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## A non-trivial zero length limit of the Nambu–Goto string



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## ABSTRACT

We show that a Nambu–Goto string has a nontrivial zero length limit which corresponds to a massless particle with extrinsic curvature. The system has the set of six first class constraints, which restrict the phase space variables so that the spin vanishes. Upon quantization, we obtain six conditions on the state, which can be represented as a wave function of position coordinates,  $x^\mu$ , and velocities,  $q^\mu$ . We have found a wave function  $\psi(x, q)$  that turns out to be a general solution of the corresponding system of six differential equations, if the dimensionality of spacetime is eight. Although classically the system is just a point particle with vanishing extrinsic curvature and spin, the quantized system is not trivial, because it is consistent in eight, but not in arbitrary, dimensions.

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## 1. Introduction

Before the occurrence of string theories, elementary particles were described as point particles. They could live in principle in arbitrary dimensions, interacting by gravitational and other three fundamental forces. In string theories, particles and fields are excitations of a string. Quantized bosonic string theory is consistent in 26 dimensions. As an approximation, a string can be treated as a point particle with extrinsic curvature and spin [1–3], the so-called rigid particle [4–17]. In the description of Ref. [3], the system has two first class constraints, inherited from the string, and four additional constraints that are second class. In this paper we consider the zero length limit of such a system, in which case all six constraints become first class, and effectively eliminate from the description all the degrees of freedom, except those of a point particle, whose extrinsic curvature and spin vanish. At first sight this could mean that we have arrived at the theory of a point particle, living, in principle, in arbitrary dimensions. But the six first class constraints are still present there, and, upon quantization, they become restrictions on possible physical states. We have found that for a rather general class of solutions the quantum description can be performed consistently in eight dimensions, but not in other dimensions.

In Section 2 we derive a particle with extrinsic curvature from a string, and in Section 3 we consider its zero length limit. We obtain the same action that had already been considered by

McKeon [13]. However, in distinction to the case of Ref. [13], our system is subjected to a constraint, inherited from the string theory, that was not taken into account in Ref. [13]. Therefore, our dynamical system is different, because it has two primary constraints, whose conservation gives additional four constraints. Altogether, we obtain six constraints that turn out to be all first class. In the presence of those constraints, the particle's center of mass momentum  $p^\mu$ , velocity  $q^\mu = \dot{x}^\mu$ , and the conjugate momentum  $\pi^\mu$  are all parallel to each other. Therefore, the particle's spin and extrinsic curvature are zero, which means that the particle's position  $x^\mu(\tau)$  describes a straight worldline, and not a helix, as in the case of a rigid particle. In Section 4 we quantize the system by imposing the six constraints as restrictions on physical states, and find a wave function that solves the latter system of equations, provided that the dimension of the space in which the particle lives, is eight. In Conclusion section we argue why this is a remarkable, nontrivial, result, revealing yet another surprising property of string theories.

## 2. The particle with curvature from a string

In the previous paper [3] it was shown that one can obtain a particle with curvature as an approximation to a string, living in a target space with an extra time-like dimension. The string equations of motion in the conformal gauge are then

$$\ddot{X}^{\hat{\mu}} + X''^{\hat{\mu}} = 0, \quad \dot{X}^{\hat{\mu}} \dot{X}_{\hat{\mu}} - X'^{\hat{\mu}} X'_{\hat{\mu}} = 0, \quad \dot{X}^{\hat{\mu}} X'_{\hat{\mu}} = 0, \quad (1)$$

where  $\hat{\mu} = (\mu, D + 1)$ ,  $\mu = 0, 1, 2, 3, \dots, D - 1$ . A possible solution is

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$$X^\mu = C^\mu \tau + \sum_n (a_n^\mu \cos \omega_n \tau + b_n^\mu \sin \omega_n \tau) e^{k_n \sigma},$$

$$X^{D+1} = \sigma, \quad \sigma \in [0, L], \quad (2)$$

where

$$\omega_n^2 - k_n^2 = 0, \quad a_n^2 = b_n^2, \quad C_\mu a_n^\mu = C_\mu b_n^\mu = a_n^\mu b_{n\mu} = 0,$$

$$C^2 = 1, \quad k_n = \frac{n\pi}{L}. \quad (3)$$

In particular, if all higher modes with  $n > 1$  vanish, we have:

$$X^\mu = C^\mu \tau + (a^\mu \cos \omega \tau + b^\mu \sin \omega \tau) e^{k\sigma},$$

$$X^{D+1} = \sigma, \quad \sigma \in [0, L], \quad (4)$$

where we have denoted  $a_1^\mu \equiv a^\mu$ ,  $b_1^\mu \equiv b^\mu$ ,  $\omega_1 \equiv \omega$ . Such a string satisfies the Dirichlet boundary condition

$$\delta X^{\hat{\mu}}|_B = 0, \quad (5)$$

such that the string ends move on a  $D$ -brane [3].

For a fixed  $\sigma$ , Eq. (4) describes a helix in  $D$  dimensions. If the string length  $L$  is small in comparison with the radius of the helix, then the string effectively behaves like a point-particle, tracing a helical worldline.

The string embedding functions can be expanded according to [1–3],

$$X^\mu(\tau, \sigma) = x^\mu(\tau) + y^\mu(\tau)k\sigma + \mathcal{O}(k^2\sigma^2),$$

$$X^{D+1}(\sigma) = \sigma, \quad (6)$$

where  $k$  is a constant. For the solution (4) this gives

$$x^\mu(\tau) = C^\mu \tau + a^\mu \cos \omega \tau + b^\mu \sin \omega \tau,$$

$$y^\mu = a^\mu \cos \omega \tau + b^\mu \sin \omega \tau. \quad (7)$$

From now on, we will consider the expansion (6), and search for the action satisfied by the variables  $x^\mu(\tau)$  and  $y^\mu(\tau)$ . In Ref. [3] we started from the Polyakov action

$$I[X^{\hat{\mu}}, \gamma^{ab}] = \frac{T}{2} \int d^2\xi \sqrt{\gamma} \gamma^{ab} \partial_a X^{\hat{\mu}} \partial_b X_{\hat{\mu}}, \quad (8)$$

where  $T$  is the string tension, and  $\xi^a = (\tau, \sigma)$ .

Using the expansion (6), the action (8) becomes [3]

$$I = \frac{LT}{2} \int d\tau \left[ \frac{1}{e} (\dot{x}^2 + Lk\dot{x}y) + e(1 + f^2)(k^2 y^2 + 1) - 2fk\dot{x}y \right]$$

$$+ \mathcal{O}(k^2 L^2), \quad (9)$$

where  $e(\tau)$  and  $f(\tau)$  comes from the expansion of  $\sqrt{\gamma} \gamma^{11}$  and  $\sqrt{\gamma} \gamma^{12}$ , respectively, whereas the expansion of  $\sqrt{\gamma} \gamma^{22}$  gives  $e(\tau)(1 + f^2(\tau)) + \mathcal{O}(\sigma)$ . The equations of motion are:

$$\delta e: \quad -\frac{1}{2e^2} (\dot{x}^2 + Lk\dot{x}y) + (1 + f^2)(1 + k^2 y^2) = 0, \quad (10)$$

$$\delta f: \quad fe(1 + k^2 y^2) - k\dot{x}y = 0, \quad (11)$$

$$\delta y: \quad -Lk \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right) + 2e(1 + f^2)y^\mu - 2fk\dot{x}^\mu = 0, \quad (12)$$

$$\delta x: \quad \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} + \frac{Lk\dot{y}^\mu}{2e} - fky^\mu \right) = 0. \quad (13)$$

In a gauge in which  $f = 0$ , the action (9) is

$$I = \frac{LT}{2} \int d\tau \left[ \frac{\dot{x}^2}{e} + e + \frac{Lk\dot{x}y}{e} + ek^2 y^2 \right]. \quad (14)$$

If we plug the equation of motion

$$y^\mu = \frac{L}{2k} \frac{1}{e} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right), \quad (15)$$

and introduce the parameters

$$m = LT, \quad \mu = \frac{L^3 T}{8}, \quad (16)$$

then the action (14) becomes [3]:

$$I[x^\mu, e] = \int d\tau \left[ \frac{m}{2} \left( \frac{\dot{x}^2}{e} + e \right) - \frac{\mu}{e} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right) \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right) \right]. \quad (17)$$

This action contains first and second order derivatives of the variables  $x^\mu(\tau)$ . According to the Ostrogradski formalism [18], higher derivative theories contain negative energies. In the presence of an interaction such a system can become unstable. A relatively recent finding is that this is not always so. As shown in Refs. [19–22], there exist interacting second order systems that are unconditionally stable. Moreover, as pointed out by Woodard [23], the presence of a sufficient number of gauge constraints can stabilize the system. As an example, Woodard cited the second derivative model of a massless point particle with rigidity, considered by Plyushchay [7].

### 3. Zero length limit

We will now consider the limit in which the string length  $L$  tends to zero. For such purpose let us introduce a new parameter  $\tau' = h(\tau)$ , and a new Lagrange multiplier  $\tilde{e}(\tau')$  according to the relation

$$d\tau e = d\tau' m \tilde{e}. \quad (18)$$

Under such a reparametrization the action (17) becomes

$$I[x^\mu, \tilde{e}] = \int d\tau' \left\{ \frac{1}{2} \left[ \left( \frac{dx}{d\tau'} \right)^2 \frac{1}{\tilde{e}} + m^2 \tilde{e} \right] - \frac{\mu}{m^3} \frac{1}{\tilde{e}} \frac{d}{d\tau'} \left( \frac{1}{\tilde{e}} \frac{dx^\mu}{d\tau'} \right) \frac{d}{d\tau'} \left( \frac{1}{\tilde{e}} \frac{dx_\mu}{d\tau'} \right) \right\}. \quad (19)$$

The parameter  $\tau'$  can be renamed into  $\tau$ , and the latter action can be written as

$$\int d\tau \left[ \frac{1}{2} \left( \frac{\dot{x}^2}{\tilde{e}} + m^2 \tilde{e} \right) - \frac{\mu}{m^3 \tilde{e}} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\tilde{e}} \right) \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{\tilde{e}} \right) \right]. \quad (20)$$

Expressing  $m$  and  $\mu$  according to Eq. (16), the coefficient in front of the second term of the latter action becomes  $\mu/m^3 = 1/(8T^2) \equiv \tilde{\mu}$ . In Eq. (20) we have a term that corresponds to the Howe–Tucker action [24], and an extra term that corresponds to the particle's curvature.

The action (20) has two important limits:

(i)  $T \rightarrow \infty$ , implying  $\mu/m^3 \equiv \tilde{\mu} = 1/(8T^2) \rightarrow 0$ . The term with curvature then disappears from the action. The term containing  $m = LT$  would become infinite, unless we also impose the limit  $L \rightarrow 0$  such that  $m = LT$  remains finite. Then Eq. (20) becomes the well-known Howe–Tucker action for massive point particle.

(ii)  $T$  finite,  $L \rightarrow 0$ . In such limit, we have  $m = LT \rightarrow 0$ , whereas  $\mu/m^3 \equiv \tilde{\mu} = 1/(8T^2)$  remains intact, and the action (20) becomes

$$I[x^\mu, \tilde{e}] = \int d\tau \left[ \frac{1}{2\tilde{e}} \dot{x}^2 - \frac{\tilde{\mu}}{\tilde{e}} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\tilde{e}} \right) \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{\tilde{e}} \right) \right]. \quad (21)$$

The latter action is identical to the action for the “massless” particle with curvature, considered by McKeon [13].

In the following we will investigate in some detail the case (ii). From now on, we will rename  $\tilde{e}$  into  $e$ , and  $\tilde{\mu}$  into  $\mu$ , and write the action (21) as

$$I[x^\mu, e] = \int d\tau \left[ \frac{1}{2e} \dot{x}^2 - \frac{\mu}{e} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right) \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right) \right]. \quad (22)$$

The canonical momenta are

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \ddot{x}^\mu} \right) = \frac{\dot{x}_\mu}{e} + \frac{2\mu}{e} \frac{d}{d\tau} \left( \frac{1}{e} \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right) \right), \quad (23)$$

$$\pi_\mu = \frac{\partial L}{\partial \ddot{x}^\mu} = -\frac{2\mu}{e^2} \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right), \quad (24)$$

$$p_e = \frac{\partial L}{\partial \dot{e}} = \frac{2\mu}{e^3} \dot{x}^\mu \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right). \quad (25)$$

The equations of motion are

$$\delta x^\mu: \quad \dot{p}_\mu = 0, \quad (26)$$

$$\delta e: \quad \frac{\partial L}{\partial e} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{e}} = \frac{\dot{x}^2}{e^2} + 3\mu \frac{1}{e} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e} \right) \frac{1}{e} \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right) - \frac{2\mu}{e} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{e^2} \frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{e} \right) \right) = 0. \quad (27)$$

The Hamiltonian is

$$H_0 = p_\mu \dot{x}^\mu + \pi_\mu \ddot{x}^\mu + p_e \dot{e} - L_0. \quad (28)$$

Let us introduce the new variables

$$\dot{x}^\mu = q^\mu, \quad \dot{e} = \beta. \quad (29)$$

From Eqs. (23), (24) we have

$$\ddot{x}^\mu = \frac{e^3}{2\mu} \pi^\mu + \frac{\dot{e}}{e} q^\mu, \quad p_e = -\frac{\pi_\mu q^\mu}{e}, \quad (30)$$

and after inserting the latter expressions into the Hamiltonian (28), we obtain

$$H_0 = e \left( \frac{p_\mu q^\mu}{e} - \frac{e^2 \pi^2}{4\mu} - \frac{q^2}{2e^2} \right) + \beta \left( p_e + \frac{\pi_\mu q^\mu}{e} \right). \quad (31)$$

In deriving the action (17) we used a gauge in which  $f = 0$ . In such a gauge the constraint (11) becomes

$$\dot{x}^\mu y_\mu = 0. \quad (32)$$

By using Eqs. (15), (24) and (29), the latter equation can be written as

$$\pi_\mu q^\mu = 0. \quad (33)$$

Our action (17) and its  $L \rightarrow 0$  limit (22) is then subjected to the constraint (33). Therefore, the Lagrangian  $\mathcal{L}_0$  must be supplemented with the above constraint:

$$\mathcal{L} = \mathcal{L}_0 - \alpha \pi_\mu q^\mu, \quad (34)$$

and the Hamiltonian  $H_0$  with

$$H = H_0 + \alpha \pi_\mu q^\mu. \quad (35)$$

The equations of motion derived from the Hamiltonian  $H$  are

$$\dot{x}^\mu = \{x^\mu, H\} = q^\mu, \quad (36)$$

$$\dot{e} = \{e, H\} = \beta, \quad (37)$$

$$\dot{q}^\mu = \{q^\mu, H\} = -\frac{e^3 \pi^\mu}{2\mu} + \alpha q^\mu + \frac{\beta q^\mu}{e}, \quad (38)$$

$$\dot{p}_\mu = \{p_\mu, H\} = 0, \quad (39)$$

$$\dot{\pi}_\mu = \{\pi_\mu, H\} = -\left( p_\mu - \frac{q_\mu}{e} + \alpha \pi_\mu + \frac{\beta \pi_\mu}{e} \right), \quad (40)$$

$$\dot{p}_e = \{p_e, H\} = -\frac{3e^2 \pi^2}{4\mu} + \frac{q^2}{2e^2} - \beta \frac{\pi_\mu q^\mu}{e^2}. \quad (41)$$

Variation of the action  $\int \mathcal{L} d\tau$  with respect to  $e$  and  $\alpha$ , gives the constraints

$$\tilde{\phi}_1 = \frac{3e^2 \pi^2}{4\mu} - \frac{q^2}{2e^2} = 0, \quad (42)$$

$$\phi_2 = \pi_\mu q^\mu = 0. \quad (43)$$

From the requirement that those constraints must be preserved in time, we obtain another three constraints,

$$\phi_3 = e p_\mu \pi^\mu, \quad (44)$$

$$\phi_4 = \frac{p_\mu q^\mu}{e} + \frac{e^2 \pi^2}{2\mu} - \frac{q^2}{e^2}, \quad (45)$$

$$\phi_5 = p^2 - \frac{p_\mu q^\mu}{e}. \quad (46)$$

The linear combination

$$\phi_1 = -\tilde{\phi}_1 + \phi_4 = \frac{p_\mu q^\mu}{e} - \frac{q^2}{2e^2} - \frac{e^2 \pi^2}{4\mu} \quad (47)$$

is an expression that enters the Hamiltonian (31).

Variation of the action  $\int \mathcal{L} d\tau$  with respect to  $\beta$  gives the constraint

$$\phi_6 = e p_e + \pi_\mu q^\mu = 0. \quad (48)$$

The equation  $\dot{\phi}_6 = \{\phi_6, H\}$  for conservation of  $\phi_6$  does not give a new constraint.

The constraints  $\phi_i$ ,  $i = 1, 2, 3, 4, 5$ , include only the momenta  $p_\mu$  and  $\pi_\mu$ , conjugated to the dynamical variables  $x^\mu$  and  $q^\mu$ , whereas the constraint  $\phi_6$  includes also the momentum  $p_e$  which, due to (43) vanishes on the constraint surface. Thus,  $p_e$  is merely an auxiliary momentum, and the constraint associated with it is treated as the “last” one.

From  $\phi_1 = 0$ ,  $\phi_4 = 0$ ,  $\phi_5 = 0$  it follows that

$$\frac{p_\mu q^\mu}{e} = p^2, \quad (49)$$

$$\frac{q^2}{e^2} = \frac{3}{2} p^2, \quad (50)$$

$$\frac{e^2 \pi^2}{\mu} = p^2. \quad (51)$$

Because  $q^\mu$  is supposed to be a time-like vector and  $\pi^\mu$  a space-like vector, and because their scalar product,  $\pi_\mu q^\mu$ , vanishes on the constraint surface, it follows that  $q^2 = 0$ ,  $\pi^2 \leq 0$ . Taking also into account that Eq. (42) implies the proportionality between  $q^2$  and  $\pi^2$ , it follows that  $\pi^2 = 0$ . Eqs. (49)–(51) then become

$$\frac{p_\mu q^\mu}{e} = 0, \quad (52)$$

$$\frac{q^2}{e^2} = 0, \quad \frac{e^2 \pi^2}{\mu} = 0, \quad (53)$$

implying  $p^2 = 0$ .

Because  $p^2 = 0$ , it follows that all constraints  $\phi_i$ ,  $i = 1, 2, \dots, 6$ , are *first class*, i.e.,  $\{\phi_i, \phi_j\} = 0$ . This can be verified by calculating the Poisson brackets between all the constraints. In fact, the constraints become  $\phi_1 = p_\mu q^\mu / e$ ,  $\phi_2 = q^\mu \pi_\mu$ ,  $\phi_3 = e p_\mu \pi^\mu$ ,  $\phi_4 = \pi^2$ ,  $\phi_5 = p^2$ ,  $\phi_6 = e p_e$ , where  $\phi_4'$ ,  $\phi_5'$  and  $\phi_6'$  are the appropriate linear combinations of the constraints  $\phi_i$ .

Eqs. (43), (44), (49) and (51) imply that  $q^\mu$ ,  $\pi^\mu$  and  $p^\mu$  are parallel. Consequently, the spin tensor  $S^{\mu\nu} = q^\mu \pi^\nu - q^\nu \pi^\mu$  vanishes. The parallelism between  $q^\mu = \dot{x}^\mu$  and  $p^\mu$  means that the 4-velocity oscillations are tangential to the worldline of the particle's center of mass. Therefore, the center of mass worldline

$$x_T(\tau) = x_0^\mu + p^\mu \tau, \quad (54)$$

and the particle's position worldline

$$x^\mu(\tau) = x_0^\mu + p^\mu \tau + a^\mu \cos \omega \tau + b^\mu \sin \omega \tau, \quad (55)$$

which are both solutions of the equations of motion (36)–(40), are not different worldlines. Both equations, (54) and (55), represent the same curve, they differ only in the choice of parameter. If in Eq. (54) we change  $\tau$  according to  $\tau \rightarrow 1 + \alpha \cos \omega \tau + \beta \sin \omega \tau$ , where  $\alpha$  and  $\beta$  are proportionality factors, defined according to  $p^\mu = \alpha a^\mu$  and  $p^\mu = \beta b^\mu$ , we obtain Eq. (55).

The six first class constraints diminish the number of independent degrees of freedom of our dynamical system. It turns out that  $q^\mu$  and  $\pi^\mu$  are not dynamical degrees of freedom at all. Since  $q^\mu$  and  $\pi^\mu$  are parallel to  $p^\mu$ , they bring nothing new to the classical dynamics system. In the following we will investigate what happens if we nevertheless pursue with the quantization of our constraint system.

#### 4. Quantization

Upon quantization the phase space variables become the operators, satisfying the commutation relations

$$[\hat{x}^\mu, \hat{p}_\nu] = i \delta^\mu_\nu, \quad [\hat{q}^\mu, \hat{\pi}_\nu] = i \delta^\mu_\nu, \quad (56)$$

$$[\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{p}^\mu, \hat{p}^\nu] = 0,$$

$$[\hat{q}^\mu, \hat{q}^\nu] = 0, \quad [\hat{\pi}^\mu, \hat{\pi}^\nu] = 0, \quad (57)$$

and the constraints become restrictions on physical states:

$$\hat{p}^\mu \frac{\hat{q}^\mu}{e} |\psi\rangle = 0, \quad (58)$$

$$\frac{1}{2} (\hat{q}^\mu \hat{\pi}_\mu + \hat{\pi}_\mu \hat{q}^\mu) |\psi\rangle = 0, \quad (59)$$

$$e \hat{\pi}_\mu \hat{p}^\mu |\psi\rangle = 0, \quad (60)$$

$$\hat{\pi}^\mu \hat{\pi}_\mu |\psi\rangle = 0, \quad (61)$$

$$\hat{p}^\mu \hat{p}_\mu |\psi\rangle = 0, \quad (62)$$

$$e \hat{p}_e |\psi\rangle = 0. \quad (63)$$

We do not impose the condition

$$\hat{q}^2 |\psi\rangle = 0, \quad (64)$$

but only

$$\langle \psi | \hat{q}^2 | \psi \rangle = 0. \quad (65)$$

In the representation in which  $\hat{x}^\mu$  and  $\hat{q}^\mu$  are diagonal, whereas  $\hat{p}_\mu = -i\partial/\partial x^\mu$ ,  $\hat{\pi}_\mu = -i\partial/\partial q^\mu$ , Eqs. (62) and (61) become massless Klein–Gordon equations in the  $x^\mu$ -space, and the  $q^\mu$ -space, respectively.

A particular solution of Eqs. (62), (61) is

$$\psi_{p,q}(x^\mu, q^\mu) = e^{ip_\mu x^\mu} e^{i\pi_\mu q^\mu}. \quad (66)$$

Here  $p_\mu$  and  $\pi_\mu$  are now eigenvalues of the corresponding operators. The eigenvalues must satisfy the relations  $p^\mu p_\mu = 0$  and  $\pi^\mu \pi_\mu = 0$ .

We will now show that a general solution of the system of Eqs. (58)–(63) that satisfies condition (65), is

$$\psi(x^\mu, q^\mu) = \int d^D p d^D \pi a(p, \pi) e^{ip_\mu x^\mu} e^{i\pi_\mu q^\mu} \delta(p^2) \times \delta(\pi^2) \delta(q^2) \delta(q^\mu \pi_\mu) \delta(p^\mu \pi_\mu) \delta(p_\mu q^\mu) \quad (67)$$

where the constraints and the condition (65) are expressed in terms of the  $\delta$ -functions.

(i) Eq. (58) gives

$$\hat{p}_\mu \hat{q}^\mu \psi = \int d^D p d^D \pi a(p, \pi) p_\mu q^\mu e^{ip_\mu x^\mu} e^{i\pi_\mu q^\mu} \times \delta(p^2) \delta(\pi^2) \delta(q^2) \delta(q\pi) \delta(p\pi) \delta(pq) = 0, \quad (68)$$

because the integral of  $p_\mu q^\mu \delta(p_\mu q^\mu)$  over  $d^D p$  gives zero. We distinguish the operators from their eigenvalues by the hat symbol.

(ii) For Eq. (59) we obtains

$$\left( \hat{q}^\mu \hat{\pi}_\mu - \frac{i}{2} D \right) \psi = 0 \quad (69)$$

$$\hat{q}^\mu \hat{\pi}_\mu \psi = (-i) \int d^D p d^D \pi a(p, \pi) e^{ip_\nu x^\nu} \times \frac{\partial}{\partial q^\mu} (e^{i\pi_\nu q^\nu} \delta(\pi^2) \delta(q\pi) \delta(q^2) \delta(pq) \delta(p^2) \delta(p\pi)). \quad (70)$$

In Eq. (69) we took into account the commutation relation (56), which gives  $\hat{\pi}_\mu \hat{q}^\mu = \hat{q}^\mu \hat{\pi}_\mu - iD$ .

We will use

$$\frac{\partial}{\partial q^\mu} \delta(f(q)) = \frac{\partial f(q)}{\partial q^\mu} \frac{\partial \delta(f(q))}{\partial f(q)}, \quad (71)$$

which in particular gives

$$\frac{\partial}{\partial q^\mu} \delta(q^\nu \pi_\nu) = \pi_\mu \frac{\partial \delta(q^\nu \pi_\nu)}{\partial (q^\nu \pi_\nu)}. \quad (72)$$

We then have

$$\begin{aligned} \frac{\partial}{\partial q^\mu} (e^{i\pi_\nu q^\nu} \delta(q\pi) \delta(q^2) \delta(pq)) \\ = i\pi_\mu e^{i\pi_\nu q^\nu} \delta(q\pi) \delta(q^2) \delta(pq) + \pi_\mu \frac{\delta(q\pi)}{\delta(q\pi)} \delta(q^2) \delta(pq) \\ + 2q_\mu \frac{\partial \delta(q^2)}{\partial q^2} \delta(q\pi) \delta(pq) + p_\mu \frac{\partial \delta(pq)}{\partial (pq)} \delta(q\pi) \delta(q^2). \end{aligned} \quad (73)$$

Inserting the latter expression into Eq. (70), we obtain

$$\begin{aligned} \hat{q}^\mu \hat{\pi}_\mu \psi = (-i) \int d^D p d^D \pi a(p, \pi) e^{ip_\nu x^\nu} \\ \times e^{i\pi_\nu q^\nu} \left( i q^\mu \pi_\mu \delta(q\pi) \delta(q^2) \delta(pq) \right. \\ \left. + q^\mu \pi_\mu \frac{\delta(q\pi)}{\delta(q\pi)} \delta(q^2) \delta(pq) + 2q^\mu q_\mu \frac{\partial \delta(q^2)}{\partial q^2} \delta(q\pi) \delta(pq) \right. \\ \left. + q^\mu p_\mu \frac{\partial \delta(pq)}{\partial (pq)} \delta(q\pi) \delta(q^2) \right) \delta(p^2) \delta(p\pi). \end{aligned} \quad (74)$$

Using the relation

$$x\delta'(x) = -\delta(x) \quad (75)$$

we obtain

$$\hat{q}^\mu \hat{\pi}_\mu \psi = 4i\psi. \quad (76)$$

Eq. (69) then becomes

$$\left(4i - \frac{iD}{2}\right)\psi = 0, \quad (77)$$

which is satisfied if  $D = 8$ .

(iii) Eq. (60) gives:

$$\hat{p}^\mu \hat{\pi}_\mu \psi = \int d^D p d^D \pi a(p, \pi) p^\mu \pi_\mu e^{ip_\nu x^\nu} e^{i\pi_\nu q^\nu} \delta(p^2) \times \delta(\pi^2) \delta(q^2) \delta(q^\mu \pi_\mu) \delta(p^\mu \pi_\mu) \delta(p_\mu q^\mu), \quad (78)$$

which vanishes, because of the expression  $p^\mu \pi_\mu \delta(p^\mu \pi_\mu)$  under the integral.

(iv) In order to calculate Eq. (61), we will use Eq. (73), in which we express the derivative of the  $\delta$ -function as

$$\delta'(x) = -\frac{\delta(x)}{x} + \Delta(x). \quad (79)$$

The latter expression gives

$$\begin{aligned} \int dx F(x) \delta'(x) &= \int dx (F(0) + F'(x)|_{x=0} x) \left(-\frac{\delta(x)}{x} + \Delta(x)\right) \\ &= -F'(x)|_{x=0} - \frac{F(x)}{x} \Big|_{x=0} + \int dx F(x) \Delta(x) \\ &= -F'(x)|_{x=0}, \end{aligned} \quad (80)$$

if we define  $\Delta(x)$  according to

$$\int dx F(x) \Delta(x) = \frac{F(x)}{x} \Big|_{x=0}, \quad (81)$$

so that after the integration the term containing  $\Delta(x)$  cancels out. Then Eq. (73) becomes

$$\begin{aligned} \frac{\partial}{\partial q^\mu} (e^{i\pi_\mu q^\mu} \delta(q\pi) \delta(q^2) \delta(pq)) \\ = e^{i\pi_\mu q^\mu} \delta(q\pi) \delta(q^2) \delta(pq) \left(2i\pi_\mu - \frac{\pi_\mu}{q^\nu \pi_\nu} - \frac{2q_\mu}{q^2} - \frac{p_\mu}{p^\nu q_\nu}\right) \\ + \text{terms with } \Delta. \end{aligned} \quad (82)$$

If we use the above expression in Eq. (70), we also obtain the same result (76).

By using Eq. (82) in Eq. (61), we obtain

$$\begin{aligned} \hat{\pi}^\mu \hat{\pi}_\mu \psi &= -\frac{\partial^2 \psi}{\partial q^\mu \partial q_\mu} \\ &= -\int d^D p d^D \pi e^{ip_\nu x^\nu} e^{i\pi_\nu q^\nu} \delta(q\pi) \delta(q^2) \delta(pq) \\ &\quad \times \left\{ \left[ \pi_\mu \left(2i - \frac{1}{q\pi}\right) - \frac{2q_\mu}{q^2} - \frac{p_\mu}{p^\nu q_\nu} \right] \right. \\ &\quad \times \left[ \pi^\mu \left(2i - \frac{1}{q\pi}\right) - \frac{2q^\mu}{q^2} - \frac{p^\mu}{p^\nu q_\nu} \right] \\ &\quad \left. + \frac{\partial}{\partial q^\mu} \left(-\frac{\pi^\mu}{q^2} - \frac{p^\mu}{pq}\right) \right\} \\ &= \frac{2}{q^2} (D - 8) \psi. \end{aligned} \quad (83)$$

All other terms, including those with  $\Delta$ , vanish.

We have found that the constraint (61) is satisfied by the wave function (67) in eight dimensions, just like the constraint (59).

(v) Eq. (62) becomes

$$\hat{p}^\mu \hat{p}_\mu \psi = -\frac{\partial^2 \psi}{\partial x^\mu \partial x_\mu} = 0, \quad (84)$$

which vanishes because of the expression  $p^2 \delta(p^2)$  under the integral over  $d^D p$ .

(vi) Eq. (63) becomes

$$-i \frac{\partial}{\partial e} \psi = 0, \quad (85)$$

which is fulfilled, because  $\psi$  does not explicitly depend on  $e$ .

A remarkable feature of the above calculations is that the wave function (67) does not solve the quantum constraints (58)–(63) and the condition (65) in arbitrary dimension  $D$ , but only in  $D = 8$ . If (67) is indeed the most general solution of the system of Eqs. (58)–(63), (65), and there is no other solution, then the system, obtained by quantizing the zero length limit of the string, is consistent in eight dimensions. Though the zero length limit is just like a point particle, the system inherits from the string a set of constraints, which upon quantization can be satisfied in eight dimensions, but not in an arbitrary number of dimensions.

If we act on  $\psi$  with the operator  $\hat{S}_{\mu\nu} = \hat{q}_\mu \hat{\pi}_\nu - \hat{q}_\nu \hat{\pi}_\mu$ , which is the generator of rotations in the  $q^\mu$ -space, we obtain

$$\begin{aligned} \hat{S}_{\mu\nu} \psi &= \int d^D p d^D \pi \left[ (q_\mu \pi_\nu - q_\nu \pi_\mu) \left(2i - \frac{1}{q\pi}\right) \right. \\ &\quad \left. + \frac{1}{q^2} (q_\mu p_\nu - q_\nu p_\mu) \right] \\ &\quad \times e^{p_\rho x^\rho} e^{\pi_\rho q^\rho} \delta(p^2) \delta(\pi^2) \delta(q\pi) \delta(q^2) \delta(p\pi) \delta(pq). \end{aligned} \quad (86)$$

The latter expression vanishes, because the  $\delta$ -functions restrict the range of the variables  $p^\mu$ ,  $q^\mu$ ,  $\pi^\mu$  on the surface, on which they are all parallel to each other, so that on the surface,  $q_\mu \pi_\nu - q_\nu \pi_\mu = 0$  and  $q_\mu p_\nu - q_\nu p_\mu = 0$ . The wave function  $\psi(x^\mu, q^\mu)$  is thus a scalar under rotations generated by  $\hat{S}^{\mu\nu}$ . The particle has vanishing spin.

## 5. Conclusion

We have found yet another surprising property of strings. So far it was well known that a bosonic string can be consistently quantized in 26 dimensions, but not in other dimensions.<sup>1</sup> In this paper we considered a zero length limit of a bosonic string. At first sight one would expect that such a system is just a point particle, whose quantized counterpart can live in arbitrary dimensions. But a thorough treatment of the constraints reveals, that upon quantization we obtain a system of equations that can be solved by a certain rather general wave function only in eight dimensions. This means that a quantized point particle that is obtained as a limit of a string must live in eight dimensions, it cannot live in four dimensions. A consequence is that, according to Kaluza–Klein theory, such a particle, in the case when the 8-dimensional space is curved, experiences the force that from the point of view of 4-dimensional subspace manifests itself as gravitation and Yang–Mills forces. This means that the zero point limit of the string leads to a theory that besides gravitation contains other fundamental forces as well.

<sup>1</sup> However, see Ref. [25], where slightly more general strings were shown to be consistent in arbitrary dimensions.

The original string theory (of strings with finite extension) also leads to gravitation and Yang–Mills fields, though within a rather different theoretical procedure.

Zero length limit of a string and the corresponding theoretical description, is merely a theoretical idealization. In reality, a string remains finite, approximately being described as a zero length string living in eight dimensions. In the approximate theory, only eight dimensions are necessary for the consistency, the remaining eighteen dimensions are superfluous. In fact the approximate theory is not consistent in 26 dimensions. The remaining eighteen dimensions are necessary for consistent description of the remaining degrees of freedom that are truncated in the approximate theory. Thus, treating a string as a point particle, decouples eighteen dimensions from the description. The point particle “sees” only eight dimensions, and, if the space is curved, feels gravitational and Yang–Mills forces. Effectively, by treating the string approximately as a point particle, we have reduced spacetime from twenty six to eight dimensions, without really compactifying the remaining eighteen dimensions; we have only eliminated them from the dynamics, and thus rendered them invisible to the particle. In other words, although there might be present additional dimensions, the particle moves only in an eight-dimensional subspace.

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