Abstract

We define the Hardy–Littlewood maximal function of \( \tau \)-measurable operators and obtain weak \((1,1)\)-type and \((p,p)\)-type inequalities for the Hardy–Littlewood maximal function.

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0. Introduction

Nelson [2] defined the measure topology of \( \tau \)-measurable operators affiliated with a semi-finite von Neumann algebra. Fack and Kosaki [1] studied generalized \( s \)-numbers of \( \tau \)-measurable operators, proved dominated convergence theorems for a gage and convexity (or concavity) inequalities.

We will study the Hardy–Littlewood maximal function of a \( \tau \)-measurable operator \( T \). More precisely, let \( \mathcal{M} \) be a semi-finite von Neumann algebra with a normal faithful semi-finite trace \( \tau \). For an operator \( T \) in \( \mathcal{M} \), the Hardy–Littlewood maximal function of \( T \) is defined by

\[
MT(x) = \sup_{r > 0} \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(|T|E_{[x-r,x+r]}(|T|)).
\]

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Classically \(Mf(x)\) is defined as
\[ Mf(x) = \sup_{r > 0} \frac{1}{m([x - r, x + r])} \int_{[x-r,x+r]} |f(t)| \, dt, \]
for the case \(f: \mathbb{R} \rightarrow \mathbb{R}\) and \(m\) a Lebesgue measure on \((-\infty, +\infty)\) (cf. [3]). A natural generalization of this is the case \(f: \mathbb{R} \rightarrow \mathbb{R}\) and \(\mu\) a Borel measure on \((-\infty, +\infty)\) where
\[ M\mu f(x) = \sup_{r > 0} \frac{1}{\mu([x - r, x + r])} \int_{[x-r,x+r]} |f(t)| \, d\mu(t). \]
Let \(\mu(A) = \tau(E_A(|T|))\), where \(A\) is a Borel subset of \((-\infty, +\infty)\). Then \(\mu\) is a Borel measure and
\[ MT(x) = \sup_{r > 0} \frac{1}{\mu([x - r, x + r])} \int_{[x-r,x+r]} t \, d\mu(t), \]
i.e., \(MT(x)\) is the Hardy–Littlewood maximal function \(M\mu f(x)\) of \(f: \mathbb{R} \rightarrow \mathbb{R}\) defined by
\[ f(t) = \begin{cases} t\chi_{\sigma(|T|)}(t), & t \in \sigma(|T|), \\ 0, & t \notin \sigma(|T|), \end{cases} \]
with respect to \(\mu\).

Via spectral theory \(|T|\) is represented as
\[ f(t) = \begin{cases} t\chi_{\sigma(|T|)}(t), & t \in \sigma(|T|), \\ 0, & t \notin \sigma(|T|), \end{cases} \]
and \(MT(|T|)\) is represented as \(MT(x)\). Then for \(T\), we consider \(MT(|T|)\) as the operator analogue of the Hardy–Littlewood maximal function in the classical case. We will give some results similar to the classical case. Hence roughly speaking, \(MT(|T|)\) stands in relation to \(T\) as \(Mf(x)\) stands in relation to \(f\) in classical analysis.

Section 1 consists of some preliminaries. In Section 2, we give some properties of the Hardy–Littlewood maximal function of \(\tau\)-measurable operators. In Section 3, we prove weak \((1,1)\)-type and \((p,p)\)-type inequalities for the Hardy–Littlewood maximal function.

1. Preliminaries

Throughout this paper, we denote by \(\mathcal{M}\) a semi-finite von Neumann algebra on the Hilbert space \(\mathcal{H}\) with a normal faithful semi-finite trace \(\tau\). The closed densely defined linear operator \(T\) in \(\mathcal{H}\) with domain \(D(T)\) is said to be affiliated with \(\mathcal{M}\) if and only if \(U^*TU = T\) for all unitary operators \(U\) which belong to the commutant \(\mathcal{M}'\) of \(\mathcal{M}\). If \(T\) is affiliated with \(\mathcal{M}\), then \(T\) is said to be \(\tau\)-measurable if for every \(\varepsilon > 0\) there exists a projection \(P \in \mathcal{M}\) such that \(P(H) \subseteq D(T)\) and \(\tau(P^\perp) < \varepsilon\) (where for any projection \(P\) we let \(P^\perp = 1 - P\)). The set of all \(\tau\)-measurable operators will be denoted by \(\overline{\mathcal{M}}\). The set \(\overline{\mathcal{M}}\) is an \(*\)-algebra with sum and product being the respective closure of the algebraic sum and product. For a positive self-adjoint operator \(T = \int_0^\infty \lambda \, dE_\lambda\) affiliated with \(\mathcal{M}\), we set
\[ \tau(T) = \sup_n \tau \left( \int_0^n \lambda \, dE_\lambda \right) = \int_0^\infty \lambda \tau(E_\lambda). \]
For $0 < p < \infty$, $L^p(\mathcal{M}; \tau)$ is defined as the set of all $\tau$-measurable operators $T$ affiliated with $\mathcal{M}$ such that
\[
\|T\|_p = \tau(|T|^p)^{1/p} < \infty.
\]

In addition, we put $L^\infty(\mathcal{M}; \tau) = \mathcal{M}$ and denote by $\|\cdot\|_\infty (= \|\cdot\|)$, the usual operator norm. It is well known that $L^p(\mathcal{M}; \tau)$ is a Banach space under $\|\cdot\|_p (1 \leq p \leq \infty)$ satisfying all the expected properties such as duality.

For a positive operator $T$, let $E(t, \infty)(T)$ be the spectral projection of $T$ corresponding to the interval $(t, \infty)$. We state for easy reference the following fact that will be applied below.

**Theorem A (Besicovitch).** Let $F$ be a bounded subset of $[0, \infty)$ and suppose to each $x \in F$ we associate a number $r(x) > 0$. Then we can take a sequence of intervals $\{[x_k - r(x_k), x_k + r(x_k)]\}$ such that
\[
F \subset \bigcup_k [x_k - r(x_k), x_k + r(x_k)]
\]
and
\[
\sum_k \chi_{[x_k - r(x_k), x_k + r(x_k)]} \leq 4, \quad \forall x \in [0, \infty).
\]

**2. Maximal function**

Let $L_{\text{loc}}(\mathcal{M}; \tau)$ be the set of all $\tau$-measurable operators such that
\[
\tau(|T|E_I(|T|)) < +\infty,
\]
for all bounded intervals $I \in [0, +\infty)$.

**Definition 1.** For $T \in L_{\text{loc}}(\mathcal{M}; \tau)$, we define the maximal function of $T$ by
\[
MT(x) = \sup_{r > 0} \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(|T|E_{[x-r,x+r]}(|T|))
\]
(let $0^0 = 0$). $M$ is called the Hardy–Littlewood maximal operator.

The maximal function of a $\tau$-measurable operator has the following property.

**Lemma 1.** Let $T \in L_{\text{loc}}(\mathcal{M}; \tau)$.

(i) If the map: $t \in [0, \infty) \rightarrow E_{(t, \infty)}(|T|)$ is strongly continuous, then $MT(x)$ is a lower semi-continuous function on $[0, \infty)$.

(ii) For all $T \in L^\infty(\mathcal{M}; \tau)$, we have
\[
\|MT(|T|)\|_\infty \leq \|T\|_\infty.
\]

**Proof.** (i) It needs to be proved that
\[
F_{MT}(t) = \{x \in [0, \infty): MT(x) > t\}, \quad \forall t > 0,
\]
is an open set. In other words, if \( \{ x_k \} \) is a sequence in \([0, \infty) \setminus F_{MT}(t)\), converging to \( x \), then \( x \in [0, \infty) \setminus F_{MT}(t) \), i.e., for all \( r > 0 \) with \( E_{[x-r,x+r]}(|T|) \neq 0 \), we have

\[
\frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(|T|E_{[x-r,x+r]}(|T|)) \leq t.
\]

Let

\[
T_k = |T|E_{[x_k-r,x_k+r]}\Delta[x-r,x+r](|T|),
\]

where

\[
[x_k-r,x_k+r]\Delta[x-r,x+r] = ([x_k-r,x_k+r]\setminus[x-r,x+r])
\]

\[
\cup ([x-r,x+r]\setminus[x_k-r,x_k+r])
\]

and \( k = 1, 2, 3, \ldots \). It is clear that

\[
T_k \leq \lambda E_{[x_k-r,x_k+r]}\Delta[x-r,x+r],
\]

for some \( \lambda > 0 \) and all \( k \).

(a) If \( \tau \) is finite, we use \( \sigma \)-strong continuity of the trace and the fact that the strong and \( \sigma \)-strong topologies agree on the unit ball of \( \mathcal{M} \), to obtain the continuity of

\[
[0, \infty) \to [0, \infty) : s \to \tau(E_{[s,\infty]}(|T|)).
\]

By (5) and the previous continuity, we get

\[
\lim_{k \to \infty} \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(|T|E_{[x_k-r,x_k+r]}(|T|)) = 0,
\]

\[
\lim_{k \to \infty} \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(E_{[x_k-r,x_k+r]}(|T|)) = \tau(E_{[x-r,x+r]}(|T|)).
\]

Hence, for \( \delta > 0 \), there exists \( k_0 \) such that

\[
\frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(|T|E_{[x_k-r,x_k+r]}(|T|))
\]

\[
= \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \frac{1}{\tau(E_{[x_k-r,x_k+r]}(|T|))} \tau(|T|E_{[x_k-r,x_k+r]}(|T|))
\]

\[
\leq \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}t < t + \delta, \quad \forall k \geq k_0.
\]

Thus for \( k \geq k_0 \), we have

\[
\frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(|T|E_{[x-r,x+r]}(|T|))
\]

\[
\leq \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(|T|E_{[x_k-r,x_k+r]\Delta[x-r,x+r]}(|T|))
\]

\[
+ \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(|T|E_{[x_k-r,x_k+r]}(|T|))
\]

\[
\leq \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(T_kE_{[x-r,x+r]}(|T|)) + t + \delta.
\]

Letting \( k \to \infty \), \( \delta \to 0 \), one obtains that

\[
\frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(|T|E_{[x-r,x+r]}(|T|)) \leq t.
\]
(b) In the general case, for $T \in L_{\text{loc}}(\mathcal{M}; \tau)$, for any $\varepsilon > 0$ and any bounded interval $I \subset [\varepsilon, \infty)$, we have that

$$\varepsilon \tau(E_{\varepsilon}(|T|)) \leq \tau(|T|) < \infty.$$  

On the other hand, we may assume $\tau(E_{[x-r,x+r]}(|T|)) < \infty$ (since otherwise the inequality (4) automatically holds). Hence $\tau(E_{[x-r-\varepsilon_0,x+r+\varepsilon_0]}(|T|)) < \infty$ for $\varepsilon_0 > 0$ small enough. Since $x_k \to x$ ($k \to \infty$), for the above $\varepsilon_0 > 0$ there exists an integer $K > 0$ such that

$$[x_k - r, x_k + r] \cup [x - r, x + r] \subset [x - r - \varepsilon_0, x + r + \varepsilon_0], \quad k \geq K.$$  

Without loss of generality we can replace $\mathcal{M}$ by

$$E_{[x-r-\varepsilon_0,x+r+\varepsilon_0]}(|T|).$$  

Hence, by the case (a), we obtain (4).

(ii) From

$$\frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau(|T|E_{[x-r,x+r]}(|T|)) \leq \|T\|_\infty,$$

we get $MT(x) \leq \|T\|_\infty$. Hence (3) follows from

$$(MT(|T|)(y, y) = \int_{\sigma(|T|)} MT(t) |E_{[x-r,x+r]}(|T|))|y, y) \leq \|T\|_\infty \int_{\sigma(|T|)} |E_{[x-r,x+r]}(|T|))|y, y) = \|T\|_\infty(y, y), \quad \forall y \in D(T). \quad \Box$$

3. Inequalities of the Hardy–Littlewood maximal function

For a $\tau$-measurable operator $T$ and a positive function $f$, we define

$$F_f(t) = \{x \in [0, \infty): f(x) > t\}$$

and

$$f_*(|T|)(t) = \tau(E_{F_f(t)}(|T|)).$$

**Lemma 2.** Let $T$ be a $\tau$-measurable operator and $f$ be a positive function on $[0, \infty)$.

(i) $f_*(|T|)$ is non-increasing on $[0, \infty)$.

(ii) If $f(|T|) \in L^p(\mathcal{M}; \tau)$, $1 < p < \infty$, then

$$\lim_{t \to \infty} t^p f_*(|T|)(t) = \lim_{t \to 0} t^p f_*(|T|)(t) = 0.$$  

(iii) If $\int_0^\infty t^p f_*(|T|)(t) dt < \infty$, then

$$\lim_{t \to \infty} t^p f_*(|T|)(t) = \lim_{t \to 0} t^p f_*(|T|)(t) = 0.$$  

**Proof.** (i) Follows immediately from the definition of $f_*(|T|)$.

(ii) From

$$t^p f_*(t)(|T|) \leq \tau(E_{F_f(t)}(|T|)) f(|T|)^p \leq \tau(f(|T|)^p),$$

we get

$$\lim_{t \to \infty} t^p f_*(|T|)(t) = \lim_{t \to 0} t^p f_*(|T|)(t) = 0.$$
we get $f_\tau(|T|)(t) = \tau(E_{f(t)}(|T|)) \to 0 \ (t \to \infty)$. Hence, we obtain that
$$
\lim_{t \to \infty} \tau(E_{f(t)}(|T|) f(|T|)^p) = 0,
$$
so that
$$
t^p f_\tau(|T|)(t) \to 0 \ (t \to \infty).
$$
Fix $\delta > 0$. Then for $t < \delta$, we have
$$
\lim_{t \to 0} t^p f_\tau(|T|)(t) = \lim_{t \to 0} t^p \left(f_\tau(|T|)(t) - f_\tau(|T|)(\delta)\right)
\leq \tau(E_{\{x \in [0, \infty) : \delta \geq f(x) > t\}}(|T|) f(|T|)^p).
$$
Letting $\delta \to 0$, we obtain
$$
t^p f_\tau(|T|)(t) \to 0 \ (t \to 0).
$$
(iii) Follows immediately from the following fact:
$$
p \int_{t/2}^{t} s^{p-1} f_\tau(|T|)(s) \, ds \geq p \int_{t/2}^{t} s^{p-1} f_\tau(|T|)(t) \, ds = f_\tau(|T|)(t) \left(t^p - \left(\frac{t}{2}\right)^p\right)
\geq f_\tau(|T|)(t) t^p \left(1 - 2^{-p}\right).
$$

**Lemma 3.** Let $1 < p < \infty$.

(i) If for $t > 0$, we have $f_\tau(|T|)(t) < \infty$, then
$$
\tau(f(|T|)^p) = - \int_{0}^{+\infty} t^p \, df_\tau(|T|)(t). \quad (8)
$$

(ii) If $T$ is a measurable operator, then
$$
\tau(f(|T|)^p) = p \int_{0}^{+\infty} t^{p-1} f_\tau(|T|)(t) \, dt. \quad (9)
$$

**Proof.** (i) Let
$$
0 < \varepsilon < 2\varepsilon < \cdots < n\varepsilon < \cdots
$$
and
$$
E_j = E_{\{x \in [0, \infty) : j\varepsilon \geq f(x) > (j-1)\varepsilon\}}(|T|), \ j = 1, 2, \ldots.
$$
Then we get
$$
\tau(f(|T|)^p) = \int_{0}^{+\infty} f(t)^p \, d\tau(E_t(|T|))
$$
\[ \lim_{\varepsilon \to 0} +\infty \sum_{j=1}^{+\infty} ((j - 1)\varepsilon)^p \tau(E_j) = -\lim_{\varepsilon \to 0} +\infty \sum_{j=1}^{+\infty} \left( f_\ast(|T|)(j\varepsilon) - f_\ast(|T|)((j - 1)\varepsilon) \right) \]

\[ = -\int_0^{+\infty} t^p df_\ast(|T|)(t). \]

(ii) If two sides of (9) are infinity, then the result follows. Let one side of (9) be finite. Then by Lemma 2 we have \( f_\ast(|T|)(t) < \infty, \ \forall t > 0. \) Therefore (8) holds. On the other hand, by (6), (7), we get

\[ -\int_0^{+\infty} t^p df_\ast(|T|)(t) = p \int_0^{+\infty} t^{p-1} \left( f_\ast(|T|)(t) \right)^\infty - t^p f_\ast(|T|)(t) \int_0^{+\infty} = p \int_0^{+\infty} t^{p-1} f_\ast(|T|)(t) dt. \]

Thus we obtain (9). \( \square \)

**Theorem 1.** For all \( t > 0 \) and \( T \in L^1(M; \tau) \), we have

\[ \tau(E_{\{x \in [0, \infty): MT(x) > t\}}(|T|)) \leq \frac{4}{t} \|T\|_1. \]

**Proof.** Let

\[ F_{MT}(t) = \{ x \in [0, \infty): MT(x) > t \}. \]

Then from the definition of \( MT(x) \), for every \( x \in F_{MT}(t) \), there is a \( r(x) > 0 \), such that

\[ \frac{1}{\tau(E_{\{x-r(x), x+r(x)\}}(|T|))} \tau(|T|E_{\{x-r(x), x+r(x)\}}(|T|)) > t. \] (10)

Take

\[ F_n = F_{MT}(t) \cap [0, n], \quad n = 1, 2, 3, \ldots. \]

We apply Theorem A to \( F_n \), to obtain a sequence of intervals \([x_k - r(x_k), x_k + r(x_k)]\) such that

\[ F_n \subset \bigcup_k [x_k - r(x_k), x_k + r(x_k)] \quad \text{and} \quad \sum_k \chi_{[x_k - r(x_k), x_k + r(x_k)]} \leq 4. \]

Notice that every \([x_k - r(x_k), x_k + r(x_k)]\) satisfies (10), so

\[ \tau(E_{F_n}(|T|)) \leq \tau(E_{\bigcup_k [x_k - r(x_k), x_k + r(x_k)]}(|T|)) \]

\[ \leq \sum_k \tau(E_{[x_k - r(x_k), x_k + r(x_k)]}(|T|)) \]

\[ \leq \sum_k \frac{1}{t} \tau(|T|E_{[x_k - r(x_k), x_k + r(x_k)]}(|T|)) \]

\[ \leq \frac{4}{t} \|T\|_1. \]
\[
= \frac{1}{t} \sum_k \tau\left( |T| E_{[x_k-r(x_k), x_k+r(x_k)]}(|T|) \right)
\]
\[
= \frac{1}{t} \sum_k \tau\left( \int_0^\infty s \chi_{[x_k-r(x_k), x_k+r(x_k)]} dE_s(|T|) \right)
\]
\[
\leq \frac{4}{t} \tau\left( \int_0^\infty s dE_s(|T|) \right) = \frac{4}{t} \tau(|T|) = \frac{4}{t} \|T\|_1.
\]
i.e.,
\[
\tau\left( E_{F_n}(|T|) \right) \leq \frac{4}{t} \|T\|_1.
\]
On the other hand, we have
\[
F_1 \subset F_2 \subset \cdots \subset F_n \cdots \quad \text{and} \quad F_{MT}(t) \subset \bigcup_{n=1}^\infty F_n.
\]
Hence, we get
\[
\tau\left( E_{F_{MT}(t)}(|T|) \right) = \lim_{n \to \infty} \tau\left( E_{F_n}(|T|) \right) \leq \frac{4}{t} \|T\|_1. \quad \square
\]

**Lemma 4.** Let \( T \in L_{\text{loc}}(\mathcal{M}; \tau) \). Then
\[
\tau\left( E_{F_{MT}(t)}(|T|) \right) \leq \frac{8}{t} \tau\left( |T| E_{(\frac{t}{2}, +\infty)}(|T|) \right), \quad \forall t > 0.
\]

**Proof.** We set
\[
T_1 = T E_{[0, \frac{t}{2}]}(|T|), \quad T_2 = T - T_1.
\]
Then since
\[
\frac{1}{\tau\left( E_{[x-r,x+r]}(|T|) \right)} \tau\left( |T| E_{[x-r,x+r]}(|T|) \right)
\]
\[
\leq \frac{1}{\tau\left( E_{[x-r,x+r]}(|T|) \right)} \tau\left( |T| E_{[0, \frac{t}{2}]} E_{[x-r,x+r]}(|T|) \right)
\]
\[
\quad + \frac{1}{\tau\left( E_{[x-r,x+r]}(|T|) \right)} \tau\left( |T| E_{(\frac{t}{2}, +\infty)} E_{[x-r,x+r]}(|T|) \right)
\]
\[
\leq \frac{1}{\tau\left( E_{[x-r,x+r]}(|T|) \right)} \tau\left( |T_1| E_{[x-r,x+r]}(|T|) \right)
\]
\[
\quad + \frac{1}{\tau\left( E_{[x-r,x+r]}(|T|) \right)} \tau\left( |T_2| E_{[x-r,x+r]}(|T|) \right),
\]
it follows that
\[
MT(x) \leq MT_1(x) + MT_2(x) \leq MT_2(x) + \frac{t}{2}.
\]
Hence,
\[ \tau\left(E_{FMT(t)}(|T|)\right) \leq \tau\left(E_{\{x \in [0, \infty); \; MT_2(x) > \frac{t}{2}\}}(|T|)\right) \]
\[ \leq \frac{8}{t} \|T_2\|_1 = \frac{8}{t} \tau(|T_2|) = \frac{8}{t} \tau\left(\left|T\right|E_{\left(\frac{t}{2}, +\infty\right)}(|T|)\right). \]

**Theorem 2.** Let \(1 < p < \infty\). Then there is a constant \(C = C(p) > 0\) such that
\[ \|MT(|T|)\|_p \leq C\|T\|_p, \; \forall T \in L^p(M; \tau). \]  \hspace{1cm} (12)

**Proof.** For \(MT(x)\) we use Lemmas 3 and 4 to obtain that
\[ \|MT(|T|)\|_p = \tau\left(\left(\int_0^\infty MT(s)^p d\left(E_s(|T|)\right)\right)^{\frac{1}{p}}\right) \]
\[ = \int_0^\infty MT(s)^p d\tau\left(E_s(|T|)\right) \]
\[ = p \int_0^\infty t^{p-1} \tau\left(E_{FMT(t)}(|T|)\right) dt \]
\[ \leq 8p \int_0^\infty t^{p-2} \tau\left(|T|E_{\left(\frac{t}{2}, +\infty\right)}(|T|)\right) dt \]
\[ = 8p \int_0^\infty t^{p-2} \left[ \int_0^\infty s^{\frac{2s}{p}} d\tau\left(E_s(|T|)\right) \right] dt \]
\[ = 8p \int_0^\infty s \left[ \int_0^{2s} t^{p-2} dt \right] d\tau\left(E_s(|T|)\right) \]
\[ = 8 \left(2^{p-1} \frac{p}{p-1}\right) \int_0^\infty s^p d\tau\left(E_s(|T|)\right) \]
\[ = 8 \left(2^{p-1} \frac{p}{p-1}\right) \|T\|_p^p. \]

**Theorem 3.** For \(T \in L^1_{\text{loc}}(M; \tau)\), \(r > 0\) define
\[ L_r T(x) = \frac{1}{\tau\left(E_{\left[x-r, x+r\right]}(|T|)\right)} \tau\left(|T|E_{\left[x-r, x+r\right]}(|T|)\right) \]
(\(let \; \frac{0}{0} = 0\)). Then we have

(i) \( \lim_{r \to 0} L_r T(x) = x\chi_{\sigma(|T|)}(x) \),

(ii) \( |T| \leq MT(|T|) \).
Proof. If $x \in \sigma([|T|])$, then for all $r > 0$, we have $E_{[x-r,x+r]}([|T|]) \neq 0$. So from

$$\tau([|T|]E_{[x-r,x+r]}([|T|])) = \int_{x-r}^{x+r} s \, d\tau(E_s([|T|])).$$

we obtain

$$x - r \leq \frac{1}{\tau(E_{[x-r,x+r]}([|T|]))} \tau([|T|]E_{[x-r,x+r]}([|T|])) \leq x + r.$$

Hence,

$$\lim_{r \to 0} L_rT(x) = x.$$

If $x \notin \sigma([|T|])$, then for enough small $r > 0$, we have $E_{[x-r,x+r]}([|T|]) = 0$, so that

$$\lim_{r \to 0} L_rT(x) = 0.$$

(ii) It follows from

$$([|T|]x,x) = \int_{\sigma([|T|])} t \, d(E_t([|T|])x,x)$$

$$\leq \lim_{r \to 0} \inf \int_{\sigma([|T|])} L_rT(t) \, d(E_t([|T|])x,x)$$

$$\leq \int_{\sigma([|T|])} MT(t) \, d(E_t([|T|])x,x)$$

$$= (MT([|T|])x,x), \quad \forall x \in D(T). \quad \Box$$

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References