Acyclic orientations of complete bipartite graphs

Douglas B. West

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801-2975, USA

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Abstract

For a complete bipartite graph, the number of dependent edges in an acyclic orientation can be any integer from \( n-1 \) to \( e \), where \( n \) and \( e \) are the number of vertices and edges in the graph.

Keywords: Bipartite graph; Acyclic orientation

In combinatorics we often ask whether an integer parameter can take on all values between its extremes. In this note we consider a question of this type for acyclic orientations of a graph. An acyclic orientation assigns an orientation to each edge of a simple graph so that no cycle is formed.

In an acyclic orientation \( H \) of a graph \( G \), an edge is dependent if reversing its orientation creates a cycle – the other edges force its orientation. This definition is due to Paul Edelman [1], who observed that the number \( f(H) \) of independent edges always satisfies \( n(G)-1 \leq f(H) \leq e(G) \) (where \( n(G) \) and \( e(G) \) denote the number of vertices and edges of \( G \)), and that these extremes are achievable when \( G \) is bipartite. Lemma 1 below includes the lower bound, and orienting all edges from one partite set to the other achieves the upper bound. Edelman asked whether \( G \) being bipartite guarantees that every number from \( n(G)-1 \) to \( e(G) \) is achievable as \( f(H) \) for some acyclic orientation \( H \) of \( G \) [3]. We call such a graph fully orientable. The Petersen graph, despite not being bipartite, is fully orientable, and we do not know of a triangle-free graph that is not fully orientable.

More generally, one can ask which values of \( f(H) \) are achievable for an arbitrary \( G \). It is not possible to make all three edges of a triangle independent; hence \( e(G) \) may not be achievable. Indeed, the strongly connected components of an acyclic orientation of \( K_n \) must be single vertices; hence every acyclic orientation of \( K_n \) is a transitive orientation and has precisely \( n-1 \) independent edges. It remains open whether for

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every graph $G$ the achievable values of $f(H)$ form a sequence of consecutive integers beginning with $n(G) - 1$.

Independent edges are precisely those whose reversal produces another acyclic orientation. This suggests a graph $AO(G)$ on the acyclic orientations of $G$, in which two acyclic orientations are adjacent if one is obtained from the other by reversal of a single independent edge; the degree of an acyclic orientation in this graph is its number of independent edges. If $AO(G)$ has a Hamiltonian path, then the acyclic orientations of $G$ can be listed in order by single-edge reversals; this condition is studied in [2].

In this note, we prove that every complete bipartite graph $K_{p,q}$ is fully orientable; Edelman [1] proved this for $p = 2$. We also provide some structural lemmas about acyclic orientations.

In an arbitrary connected graph, an acyclic orientation with $f(H) = n(G) - 1$ can be constructed by using a spanning tree $T$ found by a depth-first search from some root vertex. Orient the edges of $T$ away from the root. Since $T$ is a depth-first search tree, any edge not in $T$ joins a vertex with one of its ancestors. Orient this toward the descendent; the other direction would complete a cycle. The orientation is acyclic, and the edges not in $T$ are dependent. Lemma 1 guarantees that the edges in $T$ are independent and that no acyclic orientation has fewer independent edges.

**Lemma 1.** Every acyclic orientation of a connected simple graph $G$ contains among its independent edges a spanning tree of $G$.

**Proof.** Let $H$ be an acyclic orientation of $G$, and let $v_1, \ldots, v_n$ be a topological ordering of the vertices of $H$, meaning that every edge $v_iv_j$ in $H$ has $i < j$. Let $H'$ be the subdigraph of $H$ obtained by deleting all the dependent edges of $H$, and let $G'$ be the underlying simple graph of $H'$. If $G'$ is not connected, choose $r$ to be the largest index such that $H$ contains an edge $v_rv_j$ between two components of $G'$. Let $C$ be the component of $G'$ containing $v_r$, and choose $s$ to be the smallest index such that $v_rv_s$ is an edge not in $C$. A path from $v_r$ in $H$ that begins along an edge in $C$ never leaves $C$, by the choice of $r$. A path from $v_r$ leaving $C$ immediately by an edge other than $v_rv_s$ cannot later reach $v_s$, by the choice of the vertex ordering. Hence $v_rv_s$ is independent in $H$. The contradiction implies that $G'$ must be connected. □

Given an $n$-vertex digraph $G$ and digraphs $H_1, \ldots, H_n$, the composition $G[H_1, \ldots, H_n]$ is the digraph obtained from the disjoint union $H_1 + \cdots + H_n$ by adding an edge from each vertex of $H_i$ to each vertex of $H_j$ for each edge $v_iv_j$ in $G$.

**Lemma 2.** If $G$ and $H_1, \ldots, H_n$ are acyclic digraphs, then the composition $G' = G[H_1, \ldots, H_n]$ is acyclic. Furthermore, if $I(G)$ denotes the set of independent edges in $G$ and $n_i, r_i, t_i$, respectively, are the number of vertices, sources (indegree 0), and sinks
Proof. There is no cycle within any $H_i$. Since $G$ is acyclic, no path can leave any $H_i$ and later return to it. Hence $G'$ is acyclic, and also the independent edges of $G'$ within $H_i$ are precisely the independent edges of $H_i$. Now consider an edge $xy$ with $x \in H_j$, $y \in H_k$. If $v_j v_k$ is dependent in $G$, then a copy of the path making it dependent also makes $xy$ dependent in $G'$. If $v_j v_k$ is independent in $G$, then $xy$ is dependent if and only if $x$ has a successor in $H_j$ and $y$ has a predecessor in $H_k$. In this case, there is an $xy$-path of length three making $xy$ dependent. Otherwise, an $xy$-path would have to visit some $H_i$ other than $\{H_j, H_k\}$, which would violate the independence of $v_j v_k$. Hence the number of dependent edges from $H_j$ to $H_k$ is $(n_j - t_j)(n_k - r_k)$, and we subtract this from $n_j n_k$ to count independent edges.

In an acyclic $n$-vertex digraph having a Hamiltonian path, the independent edges are precisely the edges of the Hamiltonian path. This enables us to construct the needed orientations for the complete bipartite graph.

**Lemma 3.** Let $s_1, \ldots, s_m$ be a sequence of positive integers such that the integers with odd index sum to $p$ and the integers with even index sum to $q$. Then $K_{p,q}$ has an acyclic orientation with exactly $\sum_{i=1}^{m-1} s_i s_{i+1}$ independent edges.

**Proof.** Begin with a directed path with vertices $v_1, \ldots, v_m$ in order. Add an edge joining every pair of vertices having indices with opposite parity, directed toward the higher indices. This digraph $G$ is acyclic, and by the remark its independent edges are $\{v_i v_{i+1}: 1 \leq i \leq m-1\}$. Perform the composition $G'$ in which $v_i$ is replaced by the digraph $H_i$ consisting of an independent set of $s_i$ vertices. Each vertex of $H_i$ is both a source and a sink; $t_i = r_i = n_i = s_i$. By Lemma 2, $G'$ is an acyclic orientation of $K_{p,q}$ with the desired number of independent edges.

The construction in Lemma 3 actually yields every acyclic orientation of $K_{p,q}$. To see this, let $P$ be a maximum-length path in an acyclic orientation $H$ of $K_{p,q}$, consisting of vertices $v_1, \ldots, v_m$ in order. The subdigraph of $H$ induced by $v_1, \ldots, v_m$ must be the digraph $G$ used in the construction. Let $S_i$ be the set of vertices in $H$ having the same predecessors and successors among $v_1, \ldots, v_m$ as $v_i$. The subdigraph of $H$ induced by $\bigcup S_i$ is the construction $G'$ above with $s_i = |S_i|$. It suffices to show that every vertex belongs to some $S_i$. An arbitrary vertex $x$ of $H$ is adjacent to all the vertices of even index or all the vertices of odd index in $P$. Also, all its predecessors in $P$ must precede its successors, since $H$ is acyclic. This forces it to have the same predecessors and successors in $P$ as the unique vertex $v_i$ between its predecessors and successors on $P$, and hence $x \in S_i$.

We use instances of Lemma 3 to complete the construction.
Theorem 1. For each value of k with \( p + q - 1 \leq k \leq pq \), the complete bipartite graph \( K_{p,q} \) has an acyclic orientation with exactly k independent edges.

Proof. For \( K_{1,q} \) there is nothing to prove, so we may assume \( p,q \geq 2 \). We need only show that all integers in the desired range are achievable in the form

\[
 f(s_1, \ldots, s_m) = \sum_{i=1}^{m-1} s_i s_{i+1}
\]

for some positive integer sequence \( s_1, \ldots, s_m \) such that the odd-indexed numbers sum to \( p \) and the even-indexed numbers sum to \( q \).

We primarily use 6-term sequences of the form \( p-1-k, 1, 1, l, k, q-l-1 \), where \( 1 \leq k \leq p-1 \) and \( 1 \leq l \leq q-1 \). The odd terms sum to \( p \), the even to \( q \). All entries are non-zero, except that if \( k=p-1 \) or \( l=q-1 \), then an end-term becomes 0 and we consider instead the positive sequence with fewer terms. The value is

\[
 f = p-k+l+k(q-1),
\]
even when \( k=p-1 \) or \( l=q-1 \) and the sequence is shorter.

When \( k=l=1 \), we have \( f = q+p-1 \). For any fixed \( k, f \) covers a sequence of \( q-1 \) consecutive values as \( l \) ranges from 1 to \( q-1 \). The top value for \( k \) and the bottom value when \( k \) is replaced by \( k+1 \) are the same, since \( p-k+q-1+k(q-1) = p-k-1+1+(k+1)(q-1) \). Hence there are no gaps up to the largest value achievable in this way, which occurs when \( k=p-1 \) and \( l=q-1 \) and equals \( pq-p+1 \).

For the remaining few values, we use sequences of the form \( p-k, q-1, k, 1 \), where \( 1 \leq k \leq p \) (again the sequence shortens when \( k=p \)); the value of \( f \) here is \( pq-p+k \), and this completes the construction. 

Note added in proof. In “The number of independent edges in acyclic orientations”, D.C. Fisher, K. Fraughnaugh, L. Langley, and D.B. West have proved the following results: If the chromatic number of a graph is less than its girth, then the graph is fully orientable; this includes all bipartite graphs. On the other hand, the Grötzsch graph of order 11 and chromatic number 4 has no acyclic orientation with every edge independent.

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References