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Modular relations for the nonic analogues of the Rogers–Ramanujan functions with applications to partitions

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Abstract

We define the nonic Rogers–Ramanujan-type functions $D(q)$, $E(q)$ and $F(q)$ and establish several modular relations involving these functions, which are analogous to Ramanujan's well known forty identities for the Rogers–Ramanujan functions. We also extract partition theoretic results from some of these relations.

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1. Introduction

Throughout the paper, we assume $|q| < 1$ and for positive integers n, we use the standard notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}\left(1 - aq^j\right) \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}\left(1 - aq^n\right).$$

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The famous Rogers–Ramanujan identities ([15,19], [16, pp. 214–215]) are

$$G(q):=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}, \tag{1.1}$$

and

$$H(q):=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{1.2}$$

$G(q)$ and $H(q)$ are known as the Rogers–Ramanujan functions. S. Ramanujan [18] found forty modular relations for $G(q)$ and $H(q)$, which are called Ramanujan's forty identities. In 1921, H.B.C. Darling [9] proved one of the identities in the Proceedings of London Mathematical Society. In the same issue of the journal, L.J. Rogers [21] established 10 of the 40 identities including the one proved by Darling. In 1933, G.N. Watson [24] proved 8 of the 40 identities, 2 of which had been previously established by Rogers. In 1977, D.M. Bressoud [7], in his doctoral thesis, proved 15 more from the list of 40. In 1989, A.F.J. Biagioli [5] proved 8 of the remaining 9 identities by invoking the theory of modular forms. Recently, B.C. Berndt et al. [4] have found proofs of 35 of the 40 identities in the spirit of Ramanujan's mathematics. For each of the remaining 5 identities, they also offered heuristic arguments showing that both sides of the identity have the same asymptotic expansions as $q\to 1^-$.

Two identities analogous to the Rogers–Ramanujan identities are the so-called Göllnitz–Gordon identities [10,11], given by

$$S(q):=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2}=\frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty} \tag{1.3}$$

and

$$T(q):=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2+2n}=\frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}. \tag{1.4}$$

$S(q)$ and $T(q)$ are known as the Göllnitz–Gordon functions. Motivated by the similarity between the Rogers–Ramanujan and the Göllnitz–Gordon functions, S.S. Huang [14] and S.L. Chen and Huang [8] found 21 modular relations involving only the Göllnitz–Gordon functions, 9 relations involving both the Rogers–Ramanujan and Göllnitz–Gordon functions, and one new relation for the Rogers–Ramanujan functions. They used the methods of Rogers [21], Watson [24] and Bressoud [7] to derive the relations. They also extracted partition theoretic results from some of their relations. N.D. Baruah et al. [2] also found new proofs for the relations which involve only the Göllnitz–Gordon functions by using Schröter's formulas and some theta function identities found in Ramanujan's notebooks [17]. In the process, they also found some new relations.

In [12] and [13], H. Hahn defined the septic analogues of the Rogers–Ramanujan functions as

$$A(q):=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}}=\frac{(q^7;q^7)_\infty(q^3;q^7)_\infty(q^4;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{1.5}$$

$$B(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty(q^2;q^7)_\infty(q^5;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{1.6}$$

and

$$C(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \frac{(q^7;q^7)_\infty(q;q^7)_\infty(q^6;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{1.7}$$

where the later equalities are due to Rogers [19,20]. She found several modular relations involving only $A(q)$, $B(q)$, and $C(q)$ as well as relations that are connected with the Rogers–Ramanujan and Göllnitz–Gordon functions.

Now, we define the following nonic analogues of the Rogers–Ramanujan functions

$$D(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n}q^{3n^2}}{(q^3;q^3)_n(q^3;q^3)_{2n}} = \frac{(q^4;q^9)_\infty(q^5;q^9)_\infty(q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{1.8}$$

$$E(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n}(1-q^{3n+2})q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} = \frac{(q^2;q^9)_\infty(q^7;q^9)_\infty(q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{1.9}$$

$$F(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n+1}q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} = \frac{(q;q^9)_\infty(q^8;q^9)_\infty(q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{1.10}$$

where the later equalities are due to W.N. Bailey [1, p. 422, Eqs. (1.6), (1.8) and (1.7)]. It is worthwhile to mention that Bailey used non-standard notation in the paper where these identities first appeared. All three of these identities appear in the list of L.J. Slater [23, p. 156] as Eqs. (42), (41), and (40) in that order. However, all three contain misprints. These misprints are corrected as given in (1.8)–(1.10) by A.V. Sills [22]. The main purpose of this paper is to establish several modular relations involving $D(q)$, $E(q)$, and $F(q)$, which are analogues of Ramanujan's forty identities. We also establish several other modular relations involving quotients of $D(q)$, $E(q)$ and $F(q)$. Some of these are connected with the Rogers–Ramanujan functions, Göllnitz–Gordon functions and septic Rogers–Ramanujan-type functions. Furthermore, by the notion of colored partitions, we are able to extract partition theoretic results arising from some of our relations.

2. Definitions and preliminary results

In this section, we present some basic definitions and preliminary results on Ramanujan's theta functions. Ramanujan's general theta function is

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab| < 1. \tag{2.1}$$

In the following four lemmas, we state some basic identities satisfied by $f(a,b)$.

Lemma 2.1. *(See [3, p. 34, Entry 18(iv)].) If n is an integer, then*

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2} f\big(a(ab)^n, b(ab)^{-n}\big). \tag{2.2}$$

Lemma 2.2. *(See [3, p. 45, Entry 29].) If* $ab = cd$*, then*

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc), \tag{2.3}$$

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, ac^2d\right)f\left(\frac{b}{d}, acd^2\right). \tag{2.4}$$

Lemma 2.3. *(See [3, p. 46, Entry 30(v)].) We have*

$$f(a,b)f(-a,-b) = f\left(-a^2,-b^2\right)\phi(-ab), \tag{2.5}$$

where ϕ *is defined in* (2.8) *below.*

Lemma 2.4. *(See [3, p. 48, Entry 31 with* $k = 2$*].) We have*

$$f(a,b) = f\left(a^3b, ab^3\right) + af\left(\frac{b}{a}, a^5b^3\right). \tag{2.6}$$

Jacobi's famous triple product identity can be expressed in the following form.

Lemma 2.5. *(See [3, p. 35, Entry 19].) We have*

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.7}$$

In the next lemma, we state three special cases of $f(a,b)$.

Lemma 2.6. *(See [3, p. 36, Entry 22].) If* $|q| < 1$*, then*

$$\phi(q) := f(q,q) = \sum_{n=0}^{\infty} q^{n^2} = \left(-q;q^2\right)_\infty^2\left(q^2;q^2\right)_\infty, \tag{2.8}$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.9}$$

and

$$f(-q) := f\left(-q,-q^2\right) = \sum_{n=0}^{\infty}(-1)^n q^{n(3n-1)/2} + \sum_{n=1}^{\infty}(-1)^n q^{n(3n+1)/2} = (q;q)_\infty. \tag{2.10}$$

The product representations in (2.8)–(2.10) arise from (2.7). Also, note that if $q = e^{\pi i\tau}$, then $\phi(q) = \vartheta_3(0,\tau)$, where $\vartheta_3(z,\tau)$ denotes the classical theta-function in its standard notation [25, p. 464]. Again, if $q = e^{2\pi i\tau}$, then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind eta-function. The last equality in (2.10) is a statement of Euler's famous pentagonal number theorem.

Invoking (2.7) and (2.10) in (1.8)–(1.10), we immediately arrive at the following result.

Lemma 2.7. *We have*

$$D(q)=\frac{f(-q^4,-q^5)}{f(-q^3)}, \tag{2.11}$$

$$E(q)=\frac{f(-q^2,-q^7)}{f(-q^3)}, \tag{2.12}$$

$$F(q)=\frac{f(-q,-q^8)}{f(-q^3)}. \tag{2.13}$$

Lemma 2.8. *(See [3, pp. 39–40, Entries 24–25].) We have*

$$\chi(q)=\frac{f(q)}{f(-q^2)}=\sqrt[3]{\frac{\phi(q)}{\psi(-q)}}=\frac{\phi(q)}{f(q)}=\frac{f(-q^2)}{\psi(-q)}, \tag{2.14}$$

$$\phi(q)\phi(-q)=\phi^2(-q^2), \tag{2.15}$$

where $\chi(q):=(-q;q^2)_\infty$.

The following lemma is a consequence of (2.7) and the above lemma.

Lemma 2.9. *We have*

$$\phi(q)=\frac{f_2^5}{f_1^2f_4^2},\quad \psi(q)=\frac{f_2^2}{f_1}, \tag{2.16}$$

$$\phi(-q)=\frac{f_1^2}{f_2},\quad \psi(-q)=\frac{f_1f_4}{f_2},\quad f(q)=\frac{f_2^3}{f_1f_4}\quad \text{and}\quad \chi(q)=\frac{f_2^2}{f_1f_4}, \tag{2.17}$$

where $f_n:=f(-q^n)$*, and this notation will be used throughout the sequel.*

Lemma 2.10. *(See [3, p. 49, Corollary (ii)].) We have*

$$\psi(q)=f(q^3,q^6)+q\psi(q^9). \tag{2.18}$$

Lemma 2.11. *(See [3, p. 51, Example (v)].) We have*

$$f(q,q^5)=\chi(q)\psi(-q^3). \tag{2.19}$$

Lemma 2.12. *(See [3, p. 350, Eq. (2.3)].) We have*

$$f(q,q^2)=\phi(-q^3)/\chi(-q). \tag{2.20}$$

3. Main results

In this section, we present the modular relations for the functions $D(q)$, $E(q)$, and $F(q)$ as well as relations of these three functions with the other Rogers–Ramanujan-type functions. Proofs of these relations will be given in Sections 4–6. It is worthwhile to note that by replacing q by $-q$ in each of the following relations one can get more relations. For simplicity, we define, for positive integers n, $D_n := D(q^n)$, $E_n := E(q^n)$, $F_n := F(q^n)$.

The identities (3.1)–(3.23) involve $D(q)$, $E(q)$, and $F(q)$.

$$D_1^2E_1 + qE_1^2F_1 - qD_1F_1^2 = 1, \tag{3.1}$$

$$D_1^2F_1 - E_1^2D_1 + qF_1^2E_1 = 0, \tag{3.2}$$

$$D_3 - qE_3 - q^2F_3 = \frac{f_1}{f_9}, \tag{3.3}$$

$$D_6E_3F_3 + qE_6D_3F_3 + q^2F_6D_3E_3 = \frac{f_2f_3^3f_{27}f_{54}}{f_1f_6f_9^3f_{18}}, \tag{3.4}$$

$$D_5D_4 + q^3E_5E_4 + q^6F_5F_4 = \frac{f_2^2f_{10}^2}{f_1f_{12}f_{15}f_{20}} - q, \tag{3.5}$$

$$D_6D_3 + q^3E_6E_3 + q^6F_6F_3 = \frac{f_2^2f_9}{f_1f_{18}^2} - q, \tag{3.6}$$

$$D_{20}E_1 - q^7E_{20}F_1 + q^{13}F_{20}D_1 = \frac{f_2^2f_{10}^2}{f_3f_4f_5f_{60}} + q^2, \tag{3.7}$$

$$D_2E_1 - qE_2F_1 + qF_2D_1 = 1, \tag{3.8}$$

$$D_5F_1 + qE_5D_1 - q^3F_5E_1 = 1, \tag{3.9}$$

$$D_1D_8 + q^3E_1E_8 + q^6F_1F_8 = \frac{f_2^2f_4^2}{f_1f_3f_8f_{24}} - q, \tag{3.10}$$

$$D_{11}E_1 - q^4E_{11}F_1 + q^7F_{11}D_1 = \frac{f_1f_{11}}{f_3f_{33}} + q, \tag{3.11}$$

$$D_2D_7 + q^3E_2E_7 + q^6F_2F_7 = \frac{f_2^2f_7^2}{f_1f_6f_{14}f_{21}} - q, \tag{3.12}$$

$$D_{14}F_1 + q^4D_1E_{14} - q^9E_1F_{14} = \frac{f_1^2f_{14}^2}{f_2f_3f_7f_{42}} + q, \tag{3.13}$$

$$D_{23}F_1 + q^7E_{23}D_1 - q^{15}F_{23}E_1 = \frac{f_1f_{23}}{f_3f_{69}} + q^2, \tag{3.14}$$

$$D_{32}F_1 + q^{10}E_{32}D_1 - q^{21}F_{32}E_1 = \frac{f_1f_4f_8f_{32}}{f_2f_3f_{16}f_{96}} + q^2, \tag{3.15}$$

$$D_1D_{35} + q^{12}E_1E_{35} + q^{24}F_1F_{35} = \frac{f_5f_7}{f_3f_{105}} - q^4, \tag{3.16}$$

$$q^5 D_2 E_{19} + D_{19} F_2 - q^{12} F_{19} E_2 = \frac{f_1 f_{38}}{f_6 f_{57}} + q, \tag{3.17}$$

$$D_{38} E_1 - q^{13} E_{38} F_1 + q^{25} F_{38} D_1 = \frac{f_2 f_{19}}{f_3 f_{114}} + q^4, \tag{3.18}$$

$$D_1 D_{44} + q^{15} E_1 E_{44} + q^{30} F_1 F_{44} = \frac{f_4 f_{11}}{f_3 f_{132}} - q^5, \tag{3.19}$$

$$D_{56} E_1 - q^{19} E_{56} F_1 + q^{37} F_{56} D_1 = \frac{f_2 f_7 f_8 f_{28}}{f_3 f_4 f_{14} f_{168}} + q^6, \tag{3.20}$$

$$D_{24} F_3 + q^6 E_{24} D_3 - q^{15} F_{24} E_3 = \frac{f_1 f_4 f_{18}}{f_2 f_9 f_{36}} + q, \tag{3.21}$$

$$D_1 D_{80} + q^{27} E_1 E_{80} + q^{54} F_{80} F_1 = \frac{f_4 f_5 f_{16} f_{20}}{f_3 f_8 f_{10} f_{240}} - q^9, \tag{3.22}$$

$$D_{1955} E_1 - q^{652} E_{1955} F_1 + q^{1303} F_{1955} D_1 = q^{217}. \tag{3.23}$$

The identities (3.24)–(3.32) involve quotients of the functions $D(q)$, $E(q)$, and $F(q)$.

$$\frac{D_3 - qE_3 - q^2 F_3}{D_9 - q^3 E_9 - q^6 F_9} = \frac{f_1 f_{27}}{f_3 f_9}, \tag{3.24}$$

$$\frac{D_{11} E_1 - q - q^4 E_{11} F_1 + q^7 F_{11} D_1}{D_{33} - q^{11} E_{33} - q^{22} F_{33}} = \frac{f_1 f_{99}}{f_3 f_{33}}, \tag{3.25}$$

$$\frac{D_1 D_{35} + q^4 + q^{12} E_{35} E_1 + q^{24} F_1 F_{35}}{D_{21} - q^7 E_{21} - q^{14} F_{21}} = \frac{f_5 f_{63}}{f_3 f_{105}}, \tag{3.26}$$

$$\frac{D_{23} F_1 - q^2 + q^7 E_{23} D_1 - q^{15} F_{23} E_1}{D_{69} - q^{23} E_{69} - q^{46} F_{69}} = \frac{f_1 f_{207}}{f_3 f_{69}}, \tag{3.27}$$

$$\frac{D_2 D_{25} + q^3 + q^9 E_2 E_{25} + q^{18} F_2 F_{25}}{D_{50} F_1 - q^5 + q^{16} E_{50} D_1 - q^{33} F_{50} E_1} = \frac{f_2 f_3 f_{25} f_{150}}{f_1 f_6 f_{50} f_{75}}, \tag{3.28}$$

$$\frac{D_{73} F_2 - q^7 + q^{23} E_{73} D_2 - q^{48} F_{73} E_2}{D_{146} E_1 - q^{16} - q^{49} E_{146} F_1 + q^{97} F_{146} D_1} = \frac{f_3 f_{438}}{f_6 f_{219}}, \tag{3.29}$$

$$\frac{D_{49} F_8 - q + q^{11} E_{49} D_8 - q^{30} F_{49} E_8}{D_{392} F_1 - q^{43} + q^{130} E_{392} D_1 - q^{261} F_{392} E_1} = \frac{f_3 f_{1176}}{f_{24} f_{147}}, \tag{3.30}$$

$$\frac{D_{68} F_1 - q^7 + q^{22} E_{68} D_1 - q^{45} F_{68} E_1}{D_{17} E_4 - q - q^7 E_{17} F_4 + q^{10} F_{17} D_4} = \frac{f_{12} f_{51}}{f_3 f_{204}}, \tag{3.31}$$

$$\frac{D_1 D_{260} + q^{29} + q^{87} E_1 E_{260} + q^{174} F_1 F_{260}}{D_{65} F_4 - q^5 + q^{19} E_{65} D_4 - q^{42} F_{65} E_4} = \frac{f_{12} f_{195}}{f_3 f_{780}}. \tag{3.32}$$

The following identities are relations involving some combinations of $D(q)$, $E(q)$ and $F(q)$ with the Rogers–Ramanujan functions $G(q)$ and $H(q)$. Here, for positive integers n, we define $G_n := G(q^n)$ and $H_n := H(q^n)$.

$$\frac{D_9 - q^3E_9 - q^6E_9}{G_9G_1 + q^2H_9H_1} = \frac{f_1f_9}{f_3f_{27}}, \tag{3.33}$$

$$\frac{D_{25}D_2 + q^3 + q^9E_{25}E_2 + q^{18}F_{25}F_2}{G_5G_{10} + q^3H_5H_{10}} = \frac{f_5f_{10}}{f_6f_{75}}, \tag{3.34}$$

$$\frac{D_3D_{33} + q^4 + q^{12}E_3E_{33} + q^{24}F_3F_{33}}{G_9G_{11} + q^4H_9H_{11}} = \frac{f_{11}}{f_{99}}, \tag{3.35}$$

$$\frac{D_3D_{42} + q^5 + q^{15}E_3E_{42} + q^{30}F_3F_{42}}{G_{18}G_7 + q^5H_{18}H_7} = \frac{f_{18}f_7}{f_9f_{126}}, \tag{3.36}$$

$$\frac{D_1D_{26} + q^3 + q^9E_1E_{26} + q^{18}F_1F_{26}}{G_{13}G_2 + q^3H_{13}H_2} = \frac{f_2f_{13}}{f_3f_{78}}, \tag{3.37}$$

$$\frac{D_{13}E_2 - q - q^5E_{13}F_2 + q^8F_{13}D_2}{G_{26}H_1 - q^5G_1H_{26}} = \frac{f_1f_{26}}{f_6f_{39}}, \tag{3.38}$$

$$\frac{D_{29}E_1 - q^3 - q^{10}E_{29}F_1 + q^{19}F_{29}D_1}{G_{29}G_1 + q^6H_{29}H_1} = \frac{f_1f_{29}}{f_3f_{87}}, \tag{3.39}$$

$$\frac{D_{74}E_1 - q^8 - q^{25}E_{74}F_1 + q^{49}F_{74}D_1}{G_{37}H_2 - q^7G_2H_{37}} = \frac{f_3f_{222}}{f_2f_{37}}, \tag{3.40}$$

$$\frac{D_1D_{116} + q^{13} + q^{39}E_1E_{116} + q^{78}F_1F_{116}}{G_{29}H_4 - q^5G_4H_{29}} = \frac{f_8f_{58}}{f_3f_{348}}, \tag{3.41}$$

$$\frac{D_1D_{125} + q^{14} + q^{42}E_1E_{125} + q^{84}F_1F_{125}}{G_{25}H_5 - q^4G_5H_{25}} = \frac{f_5f_{25}}{f_3f_{375}}. \tag{3.42}$$

The following identities are relations involving some combinations of $D(q)$, $E(q)$, and $F(q)$ with the Göllnitz–Gordon functions $S(q)$ and $T(q)$. For simplicity, for positive integers n, we define $S_n := S(q^n)$ and $T_n := T(q^n)$.

$$\frac{D_{15} - q^5E_{15} - q^{10}F_{15}}{S_5S_1 + q^3T_5T_1} = \frac{f_1f_5f_{20}}{f_2f_{10}f_{45}}, \tag{3.43}$$

$$\frac{D_{60} - q^{20}E_{60} - q^{40}F_{60}}{S_5T_1 - q^2T_5S_1} = \frac{f_4f_5f_{20}}{f_2f_{10}f_{180}}, \tag{3.44}$$

$$\frac{D_{68}F_1 - q^7 + q^{22}E_{68}D_1 - q^{45}F_{68}E_1}{S_{17}T_1 - q^8T_{17}S_1} = \frac{f_1f_4f_{17}f_{68}}{f_2f_3f_{34}f_{204}}, \tag{3.45}$$

$$\frac{D_{128}E_1 - q^{14} - q^{43}E_{128}F_1 + q^{85}F_{128}D_1}{S_{16}T_2 - q^7S_2T_{16}} = \frac{f_2f_8f_{16}f_{64}}{f_3f_4f_{32}f_{384}}, \tag{3.46}$$

$$\frac{D_{60}F_3 - q^5 + q^{18}E_{60}D_3 - q^{39}F_{60}E_3}{S_{45}S_1 + q^{23}T_{45}T_1} = \frac{f_1f_4f_{45}}{f_2f_9f_{90}}, \tag{3.47}$$

$$\frac{\{S_{45}S_1 + q^{23}T_{45}T_1\}\{S_{45}T_1 - q^{22}T_{45}S_1\}}{D_6D_{30} + q^4 + q^{12}E_6E_{30} + q^{24}F_6F_{30}} = \frac{f_2f_{18}f_{90}^2}{f_1f_4f_{45}f_{180}}, \tag{3.48}$$

$$\frac{D_3D_{60}+q^7+q^{21}E_3E_{60}+q^{42}F_3F_{60}}{S_9S_5+q^7T_9T_5}=\frac{f_5f_{20}f_{36}}{f_{10}f_{18}f_{180}}, \tag{3.49}$$

$$\frac{D_1D_{224}+q^{25}+q^{75}E_1E_{224}+q^{150}F_1F_{224}}{S_{14}T_4-q^5T_{14}S_4}=\frac{f_4f_{14}f_{16}f_{56}}{f_3f_8f_{28}f_{672}}, \tag{3.50}$$

$$\frac{D_{96}E_3-q^{10}-q^{33}E_{96}F_3+q^{63}F_{96}D_3}{S_{36}S_2+q^{19}T_{36}T_2}=\frac{f_2f_8f_{36}f_{144}}{f_4f_9f_{72}f_{288}}, \tag{3.51}$$

$$\frac{D_{44}F_7-q+q^{10}E_{44}D_7-q^{27}F_{44}E_7}{S_{77}T_1-q^{38}T_{77}S_1}=\frac{f_1f_4f_{77}f_{308}}{f_2f_{21}f_{132}f_{154}}, \tag{3.52}$$

$$\frac{D_{64}E_5-q^6-q^{23}E_{64}F_5+q^{41}F_{64}D_5}{S_2S_{40}+q^{21}T_2T_{40}}=\frac{f_2f_8f_{40}f_{160}}{f_4f_{15}f_{80}f_{192}}, \tag{3.53}$$

$$\frac{D_{320}F_1-q^{35}-q^{106}E_{320}D_1-q^{213}E_1F_{320}}{S_{10}T_8-q^2S_8T_{10}}=\frac{f_8f_{10}f_{32}f_{40}}{f_3f_{16}f_{20}f_{960}}. \tag{3.54}$$

The following identities are relations involving some combinations of $D(q)$, $E(q)$, and $F(q)$ with the septic analogues $A(q)$, $B(q)$, and $C(q)$. Here also, for positive integers n, we define $A_n := A(q^n)$, $B_n := B(q^n)$ and $C_n := C(q^n)$.

$$\frac{D_9-q^3E_9-q^6F_9}{A_1A_{27}+q^4B_1B_{27}+q^{12}C_1C_{27}}=\frac{f_2f_{54}}{f_9f_{27}}, \tag{3.55}$$

$$\frac{D_{15}-q^5E_{15}-q^{10}F_{15}}{A_1A_{20}+q^3B_1B_{20}+q^9C_1C_{20}}=\frac{f_2f_{40}}{f_4f_{45}}, \tag{3.56}$$

$$\frac{D_{47}E_1-q^5-q^{16}E_{47}F_1+q^{31}F_{47}D_1}{A_{47}B_1-q^7B_{47}C_1-q^{20}C_{47}A_1}=\frac{f_2f_{94}}{f_3f_{141}}, \tag{3.57}$$

$$\frac{D_{59}F_1-q^6+q^{19}E_{59}D_1-q^{39}F_{59}E_1}{A_{59}C_1-q^8B_{59}A_1+q^{25}C_{59}B_1}=\frac{f_2f_{118}}{f_3f_{177}}, \tag{3.58}$$

$$\frac{D_{31}E_2-q^3-q^{11}E_{31}F_2+q^{20}F_{31}D_2}{A_1A_{62}+q^9B_1B_{62}+q^{27}C_1C_{62}}=\frac{f_2f_{124}}{f_6f_{93}}, \tag{3.59}$$

$$\frac{D_1D_{98}+q^{11}+q^{33}E_1E_{98}+q^{66}F_1F_{98}}{A_{14}B_7-q^4B_{14}C_7-q^5C_{14}A_7}=\frac{f_{14}f_{28}}{f_3f_{294}}, \tag{3.60}$$

$$\frac{D_6D_{39}+q^5+q^{15}E_6E_{39}+q^{30}F_6F_{39}}{A_9A_{26}+q^5B_9B_{26}+q^{15}C_9C_{26}}=\frac{f_{52}}{f_{117}}, \tag{3.61}$$

$$\frac{D_1D_{215}+q^{24}+q^{72}E_1E_{215}+q^{144}F_1F_{215}}{A_{43}C_5-q^4B_{43}A_5+q^{17}C_{43}B_5}=\frac{f_{10}f_{86}}{f_3f_{645}}, \tag{3.62}$$

$$\frac{D_2D_{115}+q^{13}+q^{39}E_2E_{115}+q^{78}F_2F_{115}}{A_{46}B_5-q^8B_{46}C_5-q^{19}C_{46}A_5}=\frac{f_{10}f_{92}}{f_6f_{345}} \tag{3.63}$$

$$\frac{D_1D_{188}+q^{21}+q^{63}E_1E_{188}+q^{126}F_1F_{188}}{A_{47}C_4-q^5B_{47}A_4+q^{19}C_{47}B_4}=\frac{f_8f_{94}}{f_3f_{564}}, \tag{3.64}$$

$$\frac{D_{230}F_1 - q^{25} + q^{76}E_{230}D_1 - q^{153}F_{230}E_1}{A_{10}B_{23} - q^8 B_{10}C_{23} - qC_{10}A_{23}} = \frac{f_{10}f_{46}}{f_3 f_{690}}. \tag{3.65}$$

Remark. From (3.3) and (3.33), we readily obtain

$$G_9G_1 + q^2 H_9H_1 = \frac{f_3^2}{f_1 f_9}, \tag{3.66}$$

which is the sixth of Ramanujan's forty identities [4].

4. Proofs of (3.1)–(3.4)

Proof of (3.1). From Entry 2(viii) [3, p. 349], we find that

$$\frac{f(-q^4,-q^5)}{f(-q,-q^8)} + q\frac{f(-q^2,-q^7)}{f(-q^4,-q^5)} = q\frac{f(-q,-q^8)}{f(-q^2-q^7)} + \frac{f^4(-q^3)}{f(-q)f^3(-q^9)}. \tag{4.1}$$

Using (2.11)–(2.13) in (4.1), we obtain

$$D_1^2E_1 + qE_1^2F_1 = qD_1F_1^2 + D_1E_1F_1\frac{f_3^4}{f_1 f_9^3}. \tag{4.2}$$

Again, from Entry 2(vi) [3, p. 349], we note that

$$f\left(-q,-q^8\right)f\left(-q^2-q^7\right)f\left(-q^4,-q^5\right) = \frac{f(-q)f^3(-q^9)}{f(-q^3)}. \tag{4.3}$$

With the aid of (2.11)–(2.13), the above identity can be written as

$$D_1E_1F_1 = \frac{f_1 f_9^3}{f_3^4}. \tag{4.4}$$

Using (4.4) in (4.2) we easily arrive at (3.1). □

Proof of (3.2). From Entry 2(vii) [3, p. 349]

$$\frac{f(-q^4,-q^5)}{f(-q^2,-q^7)} + q\frac{f(-q,-q^8)}{f(-q^4,-q^5)} = \frac{f(-q^2,-q^7)}{f(-q-q^8)}. \tag{4.5}$$

Using (2.11)–(2.13) and (4.4) in (4.5), we obtain (3.2) to complete the proof. □

Proof of (3.3). Replacing q by q^3 in Entry 2(v) [3, p. 349], we obtain

$$f\left(-q^{12},-q^{15}\right) - qf\left(-q^6,-q^{21}\right) - q^2 f\left(-q^3,-q^{24}\right) = f(-q). \tag{4.6}$$

Dividing both sides by $f(-q^9)$ and using (2.11)–(2.13), we complete the proof.

This result can also be obtained from Theorem 5.1 in Section 5 by setting $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = q = b$, $c = 1$, $d = q$, $\alpha = 1$, $\beta = 3$, and $m = 9$. □

Proof of (3.4). Replacing q by q^3 in Entry 2(iv) [3, p. 349] and using (2.18) and (2.20), we find that

$$f(q^{12},q^{15})+qf(q^6,q^{21})+q^2f(q^3,q^{24})=\frac{\phi(-q^3)}{\chi(-q)}. \tag{4.7}$$

Employing (2.5), (2.11)–(2.13), and (2.17), we complete the proof. □

5. Second proof of (3.3) and proofs of (3.5)–(3.7)

To present a second proof of (3.3) and proofs of (3.5)–(3.7), we use a formula of R. Bleckmith, J. Brillhart, and I. Gerst [6, Theorem 2], providing a representation for a product of two theta functions as a sum of m products of pair of theta functions, under certain conditions. This formula is a generalization of formulas of H. Schröter [3, p. 65–72].

Define, for $\epsilon\in\{0,1\}$ and $|ab|<1$,

$$f_\epsilon(a,b)=\sum_{n=-\infty}^{\infty}(-1)^{\epsilon n}(ab)^{n^2/2}(a/b)^{n/2}. \tag{5.1}$$

Theorem 5.1. *Let a, b, c, and d denote positive numbers with $|ab|, |cd|<1$. Suppose that there exist positive integers α, β, and m such that*

$$(ab)^\beta=(cd)^{\alpha(m-\alpha\beta)}. \tag{5.2}$$

Let $\epsilon_1,\epsilon_2\in\{0,1\}$, and define $\delta_1,\delta_2\in\{0,1\}$ by

$$\delta_1\equiv\epsilon_1-\alpha\epsilon_2\ (mod\ 2)\quad\text{and}\quad\delta_2\equiv\beta\epsilon_1+p\epsilon_2\ (mod\ 2), \tag{5.3}$$

respectively, where $p=m-\alpha\beta$. Then if R denotes any complete residue system modulo m,

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d)=\sum_{r\in R}(-1)^{\epsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2}f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^\alpha},\frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^\alpha}\right)$$
$$\times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^p},\frac{(ab)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^p}\right). \tag{5.4}$$

Second proof of (3.3). Applying Theorem 5.1 with the parameters $\epsilon_1=1, \epsilon_2=0, a=1, b=q^8$, $c=q$, $d=q^3$, $\alpha=2$, $\beta=3$, and $m=9$, we find that

$$\begin{aligned}&f(-q^{10},-q^{14})\{f(-q^{69},-q^{39})-qf(-q^{33},-q^{75})-q^{11}f(-q^3,-q^{105})\}\\&\quad+qf(-q^2,-q^{22})\{f(-q^{57},-q^{51})-q^5f(-q^{21},-q^{87})-q^7f(-q^{15},-q^{93})\}\\&\quad-\psi(-q^6)\{f(-q^{45},-q^{63})-q^9f(-q^9,-q^{99})-q^3f(-q^{27},-q^{81})\}=0,\end{aligned} \tag{5.5}$$

where we also used (2.2).

Again, applying Theorem 5.1 with $\epsilon_1=1$, $\epsilon_2=0$, $a=q^4$, $b=q^4$, $c=q$, $d=q^3$, $\alpha=2$, $\beta=3$, and $m=9$, we obtain

$$\begin{aligned}\psi(q)\phi(-q^4) &= f(-q^{10},-q^{14})\{f(-q^{57},-q^{51})-q^5f(-q^{21},-q^{87})-q^7f(-q^{93},-q^{15})\}\\&\quad+q^3f(-q^2,-q^{22})\{f(-q^{69},-q^{39})-q^{11}f(-q^3,-q^{105})-qf(-q^{33},-q^{75})\}\\&\quad+q\psi(-q^6)\{f(-q^{45},-q^{63})-q^9f(-q^9,-q^{99})-q^3f(-q^{27},-q^{81})\}.\end{aligned}\tag{5.6}$$

Multiplying (5.5) by q and adding with (5.6), we deduce that

$$\begin{aligned}\psi(q)\phi(-q^4) &= qf(-q^{10},-q^{14})\{f(-q^{69},-q^{39})-q^6f(-q^{15},-q^{93})\}-f(-q^{10},-q^{14})\\&\quad\times\left[q^2\{f(-q^{33},-q^{75})+q^3f(-q^{21},-q^{87})\}-\{f(-q^{51},-q^{57})\right.\\&\quad\left.-q^{12}f(-q^3,-q^{105})\}\right]+q^2f(-q^2,-q^{22})\{f(-q^{69},-q^{39})\\&\quad-q^6f(-q^{15},-q^{93})\}q-q^2f(-q^2,-q^{22})[\{f(-q^{33},-q^{75})\\&\quad+q^3f(-q^{21},-q^{87})\}q^2-\{f(-q^{51},-q^{57})-q^{12}f(-q^3,-q^{105})\}].\end{aligned}\tag{5.7}$$

Employing in turn $a=-q^6$ and $b=q^{21}$; $a=-q^{12}$ and $b=q^{15}$; $a=q^3$ and $b=-q^{24}$ in (2.6), we find that

$$f(-q^6,q^{21}) = f(-q^{39},-q^{69})-q^6f(-q^{15},-q^{39}),\tag{5.8}$$

$$f(-q^{12},q^{15}) = f(-q^{51},-q^{57})-q^{12}f(-q^3,-q^{105}),\tag{5.9}$$

$$f(q^3,-q^{24}) = f(-q^{33},-q^{75})+q^3f(-q^{21},-q^{87}),\tag{5.10}$$

Applying (5.8), (5.9), (5.10) in (5.7), we obtain

$$\begin{aligned}\psi(q^3)\phi(-q^4) &= \{f(-q^{10},-q^{14})+q^2f(-q^2,-q^{22})\}\\&\quad\times\{qf(-q^6,q^{21})-q^2f(q^3,-q^{24})+f(-q^{12},q^{15})\}.\end{aligned}\tag{5.11}$$

Again, putting $a=q^2$, $b=q^4$, $c=q$, $d=q^5$ in (2.3) and (2.4), we find that

$$f(q^2,q^4)f(q,q^5)+f(-q^2,-q^4)f(-q,-q^5)=2f(q^3,q^9)f(q^5,q^7),\tag{5.12}$$

and

$$f(q^2,q^4)f(q,q^5)-f(-q^2,-q^4)f(-q,-q^5)=2f(q^3,q^9)f(q^{-1},q^{13}).\tag{5.13}$$

Employing (2.19), (2.20), (2.9), and (2.2), the above two identities can be written as

$$2qf(q,q^{11})=\frac{\psi^2(-q^3)}{\psi(q^6)\chi(-q)}-f(-q^2)\chi(-q),\tag{5.14}$$

and

$$2f(q^5,q^7)=\frac{\psi^2(-q^3)}{\psi(q^6)\chi(-q)}+f(-q^2)\chi(-q).\tag{5.15}$$

Replacing q by $-q$ in (5.14) and (5.15), and then using (2.15) and (2.14), we find that

$$2f\left(-q^5,-q^7\right)=\frac{\phi(q^3)}{\chi(q)}+f(q) \tag{5.16}$$

and

$$2qf\left(-q,-q^{11}\right)=\frac{\phi(q^3)}{\chi(q)}-f(q). \tag{5.17}$$

Adding (5.16) and (5.17), we obtain

$$f\left(-q^5,-q^7\right)+qf\left(-q,-q^{11}\right)=f(q). \tag{5.18}$$

Replacing q by q^2 in (5.18), and then using the resulting identity in (5.11), we deduce that

$$\psi\left(q^3\right)\phi\left(-q^4\right)=f\left(q^2\right)\left\{qf\left(-q^6,q^{21}\right)-q^2f\left(q^3,-q^{24}\right)+f\left(-q^{12},q^{15}\right)\right\}. \tag{5.19}$$

Dividing both sides by $f(q^9)$, using (2.16), (2.17), (2.11), (2.12), and (2.13), and replacing q by $-q$, we arrive at (3.3) to finish the proof. □

Proof of (3.5). Applying Theorem 5.1 with the parameters $\epsilon_1=1$, $\epsilon_2=0$, $a=q^{10}=b$, $c=q$, $d=1$, $\alpha=5$, $\beta=1$, and $m=9$, we find that

$$\begin{aligned}\phi\left(-q^{10}\right)\psi(q)&=f\left(-q^{20},-q^{25}\right)f\left(-q^{16},-q^{20}\right)+qf_{15}f_{12}+q^3f\left(q^{10},-q^{35}\right)f\left(-q^8,-q^{28}\right)\\&\quad+q^6f\left(-q^5,-q^{40}\right)f\left(-q^4,-q^{32}\right).\end{aligned} \tag{5.20}$$

Using (2.16) and (2.17) in (5.20), we readily arrive at (3.5).

In a similar way, we can obtain the identities (3.7) and (3.6) by setting $m=9$, $\epsilon_1=1$, $\epsilon_2=0$, $a=b=q^2$, $c=q^5$, $d=1$, $\alpha=1$, $\beta=5$ and $m=9$, $\epsilon_1=1$, $\epsilon_2=0$, $a=b=q^9$, $c=1$, $d=q$, $\alpha=6$, $\beta=1$, respectively, in Theorem 5.1. □

6. Proofs of (3.8)–(3.65)

In this section, we present proofs of (3.8)–(3.23) by adopting ideas of Rogers [21] and Bressoud [7]. We replace Bressoud's notation P_n and x by $q^{n/24}f(-q^n)$ and q, respectively. Let $g_\alpha^{(p,n)}$ and $\phi_{\alpha,\beta,m,p}$ be defined as

$$\begin{aligned}g_\alpha^{(p,n)}(q)&:=g_\alpha^{(p,n)}(q)\\&=q^{\alpha(\frac{12n^2-12n+3-p}{24p})}\prod_{r=0}^{\infty}\frac{(1-(q^\alpha)^{pr+(p-2n+1)/2})(1-(q^\alpha)^{pr+(p+2n-1)/2})}{\prod_{k=1}^{p-1}(1-(q^\alpha)^{pr+k})}.\end{aligned} \tag{6.1}$$

For any positive odd integer p, integer n, and natural number α, let

$$\begin{aligned}\phi_{\alpha,\beta,m,p} &:= \phi_{\alpha,\beta,m,p}(q)\\ &= \sum_{n=1}^{p}\sum_{r,s=-\infty}^{\infty}(-1)^{r+s}q^{1/2\{p\alpha(r+m(2n-1)/2p)^2+p\beta(s+(2n-1)/2p)^2\}}, \end{aligned}\tag{6.2}$$

where α, β, and p are natural numbers, and m is an odd positive integer. Then we can obtain immediately the following propositions.

Proposition 6.1. *(See [7, Eqs. (2.12) and (2.13)].) We have*

$$g_\alpha^{(5,1)} = q^{-\alpha/60}G_\alpha, \tag{6.3}$$

$$g_\alpha^{(5,2)} = q^{-11\alpha/60}H_\alpha. \tag{6.4}$$

Proposition 6.2. *(See [12, Eqs. (6.3)–(6.5)].) We have*

$$g_\alpha^{(7,1)} = q^{-\alpha/42}\frac{f(-q^{2\alpha})}{f(-q^\alpha)}A_\alpha, \tag{6.5}$$

$$g_\alpha^{(7,2)} = q^{5\alpha/42}\frac{f(-q^{2\alpha})}{f(-q^\alpha)}B_\alpha, \tag{6.6}$$

$$g_\alpha^{(7,3)} = q^{17\alpha/42}\frac{f(-q^{2\alpha})}{f(-q^\alpha)}C_\alpha. \tag{6.7}$$

Proposition 6.3. *We have*

$$g_\alpha^{(9,1)} = q^{-\alpha/36}\frac{f(-q^{3\alpha})}{f(-q^\alpha)}D_\alpha, \tag{6.8}$$

$$g_\alpha^{(9,2)} = q^{\alpha/12}\frac{f(-q^{3\alpha})}{f(-q^\alpha)}, \tag{6.9}$$

$$g_\alpha^{(9,3)} = q^{11\alpha/36}\frac{f(-q^{3\alpha})}{f(-q^\alpha)}E_\alpha, \tag{6.10}$$

$$g_\alpha^{(9,4)} = q^{23\alpha/36}\frac{f(-q^{3\alpha})}{f(-q^\alpha)}F_\alpha. \tag{6.11}$$

Proof. Setting $p=9$, and $n=1$ in (6.1), we find that

$$\begin{aligned} g_\alpha^{(9,1)} &= q^{-\alpha/36}\prod_{r=0}^{\infty}\frac{(1-(q^\alpha)^{9r+4})(1-(q^\alpha)^{9r+5})}{\prod_{k=1}^{8}(1-(q^\alpha)^{9r+k})}\\ &= \frac{q^{-\alpha/36}}{(q^\alpha;q^{9\alpha})(q^{2\alpha};q^{9\alpha})(q^{3\alpha};q^{9\alpha})(q^{6\alpha};q^{9\alpha})(q^{7\alpha};q^{9\alpha})(q^{8\alpha};q^{9\alpha})}. \end{aligned}\tag{6.12}$$

Employing (2.7), (2.10), and (2.11) in (6.12), we arrive at (6.8).

In a similar fashion, we can prove (6.9)–(6.11). □

Lemma 6.4. *(See [7, Proposition 5.1].) We have*

$$g_\alpha^{(p,n)} = g_\alpha^{(p,n-2p)}, \quad g_\alpha^{(p,n)} = g_\alpha^{(p,-n+1)},$$
$$g_\alpha^{(p,n)} = g_\alpha^{(p,2p-n+1)}, \quad g_\alpha^{(p,n)} = -g_\alpha^{(p,n-p)},$$
$$g_\alpha^{(p,n)} = -g_\alpha^{(p,p-n+1)} \quad \text{and} \quad g_\alpha^{(p,(p+1)/2)} = 0.$$

Theorem 6.5. *(See [7, Proposition 5.4].) For odd $p > 1$,*

$$\phi_{\alpha,\beta,m,p} = 2q^{\alpha+\beta/24} f\left(-q^\alpha\right) f\left(-q^\beta\right)\left(\sum_{n=1}^{(p-1)/2} g_\beta^{(p,n)} g_\alpha^{(p,(2mn-m+1)/2)}\right). \tag{6.13}$$

Lemma 6.6. *(See [14, Lemma 5.1].) We have*

$$\phi_{\alpha,\beta,1,4} = 2q^{(\alpha+\beta)/32}\left\{S_{\beta/2}S_{\alpha/2} + q^{(\alpha+\beta)/4}T_{\beta/2}T_{\alpha/2}\right\}\frac{f_{2\alpha} f_{2\beta} f_{\alpha/2} f_{\beta/2}}{f_\alpha f_\beta}, \tag{6.14}$$

$$\phi_{\alpha,\beta,3,4} = 2q^{(9\alpha+\beta)/32}\left\{S_{\beta/2}T_{\alpha/2} - q^{(\beta-\alpha)/4}S_{\alpha/2}T_{\beta/2}\right\}\frac{f_{2\alpha} f_{2\beta} f_{\alpha/2} f_{\beta/2}}{f_\alpha f_\beta}. \tag{6.15}$$

Lemma 6.7. *(See [7, Lemma 6.5].) We have*

$$\phi_{\alpha,\beta,1,5} = 2q^{(\alpha+\beta)/40} f\left(-q^\alpha\right) f\left(-q^\beta\right)\left\{G_\beta G_\alpha + q^{(\alpha+\beta)/5}H_\beta H_\alpha\right\}, \tag{6.16}$$

$$\phi_{\alpha,\beta,3,5} = 2q^{(9\alpha+\beta)/40} f\left(-q^\alpha\right) f\left(-q^\beta\right)\left\{G_\beta H_\alpha - q^{(-\alpha+\beta)/5}H_\beta G_\alpha\right\}. \tag{6.17}$$

Lemma 6.8. *(See [12, Lemma 6.6].) We have*

$$\begin{aligned}\phi_{\alpha,\beta,1,7} = {} & 2q^{(\alpha+\beta)/56} f\left(-q^{2\alpha}\right) f\left(-q^{2\beta}\right)\\ & \times\left\{A_\beta A_\alpha + q^{(\alpha+\beta)/7}B_\beta B_\alpha + q^{(3\alpha+3\beta)/7}C_\beta C_\alpha\right\},\end{aligned} \tag{6.18}$$

$$\begin{aligned}\phi_{\alpha,\beta,3,7} = {} & 2q^{(9\alpha+\beta)/56} f\left(-q^{2\alpha}\right) f\left(-q^{2\beta}\right)\\ & \times\left\{A_\beta B_\alpha - q^{(2\alpha+\beta)/7}B_\beta C_\alpha - q^{(-\alpha+3\beta)/7}C_\beta A_\alpha\right\},\end{aligned} \tag{6.19}$$

$$\begin{aligned}\phi_{\alpha,\beta,5,7} = {} & 2q^{(25\alpha+\beta)/56} f\left(-q^{2\alpha}\right) f\left(-q^{2\beta}\right)\\ & \times\left\{A_\beta C_\alpha - q^{(-3\alpha+\beta)/7}B_\beta A_\alpha + q^{(-2\alpha+3\beta)/7}C_\beta B_\alpha\right\}.\end{aligned} \tag{6.20}$$

Lemma 6.9. *We have*

$$\begin{aligned}\phi_{\alpha,\beta,1,9} = {} & 2q^{(\alpha+\beta)/72} f\left(-q^{3\alpha}\right) f\left(-q^{3\beta}\right)\left\{D_\alpha D_\beta + q^{(\alpha+\beta)/9} + q^{(\alpha+\beta)/3}E_\alpha E_\beta\right.\\ & \left. + q^{2(\alpha+\beta)/3}F_\alpha F_\beta\right\},\end{aligned} \tag{6.21}$$

$$\phi_{\alpha,\beta,3,9} = 2q^{(9\alpha+\beta)/72} f\left(-q^{3\alpha}\right) f\left(-q^{3\beta}\right)\left\{D_\beta - q^{\beta/3}E_\beta - q^{2\beta/3}F_\beta\right\}, \tag{6.22}$$

$$\begin{aligned}\phi_{\alpha,\beta,5,9} = {} & 2q^{(25\alpha+\beta)/72} f\left(-q^{3\alpha}\right) f\left(-q^{3\beta}\right)\left\{D_\beta E_\alpha - q^{(\beta-2\alpha)/9} - q^{(\alpha+\beta)/3}E_\beta F_\alpha\right.\\ & \left. + q^{(2\beta-\alpha)/3}F_\beta D_\alpha\right\},\end{aligned} \tag{6.23}$$

$$\phi_{\alpha,\beta,7,9} = 2q^{(49\alpha+\beta)/72} f\left(-q^{3\alpha}\right) f\left(-q^{3\beta}\right)\left\{D_\beta F_\alpha - q^{(\beta-5\alpha)/9} + q^{(\beta-2\alpha)/3} E_\beta D_\alpha - q^{(2\beta-\alpha)/3} F_\beta E_\alpha\right\}. \tag{6.24}$$

Proof. Applying Theorem 6.5 with $m = 1$ and $p = 9$, we find that

$$\phi_{\alpha,\beta,1,9} = 2q^{(\alpha+\beta)/24} f\left(-q^{\alpha}\right) f\left(-q^{\beta}\right)\left\{g_\beta^{(9,1)} g_\alpha^{(9,1)} + g_\beta^{(9,2)} g_\alpha^{(9,2)} + g_\beta^{(9,3)} g_\alpha^{(9,3)} + g_\beta^{(9,4)} g_\alpha^{(9,4)}\right\}. \tag{6.25}$$

Using (6.8)–(6.11) in (6.25) and then simplifying, we arrive at (6.21). The identities (6.22)–(6.24) can be proved in a similar way by setting $m = 3$, 5, and 7, respectively, and $p = 9$ in Theorem 6.5. □

Corollary 6.10. *(See [7, Corollaries 5.5 and 5.6].) If $\phi_{\alpha,\beta,m,p}$ is defined by* (6.2)*, then*

$$\phi_{\alpha,\beta,m,1} = 0, \tag{6.26}$$

$$\phi_{\alpha,\beta,1,3} = 2q^{(\alpha+\beta)/24} f\left(-q^{\alpha}\right) f\left(-q^{\beta}\right). \tag{6.27}$$

Corollary 6.11. *(See [7, Corollary 5.11].) If α and β are even positive integers, then*

$$\phi_{\alpha,\beta,1,2} = 2q^{(\alpha+\beta)/16} \frac{f(-q^{2\alpha}) f(-q^{2\beta}) f(-q^{\alpha/2}) f(-q^{\beta/2})}{f(-q^{\alpha}) f(-q^{\beta})}. \tag{6.28}$$

Theorem 6.12. *(See [7, Corollary 7.3].) Let α_i, β_i, m_i, p_i, where $i = 1, 2$, be positive integers with m_1, m_2 be odd. Let $\lambda_1 := (\alpha_1 m_1^2 + \beta_1)/p_1$ and $\lambda_2 := (\alpha_2 m_2^2 + \beta_2)/p_2$. If the conditions*

$$\lambda_1 = \lambda_2, \quad \alpha_1\beta_1 = \alpha_2\beta_2, \quad \textit{and} \quad \alpha_1 m_1 = \pm\alpha_2 m_2 \ (\textit{mod}\ \lambda_1)$$

hold, then $\phi_{\alpha_1,\beta_1,m_1,p_1} = \phi_{\alpha_2,\beta_2,m_2,p_2}$.

In the following sequel, let N denote the set of positive integers.

Proposition 6.13. *For $u \in N$, we have*

$$\phi_{u,2u,5,9} = \phi_{2u,u,1,1}. \tag{6.29}$$

Furthermore, the identity (3.8) *holds.*

Proof. By setting $\alpha_1 = u$, $\beta_1 = 2u$, $m_1 = 5$, $p_1 = 9$, $\alpha_2 = 2u$, $\beta = u$, $m_2 = 1$, and $p_2 = 1$, we see that the equality (6.29) holds by Theorem 6.12.

Using (6.26) in (6.29), we obtain

$$\phi_{u,2u,5,9} = 0. \tag{6.30}$$

In particular, setting $u = 1$ in (6.30) and then using (6.23), we obtain (3.8). □

Proposition 6.14. *(See [7, Proposition 8.1].) Let u be an odd integer* $\geqslant 5$, *then*

$$\phi_{1,u-4,u-2,u} = 0. \tag{6.31}$$

Corollary 6.15. *The identity* (3.9) *holds.*

Proof. Setting $u = 9$ and employing (6.24), we readily obtain (3.9).

The above result can also be proved by setting $\epsilon_1 = 1$, $\epsilon_2 = 1$, $a = 1$, $b = q^5$, $c = 1$, $d = q$, $\alpha = 2$, $\beta = 2$, and $m = 9$ in Theorem 5.1. □

Proposition 6.16. *(See [14, Proposition 5.4].) For a positive integer* $u > 1$, *we have*

$$\phi_{1,u-1,1,u} = q^{1/4} f\left(1, q^2\right) f\left(-q^{u-1}, -q^{u-1}\right). \tag{6.32}$$

Corollary 6.17. *The identity* (3.10) *holds.*

Proof. Setting $u = 9$ and using (6.21), we readily obtain (3.10).

This identity (3.10) can also be established by setting $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = q^4 = b$, $c = 1$, $d = q$, $\alpha = 1$, $\beta = 1$, and $m = 9$ in Theorem 5.1. □

Proposition 6.18. *(See [7, Proposition 8.5].) Let u be an odd integer* $\geqslant 7$, *then*

$$\phi_{1,3u-16,u-4,u} = \phi_{1,3u-16,1,3}. \tag{6.33}$$

Corollary 6.19. *The identity* (3.11) *holds.*

Proof. We set $u = 9$ in (6.33) and then use (6.23) and (6.27) to arrive at the desired identity. □

Proposition 6.20. *(See [7, Proposition 8.11].) If u is an odd integer* $\geqslant 3$, *then*

$$\phi_{2,u-2,1,u} = 2q^{1/8} \prod_{n=0}^{\infty} \left(1 + q^{(n+1)}\right)^2 \left(1 - q^{n+1}\right)\left(1 - \left(q^{u-2}\right)^{2n+1}\right)^2 \left(1 - \left(q^{u-2}\right)^{2n+2}\right). \tag{6.34}$$

Corollary 6.21. *The identity* (3.12) *holds.*

Proof. Setting $u = 9$ in (6.34), we find that

$$\phi_{2,7,1,9} = q^{-1/4} \prod_{n=0}^{\infty} \frac{(1 - q^{7(2n+1)})}{(1 - q^{2n+1})} = q^{-1/4} \frac{\chi(-q^7)}{\chi(-q)}. \tag{6.35}$$

Employing (6.21), (2.14), and (2.17) in (6.35), we easily arrive at (3.12). □

This result can also be proved by applying Theorem 5.1 with $m = 9$, $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = b = q^7$, $c = 1$, $d = q$, $\alpha = 2$, and $\beta = 1$.

Proposition 6.22. *(See [7, Proposition 8.8].) Let u be an odd integer* $\geqslant 3$. *Then*

$$\phi_{1,2u-4,u-2,u} = 2q^{(u-2)/8} \prod_{n=0}^{\infty} \left(1+q^{(u-2)(n+1)}\right)^2 \left(1-q^{(u-2)(n+1)}\right)\left(1-q^{2n+1}\right)^2\left(1-q^{2n+2}\right). \tag{6.36}$$

Corollary 6.23. *The identity* (3.13) *holds.*

Proof. Setting $u=9$ in (6.36), we find that

$$\begin{aligned}\phi_{1,14,7,9} &= 2q^{7/8} \prod_{n=0}^{\infty} \left(1+q^{7(n+1)}\right)^2 \left(1-q^{7(n+1)}\right)\left(1-q^{2n+1}\right)^2\left(1-q^{2n+2}\right) \\ &= 2q^{7/8} f_2 f_7 \frac{\chi^2(-q)}{\chi^2(-q^7)}.\end{aligned} \tag{6.37}$$

Invoking (6.24), (2.14), and (2.17) in (6.37), we deduce (3.13). □

This result can also be proved by employing Theorem 5.1 with $m=9$, $\epsilon_1=0$, $\epsilon_2=1$, $a=1$, $b=q^7$, $c=q$, $d=q$, $\alpha=1$, and $\beta=2$.

Proposition 6.24. *(See [7, Proposition 8.3].) Let p be an odd integer* $\geqslant 5$. *Then*

$$\phi_{1,3u-4,u-2,u} = \phi_{1,3u-4,1,3}. \tag{6.38}$$

Corollary 6.25. *The identity* (3.14) *holds.*

Proof. Setting $u=9$ in (6.38) and using (6.24) and (6.27), we easily deduce (3.14). □

Proposition 6.26. *For* $u \in N$, *we have*

$$\phi_{u+14,u,1,2} = \phi_{1,u^2+14u,7,u+7}. \tag{6.39}$$

Furthermore, the identity (3.15) *holds.*

Proof. The equality (6.39) follows from Theorem 6.12 with $\lambda_1=\lambda_2=u+7$. Furthermore, by setting $u=2$, and using (6.24) and (6.28), we readily arrive at (3.15). □

Proposition 6.27. *For* $u \in N$, *we have*

$$\phi_{2u+2,u+4,1,3} = \phi_{2,u^2+5u+4,1,u+3}. \tag{6.40}$$

Furthermore, the identity (3.16) *holds.*

Proof. The equality (6.40) follows from Theorem 6.12 with $\lambda_1=\lambda_2=u+2$. In particular, if we set $u=6$ and use (6.21) and (6.27), we deduce the proffered identity. □

Proposition 6.28. *(See [7, Proposition 8.12].) Let u be an odd integer* $\geqslant 5$*. Then*

$$\phi_{2,3u-8,u-2,u} = \phi_{1,6u-16,1,3}. \tag{6.41}$$

Corollary 6.29. *The identity* (3.17) *holds.*

Proof. Setting $u = 9$ in (6.41), we derive the identity (3.17) with the help of (6.24) and (6.27). □

Proposition 6.30. *(See [7, Proposition 8.13].) Let u be an odd integer* $\geqslant 5$*. Then*

$$\phi_{2,3u-8,1,3} = \phi_{1,6u-16,u-4,u}. \tag{6.42}$$

Corollary 6.31. *The identity* (3.18) *holds.*

Proof. We set $u = 9$ in (6.42), and then use (6.23) and (6.27) to arrive at the desired identity. □

Proposition 6.32. *(See [12, Proposition 6.19].) For* $u \in N$

$$\phi_{2,u^2+3u,1,u+1} = \phi_{2u+6,u,1,3}, \tag{6.43}$$

Corollary 6.33. *The identity* (3.19) *holds.*

Proof. Setting $u = 8$ in (6.43), we obtain the identity (3.19) by using (3.14) and (3.4). □

Proposition 6.34. *(See [12, Proposition 6.15].) For* $u \in N$*, we have*

$$\phi_{1,u^2+10u,5,u+5} = \phi_{u+10,u,1,2}. \tag{6.44}$$

Corollary 6.35. *The identity* (3.20) *holds.*

Proof. Setting $u = 4$ in (6.44) we can easily obtain (3.20) with the aid of (6.23) and (6.28). □

Proposition 6.36. *For* $u \in N$*, we have*

$$\phi_{u+1,6u^2,7,u+7} = \phi_{u,6u(u+1),1,u}. \tag{6.45}$$

Furthermore, the identity (3.21) *holds.*

Proof. The equality (6.45) follows from Theorem 6.12 with $\lambda_1 = \lambda_2 = 6u + 7$. Now, setting $u = 2$ in (6.45) we arrive at (3.21) with the help of (6.23) and (6.28). □

Proposition 6.37. *For* $u \in N$*, we have*

$$\phi_{1,u^2+18u+80,1,u+9} = \phi_{u+8,u+10,1,2}. \tag{6.46}$$

Furthermore, the identity (3.22) *holds.*

Proof. The equality (6.46) follows from Theorem 6.12 with $\lambda_1 = \lambda_2 = u + 9$. In particular, if we set $u = 0$ and use (6.21) and (6.28), then we readily deduce (3.22). □

Proposition 6.38. *(See [12, Proposition 6.26].) For $u \in N$, we have*

$$\phi_{1,16u^3+172u^2+472u+195,2u+1,u+7} = 0. \tag{6.47}$$

Corollary 6.39. *The identity* (3.23) *holds.*

Proof. We set $u = 2$ in (6.47) and then use (6.23) and (6.26) to arrive at the proffered identity. □

Proposition 6.40. *For an odd number u, we have*

$$\phi_{u+1,u^2+4u+4,u+2,(u+2)^2} = \phi_{1,(u+1)(u+2)^2,u+2,(u+2)^2}. \tag{6.48}$$

Furthermore, the identity (3.24) *holds.*

Proof. The equality (6.48) follows from Theorem 6.12 with $\lambda_1 = \lambda_2 = u + 2$. Furthermore, by setting $u = 1$ in (6.48) we readily deduce (3.24) with the help of (6.22). □

Proposition 6.41. *Let $u \geqslant 4$ be even. Then*

$$\phi_{1,3u^2-9,u-3,2u-3} = \phi_{3,u^2-3,u-1,2u-3}. \tag{6.49}$$

Furthermore, the identity (3.25) *holds.*

Proof. The equality (6.49) follows from Theorem 6.12 with $\lambda_1 = \lambda_2 = 2p$. We set $u = 6$ in (6.49) and then employ (6.22) and (6.23) to arrive at (3.25). □

Proposition 6.42. *(See [13, Proposition 3.4.1].) For $u \in N$, we have*

$$\phi_{u+4,2u^2+9u,3,u+3} = \phi_{u,2u^+17u+36,1,u+3}. \tag{6.50}$$

Corollary 6.43. *The identity* (3.26) *holds.*

Proof. Setting $u = 6$, in (6.50) and then employing (6.21) and (6.23), we easily obtain (3.26). □

Proposition 6.44. *(See [7, Proposition 8.16].) Let u be an odd integer $\geqslant 5$. Then*

$$\phi_{1,3u^2-36,|u-6|,u} = \phi_{3,u^2-12,u-2,u}. \tag{6.51}$$

Corollary 6.45. *The identity* (3.27) *holds.*

Proof. We set $u = 9$ in (6.51) and then use (6.22) and (6.24) to arrive at the desired identity. □

Proposition 6.46. *(See [7, Corollary 9.2].) Let u be an odd integer ⩾ 3. Then*

$$\phi_{2,3u-2,1,u}\cdot\phi_{1,6u-4,1,3}=\phi_{2,3u-2,1,3}\phi_{1,6u-4,u-2,u}. \tag{6.52}$$

Corollary 6.47. *The identity* (3.28) *holds.*

Proof. Setting $u=9$ in (6.52) and then using (6.21), (6.24), and (6.27), we easily arrive at (3.28). □

Proposition 6.48. *(See [7, Proposition 8.17].) Let u be an odd integer ⩾ 5. Then*

$$\phi_{1,2u^2-16,u-4,u}=\phi_{2,u^2-8,u-2,u}. \tag{6.53}$$

Corollary 6.49. *The identity* (3.29) *holds.*

Proof. We set $u=9$ in (6.53) to arrive at (3.29) with the aid of (6.23) and (6.24). □

Proposition 6.50. *For an odd positive integer $u>4$, we have*

$$\phi_{u-3,u^2-8u+16,u-4,u-2}=\phi_{1,u^3-11u^2+40u-48,u-4,u-2}. \tag{6.54}$$

Furthermore, the identity (3.30) *holds.*

Proof. Equality (6.54) holds by Theorem 6.12 with $\lambda_1=\lambda_2=(u-4)^2$. Setting $u=11$ in (6.54) and then using (6.24) we complete the proof. □

Proposition 6.51. *For $u\in N$, we have*

$$\phi_{1,(2u+9)u,2u-1,2u+1}=\phi_{u,2u+9,5,9}. \tag{6.55}$$

Furthermore, the identity (3.31) *holds.*

Proof. Equality (6.55) holds by Theorem 6.12 with $\lambda_1=\lambda_2=3u+1$. Setting $u=4$ in (6.55) and then employing (6.23) and (6.24), we readily deduce (3.31). □

Proposition 6.52. *For $u\in N$, we have*

$$\phi_{u,260u,1,9}=\phi_{4u,65u,7,9}. \tag{6.56}$$

Furthermore, the identity (3.32) *holds.*

Proof. Equality (6.56) holds by Theorem 6.12 with $\lambda_1=\lambda_2=29u$. We set $u=1$ in (6.56) and then use (6.21) and (6.24) to arrive at (3.32). □

Proposition 6.53. *For $u\in N$, we have*

$$\phi_{5u,5u+40,3,5u+4}=\phi_{5u,5u+40,1,u+4}. \tag{6.57}$$

Furthermore, the identity (3.33) *holds.*

Proof. Equality (6.57) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = 10$. Now, we set $u = 1$ in (6.57) and then use (6.13), (6.21) and finally replace q^5 by q to arrive at (3.33). □

Proposition 6.54. *For $u \in N$, we have*

$$\phi_{4,u(u+5),1,u+4} = \phi_{u+5,4u,1,5}. \tag{6.58}$$

Furthermore, the identity (3.34) *holds.*

Proof. Equality (6.58) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = u + 1$. Setting $u = 5$, using (6.13), and (6.21), and then replacing q^2 by q, we obtain (3.34). □

Proposition 6.55. *(See [13, Proposition 3.4.11].) For $u \in N$, we have*

$$\phi_{6,u^2+5u,1,u+3} = \phi_{2u+10,3u,1,5}. \tag{6.59}$$

Corollary 6.56. *The identity* (3.35) *holds.*

Proof. We set $u = 6$ in (6.59), use (6.21) and (6.13), and then replace q^2 by q in the resulting identity to deduce (3.35). □

Proposition 6.57. *(See [13, Proposition 3.4.21].) For $u \in N$, we have*

$$\phi_{6,u^2+5u,1,u+2} = \phi_{3u+15,2u,1,5}. \tag{6.60}$$

Corollary 6.58. *The identity* (3.36) *holds.*

Proof. Setting $u = 7$ in (6.60) and using (6.21) and (6.13), and then replacing q^2 by q, we arrive at (3.36). □

Proposition 6.59. *For $u \in N$, we have*

$$\phi_{u,4u+5,1,5} = \phi_{4u^2+5u,1,1,4u+1}. \tag{6.61}$$

Furthermore, the identity (3.37) *holds.*

Proof. Equality (6.61) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = u + 1$. We set $u = 2$ in (6.61) to deduce (3.37) with the aid of (6.13) and (6.21). □

Proof of (3.38). Setting $u = 2$ in (6.55), we obtain

$$\phi_{1,26,3,5} = \phi_{2,13,5,9}. \tag{6.62}$$

Employing (6.17) and (6.23) in (6.62), we deduce (3.38). □

Proposition 6.60. *For $u \in N$, we have*

$$\phi_{u,5u+24,5,5u+4} = \phi_{5u+24,u,1,u+4}. \tag{6.63}$$

Furthermore, the identity (3.39) *holds.*

Proof. Equality (6.63) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = 6$. Setting $u = 1$ in (6.63), we readily obtain (3.39) by means of (6.13) and (6.23). □

Proposition 6.61. *For $u \in N$, we have*

$$\phi_{1,(16u+5)u,2u+1,4u+1} = \phi_{u,16u+5,3,5}. \tag{6.64}$$

Furthermore, the identity (3.40) *holds.*

Proof. Equality (6.64) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = 5u + 1$. Setting $u = 2$ in (6.64), we readily obtain (3.40) by means of (6.17) and (6.23). □

Proposition 6.62. *For $u \in N$, we have*

$$\phi_{1,(6u+5)u,1,2u+1} = \phi_{u,6u+5,3,5}. \tag{6.65}$$

Furthermore, the identity (3.41) *holds.*

Proof. Equality (6.65) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = 3u + 1$. We set $u = 4$ in (6.65) and then use (6.17) and (6.21) to deduce (3.41). □

Proposition 6.63. *For $u \in N$, we have*

$$\phi_{1,(6u-5)u,1,2u-1} = \phi_{u,6u-5,3,5}. \tag{6.66}$$

Furthermore, the identity (3.42) *holds.*

Proof. Equality (6.66) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = 3u - 1$. We set $u = 5$ in (6.66) and then use (6.17) and (6.21) to deduce (3.42). □

Proposition 6.64. *(See [14, Proposition 7.7].) For $u \in N$ and u even, we have*

$$\phi_{4,12u+21,u+1,2u+5} = \phi_{6,8u+14,1,4}. \tag{6.67}$$

Corollary 6.65. *The identity* (3.43) *holds.*

Proof. Setting $u = 2$ in (6.67) and employing (6.22) and (6.14), then replacing q^3 by q in the resulting identity, we complete the proof. □

Proposition 6.66. *(See [14, Proposition 7.5].) For $u \in N$ and u even, we have*

$$\phi_{1,24u+84,u-1,u+5} = \phi_{6,4u+14,3,4}. \tag{6.68}$$

Corollary 6.67. *The identity* (3.44) *holds.*

Proof. Setting $u = 4$ in (6.68) and using (6.22) and (6.15), and then replacing q^3 by q in the resulting identity, we deduce (3.44). □

Proposition 6.68. *(See [14, Proposition 7.3].) For $u \in N$ and u even, we have*

$$\phi_{1,8u+36,u+3,u+5} = \phi_{2,4u+18,3,4}. \tag{6.69}$$

Corollary 6.69. *The identity* (3.45) *holds.*

Proof. Setting $u = 4$ in (6.69) and using (6.24) and (6.15), we readily arrive at (3.45). □

Proposition 6.70. *(See [14, Proposition 7.4].) For $u \in N$ and u even, we have*

$$\phi_{1,16u+64,u+1,u+5} = \phi_{4,4u+16,3,4}. \tag{6.70}$$

Corollary 6.71. *The identity* (3.46) *holds.*

Proof. Setting $u = 4$ in (6.70), we obtain the identity (3.46) by means of (6.23) and (6.15). □

Proposition 6.72. *(See [14, Proposition 7.6].) For $u \in N$ and u even, we have*

$$\phi_{3,8u+28,u+3,u+5} = \phi_{2,12u+42,1,4}. \tag{6.71}$$

Corollary 6.73. *The identity* (3.47) *holds.*

Proof. Setting $u = 4$ in (6.71), we can easily arrive at (3.47) with the aid of (6.24) and (6.14). □

Proposition 6.74. *(See [14, Proposition 8.1].) For $u \in N$ and u even, we have*

$$\phi_{2,12u+18,3,4} \cdot \phi_{3,8u+12,u+1,u+3} = \phi_{6,4u+6,1,u+3} \cdot \phi_{2,12u+18,1,2}. \tag{6.72}$$

Corollary 6.75. *The identity* (3.48) *holds.*

Proof. Setting $u = 6$ in (6.72), we find that

$$\phi_{2,90,3,4} \cdot \phi_{3,60,7,9} = \phi_{6,30,1,9} \cdot \phi_{2,90,1,2}. \tag{6.73}$$

Now, setting $u = 4$ in (6.71) and then using the resulting identity in (6.73), we deduce that

$$\phi_{2,90,3,4} \cdot \phi_{2,90,1,4} = \phi_{6,30,1,9} \phi_{2,90,1,2}, \tag{6.74}$$

Employing (6.21), (6.28), (6.14) and (6.15), we deduce (3.48). □

Proposition 6.76. *For $u \in N$, we have*

$$\phi_{18u,10u,1,4} = \phi_{3u,60u,1,9}. \tag{6.75}$$

Furthermore, the identity (3.49) *holds.*

Proof. Equality (6.75) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = 7u$. Setting $u = 1$ in (6.75), we obtain (3.49) with the help of (6.14) and (6.21). □

Proposition 6.77. *(See [14, Proposition 7.9].) For $u \in N$ and u even, we have*

$$\phi_{1,32u+96,u-3,u+5} = \phi_{8,4u+12,3,4}. \tag{6.76}$$

Corollary 6.78. *The identity* (3.50) *holds.*

Proof. Setting $u = 4$ in (6.76) and then using (6.21) and (6.15), we readily arrive at (3.50). □

Proposition 6.79. *(See [14, Proposition 7.10].) For $u \in N$ and u even, we have*

$$\phi_{3,16u+32,u+1,u+5} = \phi_{4,12u+24,1,4}. \tag{6.77}$$

Corollary 6.80. *The identity* (3.51) *holds.*

Proof. We set $u = 4$ in (6.77) and then use (6.23) and (6.14) to deduce the desired identity. □

Proposition 6.81. *(See [14, Proposition 7.8].) For $u \in N$ and u even, we have*

$$\phi_{7,8u+12,u+3,u+5} = \phi_{2,28u+42,3,4}. \tag{6.78}$$

Corollary 6.82. *The identity* (3.52) *holds.*

Proof. Setting $u = 4$ in (6.78), we obtain (3.52) by invoking (6.24) and (6.15). □

Proposition 6.83. *(See [12, Proposition 6.23].) For $u \in N$, we have*

$$\phi_{u+1,4u^2,5,u+5} = \phi_{u,4u(u+1),1,u}. \tag{6.79}$$

Corollary 6.84. *The identity* (3.53) *holds.*

Proof. Setting $u = 4$ in (6.79), we ca easily deduce (3.53) with the help of (6.23) and (6.14). □

Proposition 6.85. *For $u \in N$, we have*

$$\phi_{16u,20u,3,4} = \phi_{u,320u,7,9}. \tag{6.80}$$

Furthermore, the identity (3.54) *holds.*

Proof. Equality (6.80) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = 41u$. Setting $u = 1$ in (6.80), we obtain (3.54) with the help of (6.14) and (6.24). □

Proposition 6.86. *(See [13, Proposition 3.4.12].) For $u \in N$, we have*

$$\phi_{u+6,u^2+6u,3,u+6} = \phi_{u,(u+6)^2,1,u+4}. \tag{6.81}$$

Corollary 6.87. *The identity* (3.55) *holds.*

Proof. Setting $u = 3$ in (6.81) and then using (6.22) and (6.18), we easily obtain (3.55). □

Proposition 6.88. *(See [13, Proposition 3.4.15].) For $u \in N$, we have*

$$\phi_{u+2,2u^2+3u,3,u+3} = \phi_{u,2u^2+7u+6,1,u+1}. \tag{6.82}$$

Corollary 6.89. *The identity* (3.56) *holds.*

Proof. We set $u = 6$ in (6.82) to obtain (3.56) by means of (6.22) and (6.18). □

Proposition 6.90. *(See [12, Proposition 6.20].) For $u \in N$, we have*

$$\phi_{1,8u+7,2u+3,u+4} = \phi_{1,8u+7,2u+1,u+2}. \tag{6.83}$$

Corollary 6.91. *The identity* (3.57) *holds.*

Proof. Setting $u = 5$ in (6.83), we obtain

$$\phi_{1,47,13,9} = \phi_{1,47,11,7}. \tag{6.84}$$

Using (6.13) and Lemma 6.4 in the above identity, we find that,

$$\begin{aligned} &-g_{47}^{(9,1)} g_1^{(9,3)} + g_{47}^{(9,2)} g_1^{(9,2)} + g_{47}^{(9,3)} g_1^{(9,4)} - g_{47}^{(9,4)} g_1^{(9,1)} \\ &\quad = -g_{47}^{(7,1)} g_1^{(7,2)} + g_{47}^{(7,2)} g_1^{(7,3)} + g_{47}^{(7,3)} g_1^{(7,1)}. \end{aligned} \tag{6.85}$$

Employing (6.5)–(6.11) in (6.85) and then multiplying both sides by q, we obtain (3.57). □

Proposition 6.92. *For $u \in N$ and u odd, we have*

$$\phi_{1,7u+10,u,u+2} = \phi_{1,7u+10,5,7}. \tag{6.86}$$

Furthermore, the identity (3.58) *holds.*

Proof. Equality (6.86) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = u + 5$. Setting $u = 7$, employing (6.20) and (6.24), and then replacing q^2 by q in the resulting identity, we obtain (3.58). □

Proposition 6.93. *(See [13, Proposition 3.4.19].) For $u \in N$, we have*

$$\phi_{u+4,4u^2+15u,5,u+5} = \phi_{u,4u^2+31u+60,1,u+3}. \tag{6.87}$$

Corollary 6.94. *The identity* (3.59) *holds.*

Proof. Setting $u = 4$ in (6.87) and then using (6.23) and (6.18), we deduce (3.59). □

Proposition 6.95. *For $u \in N$, we have*

$$\phi_{21u+154,u,1,u+7} = \phi_{7u,3u+22,3,3u+1}. \tag{6.88}$$

Furthermore, the identity (3.60) *holds.*

Proof. Equality (6.88) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = 22$. Setting $u = 2$, using (6.19) and (6.21), and then replacing q^2 by q, we readily deduce the required identity. □

Proposition 6.96. *(See [13, Proposition 3.4.14].) For $u \in N$, we have*

$$\phi_{2u,3u+30,1,u+6} = \phi_{2u+20,3u,1,u+4}. \tag{6.89}$$

Corollary 6.97. *The identity* (3.61) *holds.*

Proof. We set $u = 3$ in (6.89) and then use (6.21) and (6.18) to deduce (3.61). □

Proposition 6.98. *(See [13, Proposition 3.4.7].) For $u \in N$, we have*

$$\phi_{2,5u^2+23u+24,1,u+2} = \phi_{u+3,10u+16,5,7}. \tag{6.90}$$

Corollary 6.99. *The identity* (3.62) *holds.*

Proof. Setting $u = 7$ in (6.90), we obtain (3.62) with the help of (6.21) and (6.20). □

Proposition 6.100. *(See [13, Proposition 3.4.23].) For $u \in N$, we have*

$$\phi_{u,2u^2+27u+90,1,u+5} = \phi_{u+6,2u^2+15u,3,u+3}. \tag{6.91}$$

Corollary 6.101. *The identity* (3.63) *holds.*

Proof. We set $u = 4$ in (6.91) to obtain (3.63)with the aid of (6.21) and (6.19). □

Proposition 6.102. *For $u \in N$, we have*

$$\phi_{u,188u,1,9} = \phi_{4u,47u,5,7}. \tag{6.92}$$

Furthermore, the identity (3.64) *holds.*

Proof. Equality (6.92) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = 21u$. We set $u = 1$ in (6.92) to deduce (3.64) with the help of (6.20) and (6.21). □

Proposition 6.103. *For $u \in N$, we have*

$$\phi_{u,230u,7,9} = \phi_{23u,10u,3,7}. \tag{6.93}$$

Furthermore, the identity (3.65) *holds.*

Proof. Equality (6.93) holds by Theorem 6.12 with $\lambda_1 = \lambda_2 = 31u$. Setting $u = 1$ in (6.93) and then using (6.19) and (6.24), we obtain (3.65). □

7. Applications to the theory of partitions

The identities (3.1)–(3.65) have partition theoretic interpretations. We demonstrate this by deriving partition theoretic results arising from (3.1)–(3.3), (3.8), and (3.9). In the sequel, for simplicity, we adopt the standard notation

$$(a_1, a_2, \dots, a_n; q)_\infty := \prod_{j=1}^{n} (a_j; q)_\infty$$

and define

$$\left(q^{r\pm}; q^s\right)_\infty := \left(q^r, q^{s-r}; q^s\right)_\infty,$$

where r and s are positive integers and $r < s$.

We also need the notion of colored partitions. A positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called *colored partitions*. For example, if 1 is allowed to have two colors, say r (*red*), and g (*green*), then all colored partitions of 2 are 2, $1_r + 1_r$, $1_g + 1_g$, and $1_r + 1_g$. An important fact is that

$$\frac{1}{(q^u; q^v)_\infty^k}$$

is the generating function for partitions of n, where all partitions are congruent to u (*mod* v) and have k colors.

Theorem 7.1. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2, ± 3 (mod 9) with ± 1 (mod 9) having two colors and ± 3 (mod 9) having three colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 3, ± 4 (mod 9) with ± 3 (mod 9) having three colors and ± 4 (mod 9) having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, ± 4 (mod 9) with ± 2 (mod 9) having two colors and ± 3 (mod 9) having three colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2, ± 4 (mod 9) having two colors each. Then, for any positive integer $n \geqslant 1$, we have*

$$p_1(n) + p_2(n-1) - p_3(n-1) = p_4(n).$$

Proof. The identity (3.1) is equivalent to

$$\left(q^{2\pm};q^9\right)\left(q^{4\pm};q^9\right)^2+q\left(q^{1\pm};q^9\right)\left(q^{2\pm};q^9\right)^2-q\left(q^{1\pm};q^9\right)^2\left(q^{4\pm};q^9\right)=\frac{(q^3;q^3)^3}{(q^9;q^9)^3}. \tag{7.1}$$

Noting that $(q^3;q^3)_\infty=(q^{3\pm};q^9)_\infty(q^9;q^9)_\infty$, we can rewrite (7.1) as

$$\begin{aligned}&\frac{1}{(q^{1\pm};q^9)^2(q^{2\pm};q^9)(q^{3\pm};q^9)^3}+\frac{q}{(q^{1\pm};q^9)(q^{4\pm};q^9)^2(q^{3\pm};q^9)^3}\\&\quad-\frac{q}{(q^{2\pm};q^9)^2(q^{4\pm};q^9)(q^{3\pm};q^9)^3}=\frac{1}{(q^{1\pm,2\pm,4\pm};q^9)^2}.\end{aligned} \tag{7.2}$$

The four quotients of (7.2) represent the generating functions for $p_1(n)$, $p_2(n)$, $p_3(n)$, and $p_4(n)$, respectively. Hence, (7.2) is equivalent to

$$\sum_{n=0}^{\infty}p_1(n)q^n+q\sum_{n=0}^{\infty}p_2(n)q^n-q\sum_{n=0}^{\infty}p_3(n)q^n=\sum_{n=0}^{\infty}p_4(n)q^n,$$

where we set $p_1(0)=p_2(0)=p_3(0)=p_4(0)=1$. Equating coefficients of q^n on both sides yields the desired result. □

Example. It can easily be seen that $p_1(5)=24$, $p_2(4)=6$, $p_3(4)=4$, and $p_4(5)=26$, which verifies the case $n=5$ in Theorem 7.1.

Theorem 7.2. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ±1, ±2 (mod 9) with ±2 (mod 9) having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ±2, ±4 (mod 9) with ±4 (mod 9) having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ±1, ±4 (mod 9) with ±1 (mod 9) having two colors. Then, for any positive integer $n\geqslant1$, we have*

$$p_1(n)+p_2(n-1)=p_3(n).$$

Proof. The identity (3.2) is equivalent to

$$\frac{1}{(q^{1\pm};q^9)(q^{2\pm};q^9)^2}+\frac{q}{(q^{2\pm};q^9)(q^{4\pm};q^9)^2}=\frac{1}{(q^{1\pm};q^9)^2(q^{4\pm};q^9)}. \tag{7.3}$$

Note that the three quotients of (7.3) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, we have

$$\sum_{n=0}^{\infty}p_1(n)q^n+q\sum_{n=0}^{\infty}p_2(n)q^n=\sum_{n=0}^{\infty}p_3(n)q^n,$$

where we set $p_2(0)=0$. Equating coefficients of q^n on both sides yields the desired result. □

Example. Table 1 illustrates the case $n=5$ in Theorem 7.2.

Table 1

$p_1(5)=6$	$p_2(4)=3$	$p_3(5)=9$
2_r+2_r+1	4_r	5
2_r+2_g+1	4_g	$4+1_r$
2_g+2_g+1	2+2	$4+1_g$
$2_r+1+1+1$		$1_r+1_r+1_r+1_r+1_r$
$2_g+1+1+1$		$1_r+1_r+1_r+1_r+1_g$
$1+1+1+1+1$		$1_r+1_r+1_r+1_g+1_g$
		$1_r+1_r+1_g+1_g+1_g$
		$1_r+1_g+1_g+1_g+1_g$
		$1_g+1_g+1_g+1_g+1_g$

Theorem 7.3. *Let $p_1(n)$ denote the number of partitions of n into parts not congruent to* ± 12, 27 (*mod* 27)*. Let $p_2(n)$ denote the number of partitions of n into parts not congruent to* ± 6, 27 (*mod* 27)*. Let $p_3(n)$ denote the number of partitions of n into parts not congruent to* ± 3, 27 (*mod* 9)*. Then, for any positive integer $n \geqslant 2$, we have*

$$p_1(n)=p_2(n-1)+p_3(n-2).$$

Proof. The identity (3.3) is equivalent to

$$\frac{1}{(q^{1\pm,2\pm,\ldots,11\pm,13\pm};q^{27})}-\frac{q}{(q^{1\pm,2\pm,\ldots,5\pm,7\pm,\ldots,13\pm};q^{27})} \\ -\frac{q^2}{(q^{1\pm,2\pm,4\pm,\ldots,13\pm};q^{27})}=1. \tag{7.4}$$

Note that the three quotients of (7.4) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Thus, we have

$$\sum_{n=0}^{\infty}p_1(n)q^n-q\sum_{n=0}^{\infty}p_2(n)q^n-q^2\sum_{n=0}^{\infty}p_3(n)q^n=1,$$

where we set $p_1(0)=p_2(0)=p_3(0)=1$. Equating coefficients of q^n on both sides, we arrive at the desired result. □

Example. We note that $p_1(7)=15$, $p_2(6)=10$, and $p_3(5)=5$, which verifies the case $n=5$ in Theorem 7.3.

Theorem 7.4. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to* ± 1, ± 3, ± 4, ± 5, ± 6 (*mod* 18) *with* ± 6 (*mod* 18) *having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to* ± 1, ± 3, ± 6, ± 7, ± 8 (*mod* 18) *with* ± 6 (*mod* 18) *having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to* ± 2, ± 3, ± 5, ± 6, ± 7 (*mod* 18) *with* ± 6 (*mod* 18) *having two colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to* ± 1, ± 2, ± 4, ± 5, ± 7, ± 8 (*mod* 18)*. Then, for any positive integer $n \geqslant 1$, we have*

$$p_1(n)+p_2(n-1)=p_3(n-1)+p_4(n).$$

Proof. The identity (3.8) can be written as

$$\begin{aligned}&(q^{2\pm};q^9)(q^{8\pm};q^{18})+q(q^{4\pm};q^9)(q^{2\pm};q^{18})-q(q^{1\pm};q^9)(q^{4\pm};q^{18})\\&\quad=\frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}.\end{aligned}\tag{7.5}$$

Expressing all the products in (7.5) to the common base q^{18}, for examples, writing $(q;q^9)_\infty$ as $(q;q^{18})_\infty(q^{10};q^{18})_\infty$ and $(q^3;q^3)_\infty$ as $(q^{3\pm};q^{18})_\infty(q^{6\pm};q^{18})_\infty(q^9;q^{18})_\infty(q^{18};q^{18})_\infty$ and cancelling the common terms, we obtain

$$\begin{aligned}&\frac{1}{(q^{1\pm,3\pm,4\pm5\pm6\pm6\pm};q^{18})}+\frac{q}{(q^{1\pm,3\pm,6\pm6\pm7\pm8\pm};q^{18})}\\&\quad-\frac{q}{(q^{2\pm,3\pm,5\pm,6\pm6\pm7\pm};q^{18})}=\frac{1}{(q^{1\pm,2\pm,4\pm,5\pm7\pm8\pm};q^{18})}.\end{aligned}\tag{7.6}$$

Note that the four quotients of (7.6) represent the generating functions for $p_1(n)$, $p_2(n)$, $p_3(n)$, and $p_4(n)$ respectively. Thus, we have

$$\sum_{n=0}^{\infty}p_1(n)q^n+q\sum_{n=0}^{\infty}p_2(n)q^n-q\sum_{n=0}^{\infty}p_3(n)q^n=\sum_{n=0}^{\infty}p_4(n)q^n,$$

where we set $p_1(0)=p_2(0)=p_3(0)=p_4(0)=1$. Equating coefficients of q^n on both sides, we arrive at the desired result. □

Example. Table 2 illustrates the case $n=7$ in Theorem 7.4.

Table 2

$p_1(7)=8$	$p_2(6)=5$	$p_3(6)=4$	$p_4(7)=9$
6_r+1	6_r	6_r	7
6_g+1	6_g	6_g	$5+2$
$4+3$	$3+3$	$3+3$	$5+1+1$
$5+1+1$	$3+1+1+1$	$2+2+2$	$4+2+1$
$4+1+1+1+1$	$1+1+1+1+1+1$		$4+1+1+1$
$3+3+1$			$2+2+2+1$
$3+1+1+1+1$			$2+2+1+1+1$
$1+1+1+1+1+1+1$			$2+1+1+1+1+1$
			$1+1+1+1+1+1+1$

Theorem 7.5. *Let $p_1(n)$ denote the number of partitions of n into parts not congruent to ±1, ±8, ±10, ±17, ±19, ±20, 45 (mod 45) with ±15 (mod 45) having two colors. Let $p_2(n)$ denote the number of partitions of n into parts not congruent to ±4, ±5, ±10, ±13, ±14, ±22, 45 (mod 45) with ±15 (mod 45) having two colors. Let $p_3(n)$ denote the number of partitions of n into parts not congruent to ±2, ±5, ±7, ±11, ±16, ±20, 45 (mod 45) with ±15 (mod 45) having two colors. Let $p_4(n)$ denote the number of partitions of n into parts not congruent to ±3, ±6, ±10, ±11, ±15, ±21, 45 (mod 45). Then, for any positive integer $n\geqslant3$, we have*

$$p_1(n)+p_2(n-1)=p_3(n-3)+p_4(n).$$

Proof. We express (3.9) in q-products and then proceed as in the proof of Theorem 7.4 to complete the proof. □

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References

[1] W.N. Bailey, Some identities in combinatory analysis, Proc. London Math. Soc. 49 (1947) 421–435.
[2] N.D. Baruah, J. Bora, N. Saikia, Some new proofs of modular relations for the Göllnitz–Gordon functions, Ramanujan J., in press.
[3] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.
[4] B.C. Berndt, G. Choi, Y.S. Choi, H. Hahn, B.P. Yeap, A.J. Yee, H. Yesilyurt, J. Yi, Ramanujan's forty identities for the Rogers–Ramanujan functions, Mem. Amer. Math. Soc., in press.
[5] A.J.F. Biagioli, A proof of some identities of Ramanujan using modular forms, Glasgow Math. J. 31 (1989) 271–295.
[6] R. Blecksmith, J. Brillhart, I. Gerst, A fundamental modular identity and some applications, Math. Comp. 61 (1993) 83–95.
[7] D. Bressoud, Proof and generalization of certain identities conjectured by Ramanujan, Ph.D. Thesis, Temple University, 1977.
[8] S.-L. Chen, S.-S. Huang, New modular relations for the Göllnitz–Gordon functions, J. Number Theory 93 (2002) 58–75.
[9] H.B.C. Darling, Proofs of certain identities and congruences enunciated by S. Ramanujan, Proc. London Math. Soc. (2) 19 (1921) 350–372.
[10] H. Göllnitz, Partitionen mit Differenzenbedingungen, J. Reine Angew. Math. 225 (1967) 154–190.
[11] B. Gordon, Some continued fractions of Rogers–Ramanujan type, Duke Math. J. 32 (1965) 741–748.
[12] H. Hahn, Septic analogues of the Rogers–Ramanujan functions, Acta Arith. 110 (2003) 381–399.
[13] H. Hahn, Eisenstein series, analogues of the Rogers–Ramanujan functions, and partitions, Ph.D. Thesis, University of Illinois at Urbana–Champaign, 2004.
[14] S.-S. Huang, On modular relations for Göllnitz–Gordon functions with application to partitions, J. Number Theory 68 (1998) 178–216.
[15] S. Ramanujan, Proof of certain identities in combinatory analysis, Proc. Cambridge Philos. Soc. 19 (1919) 214–216.
[16] S. Ramanujan, Collected Papers, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.
[17] S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.
[18] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.
[19] L.J. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc. 25 (1894) 318–343.
[20] L.J. Rogers, On two theorems of combinatory analysis and some allied identities, Proc. London Math. Soc. 16 (1917) 315–336.
[21] L.J. Rogers, On a type of modular relation, Proc. London Math. Soc. 19 (1921) 387–397.
[22] A.V. Sills, Finite Rogers–Ramanujan type identities, Electron. J. Combin. 10 (2003) #R13, 1–122.
[23] L.J. Slater, Further identities of the Rogers–Ramanujan type, Proc. London Math. Soc. (2) 54 (1952) 147–167.
[24] G.N. Watson, Proof of certain identities in combinatory analysis, J. Indian Math. Soc. 20 (1933) 57–69.
[25] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, Cambridge University Press (Indian edition is published by the Universal Book Stall, New Delhi, 1991).