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Minimising Stop and Go Waves to Optimise Traffic Flow

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Abstract—Motivated by the problem of minimising the *"stop and go"* phenomenon in traffic flow, we consider a nonstandard problem of calculus of variations. Given a system of hyperbolic conservation laws, we introduce an integral functional where the integrating measure depends on the space derivative of the solution to the conservation law. An existence result for initial and, when present, boundary data that minimise this functional is proved. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The classical Lighthill-Whitham [1] and Richards [2] model, like other traffic flow models [3–5], is based on conservation laws, i.e., on a system of first-order partial differential equations of the type

$$\begin{aligned} \partial_t u + \partial_x[f(u)] &= 0, & (t,x) &\in [0,+\infty[\times \mathbb{R}, \\ u(0,x) &= \bar{u}(x), & x &\in \mathbb{R}, \end{aligned} \tag{1.1}$$

where the flow $f : \Omega \mapsto \mathbb{R}^n$ is smooth and $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$. Both in single [3,5,6] and many [4] population models, the components of u are related to car densities and (functions of) the car speeds. Under reasonable assumptions, the Cauchy problem for (1.1) generates a Lipschitzian solution operator, namely, the standard Riemann semigroup (SRS), see [7]. Similarly, also the initial boundary value problem (IBVP) with fixed boundary

$$\begin{aligned} \partial_t u + \partial_x[f(u)] &= 0, & (t,x) &\in [0,+\infty[\times [0,+\infty[, \\ u(0,x) &= \bar{u}(x), & x &\in [0,+\infty[, \\ u(t,0) &= \tilde{u}(t), & t &\in [0,+\infty[, \end{aligned} \tag{1.2}$$

is well posed, its solution $t \mapsto \mathcal{P}_t(\bar{u},\tilde{u})$ being an $\mathbf{L}^1$-Lipschitz function of the initial data $\bar{u}$ and the boundary data $\tilde{u}$, see [8–10].

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A reasonable criterion to optimise the traffic flow is to minimise the oscillations in the components of u. This minimisation leads to a more fluent traffic, reducing in particular phenomena such as the *"stop and go"* and cluster formation, see [11, Chapter 8;12]. Strictly related problems are considered in the current physical literature, see [13] and the references therein. Note that a more fluent traffic flow reasonably reduces both the probability of accidents and the production of pollution.

We thus consider the problem of choosing $\bar{u}$ in (1.1) and $(\bar{u}, \tilde{u})$ in (1.2) so that the functional

$$\begin{aligned} J(\bar{u}) &= \int_0^T \int_{\mathbb{R}} p(t,x)\, d|\partial_x \Psi(t, S_t\bar{u})|\, dt, && \text{in case (1.1)},\\ J(\bar{u}, \tilde{u}) &= \int_0^T \int_{\mathbb{R}} p(t,x)\, d|\partial_x \Psi(t, \mathcal{P}_t(\bar{u}, \tilde{u}))|\, dt, && \text{in case (1.2)}, \end{aligned} \tag{1.3}$$

attains a minimum over a suitable set $\mathcal{D}$ of admissible data. Above, p is a nonnegative lower semicontinuous weight and Ψ is a Lipschitz function. Under standard assumptions on f, if $\bar{u}$ and $\tilde{u}$ are of bounded total variation, then so is the map $x \mapsto \Psi(t, \mathcal{P}_t(\bar{u}, \tilde{u})(x))$ and its x-derivative $\partial_x \Psi(t, \mathcal{P}_t(\bar{u}, \tilde{u}))$ is a Radon measure. The measure $|\partial_x \Psi(t, \mathcal{P}_t(\bar{u}, \tilde{u}))|$ is the total variation of $\partial_x \Psi(t, \mathcal{P}_t(\bar{u}, \tilde{u}))$, see [14, Chapter 6] for the basic measure theoretic definitions.

The well posedness of systems of hyperbolic conservation laws was recently established, see [7] and the references therein. Since then, the problems of controlling and optimising (1.1) could be considered but were given apparently little attention in the literature. In [15,16], characterisations of the attainable set are given, while a necessary condition for the minima of some integral functional is derived in [17]. The present optimisation problem may not take advantage of the above cited references, due to the particular form of J in (1.3).

In spite of this well posedness, as is well known, S in (1.1) and $\mathcal{P}$ in (1.2) turn out to be in general *not* differentiable, see [18,19]. This additional difficulty in the minimisation of J can be dealt with using an *ad hoc* differential structure, as in [17]. Here we follow a different approach exploiting compactness and lower semicontinuity.

2. RESULT

Assume that (1.1) generates a SRS, see [7], i.e., a map $S : \mathcal{D} \times [0, +\infty[\mapsto \mathcal{D}$ such that there exist two constants δ and L with

1. $\mathcal{D} \supseteq \{u \in \mathbf{L}^1(\mathbb{R}) : \mathrm{TV}(u) \le \delta\}$;
2. S is a semigroup: $S_0 u = u$ and $S_t S_s u = S_{t+s} u$;
3. S is Lipschitzian: $\|S_{t'}u' - S_t u\|_{\mathbf{L}^1} \le L \cdot (\|u' - u\|_{\mathbf{L}^1} + |t' - t|)$;
4. if u is piecewise constant, for t small the function $S_t u$ locally coincides with the solutions to the Riemann problems at the points of jumps of u.

Similarly, the IBVP (1.2) generates a solution operator $\mathcal{P} : \mathcal{D} \times [0, +\infty[\mapsto \mathbf{L}^1 \cap \mathbf{BV}$, see [8,10], such that there exist constants δ and L with

1. $\mathcal{D} \supseteq \{(\bar{u}, \tilde{u}) \in \mathbf{L}^1(\mathbb{R})^2 : \mathrm{TV}(\bar{u}) + \|\bar{u}(0+) - \tilde{u}(0+)\| + \mathrm{TV}(\tilde{u}) \le \delta\}$;
2. $\mathcal{P}$ is Lipschitzian: $\|u'(t', \cdot) - u''(t'', \cdot)\|_{\mathbf{L}^1} \le L \cdot (\|\bar{u}' - \bar{u}''\|_{\mathbf{L}^1} + \|\tilde{u}' - \tilde{u}''\|_{\mathbf{L}^1} + |t' - t''|)$;
3. if $\bar{u}$ and $\tilde{u}$ are piecewise constant, then $\mathcal{P}_t(\bar{u}, \tilde{u})$ locally coincides with the solution to the Riemann problems at the points of jumps of $\bar{u}$ and at the boundary.

(Here, the semigroup property is lost since (1.2) is not autonomous.) In both cases, the uniform continuity of S or $\mathcal{P}$ in $\mathbf{L}^1$ allows us to assume that $\mathcal{D}$ is closed in $\mathbf{L}^1$. Examples of usual assumptions that ensure the well posedness of (1.1) or (1.2) are

(A) Df is strictly hyperbolic with each characteristic field either genuinely nonlinear or linearly degenerate, see [7,8] (and, in this case, δ is *sufficiently small*), or

(B) Df is strictly hyperbolic with coinciding shock and rarefaction curves, see [10,20] (and, in this case, δ need *not* be small).

In the case of the Cauchy problem, the recent result [21] allows us to remove from (A) the requirements on the characteristic fields. In case (1.2), the recent result [9] provides solutions in $\mathbf{L}^\infty$ but requiring that all characteristic fields be genuinely nonlinear, which is usually not verified in traffic flow models. In both cases, (A) and (B) ensure also the existence of a constant H (dependent on δ) such that

$$\begin{aligned} &\mathcal{D} \subseteq \left\{u \in \mathbf{L^1} : \mathrm{TV}(u) \le H\right\}, && \text{in case (1.1)},\\ &\mathcal{D} \subseteq \left\{(\bar u, \tilde u) \in \left(\mathbf{L^1}\right)^2 : \mathrm{TV}(\bar u) + \|\bar u(0+) - \tilde u(0+)\| + \mathrm{TV}(\tilde u) \le H\right\}, && \text{in case (1.2)}. \end{aligned} \tag{2.1}$$

In what follows, T is fixed and strictly positive. Concerning the functional J in (1.3), we assume below that the weight $p : [0,T] \times \mathbb{R} \to \mathbb{R}$ is lower semicontinuous and nonnegative. The function $\Psi : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}$ is locally Lipschitzian in u, i.e., for every compact $K \subset \mathbb{R}^n$, there exists a constant $\mathcal{L}_K$ such that for all $u, u' \in K$, $|\Psi(t,u') - \Psi(t,u)| \le \mathcal{L}_K \cdot \|u' - u\|_{\mathbb{R}^n}$. If for all times t the map $x \mapsto u(t,x)$ is in $\mathbf{BV}$, then also $x \mapsto \Psi(t,u(t,x))$ belongs to $\mathbf{BV}$. By $\partial_x \Psi(t,u)$ we mean the measure obtained as weak x-derivative of $x \mapsto \Psi(t,u(t,x))$ and $|\partial_x \Psi(t,u)|$ is its total variation. The following technical lemma is of later use.

LEMMA 2.1. *Let μ be a signed Radon measure and $\Omega \subset \mathbb{R}^n$ be an open set. If p is nonnegative and lower semicontinuous, then*

$$\int_\Omega p\, d|\mu| = \sup\left\{\int_\Omega \varphi\, d\mu : \varphi \in \mathbf{C}^0_\mathbf{c}(\Omega),\ |\varphi| \le p\right\}. \tag{2.2}$$

PROOF. Note first that

$$\sup\left\{\int_\Omega \varphi\, d\mu : \varphi \in \mathbf{C}^0_\mathbf{c}(\Omega),\ |\varphi| \le p\right\} = \sup\left\{\left|\int_\Omega \varphi\, d\mu\right| : \varphi \in \mathbf{C}^0_\mathbf{c}(\Omega),\ |\varphi| \le p\right\}.$$

Hence, by [14, Theorem 6.19], (2.2) holds for all $p \in \mathbf{C}^0_\mathbf{c}(\Omega)$, $p \ge 0$. Suppose now that p is merely lower semicontinuous and that $\int_\Omega p\, d|\mu| \ne 0$, the case $\int_\Omega p\, d|\mu| = 0$ being trivial. The inequality

$$\sup\left\{\left|\int_\Omega \varphi\, d\mu\right| : \varphi \in \mathbf{C}^0_\mathbf{c}(\Omega),\ |\varphi| \le p\right\} \le \int_\Omega p\, d|\mu|$$

is immediate. On the other side, there exists a sequence $(p_h)_{h\in\mathbb{N}}$ such that $p_h \in \mathbf{C}^0_\mathbf{c}(\Omega)$, $p_h \le p_{h+1}$ and $\lim_{h\to+\infty} p_h = p$. By the monotone convergence theorem, $\lim_{h\to+\infty} \int_\Omega p_h\, d|\mu| = \int_\Omega p\, d|\mu|$. Choose a positive M such that $M < \int_\Omega p\, d|\mu|$. For all sufficiently large $h \in \mathbb{N}$, it holds that $\int_\Omega p_h\, d|\mu| > M$ and using (2.2), we have

$$\begin{aligned} M < \int_\Omega p_h\, d|\mu| &= \sup\left\{\left|\int_\Omega \varphi\, d\mu\right| : \varphi \in \mathbf{C}^0_\mathbf{c}(\Omega),\ |\varphi| \le p_h\right\}\\ &\le \sup\left\{\left|\int_\Omega \varphi\, d\mu\right| : \varphi \in \mathbf{C}^0_\mathbf{c}(\Omega),\ |\varphi| \le p\right\}, \end{aligned}$$

which is possible only if

$$\int_\Omega p\, d|\mu| \le \sup\left\{\left|\int_\Omega \varphi\, d\mu\right| : \varphi \in \mathbf{C}^0_\mathbf{c}(\Omega),\ |\varphi| \le p\right\}.$$

∎

THEOREM 2.2. *Assume that under (A) or (B), (1.1) generates a SRS $S : [0,+\infty[\times \mathcal{D} \mapsto \mathcal{D}$ satisfying (2.1). Let $T > 0$, $p : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ a nonnegative and lower semicontinuous function, while $\Psi : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}$ is locally Lipschitz in u. Define the functional $J : \mathcal{D} \to \mathbb{R}$ as in (1.3). Then, J admits a minimum on $\mathcal{D}$.*

An entirely analogous result holds in the case of the IBVP. Here the minimisation can be achieved also in the case more suitable to traffic flow of the boundary data alone, for any given initial data.

THEOREM 2.3. *Assume that under (A) or (B), (1.2) generates a solution operator* $\mathcal{P} : [0, +\infty[\times \mathcal{D} \mapsto \mathbf{L}^1 \cap \mathbf{BV}$ *satisfying (2.1). Let* $T > 0$, $p : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ *a nonnegative and lower semicontinuous function, while* $\Psi : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}$ *is locally Lipschitz in* u. *Define the functional* $J : \mathcal{D} \to \mathbb{R}$ *as in (1.3). Then,* J *admits a minimum on* $\mathcal{D}$.

We prove below only Theorem 2.2, the other case being entirely analogous.

PROOF OF THEOREM 2.2. By Lemma 2.1, we have

$$\begin{aligned} J(\bar{u}) &= \int_0^T \sup\left\{ \int_{\mathbb{R}} \varphi(t,x)\, d(\partial_x(\Psi(t, S_t\bar{u}))) : \varphi \in \mathbf{C}_{\mathbf{c}}^\infty,\ |\varphi| \le p \right\} dt \\ &= \int_0^T \sup\left\{ - \int_{\mathbb{R}} \partial_x(\varphi(t,x))\Psi(t, S_t\bar{u})\, dx : \varphi \in \mathbf{C}_{\mathbf{c}}^\infty,\ |\varphi| \le p \right\} dt. \end{aligned} \tag{2.3}$$

Consider a sequence u_h in $\mathbf{BV}(\mathbb{R})$ such that $u_h \to \bar{u}$ in $\mathbf{L}^1(\mathbb{R})$. Since $S_t(u_h) \to S_t(\bar{u})$ in $\mathbf{L}^1(\mathbb{R})$, then $\Psi(t, S_t u_h) \to \Psi(t, S_t\bar{u})$ in $\mathbf{L}^1(\mathbb{R})$. Therefore, for all $\varphi \in \mathbf{C}_{\mathbf{c}}^\infty$ and $t \in [0,T]$, the map

$$\bar{u} \mapsto - \int_{\mathbb{R}} \partial_x \left(\varphi(t,x)\right) \Psi(t, S_t\bar{u})\, dx \tag{2.4}$$

is continuous in $\mathbf{L}^1$, hence, the integrand in (2.3) is lower semicontinuous. Moreover, for all fixed φ, by (2.1) the map (2.4) is bounded from below. Now, use Fatou's lemma and (2.3) to obtain

$$\begin{aligned} \liminf_{h\to+\infty} J(u_h) &\ge \int_0^T \liminf_{h\to+\infty} \sup\left\{ - \int_{\mathbb{R}} \partial_x(\varphi(t,x))\Psi(t, S_t u_h)\, dx : \varphi \in \mathbf{C}_{\mathbf{c}}^\infty,\ |\varphi| \le p \right\} dt \\ &\ge \int_0^T \sup\left\{ - \int_{\mathbb{R}} \partial_x(\varphi(t,x))\Psi(t, S_t\bar{u})\, dx : \varphi \in \mathbf{C}_{\mathbf{c}}^\infty,\ |\varphi| \le p \right\} dt \\ &= J(\bar{u}), \end{aligned}$$

proving the sequential lower semicontinuity of J.

By (2.1), $\mathcal{D}$ is a closed subset of the $\mathbf{L}^1$-compact set $\{u \in \mathbf{L}^1(\mathbb{R}) : \mathrm{TV}(u) \le M\}$, hence, it is compact in $\mathbf{L}^1$. ∎

3. APPLICATION

In the case of the classical Lighthill-Whitham [1] and Richards [2] model, $n = 1$, $u = \rho$ is the car density and $f(\rho) = \rho \cdot v(\rho)$ is the car flow. Recently, more refined models were introduced. For example, [3] provides a model where $n = 2$ and $u = (\rho, \rho v + \rho p(\rho))$, v being the traffic speed and p a suitable *"pressure"*. In the case of the model introduced in [5], $n = 2$ and $u = (\rho, q)$, q being a *"weighted momentum"*

$$\begin{aligned} \partial_t \rho + \partial_x(\rho \cdot v) &= 0, \\ \partial_t q + \partial_x((q - q_*) \cdot v) &= 0, \end{aligned} \qquad v(\rho, q) = \left(\frac{1}{\rho} - \frac{1}{\rho_M}\right) \cdot q, \tag{3.1}$$

where ρ_M and q_* are parameters characteristic of the road under consideration. The first is the maximal car density supported by the street, so that $\rho \in [0, \rho_M]$. The latter is strictly related to *wide jams*, see [5,6] for further details. The *fundamental diagram* (i.e., the usual flow density relation) is here replaced by the compact invariant set

$$\Omega = \left\{ (\rho, q) \in [0, \rho_M] \times [0, +\infty[: v(\rho, q) \in [0, v_M] \text{ and } \frac{q}{q_*} + \frac{\rho}{\rho_*} \ge 1 \right\},$$

v_M being the maximal possible speed and with $\rho_* > \rho_M$. The shock and rarefaction curves in (3.1) coincide, so that this system satisfies (B). The search for an optimal management of traffic flows leads to consider (1.2) with, say, a given fixed initial data, so that J in (1.3) depends on the boundary data (i.e., the car inflow) alone, which is considered as a control. By the results in [10], there exists a domain $\mathcal{D} \supseteq \{(\rho,q) \in (\mathbf{L}^1)^2 : (\rho,q)(x) \in \Omega,\ \mathrm{TV}(\rho) + \mathrm{TV}(q) \leq \delta\}$ that also satisfies (2.1) for a suitable H, such that for all $(\tilde{\rho},\tilde{q})$ in $\mathcal{D}$, there exists a solution to the IBVP for (3.1) that depends Lipschitz continuously on the boundary data. Concerning the functional J in (1.3), the choice $\Psi(t,\rho,q) = v(\rho,q)$ is particularly relevant: it amounts to minimise the (weighted) total variation of the car speed. Note that all states in Ω satisfy

$$\rho \geq \frac{1}{2}\left(\rho_M + \alpha\rho_* - \sqrt{(\rho_M + \alpha\rho_*)^2 - 4\rho_*\rho_M}\right) > 0, \qquad \text{where } \alpha = \frac{\rho_M v_M}{q_*} + 1. \tag{3.2}$$

Hence, $\Psi(t,\rho,q) = v(\rho,q)$ is Lipschitzian on Ω and Theorem 2.3 applies to

$$J(\tilde{\rho},\tilde{q}) = \int_0^T \int_{\mathbb{R}} p(t,x)\, d\left|\partial_x v\left(\mathcal{P}_t(\tilde{\rho},\tilde{q})\right)\right| dt. \tag{3.3}$$

Finally, due to [22], we note that the present construction applies also to model [6] that extends (3.1) introducing phase transitions. In this more general framework, it is possible to select invariant sets that contain the vacuum state, hence, the minimisation of functionals of type (3.3) can be accomplished without a lower bound of the type (3.2) on the car density.

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