

For η on Γ and $|\zeta|$ large each element of $i\eta A(\epsilon) + M(\zeta)$ can thus be bounded by some constant times $|\zeta|^p$. Now the entries of $(i\eta A(\epsilon) + M(\zeta))^{-1}$ are cofactors of $i\eta A(\epsilon) + M(\zeta)$ divided by $\det(i\eta A(\epsilon) + M(\zeta)) = Q_\epsilon(\eta, \zeta)$, and the latter behaves like $|\zeta|^l$ for η on Γ. The cofactors are functions of ζ which become infinite for large $|\zeta|$ at most like $|\zeta|^{(N-1)p}$. Thus for η on Γ and $|\zeta|$ large the elements of $(i\eta A(\epsilon) + M(\zeta))^{-1}$ become infinite at most like $|\zeta|^{(N-1)p-l}$; clearly then from (6) for large $|\zeta|$ the entries of $Z_\epsilon(t, \zeta)$ behave at worst like $e^{-(c_\epsilon - 1)t}|\zeta|^{(N-1)p-l}$ for $t \geqslant 0$. This estimate is used in the following theorem.

THEOREM 1. *In addition to the assumptions made above, let f have $Np - l + n + 1$ continuous derivatives which are in L_1. Then the function $V_\epsilon(t, \mathbf{x})$ defined in (5),*

$$V_\epsilon(t, \mathbf{x}) = (2\pi)^{-n/2} \int_{E^n} e^{-i(\mathbf{x}, \zeta)} Z_\epsilon(t, \zeta) \hat{f}(\zeta)\, d\zeta,$$

is a classical solution of $L_\epsilon[V_\epsilon] = 0$ *satisfying the initial conditions*

$$V_\epsilon(0, \mathbf{x}) = f(\mathbf{x}).$$

PROOF. The differentiability conditions on f imply that

$$\frac{\hat{f}(\boldsymbol{\zeta})}{|\,\boldsymbol{\zeta}\,|^{Np-l+n+1}} = 0(1)$$

as $|\,\boldsymbol{\zeta}\,| \to \infty$. To show that the integral in (5), its first p formal derivatives in $x_1, \ldots, x_n$, and its formal derivative in t are continuous functions of $(t, x) \in [0, \infty) \times E^n$ and are the corresponding derivatives of V_ϵ, it suffices to observe that the integrands of these integrals are continuous in all variables and that these integrals converge uniformly in $(t, x) \in [0, a] \times E^n$ for any $a > 0$. Indeed, for the formal $\mathbf{x}$-derivative D^α, $|\,\alpha\,| \leqslant p$, one considers the integral

$$\int (-\,i\boldsymbol{\zeta})^\alpha\, e^{-i(\mathbf{x},\boldsymbol{\zeta})}\, Z_\epsilon(t, \boldsymbol{\zeta}) \hat{f}(\boldsymbol{\zeta})\, d\boldsymbol{\zeta};$$

for large $|\,\boldsymbol{\zeta}\,|$ the integrand becomes infinite at worst like

$$|\,\boldsymbol{\zeta}\,|^{|\alpha|}\, |\,\boldsymbol{\zeta}\,|^{(N-1)p-l} \times e^{-(c_\epsilon-1)t}\, |\hat{f}(\boldsymbol{\zeta})\,|,$$

which is not worse than $|\,\boldsymbol{\zeta}\,|^{Np-l}\, e^{-(c_\epsilon-1)t}\, |\hat{f}(\boldsymbol{\zeta})\,|$. This latter is integrable since $|\hat{f}(\boldsymbol{\zeta})\,| \sim |\,\boldsymbol{\zeta}\,|^{-Np+l-n-1}$ or better, and the desired conclusion follows.

For the t-derivative of $Z_\epsilon(t, \boldsymbol{\zeta})$ we have

$$\frac{\partial}{\partial t} Z_\epsilon(t, \boldsymbol{\zeta}) = \frac{1}{2\pi} \oint_\Gamma (i\eta)\, e^{i\eta t}(i\eta A(\epsilon) + M(\boldsymbol{\zeta}))^{-1}\, A(\epsilon)\, d\eta;$$

since for large $|\,\boldsymbol{\zeta}\,|$ we can estimate $|\,\eta\,|$ by $|\,\boldsymbol{\zeta}\,|^p$, in the manner employed above we have that the entries of $(\partial/\partial t)\, Z_\epsilon(t, \boldsymbol{\zeta})$ become infinite at worst like $e^{-(c_\epsilon-1)t}\, |\,\boldsymbol{\zeta}\,|^{(N-1)p-l+p} = e^{-(c_\epsilon-1)t}\, |\,\boldsymbol{\zeta}\,|^{Np-l}$. For the formal t-derivative of the integral (5) we have

$$(2\pi)^{-n/2} \int e^{-i(\mathbf{x},\boldsymbol{\zeta})} \Big(\frac{\partial}{\partial t} Z_\epsilon(t, \boldsymbol{\zeta})\Big) \hat{f}(\boldsymbol{\zeta})\, d\boldsymbol{\zeta};$$

for large $|\,\boldsymbol{\zeta}\,|$ the integrand becomes infinite no faster than

$$e^{-(c_\epsilon-1)t}\, |\,\boldsymbol{\zeta}\,|^{Np-l}\, |\hat{f}(\boldsymbol{\zeta})\,|,$$

whence the argument above again yields the desired result.

It is obvious that V_ϵ satisfies the initial conditions.

REMARK 1. Perusal of the foregoing argument will show that the function V_ϵ obtained will be the solution of (1), (2) for positive *and negative* time provided $|\operatorname{Im}(\eta_k)| \leqslant c_\epsilon$, $k = 1, \ldots, n$, for some constant c_ϵ.

REMARK 2. The heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0$$

is of the form (1) with $\epsilon = 1$, $N = 1$, $m = 0$. The Fourier-transformed equation is

$$\frac{\partial \hat{u}}{\partial t} + (\boldsymbol{\zeta}, \boldsymbol{\zeta})\, \hat{u} = 0;$$

we have $Q_1(\eta, \boldsymbol{\zeta}) = i\eta + \zeta_1^2 + \cdots + \zeta_n^2$, with root $\eta = +\, i(\zeta_1^2 + \cdots + \zeta_n^2)$, so $\operatorname{Im}(\eta) \geqslant 0$, all $\boldsymbol{\zeta}$. Thus the heat equation satisfies the conditions we have imposed. Suppose now that the initial conditions $f(\mathbf{x})$ are such that $\hat{f}(\boldsymbol{\zeta}) = 0$ for $|\boldsymbol{\zeta}| \geqslant a$ for some constant a (this already implies that $f(\mathbf{x})$ is everywhere differentiable for $\mathbf{x}$ considered complex, and thus is an entire function). Then the integral (5) defining the solution V_ϵ may be restricted to the set $\{\boldsymbol{\zeta} : |\boldsymbol{\zeta}| \leqslant 2a\}$, so estimates on $Z_1(t, \boldsymbol{\zeta})$ are needed only for $\boldsymbol{\zeta}$ in this set. But here $|\operatorname{Im}(\eta)|$ is certainly bounded; whence so are the entries of $Z_1(t, \boldsymbol{\zeta})$. Since $\hat{f}(\boldsymbol{\zeta})$ is continuous, the integral (5) and *all* its formal derivatives converge for *all* t and $\mathbf{x}$. Thus in this case (5) is a solution of the backward heat equation. This result is similar to that of Miranker [8], who, however, considers the Fourier transform on L_2.

3. THE DEGENERATE PROBLEM

Before proceeding with the problem associated with the operator L_0 we establish a simple lemma concerning the determinant and inverse of a matrix.

LEMMA 1. *Let the square matrix D be composed of the blocks D_1, D_2, D_3, D_4, where D_1 and D_4 are square, as follows:*

$$D = \begin{vmatrix} D_1 & D_2 \\ D_3 & D_4 \end{vmatrix}.$$

If D_1 is nonsingular, then

$$\det D = \det D_1 \cdot \det (D_4 - D_3 D_1^{-1} D_2);$$

if also $D_4 - D_3 D_1^{-1} D_2$ is nonsingular,

$$D^{-1} = \begin{vmatrix} I & -D_1^{-1} D_2 (D_4 - D_3 D_1^{-1} D_2)^{-1} \\ 0 & (D_4 - D_3 D_1^{-1} D_2)^{-1} \end{vmatrix} \begin{vmatrix} D_1^{-1} & 0 \\ -D_3 D_1^{-1} & I \end{vmatrix}.$$

PROOF. The last formula can be verified directly; the former follows from the block-matrix identity

$$D = \begin{vmatrix} D_1 & 0 \\ D_3 & I \end{vmatrix} \begin{vmatrix} I & D_1^{-1}D_2 \\ 0 & D_4 - D_3D_1^{-1}D_2 \end{vmatrix}$$

and elementary rules on determinants.

We return to a consideration of the degenerate operator, and postulate $1 \leqslant m \leqslant N - 1$. The excluded cases would seem of little special interest: if $m = N$, there is no ϵ dependence at all; if $m = 0$, the degenerate problem becomes algebraic after Fourier transformation, and thus no initial data at all can be imposed. The solution of the degenerate problem analogous to that derived here would be identically zero.

We thus consider the equation

$$L_0[U] = A(0)\,U_t + \sum_{|\alpha| \leqslant p} B_\alpha D^\alpha U = 0 \tag{7}$$

with boundary conditions which we shall now determine. The Fourier-transformed equation is

$$\hat{L}_0[u] = A(0)\,u_t + M(\zeta)\,u = 0, \tag{8}$$

where $M(\zeta)$ is the matrix $\sum_{|\alpha| \leqslant p} (-i\zeta)^\alpha B_\alpha$ and $u(t, \zeta)$ is the Fourier transform of $U(t, \mathbf{x})$. The first $N - m$ rows of the transformed equation do not involve differentiations at all, i.e., are algebraic; the last m rows involve the t-derivatives of $u_{N-m+1}, \ldots, u_N$. It is clear that initial data can sensibly be imposed on only these last components of u, and we shall suppose that these conditions are precisely

$$u^2(0, \zeta) = \hat{f}^2(\zeta), \tag{9}$$

where we adopt the notation u^1 for the $(N - m)$-column composed of the first $N - m$ entries of u, in order, and u^2 for the m-column whose elements are the last m entries of u, in order. No condition can then be imposed on u^1, which will satisfy initial conditions completely determined by (9) and the Eq. (7).

Let the matrix $M(\zeta)$ be broken into block matrices M_1, M_2, M_3, M_4 as in the lemma, where M_1 is $(N - m) \times (N - m)$ and M_4 is $m \times m$. If M_1 is invertible, it is possible to reduce the given problem to one for u^2 alone. Clearly $\det M_1(\zeta)$ is a polynomial in ζ; *we assume that the values of this polynomial are uniformly bounded away from zero for real ζ; i.e.,*

$$|\det M_1(\zeta)| \geqslant k > 0.$$

Without loss of generality we can assume det $M_1(\zeta) \geqslant k > 0$. Moreover, it may well happen that for large $|\zeta|$ the behavior of det M_1 is better than just bounded away from zero; hence *we assume that for* $|\zeta|$ *large* det $M_1(\zeta) \sim |\zeta|^\gamma$ *for some* $\gamma \geqslant 0$. Since M_1 is thus assumed invertible for all ζ, the system (7′) can be written

$$u^1 = - M_1^{-1}M_2u^2, \tag{10}$$

$$u_t^2 + (M_4 - M_3M_1^{-1}M_2)\, u^2 = 0. \tag{11}$$

This last equation is a system of ordinary differential equations, depending on the parameters $\zeta_1, \ldots, \zeta_n$ for the components of u^2. With the initial conditions (9), this is a problem from which u^2 can be determined uniquely; u^1 can then be found using (10).

The solution matrix $Z'(t, \zeta)$ for the system (11) has the representation

$$Z'(t, \zeta) = \frac{1}{2\pi} \oint_{\Gamma'} e^{i\eta t}(i\eta I + M_4 - M_3M_1^{-1}M_2)^{-1}\, d\eta, \tag{12}$$

where for each ζ the path Γ' in the η plane encircles all roots η of

$$Q'(\eta, \zeta) \equiv \det (i\eta I + M_4 - M_3M_1^{-1}M_2) = 0.$$

We now *assume that the m roots* $\eta_1, \ldots, \eta_m$ *of this equation satisfy* Im $(\eta_k) \geqslant c'$ *for some (possibly negative) constant* c' *and all real* ζ. For the estimates to follow we consider a special path Γ' constructed in the same manner as the path used in Section 2. Since on Γ' $|\eta - \eta_k| \geqslant 1$, for η on Γ'

$$|Q'(\eta, \zeta)| = \prod_{k=1}^{m} |\eta - \eta_k| \geqslant 1.$$

Here again we can possibly improve our estimates by assuming that in fact

$$|Q'(\eta, \zeta)| \sim |\zeta|^r \qquad \text{as} \qquad |\zeta| \to \infty$$

for some $r \geqslant 0$. Now any element of $M(\zeta)$ is a polynomial in ζ of degree p at most, and thus the same is true of M_1, M_2, M_3, M_4. Let the maximum degree of the polynomials actually occurring in M_1 be q; then $0 \leqslant q \leqslant p$. The degree of each entry in the cofactor matrix of M_1 is at most $(N - m - 1)q$, whence for large $|\zeta|$ each entry of M_1^{-1} itself increases no faster than $|\zeta|^{(N-m-1)q-\gamma}$. Hence each entry of $M_4 - M_3M_1^{-1}M_2$ increases no faster than $|\zeta|^{(N-m-1)q-\gamma+2p}$ as $|\zeta| \to \infty$. Actually, as is easily seen, this expression may have much better behavior; we therefore assume directly that no entry of $M_4 - M_3M_1^{-1}M_2$ goes to infinity more rapidly than $|\zeta|^s$ as

$|\zeta| \to \infty$, observing that $s \leqslant (N-m-1)q - \gamma + 2p$ by what we have just established. Each η_k is a root of a polynomial whose coefficients are polynomials in ζ, say

$$Q'(\eta, \zeta) = \sum_{i=0}^{m} R_{m-i}(\zeta)\,\eta^i;$$

clearly the degree of R_i is at most is. Suppose for large $|\zeta|$ that $\eta_k(\zeta)$ increases, as $|\zeta| \to \infty$, as fast as $|\zeta|^\beta$ with $\beta > s$. Then $R_0\eta_k{}^m$ increases faster than any other term in $Q'(\eta, \zeta)$ as $|\zeta| \to \infty$, an impossibility since $Q'(\eta_k, \zeta) = 0$ for all ζ. Hence for large $|\zeta|$ each of the η_k can increase no faster than $|\zeta|^s$. As in the nondegenerate case it follows that each element of $(i\eta I + M_4 - M_3M_1^{-1}M_2)^{-1}$ becomes infinite for large $|\zeta|$ at worst like $|\zeta|^{(m-1)s-r}$. A repetition of the argument employed in the nondegenerate case shows that *for nonnegative t the entries of the matrix $Z'(t, \zeta)$ become infinite like $e^{-(c'-1)t}\,|\zeta|^{(m-1)s-r}$, at worst, as $|\zeta| \to \infty$.*

The vector

$$u^2(t, \zeta) = Z'(t, \zeta)\,\hat{f}^2(\zeta)$$

satisfies the problem (11), (9); Eq. (10) then defines u^1:

$$u^1(t, \zeta) = -\,M_1^{-1}M_2Z'(t, \zeta)\,\hat{f}^2(\zeta).$$

If we set $u = \binom{u^1}{u^2}$, in an obvious sense, u clearly satisfies (8), (9). It remains only to show that $U(t, \mathbf{x})$, the inverse Fourier transform of $u(t, \zeta)$, exists and satisfies Eq. (7); it will then satisfy the initial data

$$U^2(0, \mathbf{x}) = f^2(\mathbf{x}).$$

THEOREM 2. *Let f have at least k continuous L_1 derivatives, with*

$$k \geqslant (m-1)s - r + n + 1 + \max(p, s) + \max(\sigma, 0),$$

where $\sigma \leqslant (N-m-1)q - \gamma + p$ (see proof for definition of σ). Then $U(t, \mathbf{x})$ as defined above exists and is a solution of the problem

$$L_0[U] = 0, \qquad U^2(0, \mathbf{x}) = f^2(\mathbf{x}).$$

PROOF. It is again a matter of showing that the integrals

$$U^1(t, \mathbf{x}) = -\,(2\pi)^{-n/2}\int_{E_n} e^{-i(\mathbf{x}\cdot\zeta)}\,M_1^{-1}(\zeta)\,M_2(\zeta)\,Z'(t, \zeta)\hat{f}^2(\zeta)\,d\zeta,$$

$$U^2(t, \mathbf{x}) = (2\pi)^{-n/2}\int_{E_n} e^{-i(\mathbf{x}\cdot\zeta)}\,Z'(t, \zeta)\hat{f}^2(\zeta)\,d\zeta$$

and the formal first t-derivatives and first p $\mathbf{x}$-derivatives exist and define continuous functions on $[0, \infty) \times E_n$. For U^2 this is an exact replica of the proof in the nondegenerate case, and goes through for f with at least $(m-1)s - r + n + 1 + \max(p, s)$ continuous L_1 derivatives. For U^1 we must allow also for the growth of $M_1^{-1}M_2$ with ζ, which does not exceed that of $|\zeta|^{(N-m-1)q-\gamma+p}$. If we assume that no entry of $M_1^{-1}M_2(\zeta)$ grows faster than $|\zeta|^\sigma$ as $|\zeta| \to \infty$, the theorem follows.

Remark. The assumption that det $M_1(\zeta)$ is bounded away from zero will be used again later, but is not essential for the solvability of (7) by our method. If the real zeros of the polynomial det $M_1(\zeta)$ occur for $|\zeta| \leqslant K$ for some constant K, the preceding estimates for large $|\zeta|$ go through. Moreover, if the integrals involved in establishing that U is actually a solution are broken up into integrals over $|\zeta| > 2K$ and $|\zeta| \leqslant 2K$, and if $|\det M_1(\zeta)| \geqslant k > 0$ on the former set, all the analysis above goes through for the integrals over $|\zeta| > 2K$. Consider the matrix $i\eta I + M_4 - M_3M_1^{-1}M_2$ occurring in the integral representation of $Z'(t, \zeta)$; if we write C^* for the transposed cofactor matrix of a matrix C,

$$\begin{aligned} &i\eta I + M_4 - M_3M_1^{-1}M_2 \\ &= \frac{1}{\det M_1}\,[i\eta(\det M_1)\,I + (\det M_1)\,M_4 - M_3M_1{}^*M_2], \end{aligned}$$

whence

$$(i\eta I + M_4 - M_3M_1^{-1}M_2)^{-1} = \frac{[i\eta(\det M_1)\,I + (\det M_1)\,M_4 - M_3M_1{}^*M_2]^*}{(\det M_1)^{m-1}\,Q'(\eta;\zeta)}.$$

Now det M_1 does not depend on η and so can be brought out of the integral for $Z'(t, \zeta)$; thus for $|\zeta| \leqslant 2K$

$$|Z'(t,\zeta)| \leqslant e^{-(c'-1)t}\,(\det M_1(\zeta))^{1-m} \cdot \text{constant}.$$

Clearly the integrals for U^1 and U^2 for $|\zeta| \leqslant 2K$ will converge, along with their t and $\mathbf{x}$ derivatives, if $f \in L_1$ (so $\hat{f}(\zeta)$ is defined and bounded for $|\zeta| \leqslant 2K$), provided $(\det M_1(\zeta))^{-m}$ is locally integrable. It is clear that Theorem 2 can be put through if this is the case.

In the following analysis, however, we shall be concerned solely with the case in which det $M_1(\zeta)$ is bounded away from zero for real ζ.

4. Degeneration

In order to study the behavior of the solutions V_ϵ as $\epsilon \to 0$ it is necessary to determine the behavior of the roots of the eigenvalue equation

$\det(i\eta A(\epsilon) + M(\zeta)) = 0$. As is immediately seen, the eigenvalue equation considered can be written

$$\sum_{i=m+1}^{N} P_i(\zeta; \epsilon)\, \epsilon^{i-m}\eta^i + \sum_{i=0}^{m} P_i(\zeta; \epsilon)\, \eta^i = 0 \tag{13}$$

where the P_i are polynomials in the indicated arguments and $P_N(\zeta; \epsilon) = i^N$, $P_m(\zeta; 0) = \det M_1(\zeta) \neq 0$. Following Višik and Lyusternik [9], we introduce the *auxiliary characteristic equation*

$$\mathscr{Q}(\eta; \zeta) = \sum_{i=0}^{N-m} P_{i+m}(\zeta; 0)\, \eta^i = 0. \tag{14}$$

Let the roots of this equation be $\nu_j(\zeta)$, $j = 1,..., N - m$; since $P_m \neq 0$ for all ζ, $\nu_j(\zeta) \neq 0$, all ζ. Let the roots of the equation

$$\sum_{i=0}^{m} P_i(\zeta; 0)\, \eta^i = 0$$

be $\mu_l(\zeta)$, $l = 1,..., m$. Then the following lemma of Višik and Lyusternik ([2], p. 252) holds:

LEMMA 2. *The roots $\bar{\nu}_{j/\epsilon}$, $\bar{\mu}_l$ of equation (13) have the form*

$$\bar{\mu}_l = \mu_l + \epsilon_l, \qquad l = 1,..., m,$$

and

$$\frac{\bar{\nu}_j}{\epsilon} = \frac{\nu_j + \epsilon_j'}{\epsilon}, \qquad j = 1,..., N - m,$$

where ϵ_l and ϵ_j' go to zero with ϵ.

The proof given in Višik and Lyusternik ([2], pp. 262-263) applies without modification because of our assumption $\det M_1(\zeta) \neq 0$ for all real ζ.

As $\epsilon \to 0$ Eq. (13) goes to the degenerate eigenvalue problem

$$\det(i\eta A(0) + M(\zeta)) = \det M_1 \cdot \det(i\eta I + M_4 - M_3 M_1^{-1} M_2) = 0$$

by Lemma 1, so the roots $\mu_l(\zeta)$ introduced above are precisely the roots of $\det(i\eta I + M_4 - M_3 M_1^{-1} M_2) = 0$.

We have already assumed that $\operatorname{Im}(\bar{\mu}_l(\zeta)) \geqslant c_\epsilon$, $\operatorname{Im}(\bar{\nu}_j(\zeta)/\epsilon) \geqslant c_\epsilon$ for some constant c_ϵ which could depend on ϵ. *We now stipulate that*

$$\operatorname{Im}(\bar{\mu}_l) \geqslant c, \qquad \operatorname{Im}(\bar{\nu}_j(\zeta; \epsilon)) \geqslant d > 0$$

for all ϵ sufficiently small, where c and d are independent of ϵ. Then the condition previously imposed on the degenerate problem, $\operatorname{Im}(\mu_l) \geqslant c'$, is automatically fulfilled with $c' = c$.

LEMMA 3. *The terms involving $e^{i(\bar{\nu}_j/\epsilon)t}$ in the last m rows of $Z_\epsilon(t, \boldsymbol{\zeta})$ are of the order of ϵ uniformly in t, $\boldsymbol{\zeta}$ for $t \in [0, a]$ and $|\boldsymbol{\zeta}|$ bounded.*

PROOF. We must examine the structure of the last m rows of the matrix $[i\eta A(\epsilon) + M(\boldsymbol{\zeta})]^{-1} A(\epsilon)$ occurring in the integrand of the expression for $Z_\epsilon(t, \boldsymbol{\zeta})$. We have $\epsilon_l(\boldsymbol{\zeta}) = \bar{\mu}_l(\boldsymbol{\zeta}; \epsilon) - \mu_l(\boldsymbol{\zeta})$; but $\bar{\mu}_l$ and μ_l depend continuously on $\boldsymbol{\zeta}$ since each is a root of a certain polynomial whose coefficients, being themselves polynomials in $\boldsymbol{\zeta}$, are continuous in $\boldsymbol{\zeta}$ and whose leading coefficient is independent of $\boldsymbol{\zeta}$. Thus $\epsilon_l(\boldsymbol{\zeta})$ depends continuously on $\boldsymbol{\zeta}$, and by a totally similar argument so does $\epsilon_j'(\boldsymbol{\zeta})$. By Cauchy's theorem and the process used before we may replace the curve $\Gamma(\boldsymbol{\zeta}; \epsilon)$ in the integral expression for $Z_\epsilon(t, \zeta)$ by a curve $\Gamma''(\boldsymbol{\zeta}; \epsilon)$ whose length is bounded independently of ϵ and $\boldsymbol{\zeta}$ and such that for η on $\Gamma''(\boldsymbol{\zeta}; \epsilon)$

$$|\eta - \bar{\mu}_i| > 1, \qquad \left|\eta - \frac{\bar{\nu}_i}{\epsilon}\right| > 1, \qquad \operatorname{Im}(\eta) > c - 2.$$

It is clear that a curve $\Gamma''(\boldsymbol{\zeta}; \epsilon)$ will do for neighboring values of $\boldsymbol{\zeta}$ as well. $\Gamma_j''(\boldsymbol{\zeta}; \epsilon)$ will denote that portion of $\Gamma''(\boldsymbol{\zeta}; \epsilon)$ enclosing the root $\bar{\nu}_j/\epsilon$.

Consider the term involving $e^{i(\bar{\nu}_j/\epsilon)t}$ in an element of $[i\eta A(\epsilon) + M(\boldsymbol{\zeta})]^{-1}$ occurring in the (i, j)th place, with $N - m + 1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N - m$. This element is in fact the cofactor of the entry of $i\eta A(\epsilon) + M(\boldsymbol{\zeta})$ in the (j, i)th place divided by $\det(i\eta A(\epsilon) + M(\boldsymbol{\zeta}))$. If we momentarily set $\epsilon\eta = \delta$, so that $i\eta A(\epsilon)$ is a matrix with $N - m$ diagonal entries $i\delta$ and m diagonal entries $i\eta$, it is clear that the cofactor is a polynomial in η and δ of degree $N - m - 1$ in δ and degree $m - 1$ in η. With the substitution $\eta = (\bar{\nu}_j/\epsilon) + \eta'$, $\delta = \bar{\nu}_j + \epsilon\eta'$, which shifts the origin to $\bar{\nu}_j/\epsilon$, the cofactor becomes an expression which behaves like a polynomial of degree at most $m - 1$ in $1/\epsilon$.

We must now consider the behavior of $\det(i\eta A(\epsilon) + M(\boldsymbol{\zeta}))$. Certainly

$$\det(i\eta A(\epsilon) + M(\boldsymbol{\zeta})) = i^N \cdot \prod_{l=1}^{m} (\eta - \bar{\mu}_l(\boldsymbol{\zeta}; \epsilon)) \cdot \prod_{j=1}^{N-m} (\epsilon\eta - \bar{\nu}_j(\boldsymbol{\zeta}; \epsilon)).$$

Let us first consider a value of $\boldsymbol{\zeta}$ for which the roots $\nu_j(\boldsymbol{\zeta})$ of Eq. (14) are distinct for distinct j; then it follows that the $\bar{\nu}_j(\boldsymbol{\zeta}; \epsilon)$ are distinct for ϵ sufficiently small and that the roots $(\bar{\nu}_j(\boldsymbol{\zeta}; \epsilon)/\epsilon)$ tend apart in the complex plane like $1/\epsilon$ as $\epsilon \to 0$. Thus for η on $\Gamma_j''(\boldsymbol{\zeta}; \epsilon)$ we have $(\eta - \bar{\mu}_l) \sim (1/\epsilon)$, since the distance of Γ_j'' from the origin, and hence from the bounded quantities $\bar{\mu}_l$, goes as $1/\epsilon$; also

$$(\epsilon\eta - \bar{\nu}_k(\boldsymbol{\zeta}; \epsilon)) \sim (\nu_j(\boldsymbol{\zeta}) - \nu_k(\boldsymbol{\zeta})) \sim 1 \qquad \text{for} \qquad k \neq j,$$

since we have assumed that the $\nu_j(\zeta)$ are distinct. Since $(\epsilon\eta - \bar{\nu}_j(\zeta; \epsilon)) \sim \epsilon$, we get that $\det(i\eta A(\epsilon) + M(\zeta)) \sim (1/\epsilon)^{m-1}$. A check of this argument (the ν_j are continuous) shows that this behavior is continuous in ζ. Denoting the cofactor of the element of $i\eta A(\epsilon) + M(\zeta)$ in the (j, i)th place by C and defining q as

$$q = i^N \cdot \prod_{l=1}^{m} (\eta - \bar{\mu}_l(\zeta; \epsilon)) \cdot \prod_{\substack{k=1 \\ k \neq j}}^{N-m} (\epsilon\eta - \bar{\nu}_k(\zeta; \epsilon)),$$

we proceed to estimate the contribution of the term in $e^{i(\bar{\nu}_j/\epsilon)t}$ to the (i, j)th element of $Z_\epsilon(t, \zeta)$. To this end we note that the residue of the integrand for this term is

$$\lim_{\eta \to \nu_j/\epsilon} \frac{\epsilon e^{i\eta t} C}{q} = \epsilon e^{i(\bar{\nu}_j/\epsilon)t} \frac{C(\bar{\nu}_j/\epsilon)}{q(\bar{\nu}_j/\epsilon)},$$

where the ϵ comes from the factor $A(\epsilon)$ in the definition of Z_ϵ. Now $C \sim (1/\epsilon)^{m-1}$ and $q \sim (1/\epsilon)^m$, whence this residue goes as $\epsilon^2 e^{i(\bar{\nu}_j/\epsilon)t}$; but since the variable of integration is $\epsilon\eta$ and not η, we must have $\epsilon d\eta$ in the integral, so the (i, j)th element of $Z_\epsilon(t, \zeta)$ hehaves like $\epsilon \mid e^{i(\bar{\nu}_j/\epsilon)t} \mid$. Since $\mid e^{i(\bar{\nu}_j/\epsilon)t} \mid = e^{-(\mathrm{Im}(\bar{\nu}_i)/\epsilon)t} \leqslant 1$, this element in fact goes as ϵ uniformly for $t \in [0, \infty)$. Moreover, this behavior is continuous in ζ.

Note that in the important case $m = N - 1$ the above situation is the only one which can arise since there is then only one root ν.

Consider now a value of ζ, say ζ_1, such that two or more of the ν_j are equal, say $\nu_1 = \nu_2 = \cdots = \nu_k$; we do not at the moment care whether any of the remaining roots are equal, so long as none of them equals ν_1. For simplicity of exposition we shall assume that $k = 2$; it will be clear that the method used extends to any finite k. Again denoting the (j, i)th cofactor of $i\eta A(\epsilon) + M(\zeta)$, $N - m + 1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N - m$, by C, we wish to show that there exists a sphere $B(\zeta_1) = \{\zeta : \mid \zeta - \zeta_1 \mid < \beta(\zeta_1)\}$ of positive radius $\beta(\zeta_1)$ about ζ_1 such that the sum of the residues of $\epsilon e^{i\eta t} C/[\det(i\eta A(\epsilon) + M)]$ at $\bar{\nu}_1/\epsilon$ and $\bar{\nu}_2/\epsilon$ is $0(\epsilon)$ uniformly for $\zeta \in B(\zeta_1)$. We set

$$q = \frac{\det(i\eta A(\epsilon) + M(\zeta))}{(\epsilon\eta - \bar{\nu}_1(\zeta; \epsilon))(\epsilon\eta - \bar{\nu}_2(\zeta; \epsilon))};$$

then we have $C(\bar{\nu}_j/\epsilon) \sim (1/\epsilon)^{m-1}$ (established already) and $q(\bar{\nu}_j/\epsilon) \sim (1/\epsilon)^m$ (immediate), $j = 1, 2$. It is easily seen that $dC/d\eta\,(\bar{\nu}_j/\epsilon) \sim (1/\epsilon)^{m-2}$, $dq/d\eta\,(\bar{\nu}_j/\epsilon) \sim (1/\epsilon)^{m-1}$, $j = 1, 2$. Setting

$$F_{\epsilon, t}(\eta) = \frac{1}{\epsilon} \frac{C(\eta/\epsilon)\, e^{i(\eta/\epsilon)t}}{q(\eta/\epsilon)},$$

we see that $F_{\epsilon,t}(\bar{\nu}_j)$, $F'_{\epsilon,t}(\bar{\nu}_j)$ are $0(1)$, $j = 1, 2$, since $(t/\epsilon)\, e^{-\gamma(t/\epsilon)}$ is bounded uniformly in $t \geqslant 0$ for $\gamma > 0$ and we have assumed $\operatorname{Im}(\bar{\nu}_j(\zeta; \epsilon)) > 0$. Denoting the sum of the residues considered by $\sum$, we clearly have

$$\sum = \epsilon^2 \left\{ \frac{F_{\epsilon,t}(\bar{\nu}_1)}{\bar{\nu}_1 - \bar{\nu}_2} + \frac{F_{\epsilon,t}(\bar{\nu}_2)}{\bar{\nu}_2 - \bar{\nu}_1} \right\} = \epsilon^2 \frac{F_{\epsilon,t}(\bar{\nu}_1) - F_{\epsilon,t}(\bar{\nu}_2)}{\bar{\nu}_1 - \bar{\nu}_2}.$$

Now for each fixed ϵ and t we know there exists a $\delta(t, \epsilon)$ such that

$$\left| \frac{F_{\epsilon,t}(\sigma) - F_{\epsilon,t}(\tau)}{\sigma - \tau} \right| \leqslant | F'_{\epsilon,t}(\sigma) | + 1$$

provided $| \sigma - \tau | < \delta(t, \epsilon)$; we may clearly take $\delta(t, \epsilon)$ continuous in both arguments. For some fixed ϵ' set

$$\delta = \min_{\substack{0 \leqslant \epsilon \leqslant \epsilon' \\ 0 \leqslant t \leqslant a}} \delta(t, \epsilon);$$

then $\delta > 0$, and for $| \sigma - \tau | < \delta$ we have

$$\left| \frac{F_{\epsilon,t}(\sigma) - F_{\epsilon,t}(\tau)}{\sigma - \tau} \right| < | F'_{\epsilon,t}(\sigma) | + 1$$

for $0 \leqslant \epsilon \leqslant \epsilon'$, $0 \leqslant t \leqslant a$. Let $\beta(\zeta_1)$ be so small that $\zeta \in B(\zeta_1)$ implies $| \nu_1(\zeta) - \nu_2(\zeta) | < \delta/3$; this is possible since the $\nu_i(\zeta)$ are continuous in ζ and $\nu_1(\zeta_1) = \nu_2(\zeta_1)$. Let $\epsilon'' \leqslant \epsilon'$ be so small that $\epsilon < \epsilon''$ implies

$$| \bar{\nu}_1(\zeta; \epsilon) - \nu_1(\zeta) | = | \epsilon_1'(\zeta) | < \frac{\delta}{3}$$

for $\zeta \in B(\zeta_1)$ and $j = 1, 2$. Then $\epsilon < \epsilon''$ implies

$$\left| \frac{F_{\epsilon,t}(\bar{\nu}_1) - F_{\epsilon,t}(\bar{\nu}_2)}{\bar{\nu}_1 - \bar{\nu}_2} \right| < | F'_{\epsilon,t}(\bar{\nu}_1) | + 1$$

for $t \in [0, a]$. Thus since $F'_{\epsilon,t}(\bar{\nu}_1) = 0(1)$, we have

$$\sum = 0(\epsilon^2)$$

uniformly in $t \in [0, a]$ and $\zeta \in B(\zeta_1)$. Since we can clearly take $\beta(\zeta_1)$ continuous in ζ_1, and using the Heine-Borel theorem, we conclude that in some neighborhood of any compact set on which $\nu_1(\zeta) = \nu_2(\zeta)$ and no other ν_j equals ν_1 the terms in $e^{i(\bar{\nu}_1/\epsilon)t}$ and $e^{i(\bar{\nu}_2/\epsilon)t}$ contribute to $Z_\epsilon(t, \zeta)$ quantities which are $0(\epsilon)$ uniformly in ζ in this neighborhood and $t \in [0, a]$. As remarked above, the method of proof easily extends to cover the case of any number of roots $\nu_j(\zeta)$ equal.

Consider the set $K \equiv \{\zeta : |\zeta| \leqslant k\}$ for any $k > 0$. Inside K the roots $\nu_j(\zeta)$ can be pairwise equal only on a closed set since they are the roots of an algebraic equation whose coefficients are polynomials in ζ. We have just established that in a neighborhood of this set the terms involving quantities of the form $e^{i(\bar{\nu}_j/\epsilon)t}$ in the first $N - m$ elements of the last m rows of $Z_\epsilon(t, \zeta)$ are $0(\epsilon)$ uniformly for $t \in [0, a]$ and ζ in this neighborhood. The complement in K of this neighborhood is compact, and there we have shown that the terms in $e^{i(\bar{\nu}_j/\epsilon)t}$ in question are $0(\epsilon)$, continuous in ζ. Thus in this compact set we also have these terms $0(\epsilon)$, uniformly in $t \in [0, a]$ and ζ. It follows from these two arguments that the terms involving $e^{i(\bar{\nu}_j/\epsilon)t}$ in the first $N - m$ elements of the last m rows of $Z_\epsilon(t, \zeta)$ are $0(\epsilon)$ uniformly for $t \in [0, a]$ and $|\zeta| \leqslant k$, as was to be proved.

For an element in the (k, l)th place of $(i\eta A(\epsilon) + M(\zeta))^{-1}$, $N - m + 1 \leqslant k \leqslant N$, $N - m + 1 \leqslant l \leqslant N$, $k \neq l$, the cofactor considered will behave like $(1/\epsilon)^{m-2}$, whence the argument above goes through and yields the desired result. A more precise analysis will be necessary for the diagonal entries $k = l$.

Letting $K_1(\eta; \epsilon; \zeta)$ denote the upper left $(N - m) \times (N - m)$ matrix of $i\eta A(\epsilon) + M(\zeta)$ and denoting the other submatrices as in Lemma 1, we show that $\det K_1(\bar{\nu}_j/\epsilon; \epsilon; \zeta) = 0(\epsilon)$. For suppose first that $\det K_1(\bar{\nu}_j/\epsilon; \epsilon; \zeta)$ does not go to zero with ϵ. Then the formula

$$\det (i\eta A(\epsilon) + M(\zeta)) = \det K_1 \cdot \det (K_4 - K_3 K_1^{-1} K_2) = 0$$

established in Lemma 1 holds at $(\bar{\nu}_j/\epsilon; \epsilon; \zeta)$, so $\det (K_4 - K_3 K_1^{-1} K_2)$ is zero regardless of ϵ; i.e., $\det (i\eta I + M_4 - M_3 K_1^{-1} M_2) = 0$ for all sufficiently small $\epsilon > 0$. But on our assumption that $|\det K_1| \geqslant$ constant > 0 for all small $\epsilon > 0$, the off-diagonal entries of $i\eta I + M_4 - M_3 K_1^{-1} M_2$ are bounded for each ζ as $\epsilon \to 0$, whereas the diagonal entries are of the form $i(\bar{\nu}^j/\epsilon)$ + quantities bounded as $\epsilon \to 0$, clearly an impossibility since $\nu_j \neq 0$ for all ζ. This contradiction shows that in fact $\det K_1 \to 0$ as $\epsilon \to 0$. Since $\det K_1$, being a polynomial in ϵ and $1/\epsilon$, must behave for small ϵ like an integral power of ϵ, we have $\det K_1(\bar{\nu}_j/\epsilon) + \eta'; \epsilon; \zeta) \sim \epsilon$ (or possibly as some power of ϵ greater than one). Since $\det K_1$ is a continuous function of ζ, the $0(\epsilon)$ symbol is again continuous in ζ.

Now the cofactor of one of the last m diagonal entries of $i\eta A(\epsilon) + M(\zeta)$ can be written, again writing $\eta\epsilon = \delta$ and treating δ and η as distinct quantities in the manner employed above, as

$$\det K_1 \cdot (i\eta)^{m-1} + \text{terms of lower order in } i\eta.$$

Since a determinant is a continuous function of its entries and we have seen that $\det K_1((\bar{\nu}_j/\epsilon); \epsilon; \zeta) = 0(\epsilon)$, we have also $\det K_1((\bar{\nu}_j/\epsilon) + \eta'; \epsilon; \zeta) = 0(\epsilon)$ continuous in ζ. Thus the cofactor goes as $(1/\epsilon)^{m-2}$ (or possibly a lower power of $1/\epsilon$) continuously in ζ, as $\epsilon \to 0$, and an application of the methods used before yields the result that the last m diagonal entries of $Z_\epsilon(t, \zeta)$ are $0(\epsilon)$ uniformly for $t \in [0, a]$ and $|\zeta|$ bounded. Q.E.D.

LEMMA 4. *The term involving $e^{i\bar{\mu}_l t}$ in any of the first $N - m$ elements of the jth row of $Z_\epsilon(t, \zeta)$, $N - m + 1 \leqslant j \leqslant N$, is $0(\epsilon)$ uniformly in t, ζ for $t \in [0, a]$ and $|\zeta|$ bounded. The term involving $e^{i\bar{\mu}_l t}$ in the kth element, $N - m + 1 \leqslant k \leqslant N$, of the jth row of $Z_\epsilon(t, \zeta)$, $N - m + 1 \leqslant j \leqslant N$, is equal to the term in $e^{i\mu_l t}$ in the element of $Z'(t, \zeta)$ in the $(j + m - N, k + m - N)$-position except for a term which is $0(\epsilon)$ uniformly in t and ζ for $t \in [0, a]$ and $|\zeta|$ bounded.*

PROOF. By Cauchy's theorem and the process used before we replace the curve $\Gamma(\zeta; \epsilon)$ in the integral expression for $Z_\epsilon(t, \zeta)$ by a curve $\Gamma''(\zeta; \epsilon)$ whose length is bounded independently of ϵ and ζ and such that for η on $\Gamma''(\zeta; \epsilon)$

$$|\eta - \bar{\mu}_l| > 1, \qquad \left|\eta - \frac{\bar{\nu}_j}{\epsilon}\right| > 1, \qquad \operatorname{Im}(\eta) > c - 2.$$

It is clear that a curve $\Gamma''(\zeta; \epsilon)$ will do for neighboring values of ζ as well. Consider an element in the (j, k)th place of $(i\eta A(\epsilon) + M)^{-1} A(\epsilon)$, where $N - m + 1 \leqslant j \leqslant N$, $1 \leqslant k \leqslant N - m$. It is in fact ϵ times the cofactor of the entry in the (k, j)th place of $i\eta A(\epsilon) + M$ divided by $\det(i\eta A(\epsilon) + M)$. But $\det(i\eta A(\epsilon) + M) \geqslant 1$ on $\Gamma''(\zeta; \epsilon)$ and the cofactor is certainly bounded for $|\zeta|$ bounded, whence the element is itself $0(\epsilon)$ for $|\zeta|$ bounded. Since we have the estimate $|e^{i\eta t}| \leqslant e^{-(c-2)t}$ for η on $\Gamma''(\zeta; \epsilon)$, the desired estimate follows readily.

We now consider the entries in the (j, k)th positions of $(i\eta A(\epsilon) + M)^{-1} A(\epsilon)$ where $N - m + 1 \leqslant j \leqslant N$, $N - m + 1 \leqslant k \leqslant N$. Using the notation established in Lemma 1, we have $(i\eta A(\epsilon) + M(\zeta))_1 = i\eta\epsilon I + M_1(\zeta)$; since η is bounded uniformly in ϵ for η on $\Gamma_l''(\zeta; \epsilon)$—that part of $\Gamma''(\zeta; \epsilon)$ surrounding $\bar{\mu}_l$—and M_1 is nonsingular, it follows that for ϵ sufficiently small $(i\eta A(\epsilon) + M)_1$ is also nonsingular. Therefore for ϵ small and η on $\Gamma_l''(\zeta; \epsilon)$ $(i\eta A(\epsilon) + M)_1^{-1}$ exists and is "near" M_1^{-1}, because the inverse of a matrix, when it exists, is a continuous function of the entries of the matrix. Thus also $M_4 - M_3(i\eta A(\epsilon) + M)_1^{-1} M_2$ is "near" $M_4 - M_3 M_1^{-1} M_2$. But for η on $\Gamma_l''(\zeta; \epsilon)$ and ϵ sufficiently small $|\eta - \mu_k| \geqslant \frac{1}{2}$, $k = 1, \ldots, m$, since $\mu_k = \bar{\mu}_k - \epsilon_k'$, and it follows that $(i\eta I + M_4 - M_3(i\eta A(\epsilon) + M)_1^{-1} M_2)^{-1}$ will exist because η is bounded away from the roots of the determinant of this matrix.

From Lemma 1 we then get the formula

$$(i\eta A(\epsilon) + M)^{-1} A(\epsilon)$$
$$= \begin{pmatrix} I & -(i\eta A(\epsilon) + M)_1^{-1} M_2[i\eta I + M_4 - M_3(i\eta A(\epsilon) + M)_1^{-1} M_2]^{-1} \\ 0 & [i\eta I + M_4 - M_3(i\eta A(\epsilon) + M)_1^{-1} M_2]^{-1} \end{pmatrix}$$
$$\times \begin{pmatrix} \epsilon(i\eta A(\epsilon) + M)_1^{-1} & 0 \\ -\epsilon M_3(i\eta A(\epsilon) + M)_1^{-1} & I \end{pmatrix}. \qquad (15)$$

Clearly the block of $(i\eta A(\epsilon) + M)^{-1} A(\epsilon)$ under consideration is just $(i\eta I + M_4 - M_3(i\eta A(\epsilon) + M)_1^{-1} M_2)^{-1}$, and as $\epsilon \to 0$ this goes to $(i\eta I + M_4 - M_3 M_1^{-1} M_2)^{-1}$; since the former matrix must behave as a rational function of ϵ, it follows that these two matrices differ by a term which is $O(\epsilon)$. A review of the analysis leading to this result shows that this occurs uniformly in ζ for $|\zeta|$ bounded. By using again the estimate $|e^{i\eta t}| \leqslant e^{-(c-2)t}$ for η on $\Gamma''(\zeta; \epsilon)$, the conclusion follows immediately by an application of the method used before.

Before proceeding we first slightly sharpen Theorem 1 under our present hypotheses:

LEMMA 5. *Given any $\delta > 0$ there exists a constant k_δ independent of $t \in [0, a]$, any $a > 0$, and ϵ for ϵ sufficiently small such that*

$$\int_{|\zeta| > k_\delta} |Z_\epsilon(t, \zeta) \hat{f}(\zeta)| \, d\zeta < \delta.$$

PROOF. We re-examine the estimates preceding Theorem 1. We begin with that part Γ' of Γ surrounding the roots $\bar{\mu}_l(\zeta; \epsilon)$, $l = 1, \ldots, m$, and observe that $|e^{i\eta t}| \leqslant e^{-(c-1)t}$ holds uniformly in ϵ for η on Γ' since c is not allowed to depend on ϵ. The length of Γ' is bounded uniformly in ϵ, and for η on Γ' $\det(i\eta A(\epsilon) + M) \geqslant 1$. Thus any entry of $[i(\bar{\mu}_l + \eta') A(\epsilon) + M(\zeta)]^{-1}$ behaves no worse than a polynomial in ζ with coefficients polynomials in ϵ, so we can assert that the integral around the roots $\bar{\mu}_l$ contributes to $Z_\epsilon(t, \zeta)$ terms which grow no faster than $e^{-(c-1)t} |\zeta|^{(N-1)p}$ with t, ζ and ϵ as $|\zeta| \to \infty$ and $\epsilon \to 0$. For the part of Γ surrounding $\bar{\nu}_j/\epsilon$ we proceed as follows. Let k be the maximum number of roots ν_l (counting ν_j) which are equal to ν_j for some ζ. Then for η on Γ we have $\det(i\eta A(\epsilon) + M(\zeta)) \sim (1/\epsilon)^{m-k}$ *or some higher power of* $1/\epsilon$. Clearly, with the substitution $\eta = \bar{\nu}_j/\epsilon + \eta'$ the terms in the first $N - m$ columns of $[i\eta A(\epsilon) + M(\zeta)]^* A(\epsilon)$ ($*$ = transpose of the cofactor matrix) behave like polynomials in $1/\epsilon$ of degree at most $m - 1$; this conclusion also holds for the last m columns. As for the roots $\bar{\mu}_l$ we see that the integral about the root $\bar{\nu}_j/\epsilon$ contributes to $Z_\epsilon(t, \zeta)$ terms which grow no faster that $e^{-(\mathrm{Im}(\bar{\nu}_j)/\epsilon)t} e^t (1/\epsilon)^{k-1} |\zeta|^{(N-1)p}$ as $|\zeta| \to \infty$

and $\epsilon \to 0$. If $k = 1$ the assumption $\operatorname{Im}(\bar{\nu}_j) \geqslant 0$ is sufficient to bound the ϵ behavior; for $k > 1$ we must use the method of Lemma 3 to conclude that these terms are bounded uniformly in ϵ small. Thus Theorem 1 holds with the integral in Eq. (5) converging *uniformly in* ϵ for ϵ in some interval of the form $(0, b]$, $b > 0$, whence the desired conclusion follows.

We can now state and prove

THEOREM 3. *Each of the last m entries of $V_\epsilon(t, \mathbf{x})$ given by Eq. (5) goes to the corresponding entry of $U(t, \mathbf{x})$ given by*

$$U(t, \mathbf{x}) = (2\pi)^{-n/2} \int_{E_n} e^{-i(\mathbf{x},\zeta)} \begin{pmatrix} u^1(t, \zeta) \\ u^2(t, \zeta) \end{pmatrix} d\zeta,$$

as $\epsilon \to 0$ uniformly in $(t, \mathbf{x}) \in [0, a] \times E_n$ for any a, $0 < a < \infty$.

PROOF. We showed above that $Z_\epsilon(t, \zeta) \hat{f}(\zeta)$ is integrable uniformly in ϵ under the conditions imposed; in Section 3 we showed that $\binom{u^1}{u^2}$ is integrable. Thus given $\delta > 0$ there exists k (independent of ϵ) so large that for all $t \in [0, a]$

$$(2\pi)^{-n/2} \int_{|\zeta| > k} \left| \begin{pmatrix} u^1(t, \zeta) \\ u^2(t, \zeta) \end{pmatrix} \right| d\zeta < \delta, \qquad (2\pi)^{-n/2} \int_{|\zeta| > k} | Z_\epsilon(t, \zeta) \hat{f}(\zeta) \, d\zeta | < \delta.$$

On the other hand it follows from Lemma 3 that for ϵ sufficiently small and $t \in [0, a]$

$$(2\pi)^{-n/2} \int_{|\zeta| \leqslant k} | (Z_\epsilon(t, \zeta) \hat{f}(\zeta))^2 | = 0(\epsilon)$$

for those terms involving $e^{i(\nu_j/\epsilon)t}$; from Lemma 4 we have

$$(2\pi)^{-n/2} \int_{|\zeta| \leqslant k} | (Z_\epsilon \,|\, (t, \zeta) \hat{f}(\zeta))^2 - u(t, \zeta)^2 | = 0(\epsilon)$$

for those terms involving $e^{i\bar{\mu}_i t}$. Putting all this together with the Schwartz inequality we conclude that

$$| V_\epsilon(t, \mathbf{x}) - U(t, \mathbf{x}) | < 4\delta$$

for ϵ sufficiently small. The conclusion follows since $\delta > 0$ is arbitrary.

LEMMA 6. *The first $N - m$ components of $Z_\epsilon(t, \zeta) \hat{f}(\zeta)$ are each of the form*

$$v_{\epsilon,k}(t, \zeta) = (\hat{f}^1 + M_1^{-1} M_2 \hat{f}^2)_k \left(\sum_{j=1}^{N-m} \alpha_j^k(t, \zeta; \epsilon) \, e^{i(\bar{\nu}_j/\epsilon)t} \right) + u_k(t, \zeta) + 0(\epsilon) \tag{16}$$

$k = 1, \ldots, N - m$, where $\sum_{j=1}^{N-m} \alpha_j^k(0, \zeta; \epsilon) \to 1$ as $\epsilon \to 0$ and $0(\epsilon)$ represents a function which is of the order of ϵ uniformly in t, ζ for $t \in [0, a]$ and $|\zeta|$ bounded.

PROOF. We consider initially the terms of any of the first $N - m$ components $v_{\epsilon,k}(t, \zeta)$ involving $e^{i\bar{\mu}_l t}$. For ϵ sufficiently small we have the representation of $(i\eta A(\epsilon) + M)^{-1} A(\epsilon)$ given in Eq. (15). From this formula it follows easily, using familiar arguments, that the terms involving $e^{i\bar{\mu}_l t}$ and f^1, corresponding to the first $N - m$ colums of $Z_\epsilon(t, \zeta)$, are $0(\epsilon)$ uniformly in $t \in [0, a]$ and $|\zeta|$ bounded, for by Eq. (15) these terms arise from the expression

$$\epsilon(i\eta A(\epsilon) + M)_1^{-1} + \epsilon(i\eta A(\epsilon) + M)_1^{-1} \times M_2[M_4 - M_3(i\eta A(\epsilon) + M)_1^{-1} M_2]^{-1} M_3(i\eta A(\epsilon) + M)_1^{-1},$$

which for ϵ small is near

$$\epsilon M_1^{-1} + \epsilon M_1^{-1} M_2[M_4 - M_3 M_1^{-1} M_2]^{-1} M_3 M_1^{-1},$$

which clearly goes as ϵ, uniformly for $|\zeta|$ bounded. For the terms corresponding to the last m columns of $Z_\epsilon(t, \zeta)$ we use the same equation, observing that as $\epsilon \to 0$ $(i\eta A(\epsilon) + M)_1^{-1} \to M_1^{-1}$ uniformly in t (obviously) and ζ for $|\zeta|$ bounded. Hence these terms go to those of

$$-M_1^{-1} M_2 u^2(t, \zeta) = u^1(t, \zeta)$$

uniformly for $t \in [0, a]$ and $|\zeta|$ bounded, as before. Thus far we have established that

$$v_{\epsilon,k} = u_k(t, \zeta) + g_k(t, \zeta; \epsilon) + \text{terms in } e^{i(\bar{\nu}_j/\epsilon)t},$$

where $g_k(t, \zeta; \epsilon) \sim \epsilon$ uniformly for $t \in [0, a]$ and $|\zeta|$ bounded.

We turn now to the terms involving $e^{i(\bar{\nu}_j/\epsilon)t}$ and show that these have the representation specified by the first term of Eq. (14). Indeed, the terms in question must have the form

$$\sum_{j=1}^{N-m} \beta_j^k(t, \zeta; \epsilon)\, e^{i(\bar{\nu}_j/\epsilon)t}$$

and at $t = 0$ equal

$$(f^1 + M_1^{-1} M_2 f^2)_k - g_k(0, \zeta; \epsilon),$$

since u_k satisfies the initial condition $u_k(0, \zeta) = (-M_1^{-1} M_2 f^2)_k$. The desired conclusion follows at once from the fact that the initial conditions must be satisfied for all $\epsilon > 0$. Q.E.D.

Lemmas 3, 4, and 6 illustrate the structure of the Fourier transform of $V_\epsilon(t, \mathbf{x})$. It is clear that, should we allow $\operatorname{Im}(\bar{\nu}_j(\zeta; \epsilon)) = 0$ for all ζ and all ϵ sufficiently small, the term $\sum \alpha_i e^{i(\bar{\nu}_j/\epsilon)t}$ would not necessarily go to zero for $t > 0$ as $\epsilon \to 0$, and this behavior might carry over to the inverse Fourier transform of $v_\epsilon(t, \zeta)$. Such a degeneration would not be "regular." It was in part to exclude such a happening that we assumed $\operatorname{Im}(\bar{\nu}_j(\zeta; \epsilon)) \geqslant d > 0$ for all real ζ. This is the simplest assumption which suffices; see the Remark below for an extension.

We can now state the following theorem, wherein by way of summary all necessary hypotheses are made explicit.

THEOREM 4. *Let $V_\epsilon(t, \mathbf{x})$ be the solution of the problem (1), (2) with $1 \leqslant m \leqslant N - 1$, where $f(\mathbf{x})$ has a sufficiently large number of continuous L_1 derivatives (see Theorems 1 and 2), and let $U(t, \mathbf{x})$ be the solution of the degenerate equation (7) satisfying the initial conditions*

$$U^2(0, \mathbf{x}) = f^2(\mathbf{x}).$$

Assume that the roots η of $Q_\epsilon(\eta, \zeta) \equiv \det(i\eta A(\epsilon) + M(\zeta))$ satisfy $\operatorname{Im}(\eta) \geqslant c$ for some constant c and all sufficiently small $\epsilon > 0$, and assume that $\det M_1$ is bounded away from zero, so these solutions do in fact exist. Assume, moreover, that the roots $\bar{\nu}_j(\zeta; \epsilon)/\epsilon$ (Cf. Eq. (13)) are such that $\operatorname{Im}(\bar{\nu}_j(\zeta; \epsilon)) \geqslant d > 0$ for all real ζ. Then

$$V_\epsilon(t, \mathbf{x}) = U(t, \mathbf{x}) + \textit{boundary layer terms} + o(1),$$

where boundary layer terms enter into only the first $N - m$ elements of $V_\epsilon(t, \mathbf{x})$ and have the following behavior: for $t \in [\delta, a]$, $\delta > 0$, any $a > 0$, they converge to zero uniformly in t and $\mathbf{x}$; however, they need not go to zero for $t = 0$, although they must remain bounded at $t = 0$. The quantity $o(1)$ goes to zero with $\epsilon \to 0$ uniformly for $(t, \mathbf{x}) \in [0, a] \times E_n$.

PROOF. As concerns the last m components of $V_\epsilon(t, \mathbf{x})$, this theorem follows immediately from Theorem 3, so we need consider only the first $N - m$ components. We have already argued that for all ϵ sufficiently small there exists a constant k such that

$$\int_{|\zeta| > k} | Z_\epsilon(t, \zeta) \hat{f}(\zeta) | \, d\zeta < \delta$$

for t in the appropriate interval. Since the term involving the roots $\bar{\nu}_j/\epsilon$ in Eq. (16) is by construction just

$$\Big[\sum \oint_{\Gamma_j''} e^{i\eta t} (i\eta A(\epsilon) + M)^{-1} A(\epsilon) \, d\eta \hat{f} \Big]_l ,$$

where the sum is over all parts of $\Gamma''(\zeta; \epsilon)$ enclosing the roots $\bar{\nu}_j/\epsilon$, it follows by the same argument applied to this expression that there exists for this term a constant k with similar properties. Hence we may restrict our considerations to $|\zeta| \leqslant k$, since the inverse Fourier integrals over $|\zeta| > k$ each contribute at most δ for arbitrary $\delta > 0$. For brevity the inverse Fourier integral taken over $|\zeta| \leqslant k$ will be called the *k-inverse transform.* From Lemma 6 the k-inverse transform of the term denoted $o(\epsilon)$ is again $o(\epsilon)$ uniformly for $(t, \mathbf{x}) \in [0, a] \times E_n$. The k-inverse transforms of $u_l(t, \zeta)$ and $v_{\epsilon,l}(t, \zeta)$ are $U_l(t, \mathbf{x})$ and $V_{\epsilon,l}(t, \mathbf{x})$ to within δ. It remains only to consider the k-inverse transform of $(\hat{f}^1 + M_1^{-1}M_2\hat{f}^2)_l\,(\sum_{j=1}^{N-m} \alpha_j{}^l(t, \zeta; \epsilon)\, e^{i(\bar{\nu}_j/\epsilon)t})$, and this is just the k-inverse transform of the sum of the integrals around the $\Gamma_j''(\zeta; \epsilon)$ of $e^{i\eta t}(i\eta A(\epsilon) + M)^{-1} A(\epsilon)\hat{f}$. We again use the residue technique to estimate the contribution of the term in $e^{i(\bar{\nu}_j/\epsilon)t}$ to the integral involved in the definition of $Z_\epsilon(t, \zeta)$, restricting attention to the first $N - m$ rows. We shall present only the case where the ν_j are distinct; the case of multiple roots follows by an application of the method used in the proof of Lemma 3. Denote by c_{kl} the (l, k)th cofactor of $i\eta A(\epsilon) + M(\zeta)$, $1 \leqslant k \leqslant N - m$; it is easy to see that, at worst, $c_{kl} \sim (1/\epsilon)^m$ if $1 \leqslant l \leqslant N - m$, $c_{kl} \sim (1/\epsilon)^{m-1}$ if $N - m + 1 \leqslant l \leqslant N$. Let q be $\det(i\eta A(\epsilon) + M(\zeta))/(\epsilon\eta - \bar{\nu}_j)$; then $q \sim (1/\epsilon)^m$. Then the residue we are after is

$$\left.\frac{e^{i\eta t}\epsilon^\sigma c_{kl}}{q}\right|_{\eta=\bar{\nu}_j/\epsilon},$$

where $\sigma = 1$ if $1 \leqslant l \leqslant N - m$, $\sigma = 0$ if $N - m + 1 \leqslant l \leqslant N$. Thus $\epsilon^\sigma c_{kl} \sim (1/\epsilon)^{m-1}$, $1 \leqslant l \leqslant N$. As in the proof of Lemma 3 it is easy to see, allowing for the fact that $\epsilon\eta$ is the variable of integration, that $1/\epsilon$ times this residue, and hence the term in $e^{i(\bar{\nu}_j/\epsilon)t}$ in the (k, l)th element of $Z_\epsilon(t, \zeta)$, behaves like $e^{-(\mathrm{Im}(\nu_j)/\epsilon)t}\, 0(1)$ for each ζ, continuously in ζ. We now trivially have

$$\left| \left(\int_{|\zeta| \leqslant k} e^{-i(x,\zeta)}\, Z_\epsilon(t, \zeta) \hat{f}(\zeta)\, d\zeta \right)_k \right| \leqslant K \sum_{j=1}^{N-m} e^{-[\theta_j(\epsilon)/\epsilon]t}.$$

for some constant K, where

$$\theta_j(\epsilon) = \min_{|\zeta| \leqslant k} [\mathrm{Im}\,(\bar{\nu}_j(\zeta; \epsilon)].$$

Here the integrand on the left is understood to contain only those terms involving $e^{i(\bar{\nu}_j/\epsilon)t}$. For ϵ sufficiently small we have $\theta_j \geqslant d$ for $d > 0$ and $j = 1, 2, \ldots, N - m$. Thus the k-inverse transform converges to zero uniformly in $(t, \mathbf{x}) \in [\delta, a) \times E_n$. Clearly convergence to zero at $t = 0$ cannot be guaranteed, although the k-inverse transform is bounded there. The theorem follows.

Of course, that boundary layer terms will actually occur is not guaranteed; the theorem only states that they may be present. In particular, if the initial data are trumped up so that

$$\hat{f}^1 + M_1^{-1}M_2\hat{f}^2 \equiv 0,$$

no boundary layer terms need be anticipated, as the analysis above clearly shows.

REMARK. The assumption $\operatorname{Im}(\bar{\nu}_j(\boldsymbol{\zeta}; \epsilon)) \geqslant d > 0$ for all real $\boldsymbol{\zeta}$ can be relaxed somewhat. For example, suppose $\bar{\nu}_1(\boldsymbol{\zeta}; \epsilon) \neq \bar{\nu}_j(\boldsymbol{\zeta}; \epsilon)$, $j \neq 1$, for all real $\boldsymbol{\zeta}$ and ϵ sufficiently small, and that $\operatorname{Im}(\bar{\nu}_1(\boldsymbol{\zeta}; \epsilon)) \equiv 0$. Suppose also that for each fixed real $\zeta_2, \ldots, \zeta_n$ and ϵ sufficiently small $\bar{\nu}_1(\cdot, \zeta_2, \ldots, \zeta_n; \epsilon)$ is one-to-one (this state could possibly be brought about by a rotation in the $\boldsymbol{\zeta}$ space) for real values of the argument. In Theorem 4 we are interested in the integral

$$\int_{|\zeta| \leqslant k} e^{i\bar{\nu}_1[(\zeta;\epsilon)]t/\epsilon} h(\mathbf{x}, t, \boldsymbol{\zeta}; \epsilon)\, d\boldsymbol{\zeta},$$

where

$$h(\mathbf{x}, t, \boldsymbol{\zeta}; \epsilon) = e^{i(\mathbf{x},\zeta)}\, \alpha_1(t, \boldsymbol{\zeta}; \epsilon)\, (\hat{f}^1 + M_1^{-1}M_2\hat{f}^2)_l\, .$$

Under our present assumptions we can introduce $\nu \equiv \bar{\nu}_1, \zeta_2, \ldots, \zeta_n$ as new variables of integration, whence the integral above can be written as

$$\int e^{i\nu(t/\epsilon)}\, h_1(x, t, \nu, \zeta_2, \ldots, \zeta_n; \epsilon)\, d\nu\, d\zeta_2 \cdots d\zeta_n\, ,$$

the integral still being over a compact set. Performing the ν integration first, we are left with the integral over a compact set of

$$\int e^{i(t/\epsilon)\nu}\, h_1(x, t, \nu, \zeta_2, \ldots, \zeta_n; \epsilon)\, d\nu.$$

The integral above is clearly continuous in $\zeta_2, \ldots, \zeta_n$; by the Riemann-Lebesgue Lemma of Fourier theory this integral goes to zero as $\epsilon \to 0$ for $t > 0$. It follows that

$$\lim_{\epsilon \to 0} \int_{|\zeta| \leqslant k} e^{i\bar{\nu}_1(\zeta;\epsilon)t/\epsilon} h(\mathbf{x}, t, \boldsymbol{\zeta}; \epsilon)\, d\boldsymbol{\zeta} = 0$$

for $t > 0$, as in Theorem 4. The same method can be applied to obtain the corresponding result for the term in $e^{i(\bar{\nu}_1(\zeta;\epsilon)/\epsilon)t}$ in the last m entries of V_ϵ, which then replaces Lemma 3 for this term. Hence Theorem 4 holds in the present case.

If we have only $\mathrm{Im}\,(\bar{\nu}_1(\zeta;\epsilon)) \geqslant 0$ and $\mathrm{Re}\,(\bar{\nu}_1(\cdot, \zeta_2, ..., \zeta_n; \epsilon)$ one-to-one for real values of the argument, then the analysis above (and hence Theorem 4) goes through if we replace therein $\bar{\nu}_1$ by $\mathrm{Re}\,(\bar{\nu}_1)$ and define

$$h(\mathbf{x}, t, \zeta; \epsilon) \qquad \text{to be} \qquad e^{i(x,\zeta)}\, e^{-[\mathrm{Im}(\bar{\nu}_1)/\epsilon]t}\, \alpha_1(t, \zeta; \epsilon)\, (\hat{f}^1 + M_1^{-1} M_2 \hat{f}^2)_l\,.$$

The extensions described in this Remark can also be put through for the inhomogeneous problem discussed next.

5. The Inhomogeneous Equation

We now consider the inhomogeneous problem

$$L_\epsilon[V_\epsilon] = b, \qquad V_\epsilon(0, \mathbf{x}) = f(\mathbf{x}), \tag{17}$$

where $b(t, \mathbf{x})$ is a function of the indicated arguments with such properties as will subsequently be imposed. We shall be interested in the solution for t in some interval $[0, T]$. The solution V_ϵ of the problem (17) can be written $V_\epsilon = U_\epsilon + W_\epsilon$, where U_ϵ satisfies the homogeneous equation $L_\epsilon[U_\epsilon] = 0$ with inhomogeneous data $f(\mathbf{x})$ and W_ϵ satisfies the inhomogeneous equation $L_\epsilon[W_\epsilon] = b$ with homogeneous data. The former problem has already been considered, so we need here consider only the problem for the function W_ϵ.

If $Z_\epsilon(t, \zeta)$ is the function defined by Eq. (6), the function w_ϵ defined by

$$w_\epsilon(t, \zeta) = Z_\epsilon(t, \zeta) \int_0^t Z_\epsilon^{-1}(s, \zeta)\, A^{-1}(\epsilon)\, \hat{b}(s, \zeta)\, ds \tag{18}$$

will be the solution of the Fourier-transformed inhomogeneous system

$$A(\epsilon) \frac{\partial w_\epsilon(t, \zeta)}{\partial t} + M(\zeta)\, w_\epsilon(t, \zeta) = \hat{b}(t, \zeta) \tag{19}$$

satisfying homogeneous initial data. *We assume that $\hat{b}$ exists*, i.e., that $b \in L_1$. Taking advantage of the fact that $Z_\epsilon(t, \zeta)$ generates a semigroup, we may rewrite Eq. (18) as

$$w_\epsilon(t, \zeta) = \int_0^t Z_\epsilon(t - s, \zeta)\, A^{-1}(\epsilon)\, \hat{b}(s, \zeta)\, ds. \tag{20}$$

The solution $W_\epsilon(t, \mathbf{x})$ of $L_\epsilon[W_\epsilon] = b$ will be given by

$$W_\epsilon(t, \mathbf{x}) = (2\pi)^{-n/2} \int_{E_n} e^{-i(\mathbf{x},\zeta)}\, w_\epsilon(t, \zeta)\, d\zeta, \tag{21}$$

provided this integral and the derivatives thereof required in Eq. (17) converge uniformly on $[0, T] \times E_n$.

In Section 2 we showed that the elements of $Z_\epsilon(t, \zeta)$ behave at worst like $e^{-(c-1)t} \mid \zeta \mid^{(N-1)p-l}$ as $\mid \zeta \mid \to \infty$ for $t \geqslant 0$, whence $w_\epsilon(t, \zeta)$ becomes infinite no faster than

$$A^{-1}(\epsilon)\, e^{-(c-1)t} \mid \zeta \mid^{(N-1)p-l} \int_0^t e^{(c-1)s}\, \hat{b}(s, \zeta)\, ds.$$

By requiring that $b(t, \mathbf{x})$ be continuous in all arguments and be integrable in $\mathbf{x}$ *uniformly in t for $t \in [0, T]$ for some finite $T > 0$,* we have $\hat{b}(t, \zeta)$ bounded for $t \in [0, T]$ and $\zeta \in E_n$. Since $\int_0^t e^{(c-1)s} ds$ is bounded (uniformly) for $t \in [0, T]$, it follows that $w_\epsilon(t, \zeta)$ will become infinite as $\mid \zeta \mid \to \infty$ no faster than $\mid \zeta \mid^{(N-1)p-l-k}$ if $b(t, \mathbf{x})$ has k continuous $\mathbf{x}$-derivatives. In precisely the manner of Theorem 1 we now have

THEOREM 5. *If, in addition to the assumptions above, $k \geqslant Np - l + n + 1$, then $W_\epsilon(t, \mathbf{x})$, defined by Eq. (21), is the solution of the inhomogeneous equation $L_\epsilon[W_\epsilon] = b$ satisfying homogeneous Cauchy data.*

We turn now to the degenerate problem $L_0[W_0] = b$, $W_0(0, \mathbf{x}) = 0$. Taking account of the form of $A(0)$, the Fourier-transformed system becomes

$$M_1 w_0^1 + M_2 w_0^2 = \hat{b}^1$$
$$\frac{\partial w_0^2}{\partial t} + M_3 w_0^1 + M_4 w_0^2 = \hat{b}^2,$$

which under the assumptions made in Section 3 is equivalent to

$$w_0^1 = - M_1^{-1} M_2 w_0^2 + M_1^{-1} \hat{b}^1 \tag{22}$$

$$\frac{\partial w_0^2}{\partial t} + (M_4 - M_3 M_1^{-1} M_2)\, w_0^2 = \hat{b}^2 - M_3 M_1^{-1} \hat{b}^1. \tag{23}$$

As in the homogeneous problem, it is clear that the values of w_0^1 are completely determined by those of w_0^2. In particular, this holds for $t = 0$; i.e., w_0^1 need not satisfy homogeneous initial data, whence neither need W_0^1.

As in the analysis for the nondegenerate problem, we set

$$w_0^2(t, \zeta) = Z'(t, \zeta) \int_0^t (Z'(s, \zeta))^{-1} (\hat{b}^2(s, \zeta) - M_3 M_1^{-1}(s, \zeta))\, ds$$

$$= \int_0^t Z'(t - s, \zeta)\, (\hat{b}^2(s, \zeta) - M_3 M_1^{-1} \hat{b}^1(s, \zeta))\, ds,$$

$$w_0^1(t, \zeta) = - M_1^{-1} M_2 w_0^2(t, \zeta) + M_1^{-1} \hat{b}^1(t, \zeta),$$

$$W_0^j(t, \mathbf{x}) = (2\pi)^{-n/2} \int_{E_n} e^{-i(\mathbf{x}, \zeta)}\, w_0^j(t, \zeta)\, d\zeta, \qquad j = 1, 2,.$$

The following result is established in the same manner as Theorem 2; the notation is that of Section 3.

THEOREM 6. *Let b have at least k continuous derivatives, with*

$$k \geqslant (m-1)s - r + n + 1 + \max(p, s) + 2\max(0, N-m-1)q - \gamma + p.$$

Then $W_0(t, \mathbf{x})$ exists and is a solution of the problem

$$L_0[W_0] = b, \qquad W_0^2(0, \mathbf{x}) = 0.$$

In establishing Theorems 3 and 4 of the preceding section we used the fact that the integrals defining V_ϵ converged uniformly in ϵ for ϵ in some interval $(0, a]$. It is easy to see that such a result should not be expected for W_ϵ. Moreover, even if Lemma 5 were valid for W_ϵ, the nice behavior $W_\epsilon \to W_0$ + boundary layer terms $+ o(1)$ as $\epsilon \to 0$ could not, on the basis of the results we have heretofor established, be expected for a general $b(t, \mathbf{x})$ satisfying only the properties so far imposed. Indeed, Lemma 4 yields the estimate that the term involving $e^{i\bar{\mu}_l t}$ in any of the first $N - m$ elements of the last m rows of $Z_\epsilon(t, \zeta)\, A^{-1}(\epsilon)$ is $0(1)$ uniformly in (t, ζ) for $t \in [0, T]$ and $|\zeta|$ bounded. Thus if Lemma 5 were valid we would have the result that such terms contribute to W_ϵ quantities which are bounded as $\epsilon \to 0$ but which need not go to zero, as they did in the homogeneous case.

Nevertheless, regular degeneration will be shown to hold under a certain apparently restrictive condition on b. *This condition, henceforward assumed to hold, is just*

$$b^1(t, \mathbf{x}) \equiv 0.$$

We first show that this restriction includes the interesting case wherein the system arises from a single inhomogeneous equation $P_\epsilon[V] = g(t, \mathbf{x})$ with $\partial g(t, \mathbf{x})/\partial t$ satisfying the conditions imposed on b above in Theorems 5 and 6. To see this, suppose the single homogeneous equation $P_\epsilon[V] = 0$ has been written as the system $L_\epsilon[U] = 0$ satisfying the various requirements of Theorem 4 and such that the inhomogeneous equation $P_\epsilon[V] = g$ corresponds to the system

$$L_\epsilon[U] = \begin{pmatrix} g \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Introduce a new variable U'_{N+1} by the inhomogeneous equation

$$\frac{\partial U'_{N+1}(t, \mathbf{x})}{\partial t} = \frac{\partial g(t, \mathbf{x})}{\partial t},$$

where U'_{N+1} is to satisfy the initial condition

$$U'_{N+1}(0, \mathbf{x}) = g(0, \mathbf{x}).$$

Clearly $U'_{N+1} \equiv g$. Set $U_i' = U_i$, $i = 1,..., N$, and define a new operator L_ϵ' by

$$(L_\epsilon'[U'])_1 = (L_\epsilon[U])_1 - U'_{N+1}$$
$$(L_\epsilon'[U'])_i = (L_\epsilon[U])_i, \qquad i = 2,..., N$$
$$(L_\epsilon'[U'])_{N+1} = \frac{\partial U'_{N+1}}{\partial t}.$$

Denoting by $M'(\zeta)$ the matrix resulting from Fourier transformation on the non-t-differentiated terms, one has

$$M' = \left|\begin{array}{ccc|c} & M & & -1 \\ & & & 0 \\ & & & \vdots \\ \hline 0 & 0 & \cdots & 0 \end{array}\right|, \qquad i\eta A'(\epsilon) + M' = \left|\begin{array}{ccc|c} & i\eta A(\epsilon) + M & & -1 \\ & & & 0 \\ & & & \vdots \\ \hline 0 & 0 & \cdots & i\eta \end{array}\right|,$$

so $\det(i\eta A'(\epsilon) + M') = i\eta \det(i\eta A(\epsilon) + M)$ follows by a Laplace expansion about the last row. This equation has only the root $\eta = 0$ in addition to the roots of $\det(i\eta A(\epsilon) + M) = 0$, whence the new homogeneous system $L_\epsilon'[U'] = 0$ satisfies the requirements of Theorem 4 since the system $L_\epsilon[U] = 0$ did. The inhomogeneous equation $P_\epsilon[V] = g(t, \mathbf{x})$ corresponds to the inhomogeneous system

$$L_\epsilon'[U'] = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \dfrac{\partial g}{\partial t} \end{pmatrix}.$$

Note that the homogeneous equation $P_\epsilon[V] = 0$ does *not* correspond to the homogeneous system $L_\epsilon'[U'] = 0$ unless $g(0, \mathbf{x}) \equiv 0$. It does, of course, correspond to the system $L_\epsilon[U] = 0$.

It is now easy to see, by iterating the procedure outlined, that this procedure can readily be extended to any inhomogeneous system of N equations $L_\epsilon[U] = b$ in which $\partial b_j/\partial t, j = 1,..., N - m$, satisfies the conditions imposed on b in Theorems 5 and 6 (where the quantities N, m, etc. refer to the *new* system). The result is a new system of $2N - m$ equations $L_\epsilon'[U'] = d$ in which all the old unknowns U appear as the first N components of U' and whose right-hand side d satisfies $d^1 \equiv 0$. Thus our assumption that $b^1 \equiv 0$ is not nearly as restrictive as might appear at first sight.

Notice that the solution U of the original inhomogeneous system with inhomogeneous data is now obtained as the sum of the solutions of *three* systems: the solution U_1 of the original homogeneous system with inhomogeneous data, the vector composed of the first N components of the solution U_2 of the homogeneous re-written system with inhomogeneous initial data (namely, the initial values of the b_j, $j = 1,..., N - m$) in the last $N - m$ components only, and the vector composed of the first N components of the solution U_3 of the inhomogeneous rewritten system with homogeneous data. The sum of the latter two vectors is, in fact, the solution of the original inhomogeneous system with homogeneous data. The sum $U_1^* + U_2$ (where the vector U_1^* is U_1 with $N - m$ zeros added at the end to make it of the same dimension as U_2) is the solution of the homogeneous rewritten system with inhomogeneous data in all components:

$$(U_1^* + U_2)(0, x) = U_1^*(0, x) + U_2(0, x).$$

Clearly we could study the solution U as the vector composed of the first N components of $(U_1^* + U_2) + U_3$, investigating $(U_1^* + U_2)$ directly as the solution of the homogeneous system just described. The disadvantage of so doing, however, is that stronger restrictions are put on the initial data of U, since the system for $U_1^* + U_2$ is of larger dimension than that for U and the differentiability requirements imposed on the data increase with the size of the system (see Theorems 1 and 2).

We note that, if b has the properties we have imposed (in particular, $b^1 \equiv 0$), Lemma 5 holds with $f(\zeta)$ replaced by $A^{-1}(\epsilon)\,\hat{b}(s, \zeta) = \hat{b}(s, \zeta)$. Although Lemmas 3 and 4 are not valid for $Z_\epsilon(t, \zeta)\, A^{-1}(\epsilon)$, they do hold for those terms of $Z_\epsilon(t, \zeta)\, A^{-1}(\epsilon)$ which are not multiplied by zero in forming the product $Z_\epsilon(s, \zeta)\, A^{-1}(\epsilon)\, \hat{b}(s, \zeta)$; hence we lose nothing by taking Lemmas 3 and 4 to be valid. Thus we have

THEOREM 7. *Each of the last m entries of $W_\epsilon(t, \mathbf{x})$ goes to the corresponding entry of $W_0(t, \mathbf{x})$ uniformly in $(t, \mathbf{x}) \in [0, T] \times E_n$.*

PROOF. Exactly like that of Theorem 3.

We now establish an analogue to Lemma 6.

LEMMA 7. *The kth component of $Z_\epsilon(t-s, \zeta)\, A^{-1}(\epsilon)\, \hat{b}(s, \zeta)$, $1 \leqslant k \leqslant N-m$, is of the form*

$$\sum_{j=1}^{N-m} \beta_j^k(t-s, \zeta; \epsilon)\, e^{i(\bar{\nu}_j/\epsilon)(t-s)} + [Z'(t-s, \zeta)\, \hat{b}^2(s, \zeta)]_k + 0(\epsilon),$$

where $0(\epsilon)$ represents a function which is $0(\epsilon)$ uniformly in (t, ζ) for $t \in [0, T]$ and $|\zeta|$ bounded, and $\beta_j^k(t, \zeta; \epsilon)$ is a polynomial in t with coefficients dependent on ζ and ϵ and which is bounded as $\epsilon \to 0$.

PROOF. Proceeding as in Lemma 6 with the analogue of Eq. (15) for $(i\eta A(\epsilon) + M)^{-1}$, it is easily seen that the term of $Z_\epsilon(t-s, \zeta)\, A^{-1}(\epsilon)\, \hat{b}(s, \zeta)$ involving $e^{i\bar{\mu}_l t}$ can be written as the corresponding term of $Z'(t-s, \zeta)\, \hat{b}^2(s, \zeta)$ plus a quantity which is $0(\epsilon)$ uniformly for $t \in [0, T]$ and $|\zeta|$ bounded. As in Lemma 6, the term involving $e^{i(\bar{\nu}_j/\epsilon)t}$ arises from the integral of

$$e^{i\eta t}[i\eta A(\epsilon) + M(\zeta)]^{-1}\, \hat{b}(s, \zeta)$$

around a curve enclosing the root $\bar{\nu}_j/\epsilon$. That $\beta_j^k(t-s, \zeta; \epsilon)$ is $0(1)$ and is a polynomial in t follows by the residue technique used in proving Lemma 3; we shall present the proof only for $\nu_j(\zeta)$ distinct from $\nu_k(\zeta)$, $k \neq j$, in which case $\beta_j^k(t-s, \zeta; \epsilon)$ does not depend on t. Then, using C for the cofactor of $i\eta A(\epsilon) + M(\zeta)$ considered and q for $\det(i\eta A(\epsilon) + M(\zeta))/(\epsilon\eta - \bar{\nu}_j)$, we desire to determine the behavior of the residue $[e^{i(\bar{\nu}_j/\epsilon)t}\, C(\bar{\nu}_j/\epsilon)]/q(\bar{\nu}_j/\epsilon)$ as $\epsilon \to 0$. But $C(\bar{\nu}_j/\epsilon) \sim (1/\epsilon)^{m-1}$, $q(\bar{\nu}_j/\epsilon) \sim (1/\epsilon)^m$, whence, allowing for the fact that $\epsilon\eta$ is the variable of integration, we conclude that $\beta_j^k(t-s, \zeta; \epsilon) \sim 1$.

LEMMA 8. *If $\operatorname{Im}(\nu_j(\zeta)) > 0$ for each real ζ, $j = 1,\ldots, N-m$, the kth component of $w_\epsilon(t, \zeta)$, $k = 1,\ldots, N-m$, has the form*

$$w_{\epsilon,k}(t, \zeta) = w_{0,k}(t, \zeta) + 0(\epsilon),$$

where $0(\epsilon)$ is of the order of ϵ uniformly for $|\zeta|$ bounded and $t \in [0, T]$.

PROOF. We simply integrate the result of Lemma 7 from 0 to t, noting that in calculating an integral of the form $\int x^k e^{x/\epsilon}\, dx$ a factor ϵ is always brought down. Thus since $\operatorname{Im}(\nu_j(\zeta)) > 0$ implies $\operatorname{Im}(\bar{\nu}_j(\zeta; \epsilon)) > 0$ for all positive ϵ sufficiently small (Lemma 2), we conclude that these terms are in fact $0(\epsilon)$ uniformly in $t \in [0, T]$ and $|\zeta|$ bounded.

THEOREM 8. *Let the hypotheses of Theorem 4 of Section 4 be satisfied with $f = 0$. Let $b(t, \mathbf{x})$ be a function with the number of continuous $\mathbf{x}$-derivatives indicated in Theorems 5 and 6, and let $b(t, \mathbf{x})$ be integrable in $\mathbf{x}$ uniformly in t for all $t \in [0, T]$ for some finite $T > 0$. Assume that the first $N-m$ components of b are identically zero.*

Then

$$W_\epsilon(t, \mathbf{x}) = W_0(t, \mathbf{x}) + o(1),$$

where $o(1)$ *represents a quantity which goes to zero as* $\epsilon \to 0$, *uniformly in* $(t, \mathbf{x}) \in [0, T] \times E_n$.

PROOF. For the last m elements of W_ϵ this theorem is an immediate consequence of Theorem 7. The statement about the first $N - m$ elements of W_ϵ is established by means of Lemmas 5 and 8 in the manner of Theorem 4.

As a consequence of Theorem 8, Theorem 4, and the remarks made earlier in this paragraph, and under the various conditions imposed on the operator L_ϵ, we have the following behavior for the solution V_ϵ of the problem

$$L_\epsilon[V_\epsilon] = g(t, \mathbf{x}) \qquad V_\epsilon(0, \mathbf{x}) = f(\mathbf{x}),$$

where f and g are sufficiently smooth:

$$V_\epsilon(t, x) = V_0(t, \mathbf{x}) + \Lambda(t, \mathbf{x}; \epsilon) + o(1),$$

where V_0 is the solution of the degenerate problem discussed in Section 3. Here $o(1)$ again denotes a quantity which goes to zero as $\epsilon \to 0$, uniformly in $(t, \mathbf{x}) \in [0, T] \times E_n$, and $\Lambda(t, \mathbf{x}; \epsilon)$, the boundary-layer term, has its last m components identically zero and converges to zero uniformly in $(t, \mathbf{x}) \in [\delta, T] \times E_n$ for any $\delta > 0$.

6. EXAMPLES

The strongest of the conditions which we have had to impose to insure regular degeneration is the requirement that det $M_1(\zeta)$ be bounded away from zero for real ζ. We will now show by a simple example that this restriction is, as might be expected from its frequent appearance in our argument, not totally superfluous. First we observe that if for some ζ_1 det $M_1(\zeta_1) = 0$, then it follows from Eq. (14) that for some j, $1 \leqslant j \leqslant N - m$, $\nu_j(\zeta_1) = 0$. Hence if the condition that det M_1 is bounded away from zero for real ζ is to be done away with by allowing det M_1 to be zero at some point, we must be prepared to give up the restriction Im $(\nu_j(\zeta)) > 0$ for all real ζ under which all our theorems on degeneration have been obtained (excepting the Remark at the end of Section 4). Thus regular degeneration can doubly be expected to fail.

To illustrate we consider briefly the two-dimensional system

$$\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} u_t + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} u_x + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} u_{xx} = 0$$

with initial data

$$u(0, x) = \begin{pmatrix} x \\ g(x) \end{pmatrix}$$

where g is sufficiently differentiable. One easily calculates that

$$\det (i\eta A(\epsilon) + M(\zeta)) = (i\eta\epsilon + i\zeta + \zeta^2)(i\eta - i\zeta),$$

with roots $(-\zeta + i\zeta^2)/\epsilon$, ζ. Thus $\operatorname{Im}(\eta) \geqslant 0$ for all real ζ. Since $\det M_1(\zeta) = i\zeta + \zeta^2$, one sees that $\det M_1(0) = 0$, in violation of the requirement imposed in Sections 2-5. The root $\nu(\zeta)$ of Eq. (14) is $\nu(\zeta) = \bar{\nu}(\zeta; \epsilon) = i\zeta^2 - \zeta$, so that $\nu(0) = 0$, the necessity of which was remarked above, and $\operatorname{Im}(\bar{\nu}(\zeta; \epsilon) = 0$, in violation of the requirement imposed in Theorem 4. Note, however, that $\operatorname{Im}(\bar{\nu}(\zeta; \epsilon)) > 0$ for $\zeta \neq 0$, so this requirement is violated at only one point. Moreover, since $\operatorname{Re}(\bar{\nu}(\zeta; \epsilon)) = -\zeta$ is one-to-one, we are in the case covered by the Remark following Theorem 4, and hence the conclusion of Theorem 4 would hold if $\det M_1$ were in fact bounded away from zero. If we assume that g is continuously differentiable, it is immediately verifiable that a solution of the problem considered is

$$u = \begin{pmatrix} \dfrac{t}{\epsilon} + x \\ g(x - t) \end{pmatrix}.$$

It is obvious that regular degeneration does not obtain.

We turn now to an application of the theory and consider the Cauchy problem for the *telegraphist's equation* with constant coefficients and a small parameter ϵ multiplying the highest-order t-derivative:

$$\epsilon \frac{\partial^2 u_\epsilon}{\partial t^2} + \beta \frac{\partial u_\epsilon}{\partial t} - \Delta u_\epsilon = F,$$

where $\beta > 0$ is a constant, Δ is the Laplacian operator in n dimensions, and $F(t, \mathbf{x})$ has such properties as we shall presently enumerate. Here u_ϵ is to satisfy

$$u_\epsilon(0, \mathbf{x}) = f(\mathbf{x}), \qquad \frac{\partial u_\epsilon}{\partial t}(0, \mathbf{x}) = g(\mathbf{x}),$$

where g has at least $n + 7$ continuous L_1 derivatives and f, $n + 8$. The degenerate equation is then the heat equation

$$\beta \frac{\partial u_0}{\partial t} - \Delta u_0 = F;$$

here u_0 must satisfy $u_0(0, \mathbf{x}) = f(\mathbf{x})$ but cannot, in general, also satisfy $(\partial u_0/\partial t)(0, \mathbf{x}) = g(\mathbf{x})$.

There are several ways to write the telegraphist's equation as a system of the type we have considered. To get the least restrictive conditions on f, g and F it is desirable to write as small a system as possible; to this end we introduce the quantities

$$V_1 = \frac{\partial u_\epsilon}{\partial t}$$

$$V_2 = u_\epsilon$$

$$V_3 = \frac{\partial u_\epsilon}{\partial x_j}$$

for some fixed j, $1 \leqslant j \leqslant n$. Thus we write the telegraphist's equation as the following system of three equations:

$$\epsilon \frac{\partial V_1}{\partial t} + \beta V_1 - \frac{\partial^2 V_2}{\partial x_1^2} - \cdots - \frac{\partial^2 V_2}{\partial x_n^2} = F$$

$$\frac{\partial V_2}{\partial t} - V_1 = 0 \qquad \frac{\partial V_3}{\partial t} - \frac{\partial V_1}{\partial x_j} = 0. \tag{24}$$

Here the initial data are

$$V_1(0, \mathbf{x}) = g(\mathbf{x})$$

$$V_2(0, \mathbf{x}) = f(\mathbf{x})$$

$$V_3(0, \mathbf{x}) = \frac{\partial f}{\partial x_j}(\mathbf{x}).$$

We will thus be able to determine the degeneration of the quantities u_ϵ, $\partial u_\epsilon/\partial t$, $\partial u_\epsilon/\partial x_j$, $j = 1,..., n$. Less restrictive conditions on f and g for the regular degeneration of u_ϵ and $\partial u_\epsilon/\partial t$ could be found by omitting the last of these equations and initial conditions to form a new, smaller system. This will not be done here.

After Fourier transforming Eq. (24) we obtain for the vector $\hat{V}$ the system

$$\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial \hat{V}}{\partial t} + \begin{pmatrix} \beta & \zeta_1^2 + \zeta_2^2 + \cdots + \zeta_n^2 & 0 \\ -1 & 0 & 0 \\ i\zeta_j & 0 & 0 \end{pmatrix} \hat{V} = \begin{pmatrix} \hat{F} \\ 0 \\ 0 \end{pmatrix}, \tag{25}$$

where $\hat{V}$ is the transform of V and $\hat{F}$ that of F. Calling the matrix multiplying $\hat{V}$ in this equation $M(\boldsymbol{\zeta})$, one easily calculates that

$$\det (i\eta A(\epsilon) + M(\boldsymbol{\zeta})) = i\eta \left(i^2\eta^2\epsilon + i\eta\beta + \sum_{j=1}^{n} \zeta_j^2 \right).$$

Thus the roots of det $(i\eta A(\epsilon) + M(\boldsymbol{\zeta}))$ are $\eta = 0$,

$$\eta = \frac{i}{2\epsilon}\left(\beta \pm \sqrt{\beta^2 - 4\epsilon \sum_{j=1}^{n} \zeta_j^2}\right),$$

so clearly we have Im $(\eta) \geqslant 0$ for these roots. We also have det $M_1 = \beta > 0$. Also

$$Q'(\eta, \boldsymbol{\zeta}) \equiv \det (i\eta I + M_4 - M_3 M_1^{-1} M_2) = i\eta \left(i\eta + \frac{1}{\beta} \sum_{j=1}^{n} \zeta_j^2\right),$$

with roots 0, $i/\beta \,(\sum_{j=1}^{n} \zeta_j^2)$. It is a simple matter of calculation to see that the hypotheses of Theorems 1 and 2 will be satisfied provided g has at least $n + 7$ continuous L_1 derivatives and f at least $n + 8$, as we have assumed is the case. The single root ν of Eq. (14) for the system considered is $i\beta$.

Thus the hypotheses of Theorem 4 are satisfied, and we obtain the result that, for the solution u_ϵ of the *homogeneous* telegraphist's equation with inhomogeneous initial data,

$$u_\epsilon = u_0 + o(1)$$

$$\frac{\partial u_\epsilon}{\partial t} = \frac{\partial u_0}{\partial t} + \text{boundary layer term} + o(1)$$

$$\frac{\partial u_\epsilon}{\partial x_j} = \frac{\partial u_0}{\partial x_j} + o(1), \qquad j = 1,\ldots, n,$$

where $o(1)$ represent quantities which go to zero as $\epsilon \to 0$ uniformly for $(t, \mathbf{x}) \in [0, a] \times E_n$ for any finite $a > 0$ and the boundary layer term converges to zero uniformly in $(t, \mathbf{x}) \in [\delta, a] \times E_n$ for any $\delta > 0$ and is bounded at $t = 0$.

We now turn to a consideration of the solution of the inhomogeneous telegraphist's equation with homogeneous initial data. To this end we examine the following system of four equations in the four unknowns V_1, V_2, V_3, V_4:

$$\epsilon \frac{\partial V_1}{\partial t} + \beta V_1 - \Delta V_2 - V_4 = 0$$

$$\frac{\partial V_2}{\partial t} - V_1 = 0 \qquad \frac{\partial V_3}{\partial t} - \frac{\partial V_1}{\partial x_j} = 0 \qquad \frac{\partial V_4}{\partial t} = \frac{\partial F}{\partial t} \tag{26}$$

satisfying the initial conditions

$$V_1(0, \mathbf{x}) = 0$$
$$V_2(0, \mathbf{x}) = 0$$
$$V_3(0, \mathbf{x}) = 0$$
$$V_4(0, \mathbf{x}) = F(0, \mathbf{x});$$

here V_1, V_2, V_3 are as before and, clearly, $V_4(t, \mathbf{x}) = F(t, \mathbf{x})$. In order to employ the theorems established in Section 5, we require that for all $t \in [0, T]$ for some $T > 0$, $\partial F(t, \mathbf{x})/\partial t$ is continuous and integrable in $\mathbf{x}$ uniformly in t, that F has at least $n + 10$ continuous L_1 $\mathbf{x}$-derivatives, and that $\partial F/\partial t\,(0, \mathbf{x})$ has at least $n + 12$ continuous $\mathbf{x}$-derivatives.

The Fourier transform of the system of Eqs. (24) is the system

$$\begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial \hat{V}}{\partial t} + \begin{pmatrix} \beta & \sum_{j=1}^{n} \zeta_j^2 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ i\zeta_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \hat{V} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{\partial F}{\partial t} \end{pmatrix},$$

where $\hat{V}$ is the Fourier transform of the vector V with components V_1, V_2, V_3, V_4. We have already, in Section 5, shown that the homogeneous system obtained from the system of Eqs. (26) satisfies our various requirements (except, of course, differentiability of data) since the system (24) does; it is now a matter of direct calculation to see that F has sufficient L_1 derivatives for Theorem 4 to hold for the homogeneous system derived from (26). We now consider the inhomogeneous system (26) with homogeneous initial data. Our differentiability requirements on $\partial F/\partial t$ are, by Theorems 5 and 6, sufficient to guarantee the existence of solutions to both the nondegenerate and degenerate problems. Hence Theorem 8 holds. Combining the results of Theorems 4 and 8 for the solution of the inhomogeneous system (26) and Theorem 4 for the solution of the homogeneous system corresponding to Eq. (24), we have the result that for ϵ small the solution u_ϵ of the inhomogeneous telegraphist's equation with inhomogeneous initial data has the behavior

$$u_\epsilon = u_0 + o(1)$$

$$\frac{\partial u_\epsilon}{\partial t} = \frac{\partial u_0}{\partial t} + \text{boundary layer term} + o(1)$$

$$\frac{\partial u_\epsilon}{\partial x_j} = \frac{\partial u_0}{\partial x_j} + o(1), \qquad j = 1, \ldots, n,$$

where again $o(1)$ goes to zero with ϵ uniformly for $(t, \mathbf{x}) \in [0, T] \times E_n$ and where the boundary layer term converges to zero uniformly in $(t, \mathbf{x}) \in [\delta, T] \times E_n$ for any $\delta > 0$ and is bounded at $t = 0$.

These results are similar to those of M. Zlámal, but impose greater differentiability requirements on the data and inhomogeneous term F. Also, Zlámal obtains these equations with $O(\epsilon)$ instead of $o(1)$; on this head it should be remarked that, in the case of only one ϵ in $A(\epsilon)$, i.e., $m = N - 1$, the analysis here presented apparently could be strengthened to obtain this behavior provided the roots $\bar{\nu}_j(\zeta; \epsilon)$ satisfy the slightly stronger requirement $\mathrm{Im}\,(\bar{\nu}_j(\zeta; \epsilon)) \geqslant d > 0$, as is the case in the telegraphist's equation. However, we have not had to require the initial data f, g or the inhomogeneous term F to have compact support, as Zlámal does.

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