ELSEVIER

Discrete Mathematics 138 (1995) 393–396

DISCRETE MATHEMATICS

Acyclic orientations of complete bipartite graphs ☆

Douglas B. West

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801-2975, USA

Received 2 December 1992; revised 10 August 1993

Abstract

For a complete bipartite graph, the number of dependent edges in an acyclic orientation can be any integer from $n-1$ to e, where n and e are the number of vertices and edges in the graph.

Keywords: Bipartite graph; Acyclic orientation

In combinatorics we often ask whether an integer parameter can take on all values between its extremes. In this note we consider a question of this type for acyclic orientations of a graph. An *acyclic orientation* assigns an orientation to each edge of a simple graph so that no cycle is formed.

In an acyclic orientation H of a graph G, an edge is *dependent* if reversing its orientation creates a cycle – the other edges force its orientation. This definition is due to Paul Edelman [1], who observed that the number $f(H)$ of independent edges always satisfies $n(G)-1 \leqslant f(H) \leqslant e(G)$ (where $n(G)$ and $e(G)$ denote the number of vertices and edges of G), and that these extremes are achievable when G is bipartite. Lemma 1 below includes the lower bound, and orienting all edges from one partite set to the other achieves the upper bound. Edelman asked whether G being bipartite guarantees that every number from $n(G)-1$ to $e(G)$ is achievable as $f(H)$ for some acyclic orientation H of G [3]. We call such a graph *fully orientable*. The Petersen graph, despite not being bipartite, is fully orientable, and we do not know of a triangle-free graph that is not fully orientable.

More generally, one can ask which values of $f(H)$ are achievable for an arbitrary G. It is not possible to make all three edges of a triangle independent; hence $e(G)$ may not be achievable. Indeed, the strongly connected components of an acyclic orientation of K_n must be single vertices; hence every acyclic orientation of K_n is a transitive orientation and has precisely $n-1$ independent edges. It remains open whether for

☆ Research supported in part by NSA/MSP Grant MDA904-90-H-4011.

SSDI 0012-365X(94)00222-3

every graph G the achievable values of $f(H)$ form a sequence of consecutive integers beginning with $n(G)-1$.

Independent edges are precisely those whose reversal produces another acyclic orientation. This suggests a graph AO(G) on the acyclic orientations of G, in which two acyclic orientations are adjacent if one is obtained from the other by reversal of a single independent edge; the degree of an acyclic orientation in this graph is its number of independent edges. If AO(G) has a Hamiltonian path, then the acyclic orientations of G can be listed in order by single-edge reversals; this condition is studied in [2].

In this note, we prove that every complete bipartite graph $K_{p,q}$ is fully orientable; Edelman [1] proved this for $p=2$. We also provide some structural lemmas about acyclic orientations.

In an arbitrary connected graph, an acyclic orientation with $f(H)=n(G)-1$ can be constructed by using a spanning tree T found by a depth-first search from some root vertex. Orient the edges of T away from the root. Since T is a depth-first search tree, any edge not in T joins a vertex with one of its ancestors. Orient this toward the descendent; the other direction would complete a cycle. The orientation is acyclic, and the edges not in T are dependent. Lemma 1 guarantees that the edges in T are independent and that no acyclic orientation has fewer independent edges.

Lemma 1. *Every acyclic orientation of a connected simple graph G contains among its independent edges a spanning tree of G.*

Proof. Let H be an acyclic orientation of G, and let $v_1, \ldots, v_n$ be a topological ordering of the vertices of H, meaning that every edge v_iv_j in H has $i<j$. Let H' be the subdigraph of H obtained by deleting all the dependent edges of H, and let G' be the underlying simple graph of H'. If G' is not connected, choose r to be the largest index such that H contains an edge v_rv_j between two components of G'. Let C be the component of G' containing v_r, and choose s to be the smallest index such that v_rv_s is an edge not in C. A path from v_r in H that begins along an edge in C never leaves C, by the choice of r. A path from v_r leaving C immediately by an edge other than v_rv_s cannot later reach v_s, by the choice of the vertex ordering. Hence v_rv_s is independent in H. The contradiction implies that G' must be connected. □

Given an n-vertex digraph G and digraphs $H_1, \ldots, H_n$, the *composition* $G[H_1, \ldots, H_n]$ is the digraph obtained from the disjoint union $H_1 + \cdots + H_n$ by adding an edge from each vertex of H_i to each vertex of H_j for each edge v_iv_j in G.

Lemma 2. *If G and $H_1, \ldots, H_n$ are acyclic digraphs, then the composition $G'=G[H_1, \ldots, H_n]$ is acyclic. Furthermore, if $I(G)$ denotes the set of independent edges in G and n_i, r_i, t_i, respectively, are the number of vertices, sources (indegree 0), and sinks* *(outdegree* 0) *in* H_i, *then*

$$f(G')=\sum_{i=1}^{n} f(H_i)+\sum_{v_jv_k\in I(G)} (t_jn_k+n_jr_k-t_jr_k).$$

Proof. There is no cycle within any H_i. Since G is acyclic, no path can leave any H_i and later return to it. Hence G' is acyclic, and also the independent edges of G' within H_i are precisely the independent edges of H_i. Now consider an edge xy with $x\in H_j$, $y\in H_k$. If v_jv_k is dependent in G, then a copy of the path making it dependent also makes xy dependent in G'. If v_jv_k is independent in G, then xy is dependent if and only if x has a successor in H_j and y has a predecessor in H_k. In this case, there is an xy-path of length three making xy dependent. Otherwise, an xy-path would have to visit some H_i other than $\{H_j, H_k\}$, which would violate the independence of v_jv_k. Hence the number of dependent edges from H_j to H_k is $(n_j-t_j)(n_k-r_k)$, and we subtract this from n_jn_k to count independent edges. □

In an acyclic n-vertex digraph having a Hamiltonian path, the independent edges are precisely the edges of the Hamiltonian path. This enables us to construct the needed orientations for the complete bipartite graph.

Lemma 3. *Let* $s_1, \ldots, s_m$ *be a sequence of positive integers such that the integers with odd index sum to* p *and the integers with even index sum to* q. *Then* $K_{p,q}$ *has an acyclic orientation with exactly* $\sum_{i=1}^{m-1} s_is_{i+1}$ *independent edges.*

Proof. Begin with a directed path with vertices $v_1, \ldots, v_m$ in order. Add an edge joining every pair of vertices having indices with opposite parity, directed toward the higher indices. This digraph G is acyclic, and by the remark its independent edges are $\{v_iv_{i+1}: 1\leqslant i\leqslant m-1\}$. Perform the composition G' in which v_i is replaced by the digraph H_i consisting of an independent set of s_i vertices. Each vertex of H_i is both a source and a sink; $t_i=r_i=n_i=s_i$. By Lemma 2, G' is an acyclic orientation of $K_{p,q}$ with the desired number of independent edges. □

The construction in Lemma 3 actually yields every acyclic orientation of $K_{p,q}$. To see this, let P be a maximum-length path in an acyclic orientation H of $K_{p,q}$, consisting of vertices $v_1, \ldots, v_m$ in order. The subdigraph of H induced by $v_1, \ldots, v_m$ must be the digraph G used in the construction. Let S_i be the set of vertices in H having the same predecessors and successors among $v_1, \ldots, v_m$ as v_i. The subdigraph of H induced by $\bigcup S_i$ is the construction G' above with $s_i=|S_i|$. It suffices to show that every vertex belongs to some S_i. An arbitrary vertex x of H is adjacent to all the vertices of even index or all the vertices of odd index in P. Also, all its predecessors in P must precede its successors, since H is acyclic. This forces it to have the same predecessors and successors in P as the unique vertex v_i between its predecessors and successors on P, and hence $x\in S_i$.

We use instances of Lemma 3 to complete the construction.

Theorem 1. *For each value of k with $p+q-1 \leqslant k \leqslant pq$, the complete bipartite graph $K_{p,q}$ has an acyclic orientation with exactly k independent edges.*

Proof. For $K_{1,q}$ there is nothing to prove, so we may assume $p,q \geqslant 2$. We need only show that all integers in the desired range are achievable in the form $f(s_1,\dots,s_m)=\sum_{i=1}^{m-1} s_i s_{i+1}$ for some positive integer sequence $s_1,\dots,s_m$ such that the odd-indexed numbers sum to p and the even-indexed numbers sum to q.

We primarily use 6-term sequences of the form $p-1-k,\ 1, 1, l, k, q-l-1$, where $1 \leqslant k \leqslant p-1$ and $1 \leqslant l \leqslant q-1$. The odd terms sum to p, the even to q. All entries are non-zero, except that if $k=p-1$ or $l=q-1$, then an end-term becomes 0 and we consider instead the positive sequence with fewer terms. The value is $f=p-k+l+k(q-1)$, even when $k=p-1$ or $l=q-1$ and the sequence is shorter.

When $k=l=1$, we have $f=q+p-1$. For any fixed k, f covers a sequence of $q-1$ consecutive values as l ranges from 1 to $q-1$. The top value for k and the bottom value when k is replaced by $k+1$ are the same, since $p-k+q-1+k(q-1)=p-k-1+1+(k+1)(q-1)$. Hence there are no gaps up to the largest value achievable in this way, which occurs when $k=p-1$ and $l=q-1$ and equals $pq-p+1$.

For the remaining few values, we use sequences of the form $p-k, q-1, k, 1$, where $1 \leqslant k \leqslant p$ (again the sequence shortens when $k=p$); the value of f here is $pq-p+k$, and this completes the construction. □

Note added in proof. In "The number of independent edges in acyclic orientations", D.C. Fisher, K. Fraughnaugh, L. Langley, and D.B. West have proved the following results: If the chromatic number of a graph is less than its girth, then the graph is fully orientable; this includes all bipartite graphs. On the other hand, the Grötzsch graph of order 11 and chromatic number 4 has no acyclic orientation with every edge independent.

Acknowledgement

The author thanks Paul Edelman for communicating the problem, Albert Li for interesting discussions about it, and a referee for suggesting stronger versions of the Lemmas.

References

[1] P. Edelman, private communication.
[2] C.D. Savage, M.B. Squire and D.B. West, Gray code results for acyclic orientations, Congr. Numer. 96 (1993) 185–204.
[3] D.B. West, SIAM Discrete Math. Activity Group Newsletter, 2 (1991/2) 4, 9–12.