ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 215 (2007) 540–568
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

A Segal conjecture for p-completed classifying spaces

Kári Ragnarsson [1]

Department of Mathematical Sciences, University of Aberdeen, Aberdeen, AB24 3UE, United Kingdom

Received 14 February 2007; accepted 4 April 2007

Available online 21 April 2007

Communicated by David J. Benson

Abstract

We formulate and prove a new variant of the Segal conjecture describing the group of homotopy classes of stable maps from the p-completed classifying space of a finite group G to the classifying space of a compact Lie group K as the p-adic completion of the Grothendieck group $A_p(G, K)$ of finite principal (G, K)-bundles whose isotropy groups are p-groups. Collecting the result for different primes p, we get a new and simple description of the group of homotopy classes of stable maps between (uncompleted) classifying spaces of groups. This description allows us to determine the kernel of the map from the Grothendieck group $A(G, K)$ of finite principal (G, K)-bundles to the group of homotopy classes of stable maps from BG to BK.

Keywords: Stable maps; Classifying spaces; Segal conjecture

Introduction

Inspired by Atiyah's description [3] of the representable complex periodic K-theory of a finite group G as the completion of the complex representation ring $R[G]$ at its augmentation ideal,

$$\hat{R}[G] \xrightarrow{\cong} KU^0(BG),$$

E-mail address: kari@maths.abdn.ac.uk.

[1] The author was supported by EPSRC grant GR/S94667/01 during most of this work.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2007.04.006

Segal conjectured that the zeroth stable cohomotopy group of a finite group G could analogously be described as the completion of the Burnside ring $A(G)$ of isomorphism classes of virtual finite G-sets [14], with respect to its augmentation ideal $I(G)$,

$$A(G)^{\wedge}_{I(G)} \xrightarrow{\cong} \pi^0_S(BG_+).$$

After many partial results by various authors, this conjecture was eventually settled in the affirmative by Carlsson in [7]. In this account we mention only the three major contributions which in the end combined to form a solution of the conjecture, but the reader is encouraged to look up [7] for an account of the history of the Segal conjecture. Adams, Gunawardena and Miller proved the conjecture for elementary abelian p-groups in [2]. May and McClure simplified the conjecture in [10], where they proved that if the conjecture holds for all finite p-groups, then it holds for all finite groups. The solution of the conjecture was completed by Carlsson in [7], where he supplied an inductive proof showing that if the Segal conjecture is true for elementary abelian p-groups then it is true for all finite p-groups.

The Segal conjecture was extended by Lewis, May and McClure in [8] to describe the group of homotopy classes of stable maps between classifying spaces of finite groups, and later by May, Snaith and Zelewski in [11] to allow the classifying space of a compact Lie group in the target. For a finite group G and a compact Lie group K, they consider the Burnside module $A(G, K)$. This is the Grothendieck group completion of isomorphism classes of principal (G, K)-bundles, which are principal K-bundles with finitely many orbits in the category of G-spaces. The Burnside module $A(G, K)$ is an $A(G)$-module which as a $\mathbb{Z}$-module is free with one basis element $[H, \varphi]$ for each conjugacy class of pairs (H, φ) consisting of a subgroup $H \leqslant G$ and a group homomorphism $\varphi : H \to K$. Letting $\{BG_+, BK_+\}$ denote the group of homotopy classes of stable maps $\Sigma^\infty BG_+ \to \Sigma^\infty BK_+$, there is a natural homomorphism

$$\alpha : A(G, K) \to \{BG_+, BK_+\},$$

which sends a basis element $[H, \varphi]$ to the stable map $\Sigma^\infty B\varphi_+ \circ tr_H$, where tr_H is the transfer map discussed in Section 1.2. In [8] and [11] it is shown that the Segal conjecture implies that this map is a completion with respect to $I(G)$. The original Segal conjecture is the special case when K is the trivial group.

The $I(G)$-adic completion of $A(G, K)$ is in general difficult to describe, but in the special case where G is a finite p-group S, May and McClure [10] gave a simple description. Letting $\widetilde{A}(S, K)$ denote the quotient module obtained from $A(S, K)$ by quotienting out basis elements of the form $[P, ct]$ where ct is a constant homomorphism, they noticed that the $I(S)$-adic topology on $\widetilde{A}(S, K)$ coincides with the p-adic topology, and therefore the induced map

$$\tilde{\alpha} : \widetilde{A}(S, K) \to \{BS, BK\}$$

is a p-adic completion.

The main result of this paper can be regarded as an extension of this result of May and McClure, where we drop the condition that G be a p-group but p-complete BG instead. In other words we describe $\{BG^{\wedge}_p, BK\}$ in similarly simple terms as a p-completion of a certain submodule of $\widetilde{A}(G, K)$. Let $A_p(G, K)$ be the submodule of $A(G, K)$ generated by those (G, K)-bundles whose isotropy groups are p-groups, and let $\widetilde{A}_p(G, K)$ denote the corresponding submodule of $A_p(G, K)$. We prove the following theorem, which appears in the text as Theorem 3.1.

Theorem A. *For a finite group G and a compact Lie group K, the homomorphism*

$$\widetilde{A}_p(G,K) \xrightarrow{\tilde{\alpha}_p} \{BG, BK\} \xrightarrow{\iota_p^*} \{BG_p^\wedge, BK\}$$

induced by α is a p-adic completion.

Here $\iota_p : \mathbb{B}G_p^\wedge \hookrightarrow \mathbb{B}G$ is the natural wedge summand inclusion obtained from the natural splitting $\Sigma^\infty BG \simeq \bigvee_q \Sigma^\infty BG_q^\wedge$, as discussed in Section 1.1.

Since α is natural in G and finite K we can state the finite group version of this theorem in terms of isomorphisms of categories. This is done as Corollary 3.3 after we have developed the necessary framework.

As a consequence of the stable splitting of the classifying space of a finite group into its q-completed components, where q runs over all primes, one easily obtains a new description of $\{BG, BK\}$. The result is similar in spirit to Minami's description of $A(G)_I^\wedge$ for a compact Lie group G in [12]. Indeed, when K is the trivial group one recovers Minami's result for finite G. This result is Theorem 3.4 in the text.

Theorem B. *For a finite group G and a compact Lie group K, the homomorphism α induces an isomorphism of $\mathbb{Z}$-modules*

$$\bigoplus_p \widetilde{A}_p(G,K)_p^\wedge \xrightarrow{\cong} \{BG, BK\}.$$

This description is much simpler than the $I(G)$-adic completion, but it has the drawback that it is not obvious how to decompose an element in $\{BG, BK\}$ into p-completed parts in $\widetilde{A}_p(G,K)_p^\wedge$. This matter is taken up in Section 5 where we describe the map

$$\widetilde{A}(G,K) \to \{BG, BK\} \to \{BG_p^\wedge, BK_p^\wedge\} \xrightarrow{\cong} \widetilde{A}_p(G,K)_p^\wedge$$

for every prime p. Loosely speaking, this map sends an element X of $A(G,K)$ to an element of $\widetilde{A}_p(G,K)_p^\wedge$ whose H-fixed point sets for subgroups $H \leqslant K \times G$ agree with the H-fixed point sets of X if H is a finite p-group, and are empty otherwise. (This description should not be taken too literally as elements of $A(G,K)$ and $\widetilde{A}_p(G,K)_p^\wedge$ are not actual $(K \times G)$-bundles and do not have fixed point sets. A precise statement can be found in Section 5.) As a consequence we deduce the following description of the kernel of α.

Theorem C. *Let G be a finite group and K a compact Lie group.*

(a) *The kernel of the map*

$$\alpha : A(G,K) \to \{BG_p^\wedge, BK\}$$

consists of those virtual (G,K)-bundles $[X - Y]$ whose fixed-point sets under the action of any finite p-subgroup $P \leqslant K \times G$ satisfy

$$|W(P)\backslash X^P| = |W(P)\backslash Y^P|,$$

where $W(P) = N_{K\times G}(P)/P$.

(b) *The kernel of the map*

$$\alpha : A(G, K) \to \{BG, BK\}$$

consists of those virtual (G, K)*-bundles* $[X - Y]$ *such that for every prime* p *and every* p*-subgroup* $P \leqslant K \times G$*, the* P*-fixed-point sets satisfy*

$$\left|W(P)\backslash X^P\right| = \left|W(P)\backslash Y^P\right|.$$

A more informative version of this theorem appears in the text as Corollary 5.8 after some relevant concepts and notation have been introduced. In particular, not all p-subgroups P need be considered, and there is a justification why the quotients of the fixed-point sets appearing are finite sets. Note also that when K is finite one does not need to take the quotient by the $W(P)$-action (cf. Remark 5.9).

The Segal conjecture has been presented here in its weak form. There is a stronger form of the conjecture, also due to Segal, describing higher stable cohomotopy groups of classifying spaces of finite groups as a completion of certain equivariant stable cohomotopy groups, constructed by Segal in [13]. The proofs of the Segal conjecture mentioned above are in fact proofs of the stronger version. Indeed, Carlsson's inductive argument would not have been possible without the presence of higher homotopy groups. The Lewis–May–McClure generalization of the conjecture also extends the stronger version of the Segal conjecture to describe spectra of stable maps between classifying spaces. We refer the reader to [8] for this description. This raises an obvious, interesting question. Namely, whether the results in this paper can be extended to also describe higher homotopy groups of spectra of stable maps between p-completed classifying spaces.

This paper is divided into four sections. In Section 1 we discuss some background material necessary for the main discussion. In Section 2 we introduce subconjugacy, which gives a convenient filtration of Burnside modules. In Section 3 we state and prove the new variants of the Segal conjecture for p-completed classifying spaces. Alternative formulations of the main results, arising from different ways to overcome a technical nuisance involving basepoints, are discussed in Section 4. Finally, in Section 5 we discuss how to decompose a stable map into its p-completed components and describe the kernel of the $I(G)$-adic completion map.

1. Preliminaries

In this section we discuss background material which will be needed in later sections. We begin by giving an overview of notation and conventions which will be used throughout this paper.

Unless otherwise specified, p is a fixed prime and all cohomology is taken with $\mathbb{F}_p$-coefficients. For a space or spectrum X, we let $X^\wedge_p$ denote the Bousfield–Kan p-completion [6], and for a $\mathbb{Z}$-module M, we let $M^\wedge_p$ denote the p-adic completion of M. Since we only consider finitely generated modules in this paper we have $M \cong M \otimes \mathbb{Z}^\wedge_p$, where $\mathbb{Z}^\wedge_p$ denotes the p-adic integers.

For a space X, let X_+ denote the pointed space obtained by adding a disjoint basepoint to X. We use the shorthand notation $\Sigma^\infty_+ X := \Sigma^\infty X_+$ and recall that there is a natural equivalence $\Sigma^\infty_+ X \simeq \Sigma^\infty X \vee \mathbb{S}$, where $\mathbb{S}$ denotes the sphere spectrum. All stable homotopy will take place in the homotopy category of spectra, which we denote by $\mathcal{S}\mathfrak{pectra}$.

Let $\mathfrak{Gr}$ denote the category of finite groups. We will use ι_H to denote a subgroup inclusion $H \hookrightarrow G$, specifying the supergroup G when there is danger of confusion. Conjugations will come up frequently. For a group element $g \in G$ we let c_g denote the conjugation isomorphism $x \mapsto gxg^{-1}$. For a subgroup $H \leqslant G$, we let ${}^gH$ denote $c_g(H)$, and H^g denote $c_g^{-1}(H)$.

We use the shorthand notations $\mathbb{B} := \Sigma^\infty B(-)$ and $\mathbb{B}_+ := \Sigma_+^\infty(B(-))$, regarded as functors $\mathfrak{Gr} \to \mathcal{S}\mathfrak{pectra}$. Since $\Sigma^\infty(BG_p^\wedge) \simeq (\Sigma^\infty BG)_p^\wedge$ for a finite group G, we will write $\mathbb{B}G_p^\wedge$ without danger of confusion. We denote by $\mathbb{B}\mathfrak{Gr}$ the category whose objects are the finite groups and whose morphisms are homotopy classes of stable maps between classifying spaces,

$$\mathrm{Mor}_{\mathbb{B}\mathfrak{Gr}}(G_1, G_2) = \{BG_1, BG_2\}.$$

Similarly we let $\mathbb{B}\mathfrak{Gr}_p^\wedge$ be the category with the same objects, but whose morphisms are homotopy classes of stable maps between p-completed classifying spaces

$$\mathrm{Mor}_{\mathbb{B}\mathfrak{Gr}_p^\wedge}(G_1, G_2) = \bigl\{(BG_1)_p^\wedge, (BG_2)_p^\wedge\bigr\}.$$

1.1. Bousfield–Kan p-completion of $\mathbb{B}G$

In this section we list the basic properties of Bousfield–Kan completion of spectra at a prime. We recall how the suspension spectrum of the classifying space of a finite group decomposes as a wedge sum of its p-completions as p runs over all primes and see that this splitting is natural.

Bousfield–Kan completion at a prime p is an endofunctor $(-)_p^\wedge$ defined on either the category of spaces or spectra, depending on the context. In either case its defining property is that for a map $f : X \to Y$, the p-completed map $f_p^\wedge : X_p^\wedge \to Y_p^\wedge$ is a weak homotopy equivalence if and only if f induces an isomorphism in homology with $\mathbb{F}_p$ coefficients.

The p-completion functor comes with a natural transformation $\eta_p : Id \Rightarrow (-)_p^\wedge$. We say that a space or spectrum X is *p-complete* if the map $\eta_p : X \to X_p^\wedge$ is a weak equivalence. We say that X is *p-good* if $X_p^\wedge$ is p-complete. Classifying spaces of finite groups and their suspension spectra are p-good. Classifying spaces of finite p-groups and their suspension spectra are p-complete.

As noted earlier, we have $(\Sigma^\infty BG)_p^\wedge \simeq \Sigma^\infty(BG_p^\wedge)$ for a finite group G, and so we can denote both of these by $\mathbb{B}G_p^\wedge$ without danger of confusion. However, p-completion does not commute with suspension in general, as we have

$$\Sigma^\infty\bigl((BG_+)_p^\wedge\bigr) \simeq \Sigma^\infty\bigl(BG_p^\wedge\bigr)_+ \simeq \mathbb{S} \vee \Sigma^\infty BG_p^\wedge$$

while

$$\bigl(\Sigma^\infty BG_+\bigr)_p^\wedge \simeq \mathbb{S}_p^\wedge \vee \Sigma^\infty BG_p^\wedge.$$

This difference is one of the underlying reasons for the basepoint issues one has to circumvent in the formulation of the p-completed Segal conjectures.

When G is a finite group, $\mathbb{B}G$ is a torsion spectrum, so by Sullivan's arithmetic square the natural maps $\eta_q : \mathbb{B}G \to \mathbb{B}G_q^\wedge$ induce a natural homotopy equivalence

$$h := \bigvee_q \eta_q : \mathbb{B}G \xrightarrow{\simeq} \bigvee_q \mathbb{B}G_q^\wedge,$$

where the wedge sum runs over all primes q. We obtain a natural inclusion of $\mathbb{B}G_p^\wedge$ as a wedge summand of $\mathbb{B}G$, defined as the composite

$$\iota_p : \mathbb{B}G_p^\wedge \hookrightarrow \bigvee_q \mathbb{B}G_q^\wedge \xrightarrow{h^{-1}} \mathbb{B}G,$$

with left homotopy inverse η_p.

Lemma 1.1. *For a finite group G and a compact Lie group K, the p-completion functor sends $f \in \{BG, BK\}$ to $\eta_p \circ f \circ \iota_p \in \{BG_p^\wedge, BK_p^\wedge\}$.*

Proof. By naturality of η_p we have $f_p^\wedge \circ \eta_p = \eta_p \circ f$, from which it follows that $f_p^\wedge \simeq f_p^\wedge \circ \eta_p \circ \iota_p = \eta_p \circ f \circ \iota_p$. □

Bousfield–Kan p-completion of spectra can also be thought of as Bousfield localization with respect to the homology theory $H_*(-, \mathbb{F}_p)$. Bousfield localization is discussed in [5]. Taking this point of view, one of Bousfield's results in [5] is that when X is a connective spectrum, the p-completion of X is homotopy equivalent to the function spectrum $F(S^{-1}\mathbb{Z}/p^\infty, X)$, where $S^{-1}\mathbb{Z}/p^\infty$ is the desuspension of the Moore spectrum $S\mathbb{Z}/p^\infty$, with $Z/p^\infty = \mathbb{Z}[1/p]/\mathbb{Z}$.

Lemma 1.2. *Let X and Y be spectra such that Y and $F(X, Y)$ are both connective. Then there is a natural homotopy equivalence*

$$\mathrm{Ad} : F\big(X, Y_p^\wedge\big) \xrightarrow{\simeq} F(X, Y)_p^\wedge.$$

Proof. The map is just the adjunction

$$F\big(X, F\big(S^{-1}\mathbb{Z}/p^\infty, Y\big)\big) \cong F\big(S^{-1}\mathbb{Z}/p^\infty, F(X, Y)\big). \qquad \square$$

In this paper we study stable maps from the classifying space of a finite group to the classifying space of a compact Lie group. These exhibit very unusual behaviour under p-completion as seen in the following lemma.

Lemma 1.3. *For a finite group G and a compact Lie group K the maps in the sequence below are all homotopy equivalences*

$$F\big(\mathbb{B}G_p^\wedge, \mathbb{B}K\big) \xrightarrow{\eta_p \circ -} F\big(\mathbb{B}G_p^\wedge, \mathbb{B}K_p^\wedge\big) \xrightarrow{- \circ \eta_p} F\big(\mathbb{B}G, \mathbb{B}K_p^\wedge\big) \xrightarrow{\mathrm{Ad}} F(\mathbb{B}G, \mathbb{B}K)_p^\wedge.$$

In particular they induce isomorphisms of groups of homotopy classes of maps.

Proof. We obtain the homotopy equivalence Ad because $\mathbb{B}K$ is connective by construction as a suspension spectrum, and $F(\mathbb{B}G, \mathbb{B}K)$ is connective by [8]. The map $- \circ \eta_p$ is a homotopy equivalence since the cofibre of η_p is $H_*(-, \mathbb{F}_p)$-acyclic [5].

The map $\eta_p \circ -$ factors as the composition

$$F\big(\mathbb{B}G_p^\wedge, \mathbb{B}K\big) \xrightarrow{\eta_p} F\big(\mathbb{B}G_p^\wedge, \mathbb{B}K\big)_p^\wedge \xrightarrow{\mathrm{Ad}} F\big(\mathbb{B}G_p^\wedge, \mathbb{B}K_p^\wedge\big).$$

Let S be a Sylow subgroup of G. As explained in the next subsection, the spectrum $F(\mathbb{B}G^{\wedge}_p, \mathbb{B}K)$ is a retract of the spectrum $F(\mathbb{B}S, \mathbb{B}K)$, which is p-complete and connective by [8]. Hence $F(\mathbb{B}G^{\wedge}_p, \mathbb{B}K)$ is p-complete and connective, and so the two maps in the factorization are both homotopy equivalences. □

When working with "pointed" classifying spectra one similarly has a sequence of homotopy equivalences

$$F\big((\mathbb{B}_+G)^{\wedge}_p, (\mathbb{B}_+K)^{\wedge}_p\big) \xrightarrow{-\circ\eta_p} F\big(\mathbb{B}_+G, (\mathbb{B}_+K)^{\wedge}_p\big) \xrightarrow{\mathrm{Ad}} F(\mathbb{B}_+G, \mathbb{B}_+K)^{\wedge}_p.$$

The function spectrum $F((\mathbb{B}_+G)^{\wedge}_p, \mathbb{B}_+K)$ is no longer homotopy equivalent because it contains an uncompleted wedge summand $F(\mathbb{S}^{\wedge}_p, \mathbb{S})$ (but p-completing this summand makes it homotopy equivalent).

1.2. Transfers

Recall (see for example [1]) that a subgroup inclusion $H \hookrightarrow G$, has a transfer map

$$tr_H : \mathbb{B}_+G \to \mathbb{B}_+H$$

such that the composition $\mathbb{B}_+\iota_H \circ tr_H$ acts as multiplication by $[G : H]$ in singular cohomology (with any coefficients). We will for the most part prefer to work with the reduced transfer map $\mathbb{B}G \to \mathbb{B}H$, which we also denote by tr_H to reduce notation. The composition $\mathbb{B}\iota \circ tr_H$ involving the reduced transfer also acts as multiplication by $[G : H]$ in singular cohomology.

When S is a Sylow subgroup of G, the index $[G : S]$ is a unit in $\mathbb{F}_p$ and therefore $\mathbb{B}\iota_S \circ tr_S$ induces an isomorphism on $H^*(\mathbb{B}G) = H^*(BG)$. After p-completion, $\mathbb{B}\iota_S \circ tr_S$ consequently becomes a homotopy equivalence

$$\mathbb{B}G^{\wedge}_p \xrightarrow{tr^{\wedge}_p} \mathbb{B}S \xrightarrow{\mathbb{B}\iota^{\wedge}_p} \mathbb{B}G^{\wedge}_p$$

and so the p-completions of $\mathbb{B}\iota_S$ and tr_S make $\mathbb{B}G^{\wedge}_p$ a wedge summand of $\mathbb{B}S^{\wedge}_p \simeq \mathbb{B}S$.

When H is a subgroup of G, we will often denote the p-completed transfer $(tr_H)^{\wedge}_p$ by tr_H. Similarly, when $\varphi : H \to G$ is a group homomorphism, we will often denote the p-completed map $(\mathbb{B}\varphi)^{\wedge}_p$ by $\mathbb{B}\varphi$. This is done to reduce notation. As we always distinguish clearly between $\mathbb{B}G^{\wedge}_p$ and $\mathbb{B}G$ (except when S is a p-group and the spectra are homotopy equivalent anyway), this should not cause any confusion.

1.3. Linear categories

For a ring R, an *R-linear category* is a category in which the morphism sets are R-modules and composition is bilinear. It is perhaps more common to refer to $\mathbb{Z}$-linear categories as *pre-additive* categories, but we use different terminology to emphasize the role of the ground ring R. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is a *functor of R-linear categories* if the categories $\mathcal{C}$ and $\mathcal{D}$ are R-linear and for every pair of objects $c, c' \in \mathcal{C}$ the map

$$F : \mathrm{Mor}_{\mathcal{C}}(c, c') \to \mathrm{Mor}_{\mathcal{D}}\big(F(c), F(c')\big)$$

is a morphism of R-modules. An *isomorphism of R-linear categories* is a functor of R-linear categories that is also an isomorphism of categories.

Example 1.4. If R is a ring, then we can regard R as an R-linear category with one object $\circ_R$ such that

$$\mathrm{Mor}(\circ_R, \circ_R) = R.$$

We will sometimes take this point of view for the rings $\mathbb{Z}$ and $\mathbb{Z}_p^\wedge$.

1.4. The Burnside category

For a finite group G and a compact Lie group K, let $\mathcal{M}\mathfrak{or}(G, K)$ be the set of isomorphism classes of finite principal (G, K)-bundles over finite G-sets. A (G, K)-bundle here means a K-bundle in the category of G-spaces, finiteness means the base space is finite, and principality means that the K-action is free. When K is finite these are just K-free, finite $(K \times G)$-sets. The set $\mathcal{M}\mathfrak{or}(G, K)$ is an abelian monoid under the disjoint union operation and so we can consider its Grothendieck group completion which we call the *Burnside module of G and K*, and denote by $A(G, K)$. The Burnside modules are abelian groups, and hence $\mathbb{Z}$-modules, by construction. Their module structure is well understood and will be described below after a preliminary definition.

Definition 1.5. Let G be a finite group and K a compact Lie group. A (G, K)*-pair* is a pair (H, φ) consisting of a subgroup $H \in G$ and a homomorphism $\varphi : H \to K$. We say two (G, K)-pairs (H, φ) and (H', φ') are (G, K)*-conjugate* if there exist elements $g \in G$ and $k \in K$ such that $c_g(H) = H'$ and the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & K \\ {\scriptstyle\cong}\downarrow{\scriptstyle c_g} & & \downarrow{\scriptstyle c_k} \\ H' & \xrightarrow{\varphi'} & K. \end{array}$$

Let $C(G, K)$ denote the set of conjugacy classes of (G, K)-pairs.

When there is no danger of confusion we say *conjugacy* instead of (G, K)-conjugacy. We denote the conjugacy class of a (G, K)-pair (H, φ) by $[H, \varphi]_G^K$, or just $[H, \varphi]$ when there is no danger of confusion.

Lemma 1.6. *For a finite group G and a compact Lie group K, the number of conjugacy classes of (G, K)-pairs is finite.*

Proof. One proves this easily using the folklore result (see for example [9, Theorem 2.3(1)] for a proof) that for any finite group F there are only finitely many conjugacy classes of subgroups of K isomorphic to F. $\square$

A (G, K)-pair (H, φ) gives rise to a finite principal (G, K)-bundle

$$(K \times G)/\Delta_H^\varphi \to G/H,$$

where

$$\Delta_H^{\varphi} := \{(\varphi(h), h) \mid h \in H\} \leqslant K \times G \tag{1}$$

is the (transposed) graph of $\varphi : H \to K$. Two (G, K)-pairs give rise to isomorphic (G, K)-bundles if and only if they are conjugate.

The ring $U(K \times G)$ of finite $(K \times G)$-complexes is introduced in [14] where a basis over $\mathbb{Z}$ is also described. One can identify $A(G, K)$ with the submodule of $U(G \times K)$ generated by those $(K \times G)$-complexes whose isotropy groups are of the form Δ_H^{φ}. This leads to the following description of $A(G, K)$.

Proposition 1.7. *(See [14].) For a finite group G and a compact Lie group K, the Burnside module $A(G, K)$ is a finitely generated, free $\mathbb{Z}$-module with one basis element for each conjugacy class of (G, K)-pairs.*

Proof. The ring $U(K \times G)$ is free as a $\mathbb{Z}$-module with one basis element for every conjugacy class of subgroups of $K \times G$. The Burnside module $A(G, K)$ can be identified with the submodule generated by the subgroups of the form Δ_H^{φ} where (H, φ) is a (G, K)-pair. □

By a slight abuse of notation we also denote the basis element corresponding to a (G, K)-pair (H, φ) by $[H, \varphi]_G^K$ (or $[H, \varphi]$). For each basis element $[H, \varphi]$ we define a morphism

$$c_{[H,\varphi]} : A(G, K) \to \mathbb{Z}$$

by letting $c_{[H,\varphi]}(X)$ be the coefficient at $[H, \varphi]$ in the basis decomposition of X. In other words we demand that

$$X = \sum_{[H,\varphi]} c_{[H,\varphi]}(X) \cdot [H, \varphi]$$

for all $X \in A(G, K)$. When appropriate, we will also let $c_{[H,\varphi]}$ denote the analogous homomorphism $A(G, K)_p^\wedge \to \mathbb{Z}_p^\wedge$.

The Burnside modules form the morphism sets of a certain linear category as described below. For finite groups G_1 and G_2, and a compact Lie group K, there is a pairing

$$\mathcal{M}\mathfrak{or}(G_2, K) \times \mathcal{M}\mathfrak{or}(G_2, G_1) \to \mathcal{M}\mathfrak{or}(G_1, K),$$

given by

$$(X, Y) \mapsto G_2 \backslash (X \times Y),$$

where G_2 acts by the diagonal. This extends to a bilinear pairing

$$A(G_2, K) \times A(G_1, G_2) \to A(G_1, K),$$

suggestively denoted by

$$(X, Y) \mapsto X \circ Y.$$

This pairing can be described on basis elements by the double coset formula

$$[H_2, \varphi_2] \circ [H_1, \varphi_1] = \sum_{x \in H_2 \backslash G_2 / \varphi_1(H_1)} \left[\varphi_1^{-1}\left(\varphi_1(H_1) \cap H_1^x\right), \varphi_2 \circ c_x \circ \varphi_1\right].$$

It is easy to see that this composition pairing satisfies the associativity law and therefore we can make the following definition.

Definition 1.8. The *Burnside category* is the $\mathbb{Z}$-linear category $A\mathfrak{Gr}$ whose objects are the finite groups and whose morphism sets are the Burnside modules,

$$\mathrm{Mor}_{A\mathfrak{Gr}}(G_1, G_2) = A(G_1, G_2),$$

where composition is given by the pairing described above.

Similarly, the *p-completed Burnside category* is the $\mathbb{Z}_p^\wedge$-linear category $A\mathfrak{Gr}_p^\wedge$ whose objects are the finite groups and whose morphism sets are the p-completed Burnside modules,

$$\mathrm{Mor}_{A\mathfrak{Gr}_p^\wedge}(G_1, G_2) = A(G_1, G_2)_p^\wedge.$$

Since we are focusing on p-local properties it will suit us, for a finite group G and a compact Lie group K, to study the submodule $A_p(G, K)$ of the Burnside module $A(G, K)$ obtained by considering only (G, K)-bundles whose isotropy groups are all p-groups. Alternatively, this is the submodule generated by those basis elements $[P, \varphi]$, where P is a p-group. By the double coset formula, we see that these modules are preserved by the composition pairing as described in the following lemma.

Lemma 1.9. *For finite groups G_1 and G_2, and a compact Lie group K, we have*

$$A(G_2, K) \circ A_p(G_1, G_2) \subset A_p(G_1, K).$$

In particular

$$A_p(G_2, K) \circ A_p(G_1, G_2) \subset A_p(G_1, K).$$

Therefore we can make the following definition.

Definition 1.10. The *p-isotropy Burnside category* is the $\mathbb{Z}$-linear category $A_p\mathfrak{Gr}$ whose objects are the finite groups and whose morphism sets are the p-Burnside modules,

$$\mathrm{Mor}_{A_p\mathfrak{Gr}}(G_1, G_2) = A_p(G_1, G_2).$$

Similarly, the *p-completed p-Burnside category* is the $\mathbb{Z}_p^\wedge$-linear category $A_p\mathfrak{Gr}_p^\wedge$ whose objects are the finite groups and whose morphism sets are the p-completed p-Burnside modules,

$$\mathrm{Mor}_{A_p\mathfrak{Gr}_p^\wedge}(G_1, G_2) = A_p(G_1, G_2)_p^\wedge.$$

We conclude this discussion by introducing an augmentation for the Burnside category. This will prove very useful for understanding the filtration of Burnside modules introduced in Section 2.

Definition 1.11. For a finite group G and a compact Lie group K, the *orbit augmentation* of $A(G,K)$ is the homomorphism

$$\epsilon : A(G,K) \to \mathbb{Z}$$

obtained as the group completion of the monoid morphism

$$\mathcal{M}\mathfrak{or}(G,K) \to \mathbb{Z}, \quad X \mapsto |K\backslash X|.$$

The *orbit augmentation* of $A(G,K)^{\wedge}_p$ is the homomorphism

$$\epsilon : A(G,K)^{\wedge}_p \to \mathbb{Z}^{\wedge}_p$$

obtained upon p-completion.

When K is finite we have $\epsilon(X) = |X|/|K|$, since X is K-free. Note also that

$$\epsilon\big([H,\varphi]\big) = |G|/|H|$$

for a (G,K)-pair (H,φ). One easily checks that the orbit aumentations send the composition pairing to multiplication, and consequently we obtain a $\mathbb{Z}$-linear functor

$$\epsilon : A\mathfrak{Gr} \to \mathbb{Z}$$

and a $\mathbb{Z}^{\wedge}_p$-linear functor

$$\epsilon : A\mathfrak{Gr}^{\wedge}_p \to \mathbb{Z}^{\wedge}_p.$$

We refer to these as the *orbit augmentation functors* of the respective Burnside categories.

1.5. The Segal conjecture

For a finite group G and a compact Lie group K, there is a homomorphism of $\mathbb{Z}$-modules

$$\alpha : A(G,K) \to \{BG_+, BK_+\}$$

sending a basis element $[H,\varphi]$ to the stable map $\mathbb{B}_+\varphi \circ tr_H$. This assignment is functorial for G and finite K, and so we get a functor of $\mathbb{Z}$-linear categories

$$\alpha : A\mathfrak{Gr} \to \mathbb{B}\mathfrak{Gr},$$

which is the identity on objects.

For a finite group G and a compact Lie group K, we have an obvious bilinear map

$$A(G,1) \times A(G,K) \to A(G,K),$$

where 1 denotes the trivial group, extending the map sending a G-set X and a (G,K)-bundle Y to the (G,K)-bundle $X \times Y$, where G acts via the diagonal and K acts only on the second coordinate.

When $K = 1$, this gives a ring structure on $A(G) = A(G,1)$, and the resulting ring is called the *Burnside ring of* G. This can also be described as the Grothendieck group of isomorphism classes of left G-sets. For general compact Lie groups K, the Burnside module $A(G,K)$ becomes a module over $A(G)$ under the action described above.

The orbit augmentation functor ϵ induces a $\mathbb{Z}$-algebra augmentation

$$\epsilon : A(G) \to \mathbb{Z},$$

which extends the "counting map" sending a G-set X to its cardinality $|X|$. We denote the augmentation ideal by $I(G)$.

The Segal conjecture states that for finite group G, the homomorphism $A(G) \to \pi_S^0(BG_+)$, which we describe as the composite

$$A(G) = A(G,1) \xrightarrow{\alpha} \{BG_+, B1_+\} \cong \pi_S^0(BG_+),$$

is a completion with respect to the ideal $I(G)$. Lewis, May and McClure showed in [8] that an extended version of the conjecture, describing the homotopy classes of stable maps between classifying spaces of finite groups, follows from the original version. In a further generalization, May, Snaith and Zelewski showed in [11] that one can allow the target group to be a compact Lie group. Since the Segal conjecture was proved by Carlsson in [7], we can state these extensions as a theorem.

Theorem 1.12 *(Segal Conjecture [7,8,11]). For a finite groups G and a compact Lie group K, the map*

$$\alpha : A(G,K) \to \{BG_+, BK_+\}$$

induces an isomorphism

$$\alpha^\wedge_{I(G)} : A(G,K)^\wedge_{I(G)} \xrightarrow{\cong} \{BG_+, BK_+\},$$

where

$$A(G,K)^\wedge_{I(G)} = \varprojlim_k \big(A(G,K)/I(G)^k A(G,K)\big)$$

denotes the $I(G)$-completion of $A(G,K)$.

This is a magnificent result, but in general the $I(G)$-adic completions are difficult to calculate. However, when the groups involved are p-groups, the situation is simplified. For a finite group

G and a compact Lie group K, let $\widetilde{A}(G,K)$ be the module obtained from $A(G,K)$ by quotienting out all basis elements of the form $[H, ct]$, where ct is the constant homomorphism. May and McClure noticed in [10] that when G is a p-group, the $I(G)$-adic topology on $\widetilde{A}(G,K)$ is equivalent to the p-adic topology on $\widetilde{A}(G,K)$. (This is not true on $A(G,K)$.) The result is the following version of the Segal conjecture.

Theorem 1.13 *(Segal Conjecture [7,8,10,11]). Let S be a finite p-group and K be a compact Lie group. Then the map*

$$\alpha : A(S,K) \to \{BS_+, BK_+\}$$

induces an isomorphism

$$\tilde{\alpha}_p^\wedge : \widetilde{A}(S,K)_p^\wedge \to \{BS, BK\}.$$

An explanation of the target of this isomorphism is in order. Recall that $\Sigma_+^\infty BK \simeq \Sigma^\infty BK \vee \mathbb{S}$. Now, the submodule of $\{BS_+, BK_+\}$ generated by the maps $\alpha([P, ct])$, where ct is the constant homomorphism $P \to K$, consists of the maps that factor through the $\mathbb{S}$-term of $\Sigma_+^\infty BK$. Therefore it is appropriate to replace $\{BS_+, BK_+\}$ with

$$\{BS, BK\} \cong \{BS_+, BK_+\} / \{BS_+, S^0\}$$

when passing from $A(S,K)$ to $\widetilde{A}(S,K)$ to obtain the homomorphism $\tilde{\alpha} : \widetilde{A}(S,K) \to \{BS, BK\}$. The homomorphism $\tilde{\alpha}_p^\wedge$ is the p-completion of $\tilde{\alpha}$.

Since α and $\tilde{\alpha}$ preserve composition when restricted to finite groups, both versions of the Segal conjecture described in this section can be formulated in terms of isomorphisms of linear categories.

1.6. Restrictions and transfers

Let G be a finite group and K be a compact Lie group. If S is a subgroup of G, we can regard a (S,K)-pair as a (G,K)-pair, or restrict a (G,K)-bundle to a (S,K)-bundle. This gives us a way to traverse between $A(S,K)$ and $A(G,K)$, which will be useful later when we take S to be a Sylow subgroup. In this section we briefly describe these operations and interpret them in terms of stable maps.

Given a (S,K)-pair (H,φ), we can regard H as a subgroup of G to get a (G,K)-pair $(H,\varphi)_G^K$. This assignment extends to a homomorphism

$$\Phi : A(S,K) \to A(G,K).$$

It is easy to check that

$$[H,\varphi]_G^K = [H,\varphi]_S^K \circ [S, id]_G^S$$

using the double coset formula. Furthermore, $[S, id]_G^S$ corresponds to $[G]_G^S$, the isomorphism class of G, regarded as a $(S \times G)$-set under the action $(s,g).x = gxs^{-1}$.

The homomorphism α sends $[S, id]_G^S$ to the transfer map $tr_S : \mathbb{B}_+G \to \mathbb{B}_+S$. Thus we get an induced homomorphism

$$\Phi : \{BS_+, BK_+\} \to \{BG_+, BK_+\}, \quad f \mapsto f \circ tr_S.$$

Note that α commutes with Φ by construction.

A (G, K)-bundle X, can be regarded as a (S, K)-bundle $X|_{(S,K)}$ via restriction. This induces a homomorphism

$$\Gamma : A(G, K) \to A(S, K).$$

It is easy to see that

$$[X|_{(S,K)}] \cong [X] \circ [G]_S^G,$$

where $[X]$ denotes the isomorphism class of X, and $[G]_S^G$ is the isomorphism class of G regarded as a $(G \times S)$-set via the action $(g, s).x = gxs^{-1}$. Since $[G]_S^G$ corresponds to $[S, \iota]_S^G$ in $A(S, G)$ we see that Γ is given by

$$[X] \mapsto [X] \circ [S, \iota]_S^G.$$

Applying α we get an induced homomorphism of stable maps

$$\Gamma : \{BG_+, BK_+\} \to \{BS_+, BK_+\}, \quad f \mapsto f \circ \mathbb{B}_+\iota_S.$$

Note that α also commutes with Γ by construction.

To accommodate the basepoint issue we need to consider the reduced operations $\widetilde{\Phi}$ and $\widetilde{\Gamma}$ that fit into the diagrams

$$\begin{array}{ccccc} A(S,K) & \xrightarrow{\Phi} & A(G,K) & \xrightarrow{\Gamma} & A(S,K) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \widetilde{A}(S,K) & \xrightarrow{\widetilde{\Phi}} & \widetilde{A}(G,K) & \xrightarrow{\widetilde{\Gamma}} & \widetilde{A}(S,K) \end{array}$$

and

$$\begin{array}{ccccc} \{BS_+, BK_+\} & \xrightarrow{\Phi} & \{BG_+, BK_+\} & \xrightarrow{\Gamma} & \{BS_+, BK_+\} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \{BS, BK\} & \xrightarrow{\widetilde{\Phi}} & \{BG, BK\} & \xrightarrow{\widetilde{\Gamma}} & \{BS, BK\}, \end{array}$$

where the vertical arrows are the obvious projection maps. The reduced operations $\widetilde{\Phi}$ and $\widetilde{\Gamma}$ commute with $\tilde{\alpha}$.

Since we are working p-locally we need to consider the p-completed operations

$$\widetilde{\Phi}_p^\wedge : \{BS_p^\wedge, BK\} \to \{BG_p^\wedge, BK\}, \quad f \mapsto f \circ (tr_S)_p^\wedge$$

and

$$\widetilde{\Gamma}_p^\wedge : \{BG_p^\wedge, BK\} \to \{BS_p^\wedge, BK\}, \quad f \mapsto f \circ (\mathbb{B}\iota_S)_p^\wedge.$$

These fit into the commutative diagram

$$\begin{array}{ccccc} \widetilde{A}(S,K) & \xrightarrow{\widetilde{\Phi}} & \widetilde{A}(G,K) & \xrightarrow{\widetilde{\Gamma}} & \widetilde{A}(S,K) \\ \downarrow{\scriptstyle \iota_p^*\circ\tilde{\alpha}} & & \downarrow{\scriptstyle \iota_p^*\circ\tilde{\alpha}} & & \downarrow{\scriptstyle \iota_p^*\circ\tilde{\alpha}} \\ \{BS_p^\wedge, BK\} & \xrightarrow{\widetilde{\Phi}_p^\wedge} & \{BG_p^\wedge, BK\} & \xrightarrow{\widetilde{\Gamma}_p^\wedge} & \{BS_p^\wedge, BK\}, \end{array}$$

where the vertical maps are the functor $\tilde{\alpha}$ followed by precomposition with ι_p, the natural inclusion of wedge summands described in Section 1.1. The naturality of ι_p ensures the commutativity of the diagram and in fact this is the reason why ι_p was introduced so carefully.

Notice that

$$[G_G^S] \circ [G_S^G] \cong [G_S^S],$$

where $[G_S^S]$ denotes the isomorphism class of G regarded as a $(S \times S)$-set via the action $(s_1, s_2).x = s_1 x s_2^{-1}$. Therefore the composite

$$\Gamma \circ \Phi : A(S,K) \to A(S,K)$$

is given by

$$\Gamma \circ \Phi(X) = X \circ [G]_S^S.$$

This operation will be studied in Section 2.

We also consider the opposite composite

$$\Phi \circ \Gamma : A(G,K) \to A(G,K).$$

Its corresponding operation on stable maps is given by

$$\Phi \circ \Gamma(f) = f \circ \mathbb{B}_+\iota_S \circ tr_S.$$

The interesting case is when S is a Sylow p-subgroup of G. Then $\mathbb{B}S \simeq \mathbb{B}S_p^\wedge$ and the composite $\mathbb{B}\iota_S \circ tr_S$ is a homotopy equivalence after p-completion. Therefore $\widetilde{\Phi}_p^\wedge \circ \widetilde{\Gamma}_p^\wedge$ is an automorphism of $\{BG_p^\wedge, BK\}$ which factors through $\{BS, BK\}$. We conclude that $\{BG_p^\wedge, BK\}$ is a direct summand of $\{BS, BK\}$ isomorphic to

$$\widetilde{\Gamma}_p^\wedge \circ \widetilde{\Phi}_p^\wedge(\{BS, BK\}) = \tilde{\alpha}_p^\wedge\big(\pi\big(A(S,K)_p^\wedge \circ [G]_S^S\big)\big).$$

This observation gives us a strategy to prove Theorem A. Over the next two sections we show that $A(S,K)_p^\wedge \circ [G]_S^S$ is isomorphic to $A_p(G,K)_p^\wedge$ in a way that is compatible with $\tilde{\alpha}_p$.

2. Subconjugacy

From now on, fix a finite group G with Sylow p-subgroup S and a compact Lie group K. In this section we find an explicit basis for the submodule

$$A(S,K)^\wedge_p \circ [G] \subseteq A(S,K)^\wedge_p,$$

where $[G] \in A(S,S)^\wedge_p$ is the isomorphism class of G regarded as a $(S \times S)$-set under the action $(s_1,s_2).x = s_1 x s_2^{-1}$. From this basis it is easy to obtain a basis for $\{BG^\wedge_p, BK\}$ and prove Theorem A.

The main tool we will use is a filtration induced by the following preorder on conjugacy classes of (S,K)-pairs.

Definition 2.1. Let (P,φ) and (Q,ψ) be two (S,K)-pairs. We say that (Q,ψ) is *subconjugate* to (P,φ), and write

$$(Q,\psi) \precsim (P,\varphi),$$

if there exist elements $g \in G$ and $k \in K$ such that the following diagram commutes

$$\begin{array}{ccc} Q & \xrightarrow{\psi} & \psi(Q) \\ {\scriptstyle c_g}\downarrow & & \downarrow{\scriptstyle c_k} \\ P & \xrightarrow{\varphi} & \varphi(P). \end{array}$$

It is clear that subconjugacy is preserved by (S,K)-conjugacy and therefore we can pass to conjugacy classes.

Definition 2.2. We say that a (S,K)-conjugacy class of pairs $[Q,\psi]$ is subconjugate to an (S,K)-conjugacy class $[P,\varphi]$ and write

$$[Q,\psi] \precsim [P,\varphi]$$

if the subconjugacy relation

$$(Q,\psi) \precsim (P,\varphi)$$

holds between any (and hence all) representatives of the classes.

It is also clear that subconjugacy is a transitive relation and so induces an equivalence relation as described in the following definition.

Definition 2.3. We say that $[Q,\psi]$ is (G,K)-*conjugate* to $[P,\varphi]$, and write

$$[Q,\psi] \sim [P,\varphi],$$

if $[Q,\psi] \precsim [P,\varphi]$ and $[P,\varphi] \precsim [Q,\psi]$. We say that $[Q,\psi]$ is *strictly subconjugate* to $[P,\varphi]$, and write

$$[Q,\psi] \precnsim [P,\varphi],$$

if $[Q,\psi]$ is subconjugate to $[P,\varphi]$, but not (G,K)-conjugate to $[P,\varphi]$.

Let I denote the set of (G,K)-conjugacy classes of (S,K)-pairs. This is a poset under subconjugacy and we proceed to construct an I-indexed filtration of $A(S,K)^\wedge_p$.

Definition 2.4. For an (S,K)-pair (P,φ), let $M(\precsim [P,\varphi])$ denote the submodule of $A(S,K)^\wedge_p$ generated by the basis elements $[Q,\psi]$ such that

$$[Q,\psi] \precsim (P,\varphi),$$

and let $M(\precnsim [P,\varphi])$ denote the submodule generated by the basis elements $[Q,\psi]$ such that

$$[Q,\psi] \precnsim (P,\varphi).$$

We will show that this filtration is preserved by composition with $[G]$. First note that by the double coset formula one has

$$[G] = \sum_{x\in S\backslash G/S} \bigl[S\cap S^x, c_x\bigr].$$

This prompts us to make a formalization.

Definition 2.5. Let R be the submodule of $A(S,S)^\wedge_p$ generated by the basis elements $[P,\varphi]$, where the homomorphism $\varphi: P \to S$ is induced by conjugation by an element in G.

Clearly $[G] \in R$. Using the double coset formula, one easily sees that R is in fact a subring of $A(S,S)^\wedge_p$. The next lemma shows that the modules $M(\precsim [P,\varphi])$ and $M(\precnsim [P,\varphi])$ defined above are right R-modules. They are in fact also left R-modules but this is not needed.

Lemma 2.6. *The following hold for every (S,K)-pair (P,φ):*

(a) $M(\precsim [P,\varphi]) \circ R \subseteq M(\precsim [P,\varphi])$,
(b) $M(\precnsim [P,\varphi]) \circ R \subseteq M(\precnsim [P,\varphi])$.

Proof. We begin by proving part (a). It suffices to show that for every basis element $[Q,\psi] \precsim [P,\varphi]$ of $M(\precsim [P,\varphi])$ and every basis element $[T,c_g]$ of R, one has

$$[Q,\psi]\circ[T,c_g] \in M\bigl(\precsim [P,\varphi]\bigr).$$

By the double coset formula we can write

$$[Q,\psi]\circ[T,c_g] = \sum_{x\in Q\backslash S/{}^gT} \bigl[T\cap Q^{xg}, \psi\circ c_x\circ c_g\bigr]$$

so it suffices to note that

$$\left[T \cap Q^{xg}, \psi \circ c_x \circ c_g\right] \precsim [Q, \psi] \precsim [P, \varphi]$$

for every $x \in S$.

The same argument proves part (b), for if we assume that $[Q, \psi]$ is strictly subconjugate to $[P, \varphi]$ at the beginning, we get

$$\left[T \cap Q^{xg}, \psi \circ c_x \circ c_g\right] \precsim [Q, \psi] \precnsim [P, \varphi]$$

at the end. □

In particular this lemma shows that the operation

$$\Gamma \circ \Phi = (-) \circ [G] : A(S, K)^{\wedge}_p \to A(S, K)^{\wedge}_p$$

introduced in Section 1.6 preserves the subconjugacy filtration.

Lemma 2.7. *If the basis elements $[P, \varphi]$ and $[Q, \psi]$ of $A(S, K)$ are (G, K)-conjugate, then*

$$\Gamma \circ \Phi\big([P, \varphi]\big) = \Gamma \circ \Phi\big([Q, \psi]\big).$$

Proof. Clearly (P, φ) and (Q, ψ) are also (G, K)-conjugate when regarded as (G, K)-pairs, so, using the notation of Section 1.6,

$$\Phi\big([P, \varphi]\big) = \Phi\big([Q, \psi]\big).$$

Consequently

$$\Gamma\big(\Phi\big([P, \varphi]\big)\big) = \Gamma\big(\Phi\big([Q, \psi]\big)\big). \qquad \square$$

Proposition 2.8. *Pick a representative $[P_i, \varphi_i]$ for each $i \in I$. The collection*

$$C = \big\{\Gamma \circ \Phi\big([P_i, \varphi_i]\big) \bigm| i \in I\big\}$$

forms a $\mathbb{Z}^{\wedge}_p$-basis for $\Gamma \circ \Phi(A(S, K)^{\wedge}_p)$.

Proof. By Lemma 2.7 it is clear that C spans $\Gamma \circ \Phi(A(S, K)^{\wedge}_p)$, so it suffices to prove linear independence.

For a (S, K)-pair (P, φ) we have

$$\Gamma \circ \Phi\big([P, \varphi]\big) = [P, \varphi] \circ [G] \in M\big(\precsim [P, \varphi]\big)$$

by Lemma 2.6. But note that

$$\epsilon\big(M\big(\precsim [P, \varphi]\big)\big) = \epsilon\big([P, \varphi]\big) \cdot \mathbb{Z}^{\wedge}_p$$

and

$$\epsilon\big(M\big(\precsim[P,\varphi]\big)\big) = p\cdot\epsilon\big([P,\varphi]\big)\cdot\mathbb{Z}_p^\wedge.$$

Since

$$\epsilon\big(\Gamma\circ\Phi\big([P,\varphi]\big)\big) = \epsilon\big([P,\varphi]\circ[G]\big) = \epsilon\big([P,\varphi]\big)\cdot\epsilon\big([G]\big)$$

and p does not divide $\epsilon([G]) = |G|/|S|$, we deduce that

$$\Gamma\circ\Phi\big([P,\varphi]\big)\in M\big(\precsim[P,\varphi]\big)\setminus M\big(\precnsim[P,\varphi]\big). \tag{2}$$

Now, let $c_i\in\mathbb{Z}_p^\wedge$ for each $i\in I$ and assume that

$$\sum_{i\in I} c_i\cdot\big(\Gamma\circ\Phi\big([P_i,\varphi_i]\big)\big) = 0. \tag{3}$$

Put

$$I' = \{i\in I \mid c_i\neq 0\}.$$

If I' is nonempty, then let j be a maximal element of I' regarded as a poset under subconjugacy. By (2) there is a (S,K)-pair $(Q,\psi)\underset{(G,K)}{\sim}(P_j,\varphi_j)$ such that

$$c_{[Q,\psi]}\big(\Gamma\circ\Phi\big([P_j,\varphi_j]\big)\big)\neq 0.$$

On the other hand, for $i\in I'\setminus\{j\}$, the maximality of j implies that $[Q,\psi]$ is not subconjugate to (P_i,φ_i). Hence

$$c_{[Q,\psi]}\big(M\big(\precsim[P_i,\varphi_i]\big)\big) = 0$$

and in particular

$$c_{[Q,\psi]}\big(\Gamma\circ\Phi\big([P_i,\varphi_i]\big)\big) = 0.$$

Now we get

$$\begin{aligned}
&c_{[Q,\psi]}\bigg(\sum_{i\in I} c_i\cdot\big(\Gamma\circ\Phi\big([P_i,\varphi_i]\big)\big)\bigg)\\
&\quad=\sum_{i\in I} c_i\cdot c_{[Q,\psi]}\big(\Gamma\circ\Phi\big([P_i,\varphi_i]\big)\big)\\
&\quad=\sum_{i\in I\setminus I'}\underbrace{c_i}_{=0}\cdot c_{[Q,\psi]}\big(\Gamma\circ\Phi\big([P_i,\varphi_i]\big)\big)+\sum_{i\in I'\setminus\{j\}} c_i\cdot\underbrace{c_{[Q,\psi]}\big(\Gamma\circ\Phi\big([P_i,\varphi_i]\big)\big)}_{=0}\\
&\qquad+\underbrace{c_j}_{\neq 0}\cdot\underbrace{c_{[Q,\psi]}\big(\Gamma\circ\Phi\big([P_i,\varphi_i]\big)\big)}_{\neq 0}\\
&\quad\neq 0,
\end{aligned}$$

contradicting (3). Therefore I' must be empty and we conclude that the collection is linearly independent. $\square$

3. p-Completed Segal conjectures

In this section we state and prove the new variants of the Segal conjecture promised in the introduction. We begin by describing the group of homotopy classes of stable maps from the p-completed classifying space of a finite group to the classifying space of a compact Lie group. The map $(\iota_p^* \circ \tilde{\alpha}_p)_p^\wedge$ in the statement of the theorem is the algebraic p-completion of the composite

$$\widetilde{A}(G,K) \xrightarrow{\tilde{\alpha}_p} \{BG, BK\} \xrightarrow{-\circ\iota_p} \{BG_p^\wedge, BK\},$$

where $\iota_p : \mathbb{B}G_p^\wedge \hookrightarrow \mathbb{B}G$ is the natural wedge-summand inclusion discussed in Section 1.1. Note that the target $\{BG_p^\wedge, BK\}$ is p-complete because it is a direct summand of $\{BS, BK\}$ which is p-complete by the Segal conjecture.

Theorem 3.1. *For a finite group G and a compact Lie group K, the map*

$$(\iota_p^* \circ \tilde{\alpha}_p)_p^\wedge : \widetilde{A}_p(G,K)_p^\wedge \to \{BG_p^\wedge, BK\}$$

is an isomorphism of $\mathbb{Z}_p^\wedge$-modules, natural in G and finite K.

Proof. Let S be a Sylow subgroup of G, and let I be the set of (G,K)-conjugacy classes of (S,K)-pairs. For each $i \in I$, pick a representative (P_i, φ_i) and consider the submodule $M_I \subseteq A(S,K)_p^\wedge$ generated by the collection $\{[P_i, \varphi_i] \mid i \in I\}$. Since every p-subgroup of G is conjugate to a subgroup of S, the homomorphism $\Phi : A(S,K) \to A(G,K)$ described in Section 1.6 induces an isomorphism

$$\Phi_p^\wedge : M_I \xrightarrow{\cong} A_p(G,K)_p^\wedge.$$

Letting $\widetilde{M}_I$ denote the image of M_I in $\widetilde{A}(S,K)_p^\wedge$, this descends to an isomorphism

$$\widetilde{\Phi}_p^\wedge : \widetilde{M}_I \xrightarrow{\cong} \widetilde{A}_p(G,K)_p^\wedge.$$

Proposition 2.8 implies that there is an isomorphism

$$\widetilde{\Gamma}_p^\wedge \circ \widetilde{\Phi}_p^\wedge : \widetilde{M}_I \xrightarrow{\cong} \widetilde{\Gamma}_p^\wedge \circ \widetilde{\Phi}_p^\wedge(\widetilde{M}_I)$$

and that

$$\widetilde{\Gamma}_p^\wedge \circ \widetilde{\Phi}_p^\wedge(\widetilde{M}_I) = \widetilde{\Gamma}_p^\wedge \circ \widetilde{\Phi}_p^\wedge\big(\widetilde{A}(S,K)_p^\wedge\big).$$

By the Segal conjecture, and the commutativity of α with $\widetilde{\Gamma}_p^\wedge$ and $\widetilde{\Phi}_p^\wedge$, there is an isomorphism

$$\tilde{\alpha}_p^\wedge : \widetilde{\Gamma}_p^\wedge \circ \widetilde{\Phi}_p^\wedge\big(\widetilde{A}(S,K)_p^\wedge\big) \xrightarrow{\cong} \widetilde{\Gamma}_p^\wedge \circ \widetilde{\Phi}\big(\{BS, BK\}\big).$$

Since the composition $\mathbb{B}\iota_S \circ tr_S$ is a homotopy self-equivalence of $\mathbb{B}G^\wedge_p$, the homomorphisms $\widetilde{\Phi}^\wedge_p$ and $\widetilde{\Gamma}^\wedge_p$ induce an automorphism

$$\widetilde{\Phi}^\wedge_p \circ \widetilde{\Gamma}^\wedge_p : \{BG^\wedge_p, BK\} \xrightarrow{\cong} \{BG^\wedge_p, BK\}.$$

We conclude that $\widetilde{\Phi}^\wedge_p$ is a surjection, so

$$\{BG^\wedge_p, BK\} = \widetilde{\Phi}^\wedge_p\big(\{BS, BK\}\big),$$

and that $\widetilde{\Gamma}^\wedge_p$ is an injection. Hence there is an isomorphism

$$\widetilde{\Gamma}^\wedge_p : \{BG^\wedge_p, BK\} \xrightarrow{\cong} \widetilde{\Gamma}^\wedge_p \circ \widetilde{\Phi}^\wedge_p\big(\{BS, BK\}\big).$$

Now we have a commutative diagram

$$\begin{array}{ccccccc}
\widetilde{M}_I & \xrightarrow[\cong]{\Gamma\circ\Phi} & \Gamma\circ\Phi(\widetilde{M}_I) & \xrightarrow{=} & \Gamma\circ\Phi(\widetilde{A}(S,K)^\wedge_p) & \xrightarrow[\cong]{\tilde{\alpha}^\wedge_p} & \Gamma\circ\Phi(\{BS,BK\}) \\
& \underset{\cong}{\overset{\Phi}{\searrow}} & \uparrow{\scriptstyle \Gamma} & & & & {\scriptstyle \Gamma}\uparrow{\scriptstyle \cong} \\
& & \widetilde{A}_p(G,K)^\wedge_p & & \xrightarrow{(\iota^*_p\circ\tilde{\alpha}_p)^\wedge_p} & & \{BG^\wedge_p, BK\},
\end{array}$$

where we have written Φ and Γ instead of $\widetilde{\Phi}^\wedge_p$ and $\widetilde{\Gamma}^\wedge_p$. Concentrating first on the left side of this diagram, we see that since Φ and $\Gamma \circ \Phi$ are both isomorphisms, Γ must be an isomorphism. Turning our attention to the right side of the diagram, we see that the bottom arrow forms the bottom part of a rectangular commutative diagram where all the other arrows are isomorphism. Hence the bottom arrow is itself an isomorphism. □

Using Lemma 1.6 we obtain an immediate reformulation.

Corollary 3.2. *For a finite group G and a compact Lie group K, the map*

$$(\tilde{\alpha}_p)^\wedge_p : \widetilde{A}_p(G,K)^\wedge_p \to \{BG, BK\}^\wedge_p \simeq \{BG^\wedge_p, BK^\wedge_p\},$$

is an isomorphism of $\mathbb{Z}^\wedge_p$-modules, natural in G and finite K.

The naturality property in Corollary 3.2 can also be stated as follows.

Corollary 3.3. *The functor*

$$(\tilde{\alpha}_p)^\wedge_p : A_p\mathrm{Gr}^\wedge_p \to \mathbb{B}\mathrm{Gr}^\wedge_p$$

is an isomorphism of $\mathbb{Z}^\wedge_p$-linear categories.

Finally we can collect our results for different primes p and prove Theorem B of the introduction.

Theorem 3.4. *For a finite group G and a compact Lie group K, the map*

$$\bigoplus_q (\iota_q^* \circ \tilde{\alpha}_q)_q^\wedge : \bigoplus_q \widetilde{A}_q(G, K)_q^\wedge \to \bigoplus_q \{BG_q^\wedge, BK\} \cong \{BG, BK\},$$

where the sums run over all primes q, is an isomorphism of $\mathbb{Z}$-modules, natural in G and finite K.

Proof. The stable splitting $\mathbb{B}G \simeq \bigvee_q \mathbb{B}G_q^\wedge$ induces a splitting

$$\{BG, BK\} \cong \bigoplus_q \{BG_q^\wedge, BK\}.$$

The result now follows from Theorem 3.1. □

One can also restrict this theorem to finite groups and formulate a corollary about an isomorphism of $\mathbb{Z}$-linear categories, but we refrain from doing so.

4. Alternative formulations

The basepoint issue mentioned before in this paper is more a technical nuisance for formulating statements than an actual problem in proving them. The main issue with the added basepoint is that for a finite group G, the suspension spectrum of the p-completion of BG_+ is $\Sigma^\infty((BG_+)_p^\wedge) \simeq \mathbb{B}G_p^\wedge \vee \mathbb{S}$, which is not p-complete as $\mathbb{S}$ is not p-complete. Consequently, for a compact Lie group K, the module $\{(BG_+)_p^\wedge, BK_+\}$ need not be p-complete as maps factoring through the sphere spectrum can contribute a non-completed part.

In this paper we have so far opted to remove the offending sphere spectrum, at the cost of disregarding maps factoring through the sphere spectrum, which results in introducing the aesthetically unpleasant quotients of double Burnside modules. However, there are other ways to smooth over the basepoint issue. In this section we consider two alternative approaches. The first is to p-complete the added sphere spectrum rather than to remove it, and thus force everything in sight to be p-complete. The second approach is to deal separately with the failure of $\{(BG_+)_p^\wedge, BK_+\}$ to be p-complete, which turns out to be precisely one copy of $\mathbb{Z}$ corresponding to maps from one sphere spectrum to the other.

For each approach we give the corresponding versions of Theorem 1.12 and state the generalizations to p-completed classifying spaces of finite groups. These generalizations are obtained using the same filtration argument already presented and so the proofs are omitted.

4.1. p-completed suspension spectra

Instead of looking at the suspension spectrum of the p-completion of BG_+, for a finite group G, we can look at the p-completion of the suspension spectrum of BG_+. Thus we obtain the p-complete spectrum $(\mathbb{B}_+G)_p^\wedge \simeq \mathbb{B}G_p^\wedge \vee \mathbb{S}_p^\wedge$. Taking this approach for a finite p-group S and a compact Lie group K, the appropriate version of Theorem 1.12 is that the map

$$\alpha_p^\wedge : A(S, K)_p^\wedge \xrightarrow{\cong} \big[(\mathbb{B}_+S)_p^\wedge, (\mathbb{B}_+K)_p^\wedge\big]$$

is an isomorphism of $\mathbb{Z}_p^\wedge$-modules. Using the same transfer and filtration arguments as in the proof of Theorem 3.1 we obtain the following generalization.

Theorem 4.1. *For a finite group G and a compact Lie group K, the map*

$$(\alpha_p)^\wedge_p : A_p(G,K)^\wedge_p \xrightarrow{\cong} \left[(\mathbb{B}_+G)^\wedge_p, (\mathbb{B}_+K)^\wedge_p\right]$$

is an isomorphism of $\mathbb{Z}^\wedge_p$*-modules*

While this approach has the advantage of retaining more information from the double Burnside module $A_p(G,K)$, it has the drawback that we obtain no analogue of Theorem 3.4 as $\mathbb{B}_+G$ is not a wedge sum of its completions (since the sphere spectrum is not a torsion spectrum).

4.2. Precise approach

The statement of the Segal conjecture for stable maps from classifying spaces of p-groups presented in Theorem 1.12 is actually a simplified version of a more detailed result obtained by May and McClure in [10].

For a finite group G with Sylow subgroup S and a compact Lie group K, let $I_p(G,K)$ be the kernel of the map $\epsilon : A_p(G,K) \to \mathbb{Z}$. As a $\mathbb{Z}$-module, $I_p(G,K)$ has basis $\{[P,\varphi] - |S|/|P| \cdot [S, ct]\}$. There is a splitting $A_p(G,K) = \mathbb{Z} \oplus I_p(G,K)$, where the $\mathbb{Z}$ term is generated by $[S, ct]$. Since $I(G) \cdot A(G,K) = I(G,K)$ this induces a splitting $A_p(G,K)^\wedge_{I(G)} = \mathbb{Z} \oplus I_p(G,K)^\wedge_{I(G)}$.

May and McClure showed that for a finite p-group S, the $I(S)$-adic topology on $I(S,K)$ coincides with the p-adic topology. Therefore the Segal conjecture says that the map

$$(\alpha_p)^\wedge_{I(S)} : \mathbb{Z} \oplus I(S,K)^\wedge_p \to \{BS_+, BK_+\}$$

is an isomorphism. Again, using the transfer and filtration arguments from Sections 2 and 3, one can prove the following generalization.

Theorem 4.2. *For a finite group G and a compact Lie group K, the map*

$$(\alpha_p)^\wedge_{I(G)} : \mathbb{Z} \oplus I_p(G,K)^\wedge_p \to \left\{(BG_+)^\wedge_p, BK_+\right\}$$

is an isomorphism of $\mathbb{Z}$*-modules.*

Collecting this result for different primes one can obtain an isomorphism of $\mathbb{Z}$-modules.

$$\mathbb{Z} \oplus \bigoplus_p I_p(G,K)^\wedge_p \xrightarrow{\cong} \{BG_+, BK_+\}.$$

Unlike the isomorphism in 3.4 this isomorphism is not natural in G.

Taking $K = 1$ one recovers a special case of Minami's description of the $I(G)$-adic completion of the Burnside ring in [12] (he allows G to be compact). It should be noted that even though one can regard $A(G,K)$ as a submodule of $A(K \times G)$ when K is finite, the results in this paper do not follow from Minami's results, as his result involves the $I(K \times G)$-adic completion whereas we are interested in the $I(G)$-adic completion, and the two actions are hard to reconcile.

5. Decompositions

In this section we address the following question. Given a (G, K)-pair (H, φ), what is the element in $\widetilde{A}_p(G, K)^\wedge_p$ corresponding to the p-completed stable map $\tilde{\alpha}([H, \varphi])$? In other words, we describe the homomorphism

$$\tilde{\pi}_p : \widetilde{A}(G, K) \xrightarrow{(-)^\wedge_p \circ \tilde{\alpha}} \{BG^\wedge_p, BK^\wedge_p\} \xrightarrow{((\tilde{\alpha}_p)^\wedge_p)^{-1}} \widetilde{A}_p(G, K)^\wedge_p.$$

For technical reasons it is more convenient to first describe the homomorphism

$$\pi_p : A(G, K) \xrightarrow{(-)^\wedge_p \circ \alpha} [(\mathbb{B}_+G)^\wedge_p, (\mathbb{B}_+K)^\wedge_p] \xrightarrow{((\alpha_p)^\wedge_p)^{-1}} A_p(G, K)^\wedge_p$$

and then interpret the results for the homomorphism $\tilde{\pi}$ obtained by removing the additional sphere spectrum.

Let $1_p \in A_p(G, G)^\wedge_p$ be the pre-image of the identity of $(\mathbb{B}_+G)^\wedge_p$ under the isomorphism

$$(\alpha_p)^\wedge_p : A_p(G, G)^\wedge_p \xrightarrow{\cong} [(\mathbb{B}_+G)^\wedge_p, (\mathbb{B}_+G)^\wedge_p].$$

This immediately gives us a way to describe π_p, for if $X \in A(G, K)$ then we can regard $X \circ 1_p$ as an element of $A_p(G, K)^\wedge_p$, and we have

$$(\alpha_p)^\wedge_p(X \circ 1_p) = \alpha(X)^\wedge_p \circ (\alpha_p)^\wedge_p(1_p) = \alpha(X)^\wedge_p \circ id = \alpha(X)^\wedge_p.$$

Therefore

$$\pi_p(X) = X \circ 1_p.$$

We proceed to determine 1_p using filtration methods similar to those in Section 2. Since many details are similar, they are left to the reader.

Fix a Sylow subgroup S of G. Like in Section 2, we say that a (G, G)-pair (F, ψ) is subconjugate to a (G, G)-pair (H, φ), and write $(F, \psi) \precsim (H, \varphi)$ if there exist elements $g_1, g_2 \in G$ making the following diagram commute

$$\begin{array}{ccc} F & \xrightarrow{\psi} & \psi(F) \\ \downarrow{\scriptstyle c_{g_1}} & & \downarrow{\scriptstyle c_{g_2}} \\ H & \xrightarrow{\varphi} & \varphi(H). \end{array}$$

This time the induced equivalence relation is just (G, G)-conjugacy, and so subconjugacy induces an order relation on (G, G)-conjugacy classes of (G, G)-pairs.

For a basis element $[P, \varphi]$ of $A_p(G, G)$, let $M(\precsim [P, \varphi])$ denote the submodule of $A_p(G, G)$ generated by basis elements that are subconjugate to $[P, \varphi]$, and similarly let $M(\precnsim [P, \varphi])$ denote the submodule generated by strictly subconjugate basis elements. We make the following key observation, which is proved just like Proposition 2.8.

Lemma 5.1. *For every (G,G)-pair (P,φ), where P is a p-group,*

$$[P,\varphi]\circ[S,\iota_S]\in M\big(\precsim[P,\varphi]\big)\setminus M\big(\precnsim[P,\varphi]\big).$$

Let R denote the subring of $A_p(G,G)$ generated by basis elements of the form $[P,\iota_P]$, where ι_P denotes the inclusion $P\leqslant G$. Note that $R=M(\precsim[S,\iota])$. The above lemma allows us to prove the following.

Proposition 5.2. $1_p\in R$.

Proof. Take a maximal $[Q,\psi]$ such that $c_{[Q,\psi]}(1_p)\neq 0$. By Lemma 5.1 we have

$$c_{[Q,\psi]}\big([Q,\psi]\circ[S,\iota_S]\big)\neq 0$$

(otherwise $[Q,\psi]\circ[S,\iota_S]\in M(\precnsim[Q,\psi])$).

If $[P,\varphi]$ is a basis element different from $[Q,\psi]$ such that $c_{[P,\varphi]}(1_p)\neq 0$, then, by maximality, $[Q,\psi]$ is not subconjugate to $[P,\varphi]$ and hence $c_{[Q,\psi]}(X)=0$ for all $X\in M(\precsim[P,\varphi])$. In particular,

$$c_{[Q,\psi]}\big([P,\varphi]\circ[S,\iota_S]\big)=0.$$

Now we deduce that

$$\begin{aligned}
c_{[Q,\psi]}\big(1_p\circ[S,\iota_S]\big)&=c_{[Q,\psi]}\bigg(\sum_{[P,\varphi]}c_{[P,\varphi]}(1_p)\cdot[P,\varphi]\circ[S,\iota_S]\bigg)\\
&=\sum_{[P,\varphi]}c_{[P,\varphi]}(1_p)\cdot c_{[Q,\psi]}\big([P,\varphi]\circ[S,\iota_S]\big)\\
&=c_{[Q,\psi]}(1_p)\cdot c_{[Q,\psi]}\big([Q,\psi]\circ[S,\iota_S]\big)\\
&\neq 0.
\end{aligned}$$

On the other hand, we have $1_p\circ[S,\iota_S]=[S,\iota_S]$. Therefore we must have $[Q,\psi]=[S,\iota_S]$. We deduce that $[S,\iota_S]$ is the unique maximal element such that $c_{[S,\iota_S]}(1_p)\neq 0$, and hence $1_p\in M(\precsim[S,\iota_S])=R$. □

Let n be the number of conjugacy classes of p-subgroups of G and pick one representative P_i for each conjugacy class, labelled from 1 to n so that $P_1=S$, and $i\geqslant j$ if P_j is conjugate to a subgroup of P_i. Since $1_p\in R$, we can write

$$1_p=\sum_{j=0}^{n}a_j\cdot[P_j,\iota_{P_j}].$$

For subgroups H and F of G, let $N_G(F,H)$ denote the *transporter*

$$N_G(K,H)=\big\{g\in G\;\big|\;K^g\leqslant H\big\}.$$

Proposition 5.3. *The multiplicative identity of $A_p(G,G)^\wedge_p$ is given by*

$$1_p = \sum_{j=1}^{n} a_j \cdot [P_j, \iota_{P_j}],$$

where the coefficients a_j satisfy the equations

$$\sum_{j=1}^{n} a_j \cdot \frac{|N_G(P_i, P_j)|}{|P_j|} = 1 \tag{4}$$

for $i = 1, \dots, n$.

Proof. In this proof we will denote all inclusions of groups by ι. For every i we have $1_p \circ [P_i, \iota] = [P_i, \iota]$. In particular, $c_{[P_i,\iota]}(1_p \circ [P_i, \iota]) = 1$. By the double coset formula, we have

$$[P_j, \iota] \circ [P_i, \iota] = \sum_{x \in P_j \backslash G / P_i} \big[P_i \cap P_j^x, c_x\big].$$

It is easy to check that $[P_i \cap P_j^x, c_x] = [P_i \cap P_j^x, \iota]$, and

$$c_{[P_i,\iota]}\big([P_i \cap P_j^x, \iota]\big) = \begin{cases} 1 & \text{if } {}^x P_i \leqslant P_j, \\ 0 & \text{otherwise.} \end{cases}$$

Noting that ${}^x P_i \leqslant P_j$ if and only if $x \in N_G(P_i, P_j)$, we see that

$$c_{[P_i,\iota]}\big([P_j, \iota] \circ [P_i, \iota]\big) = \big|P_j \backslash N_G(P_i, P_j)/P_i\big|.$$

If $x \in N_G(P_i, P_j)$ and $g \in P_i$, then there is a $h \in P_j$ such that $xg = hx$. Therefore

$$P_j \backslash N_G(P_i, P_j)/P_i \cong P_j \backslash N_G(P_i, P_j),$$

and

$$c_{[P_i,\iota]}\big([P_j, \iota] \circ [P_i, \iota]\big) = \frac{|N_G(P_i, P_j)|}{|P_j|}.$$

Now we get

$$\begin{aligned} 1 &= c_{[P_i,\iota]}\big(1_p \circ [P_i, \iota]\big) \\ &= c_{[P_i,\iota]}\left(\sum_{j=0}^{n} a_j \cdot [P_j, \iota] \circ [P_i, \iota]\right) \\ &= \sum_{j=1}^{n} a_j \cdot c_{[P_i,\iota]}\big([P_j, \iota] \circ [P_i, \iota]\big) = \sum_{j=1}^{n} a_j \cdot \frac{|N_G(P_i, P_j)|}{|P_j|}. \qquad \square \end{aligned}$$

Remark 5.4. Since $|N_G(P_i, P_j)| = 0$ when $i < j$, the equations in (4) constitute a lower triangular matrix equation. The elements on the diagonal are $|N_G(P_i)/P_i| \neq 0$, so the matrix is of maximal rank and the equations suffice to determine 1_p uniquely. We do not need to prove that the equation is solvable (this could be an issue because the elements on the diagonal need not be units in $\mathbb{Z}_p^\wedge$) since we already know that $A_p(G,G)_p^\wedge$ has a multiplicative identity.

We now proceed to describe π_p in terms of fixed-point sets. For every (G,K)-pair (H,ψ), there is a homomorphism

$$\chi_{[H,\psi]} : A(G,K) \to \mathbb{Z},$$

depending only on the conjugacy class of (H,ψ), which sends a (G,K)-bundle X to

$$\chi_{[H,\psi]}(X) = \chi\big(W\big(\Delta_H^\psi\big)\backslash X^{\Delta_H^\psi}\big),$$

where χ is the Euler characteristic, $X^{\Delta_H^\psi}$ is the fixed-point space, and

$$W\big(\Delta_H^\psi\big) = N_{K\times G}\big(\Delta_H^\psi\big)/\Delta_H^\psi.$$

Note that for a (G,K)-pair (H',ψ') we have

$$W\big(\Delta_H^\psi\big)\backslash\big((K\times G)/\Delta_{H'}^{\psi'}\big)^{\Delta_H^\psi} \cong N_{K\times G}\big(\Delta_H^\psi\big)\backslash N_{K\times G}\big(\Delta_H^\psi, \Delta_{H'}^{\psi'}\big)/\Delta_{H'}^{\psi'}$$

and that $N_{K\times G}(\Delta_H^\psi)\backslash N_{K\times G}(\Delta_H^\psi, \Delta_{H'}^{\psi'})$ is finite, so we in fact have

$$\chi_{[H,\psi]}(X) = \big|W\big(\Delta_H^\psi\big)\backslash X^{\Delta_H^\psi}\big|$$

for a finite principal (G,K)-bundle X over a finite G-set.

Let $C_p(G,K)$ be the set of conjugacy classes of (G,K)-pairs (P,ψ) where P is a p-group.

Proposition 5.5. *(See [14].) The homomorphism*

$$\chi_p : A_p(G,K) \to \prod_{C_p(G,K)} \mathbb{Z}, \quad X \mapsto \prod_{C_p(G,K)} \chi_{[P,\psi]}(X)$$

is an injection.

We will describe π_p in terms of this embedding. To this end we need the following reformulation of a result of Benson–Feshbach.

Lemma 5.6. *(See [4].) For $X \in A(G,K)$, a (G,K)-pair (H,ψ) and a subgroup $F \leqslant G$, we have*

$$\chi_{[H,\psi]}\big(X \circ [F, \iota_F]_G^G\big) = \frac{|N_G(H,F)|}{|F|}\chi_{[H,\psi]}(X).$$

In particular $\chi_{[H,\psi]}(X \circ [F,\iota_F]_G^G) = 0$ if H is not subconjugate to F.

Proof. This follows from Proposition 3.1 in [4] (which is actually proved in the case where K is finite and without taking the quotient by $W(\Delta_H^\psi)$ but the same argument works in this setting). $\square$

Theorem 5.7. *The homomorphism π_p sends $X \in A(G,K)$ to the unique element $X_p \in A_p(G,K)_p^\wedge$ such that*

$$\chi_p(X_p) = \chi_p(X).$$

In other words, π_p is the unique homomorphism fitting into the commutative diagram

$$\begin{array}{ccc} A(G,K) & \xrightarrow{\pi_p} & A_p(G,K)_p^\wedge \\ & {\scriptstyle \chi_p}\searrow \quad \swarrow{\scriptstyle \chi_p} & \\ & \prod_{C_p(G,K)} \mathbb{Z}_p^\wedge . & \end{array}$$

The homomorphism $\tilde{\pi}_p$ is the homomorphism obtained from π_p by quotienting out all trivial basis elements.

Proof. For $X \in A(G,K)$ and a (G,K)-pair (Q,ψ) with Q a p-group we have

$$\begin{aligned} \chi_{[Q,\psi]}\big(\pi_p(X)\big) &= \chi_{[Q,\psi]}(X \circ 1_p) \\ &= \sum_{j=0}^{n} a_j \chi_{[Q,\psi]}\big(X \circ [P_j, \iota_{P_j}]\big) \\ &= \sum_{j=0}^{n} a_j \frac{|N_G(Q,P_j)|}{|P_j|} \chi_{[Q,\psi]}(X) \\ &= \chi_{[Q,\psi]}(X), \end{aligned}$$

where the last step follows from Proposition 5.3 since Q is conjugate to one of the groups P_i. $\square$

Letting p run over all primes we obtain the following corollary, from which Theorem C in the introduction follows.

Corollary 5.8. *The kernel of the map*

$$\alpha : A(G,K) \to \{BG_+, BK_+\} \cong A(G,K)_{I(G)}^\wedge$$

is the submodule consisting of elements X such that $\chi_{[H,\psi]}(X) = 0$ for all (G,K)-pairs $[H,\psi]$ where the order of H is a prime power.

Remark 5.9. If K is finite one may replace the morphism $\chi_{[H,\psi]}$ with the morphism sending $X \mapsto |X^{\Delta_H^\psi}|$ throughout this section and the results still hold true.

References

[1] J.F. Adams, Infinite Loop Spaces, Ann. of Math. Stud., vol. 90, Princeton Univ. Press, University of Tokyo Press, 1978.
[2] J.F. Adams, J.H.C. Gunawardena, H.R. Miller, The Segal conjecture for elementary abelian p-groups, Topology 24 (1985) 435–460.
[3] M.F. Atiyah, Character and cohomology of finite groups, Publ. Math. Inst. Hautes Études Sci. 9 (1961) 23–64.
[4] D. Benson, M. Feshbach, Stable splittings of classifying spaces, Topology 31 (1992) 157–176.
[5] A.K. Bousfield, The localization of spectra with respect to homology, Topology 18 (1979) 257–281.
[6] A.K. Bousfield, D.M. Kan, Homotopy Limits, Completions and Localizations, Lecture Notes in Math., vol. 304, Springer-Verlag, 1972.
[7] G. Carlsson, Equivariant stable homotopy and Segal's Burnside ring conjecture, Ann. of Math. 120 (1984) 189–224.
[8] L.G. Lewis, J.P. May, J.E. McClure, Classifying G-spaces and the Segal conjecture, in: Current Trends in Algebraic Topology, Part 2, London, Ontario, 1981, in: CMS Conf. Proc., vol. 2, 1981, pp. 165–179.
[9] J. Martino, S. Priddy, A classification of the stable type of $BG_p^\wedge$ for compact Lie groups, Topology 38 (1999) 1–21.
[10] J.P. May, J.E. McClure, A reduction of the Segal conjecture, in: Current Trends in Algebraic Topology, Part 2, London, Ontario, 1981, in: CMS Conf. Proc., vol. 2, 1981, pp. 209–222.
[11] J.P. May, V.P. Snaith, P. Zelewski, A further generalization of the Segal conjecture, Quart. J. Math. Oxford 40 (1989) 457–473.
[12] N. Minami, On the $I(G)$-adic topology of the Burnside ring of compact Lie groups, Publ. RIMS Kyoto Univ. 20 (1984) 447–460.
[13] G.B. Segal, Equivariant stable homotopy theory, in: Actes Congrès Intern. des Math., Tome 2, Nice, 1970, pp. 59–63.
[14] T. tom Dieck, Transformation Groups, de Gruyter Stud. Math., vol. 8, de Gruyter, 1987.