Variational analysis of extended generalized equations via coderivative calculus in Asplund spaces

Boris S. Mordukhovich a,*, Nguyen Mau Nam b

a Department of Mathematics, Wayne State University, Detroit, MI 48202, USA
b Department of Mathematics, The University of Texas–Pan American, Edinburg, TX 78539-2999, USA

Received 5 February 2008
Available online 27 May 2008
Submitted by B. Cascales
Dedicated to Isaac Namioka in honor of his 80th birthday

Abstract
This paper is devoted to the development of variational analysis and generalized differentiation in the framework of Asplund spaces. We mainly concern the study of a special class of set-valued mapping given in the form

\[ S(x) = \{ y \in Y \mid 0 \in F(x, y) + Q(x, y) \}, \quad x \in X, \]

where both \( F \) and \( Q \) are set-valued mappings between Asplund spaces. Models of this type are associated with solutions maps to the so-called (extended) generalized equations and play a significant role in many aspects of variational analysis and its applications to optimization, stability, control theory, etc. In this paper we conduct a local variational analysis of such extended solution maps \( S \) and their remarkable specifications based on dual-space generalized differential constructions of the coderivative type. The major part of our analysis revolves around coderivative calculus largely developed and implemented in this paper and then applied to establishing verifiable conditions for robust Lipschitzian stability of extended generalized equations and related objects.

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Keywords: Variational analysis; Asplund spaces; Generalized differentiation; Coderivatives; Generalized equations; Lipschitzian stability

1. Introduction

Variational analysis has been well recognized as a fruitful area of mathematics that on one hand deals with optimization-related and equilibrium problems, while on the other hand implies variational principles and optimization ideas as well as approximation and perturbation techniques to a broad spectrum of problems, which may not possess any variational structure. Since nonsmooth functions, sets with nonsmooth boundaries, and set-valued map-
pings naturally and frequently appear in the framework of variational analysis and optimization, tools of generalized differentiation lay at the very heart of variational analysis, related areas, and their numerous applications.

Many problems arising in variational analysis and its applications (particularly to optimal control, partial differential equations, nonlinear dynamics, economics, mechanics, etc.) require developing variational analysis in infinite-dimensional spaces; see, e.g., the books by Borwein and Zhu [4], Deville, Godefroy and Zizler [5], Mordukhovich [14,15], Schirotzek [24], and references therein.

The class of Asplund spaces plays a remarkable role in infinite-dimensional variational analysis and generalized differentiation. Recall that this subclass of Banach spaces was introduced by Asplund [1] as “strong differentiability spaces.” The name “Asplund spaces” was coined by Namioka and Phelps [19] who conducted a deep study of the Asplund property. This beautiful class of spaces has then been comprehensively investigated in geometric theory of Banach spaces; see, e.g., the books by Deville, Godefroy and Zizler [5], Fabian [7], Phelps [21], and references therein containing various characterizations of Asplund spaces and efficient conditions ensuring the Asplund property.

One of the most important characterizations of Asplund spaces used in variational analysis is that a Banach space $X$ is Asplund if and only if each of its separable subspaces has a separable dual. The class of Asplund spaces is sufficiently broad including every Banach space with a Fréchet smooth bump function (in particular, every space with an equivalent Fréchet smooth norm and every reflexive space) as well as any Banach space with a separable dual. Although the Asplund property is generally related to the Fréchet-type differentiability, there are Asplund spaces that fail to have even a Gâteaux smooth norm; see Haydon’s examples in [5, Chapter VII].

A systematic study and applications of Asplund spaces in the framework of variational analysis and generalized differentiation have been started by Mordukhovich and Shao [16,17], while earlier significant results on the so-called “fuzzy sum rule” for Fréchet subdifferentials were obtained by Ioffe [11] and Fabian [6]. After the first variational characterization of Asplund spaces established in [16] in the form of the extremal principle, subsequent results in this direction were derived by Fabian and Mordukhovich [8], Zhu [27], and the others; see [4, Chapter 6] and [14, Chapter 2] for more discussions and references.

The recent two-volume monograph by Mordukhovich [14,15] contains a thorough development and exposition of the key issues of variational analysis and generalized differentiation, mainly in the framework of Asplund spaces, with numerous applications to problems in optimization, equilibrium, stability, ODE and PDE control, economics, and mechanics. Among important topics considered particularly in [14, Section 4.4] and [15, Sections 5.2 and 5.3] are those concerning sensitivity/stability analysis and necessary optimality conditions for optimization and equilibrium problems with the so-called equilibrium constraints. Variational systems of this type were introduced in the seminal work by Robinson [22] under the name of parameterized generalized equations (GE) given by

$$0 \in f(x, y) + Q(y) \quad (1.1)$$

with the decision variable $y \in Y$ and the parameter $x \in X$, where $f : X \times Y \to Z$ is a single-valued mapping while $Q : Y \Rightarrow Z$ is a set-valued one. Robinson actually introduced and studied model (1.1) for the setting when $Q(y) = N(y; \Omega)$ is the normal cone to a convex set $\Omega$ at $y \in \Omega$, in which case the generalized equation (1.1), or “variational condition” in the terminology of Rockafellar and Wets [23], reduces to the parametric variational inequality:

$$\text{find } y \in \Omega \text{ such that } \langle f(x, y), v - y \rangle \geq 0 \text{ for all } v \in \Omega. \quad (1.2)$$

The classical parametric complementarity system corresponds to (1.2) when $\Omega$ is the nonnegative orthant in $\mathbb{R}^n$. It is well known that the latter model covers sets of optimal solutions with the associated Lagrange multipliers and sets of Karush–Kuhn–Tucker (KKT) vectors satisfying first-order necessary optimality conditions in parametric problems of nonlinear programming with smooth data. More general models with parameter-dependent field mappings $Q(x, y)$ in (1.1) have also been, but to much lesser extent, considered in the literature. They are related to the so-called “quasivariational inequalities” in the extended framework of (1.2). We refer the reader to the recent texts [10,14,15,20] and bibliographies therein that contain various results, discussions, and applications regarding the parametric generalized equations and variational systems of types (1.1) and (1.2) as well as their modifications and remarkable specifications in both finite and infinite dimensions. Note that in infinite-dimensional spaces models of these types are closely associated with variational problems arising in partial differential equations.

It occurs nevertheless that generalized equation and variational inequality models of types (1.1) and (1.2) with single-valued base mappings $f(x, y)$, respectively, do not encompass a number of variational systems important in optimization theory and applications. Consider, e.g., the parametric optimization problem
minimize $\varphi(x, y) + \psi(x, y)$ over $y \in Y$, \hspace{1em} (1.3)
described by a cost function $\varphi$ and a constraint function $\psi$ that generally take their values in the extended real line $\mathbb{R} := (-\infty, \infty]$. The stationary point multifunction associated with (1.3) is defined by

$$S(x) := \{ y \in Y \mid 0 \in \partial_x \varphi(x, y) + \partial_y \psi(x, y) \}$$ \hspace{1em} (1.4)

via collections of partial subgradients (i.e., partial subdifferentials in an appropriate sense) of the cost and constraint functions with respect to the decision variable. If the cost function $\varphi$ in (1.3) is smooth, then $\partial_x \varphi(x, y) = \{ \nabla_y \varphi(x, y) \}$ and thus (1.4) can be written as the solution map to a generalized equation of type (1.1) with the base mapping $f(x, y) = \nabla_y \varphi(x, y)$ and the parameter-dependent field mapping $Q(x, y) = \partial_y \psi(x, y)$. In this case a local sensitivity analysis of stationarity point multifunctions was conducted by Levy and Mordukhovich [12] in the finite-dimensional setting. However, in the case of nonsmooth optimization in (1.3) the stationary point multifunction (1.4) cannot be written as the standard GE (1.1), even with a parameter-dependent field, while requiring the extended formalism

$$0 \in F(x, y) + Q(x, y),$$ \hspace{1em} (1.5)

where both the base mapping $F$ and the field mapping $Q$ are set-valued.

Another interesting and important class of variational systems that can be written in the extended GE form (1.5) but not in the conventional one (1.1) is described by the so-called set-valued/generalized variational inequalities:

$$\text{find } y \in \Omega \text{ such that there is } y^* \in F(x, y) \text{ with } [y^*, v - y] \geq 0 \text{ for all } v \in \Omega,$$ \hspace{1em} (1.6)

which provides a set-valued extension of (1.2). We refer the reader to the handbook by Yao and Chadi [26] for the theory and applications of (1.6) and related models. Let us finally mention the recent paper by Bao, Gupta and Mordukhovich [2] devoted to necessary optimality conditions for multiobjective optimization problems with equilibrium constraints described by extended GE of type (1.5).

The main intention of this paper is to conduct a local variational analysis of the extended generalized equations (1.5) governed by set-valued mappings $F$ and $Q$ and their specifications, with the emphasis on robust Lipschitzian stability of the solution maps

$$S(x) := \{ y \in Y \mid 0 \in F(x, y) + Q(x, y) \}$$ \hspace{1em} (1.7)

with respect to the parameter $x \in X$ in the case of Asplund spaces $X$ and $Y$. Our analysis is based on dual-space generalized differentiation revolving around the coderivative notion for (set-valued) mappings. The latter concept was originally introduced by Mordukhovich [13] motivated by applications to optimal control and then has been widely used in variational analysis and its numerous applications allowing us, in particular, to establish complete dual characterizations of Lipschitzian and related properties of mappings between finite-dimensional and infinite-dimensional spaces; see, e.g., the books [14,23] for more details and references. To conduct such an analysis, we develop new calculus rules for coderivatives and related objects, which are certainly of independent interests.

The rest of the paper is organized as follows. In Section 2 we present basic definitions and preliminaries from variational analysis and generalized differentiation broadly used in formulations and proofs of the main results in the subsequent sections.

Section 3 is devoted to various intersection rules for generalized normals to nonconvex sets and coderivatives of set-valued mappings, which play a crucial role in deriving calculus and stability results of this paper. These rules are used to obtain upper estimates for coderivatives of solution maps (1.7) to extended generalized equations and their specifications and to establish related calculus results in Section 4.

The final Section 5 contains some applications of the calculus results obtained above as well as the coderivative characterizations of Lipschitzian properties of general multifunctions between Asplund spaces to derive efficient conditions that ensure robust Lipschitzian stability of solution maps to extended generalized equations (1.5) and also for a broad class of restrictive range mappings that naturally and frequently arise in many situations. To establish these results, we develop and employ—besides generalized differential calculus—certain preservation/calculus rules for the so-called sequential normal compactness properties of sets and mappings, which are automatic in finite dimensions while being of crucial importance for variational analysis and optimization in infinite-dimensional spaces.

Our notation is basically standard; cf. [14,23]. Unless otherwise stated, all the spaces under consideration are Asplund with their norms denoted by $\| \cdot \|$. Given a (generic) Banach space $X$ and its topological dual $X^*$ equipped with the weak* topology $w^*$, denote their closed unit balls by $\mathbb{B}$ and $\mathbb{B}^*$, respectively. Recall that the symbol
\[
\limsup_{x \to \bar{x}} F(x) := \left\{ x^* \in X^* \mid \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \rightharpoonup x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N} \right\}
\]  

stands for the sequential Painlevé–Kuratowski upper/outer limit of a set-valued mapping \( F : X \rightrightarrows X^* \) in the norm topology of \( X \) and weak* topology of \( X^* \), where \( \mathbb{N} := \{1, 2, \ldots \} \).

2. Basic definitions and preliminaries

In this section we define, for the reader’s convenience, some basic constructions and properties from variational analysis and generalized differentiation needed in what follows. All these are taken from the book by Mordukhovich [14], where the reader can find more details, discussions, and references. The reader may also consult the books by Borwein and Zhu [4], Rockafellar and Wets [23], and Schirotzek [24] for related and additional material.

Since all the spaces under consideration are Asplund, which is our standing assumption, we adjust the given definitions and properties to this case referring the reader to [14] for the corresponding modifications in other (including arbitrary) Banach space settings.

Given a nonempty set \( \Omega \subset X \), define the Fréchet normal cone to \( \Omega \) at \( \bar{x} \in \Omega \) by

\[
\widehat{N}(\bar{x}; \Omega) := \left\{ x^* \in X^* \mid \limsup_{x \to \bar{x}} \frac{(x^*, x - \bar{x})}{\|x - \bar{x}\|} \leq 0 \right\},
\]

where the symbol \( x \to \bar{x} \) signifies that \( x \to \bar{x} \) with \( x \in \Omega \). Construction (2.1) looks as an adaptation of the idea of Fréchet derivative to the case of sets; that is, where the name comes from. However, this construction does not have a number of natural properties expected from an appropriate notion of normals. In particular, we may have \( \widehat{N}(\bar{x}; \Omega) = \{0\} \) for boundary points of \( \Omega \) even in finite dimensions \( (X = \mathbb{R}^2) \), and required calculus rules often fail for (2.1). Letting for convenience \( \widehat{N}(\bar{x}; \Omega) = \emptyset \) if \( x \notin \Omega \) and employing the outer limit (1.8) to \( \widehat{N}(\cdot; \Omega) \), we define its sequential regularization of \( \widehat{N}(\cdot; \Omega) \) by

\[
N(\bar{x}; \Omega) := \limsup_{x \to \bar{x}} \widehat{N}(x; \Omega)
\]

known as the (basic, limiting, Mordukhovich) normal cone to \( \Omega \) at \( \bar{x} \in \Omega \). Both constructions (2.1) and (2.2) reduce to the classical normal cone of convex analysis for convex sets \( \Omega \). In contrast to (2.1), the basic normal cone (2.2) is often nonconvex while satisfying the required properties and calculus rules in the Asplund space setting, together with the corresponding coderivative constructions for set-valued mappings and subdifferential constructions for extended-real-valued functions generated by it. All this calculus is mainly due to the extremal/variational principles of variational analysis; see [14].

Given a set-valued mapping/multifunction \( F : X \rightrightarrows Y \) with the graph

\[
gph F := \left\{ (x, y) \in X \times Y \mid y \in F(x) \right\}
\]

and following the pattern introduced in [13], define the coderivative constructions for \( F \) used in this paper. The Fréchet coderivative of \( F \) at \((\bar{x}, \bar{y}) \in \text{gph } F\) is given by

\[
\hat{D}^* F(\bar{x}, \bar{y})(y^*) := \left\{ x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}(\bar{x}, \bar{y}; \text{gph } F) \right\}, \quad y^* \in Y^*,
\]

and the normal coderivative of \( F \) at the reference point is given by

\[
D^* \Sigma F(\bar{x}, \bar{y})(y^*) := \left\{ x^* \in X^* \mid (x^*, -y^*) \in N(\bar{x}, \bar{y}; \text{gph } F) \right\}, \quad y^* \in Y^*.
\]

We also need the following modification of the normal coderivative (2.4) called the mixed coderivative of \( F \) at \((\bar{x}, \bar{y}) \) and defined by

\[
D^*_{\Sigma} F(\bar{x}, \bar{y})(y^*) := \left\{ x^* \in X^* \mid \exists (x_k, y_k) \rightharpoonup (\bar{x}, \bar{y}) \text{ with } (x_k^*, -y_k^*) \in \widehat{N}(x_k, y_k; \text{gph } F) \right\},
\]

where \( \rightharpoonup \) stands for the norm convergence in the dual space; we usually omit the symbol \( \| \cdot \| \) indicating the norm convergence simply by “\( \rightharpoonup \)” and also skip \( \bar{y} = f(\bar{x}) \) in the coderivative notation if \( F = f : X \to Y \) is a single-valued mapping. Recall that \( F : X \rightrightarrows Y \) is strongly coderivatively normal at \((\bar{x}, \bar{y})\) if
\[ D^{\ast}_{M} F(\tilde{x}, \tilde{y})(y^\ast) = D^{\ast}_{N} F(\tilde{x}, \tilde{y})(y^\ast) \quad \text{for all } y^\ast \in Y^\ast. \] (2.6)

Besides the obvious case of \( \dim Y < \infty \) (when both coderivatives (2.4) and (2.5) reduce to the original construction introduced in [13]), property (2.6) holds in many other situations listed, e.g., in [14, Proposition 4.9], while it may fail even for Lipschitzian single-valued mappings with values in arbitrary Hilbert spaces; see [14, Example 1.35]. In general all the three coderivatives defined above are positively homogeneous multifunctions from \( Y^\ast \) to \( X^\ast \) satisfying the inclusions

\[ \tilde{D}^{\ast} F(\tilde{x}, \tilde{y})(y^\ast) \subseteq D^{\ast}_{M} F(\tilde{x}, \tilde{y})(y^\ast) \subseteq D^{\ast}_{N} F(\tilde{x}, \tilde{y})(y^\ast) \quad \text{for all } y^\ast \in Y^\ast, \]

where the equalities hold if, in particular, either \( F \) is graph-convex or \( F = f \) is single-valued and smooth around \( \tilde{x} \) (or merely strictly differentiable at this point). At the latter case we have the following relations with the classical derivative \( \nabla f(\tilde{x}) \):

\[ \tilde{D}^{\ast} f(\tilde{x})(y^\ast) = D^{\ast}_{M} f(\tilde{x})(y^\ast) = D^{\ast}_{N} f(\tilde{x})(y^\ast) = \{ \nabla f(\tilde{x})^\ast y^\ast \} \quad y^\ast \in Y^\ast, \] (2.7)

which show that the coderivative notion is a natural extension of the adjoint derivative operator to nonsmooth and set-valued mappings.

Among the most important ingredients of variational analysis and generalized differentiation in infinite-dimensional spaces are the so-called “normal compactness” properties of sets and mappings, which are automatic in finite dimensions while playing a crucial role in infinite-dimensional variational theory and applications. In this paper we need the following general versions of such properties defined in the products of Asplund spaces, which are well known to be also Asplund.

Given a set \( \Omega \subset \prod_{j=1}^{m} X_j \) and an index set \( J \subset \{1, \ldots, m\} \), we say that \( \Omega \) is partially sequentially normally compact (PSNC) at \( \tilde{x} \in \Omega \) with respect to \( \{X_j \mid j \in J\} \) if for any sequences \( x_k \overset{\Omega}{\to} \tilde{x} \) and \( x_k^\ast = (x_{1k}^\ast, \ldots, x_{mk}^\ast) \in \hat{N}(x_k; \Omega) \) one has

\[ [x_{jk}^\ast \overset{\text{w}}{\to} 0, \ j \in J, \ \text{and} \ \|x_{jk}^\ast\| \to 0, \ j \in \{1, \ldots, m\} \setminus J] \Rightarrow \|x_{jk}^\ast\| \to 0, \ j \in J, \ \text{as } k \to \infty. \]

The set \( \Omega \) is strongly PSNC at \( \tilde{x} \) with respect to \( \{X_j \mid j \in J\} \) if for any sequences \( x_k \overset{\Omega}{\to} \tilde{x} \), and \( x_k^\ast = (x_{1k}^\ast, \ldots, x_{mk}^\ast) \in \hat{N}(x_k; \Omega) \) we have

\[ [x_{jk}^\ast \overset{\text{w}}{\to} 0, \ j \in 1, \ldots, m] \Rightarrow \|x_{jk}^\ast\| \to 0, \ j \in J, \ \text{as } k \to \infty. \]

In the extreme case of \( J = \{1, \ldots, m\} \), both PSNC properties defined above do not depend on the product structure and reduce to the so-called sequential normal compactness (SNC) property of \( \tilde{x} \). The latter one is always implied by the “compactly epi-Lipschitzian” property of Borwein and Strójwas [3], while these two properties are closely interrelated; see a comprehensive study and (counter)examples in Fabian and Mordukhovich [9].

Given a set-valued mapping \( F : X \rightrightarrows Y \), we associate it with the graph \( \text{gph } F \) that belongs to the product space \( X \times Y \). Thus the PSNC/SNC properties of the graph in the above senses induce the corresponding properties of \( F \). In this vein we say that the mapping \( F \) is PSNC at \( (\tilde{x}, \tilde{y}) \in \text{gph } F \) if its graph is PSNC at this point with respect to \( X \), and similarly for the strong PSNC property of \( F \). Recall that the PSNC property of \( F \) at \( (\tilde{x}, \tilde{y}) \) always holds when \( F \) is Lipschitz-like (or has the Aubin property) around \( (\tilde{x}, \tilde{y}) \) with some modulus \( \ell \geq 0 \), in the sense that there are neighborhoods \( U \) of \( \tilde{x} \) and \( V \) of \( \tilde{y} \) such that

\[ F(x) \cap V \subseteq F(u) + \ell\|x - u\|B \quad \text{whenever } x, u \in U. \] (2.8)

The infimum of all moduli \( \{\ell\} \) in (2.8) is called the exact Lipschitzian bound of \( F \) around \( (\tilde{x}, \tilde{y}) \) and is denoted by \( \text{lip } F(\tilde{x}, \tilde{y}) \).

When \( V = Y \) in (2.8), this property reduces to the classical (Hausdorff) local Lipschitzian behavior of \( F \) around \( \tilde{x} \). Note that these Lipschitzian properties are robust with respect to small perturbations of the initial data. A general Lipschitzian type sufficient condition for the strong PSNC property of \( F \) is established in [14, Theorem 1.75]. It is well known that the Lipschitz-like property of \( F \) around \( (\tilde{x}, \tilde{y}) \) is equivalent to the two other fundamental properties in nonlinear analysis applied to the inverse mapping \( F^{-1} : Y \rightrightarrows X \); namely, to the metric regularity of \( F^{-1} \) and to the linear openness of the inverse mapping around \( (\tilde{y}, \tilde{x}) \).
The following coderivative characterization of the Lipschitz-like property obtained by Mordukhovich [14, Theorem 4.10] is the basis for applications of the coderivative calculus developed in this paper to robust Lipschitzian stability of the extended generalized equations (1.5) and their specifications. Note that this result provides not only necessary and sufficient conditions for the Lipschitz-like property for general set-valued mappings but also establishes lower and upper estimates of the exact Lipschitzian bound. Thus we get the precise coderivative formula for computing \( \text{lip } F(\bar{x}, \bar{y}) \) when the coderivative norms \( \|D^*_MF(\bar{x}, \bar{y})\| \) and \( \|D^*_NF(\bar{x}, \bar{y})\| \) agree; see [14, Proposition 4.9] for efficient conditions ensuring this property in infinite-dimensional spaces. Recall that the norm of a positively homogeneous mapping \( G : X \rightrightarrows Y \) (which is the case of the above coderivatives as set-valued mappings from \( Y^* \) to \( X^* \)) is defined by
\[
\|G\| := \sup\{\|y\| \mid y \in G(x), \|x\| \leq 1\}.
\]

**Theorem 2.1** (Dual characterization of the Lipschitz-like property for general multifunctions). Let a set-valued mapping \( F : X \rightrightarrows Y \) be closed-graph around a given point \((\bar{x}, \bar{y}) \in \text{gph } F\). Then \( F \) is Lipschitz-like around this point if and only if
\[
D^*_MF(\bar{x}, \bar{y})(0) = \{0\}
\]
and the mapping \( F \) is PSNC at \((\bar{x}, \bar{y})\). Furthermore,
\[
\|D^*_MF(\bar{x}, \bar{y})\| \leq \text{lip } F(\bar{x}, \bar{y}) \leq \|D^*_NF(\bar{x}, \bar{y})\|,
\]
where the upper estimate holds if \( \dim X < \infty \).

3. Intersection rules for coderivatives of multifunctions

In this section we establish new intersection rules for Fréchet and normal coderivatives of set-valued mappings between Asplund spaces that are of independent interest while playing an important role to derive other calculus and stability results in the subsequent sections. The intersection rules obtained below are given in the “fuzzy form” for the Fréchet coderivative (2.3) and in the “exact/pointbased form” for the normal coderivative defined in (2.4).

First let us recall the following fuzzy relation for Fréchet normals to set intersections, which was derived in [14, Lemma 3.1] from the extremal principle; see [14] for more details and discussions. It has been recently proved in [25] that this fuzzy intersection result is actually a characterization of Asplund spaces. Recall that a set \( \Omega \subset X \) is said to be closed around \( \bar{x} \in \Omega \) if there is \( \gamma > 0 \) that the set \( \Omega \cap (\bar{x} + \gamma B) \) is closed in \( X \).

**Lemma 3.1** (Fuzzy relation for Fréchet normals to set intersections with no qualification). Let \( \Omega_1, \Omega_2 \subset X \) be arbitrary sets that are locally closed around \( \bar{x} \in \Omega_1 \cap \Omega_2 \), and let \( x^* \in \bar{N}(\bar{x}; \Omega_1 \cap \Omega_2) \). Then for any \( \varepsilon > 0 \) sufficiently small there are \( \lambda > 0 \), \( x_i \in \Omega_i \cap (\bar{x} + \varepsilon B) \), and \( x_i^* \in \bar{N}(x_i; \Omega_i) + \varepsilon B^* \) as \( i = 1, 2 \) such that
\[
\lambda x^* = x_1^* + x_2^* \quad \text{with} \quad \max\{\lambda, \|x_i^*\|\} = 1.
\]

Note that Lemma 3.1 does not provide a rule for representing Fréchet normals to set intersections, since the multiplier \( \lambda \) in (3.1) may be equal to zero. On the other hand, the above result does not involve any qualification condition. Let us formulate appropriate (fuzzy and limiting) qualification conditions, which ensure that \( \lambda = 1 \) in (3.1) and allow us to derive intersection rules for all the coderivatives under consideration. We will see in what follows that the fuzzy qualification condition combined with the corresponding SNC/PSNC properties of sets and mappings imply the limiting one, while the latter condition and its specifications are much more convenient for the further usage and applications, especially in the case of mappings with the natural product structure of their graphs.

**Definition 3.2** (Major qualification conditions). Let \( \Omega_1, \Omega_2 \subset X \), and let \( \bar{x} \in \Omega_1 \cap \Omega_2 \). We say that:

(i) The pair \( \{\Omega_1, \Omega_2\} \) satisfies the fuzzy qualification condition around \( \bar{x} \) if there is \( \gamma > 0 \) such that
\[
(\bar{N}(x_1; \Omega_1) + \gamma B^*) \cap (\bar{N}(x_2; \Omega_2) + \gamma B^*) \subset \frac{1}{2} B^*
\]
for all \( x_i \in \Omega_i \cap (\bar{x} + \gamma B) \), \( i = 1, 2 \).
(ii) The pair \( \{ \Omega_1, \Omega_2 \} \) satisfies the limiting qualification condition at \( \bar{x} \) if for any sequence \( x_{i_k} \overset{\Omega_1}{\to} \bar{x} \) and \( x_{i_k}^* \overset{w^*}{\to} x_i^* \) with \( x_{i_k}^* \in \bar{N}(x_{i_k}; \Omega_i) \) as \( i = 1, 2 \) and \( k \to \infty \) we have
\[
\|x_{i_k}^* + x_{i_k}^*\| \to 0 \quad \Rightarrow \quad x_i^* = x_i^* = 0.
\]

(iii) Given two set-valued mappings \( F_1 : X \rightrightarrows Y \) and \( F_2 : X \rightrightarrows Y \), we say that the pair \( \{ F_1, F_2 \} \) satisfies the fuzzy (respectively limiting) qualification condition around (respectively at) the point \((\bar{x}, \bar{y}) \in gph F_1 \cap gph F_2\) if the corresponding condition from (i) and (ii) holds for the set pair \( \{ gph F_1, gph F_2 \} \).

It is easy to see that the limiting qualification condition from Definition 3.2(ii) is implied by the normal qualification condition for sets \( \{ \Omega_1, \Omega_2 \} \) at \( \bar{x} \in \Omega_1 \cap \Omega_2 \) expressed via the basic/limiting normal cone (2.2) by
\[
N(\bar{x}; \Omega_1) \cap (-N(\bar{x}; \Omega_2)) = \{0\}.
\]

The opposite implication does not generally hold unless \( X \) is finite-dimensional. Applying (3.3) to the graphs \( \{ gph F_1, gph F_2 \} \) of \( F_1 : X \rightrightarrows Y \), we get the corresponding normal qualification condition for set-valued mappings \( \{ F_1, F_2 \} \), which can be explicitly expressed via the normal coderivative (2.4).

To establish the main results of this section on intersection rules for coderivatives of set-valued mappings, we need another lemma showing that the qualification and PSNC/SNC conditions discussed above imply the proper fuzzy intersection rule for Fréchet normals with a uniform boundedness estimate of the corresponding normals in the decomposition.

**Lemma 3.3 (Fuzzy intersection rule for Fréchet normals).** Let the sets \( \Omega_1 \) and \( \Omega_2 \) be locally closed around \( \bar{x} \in \Omega_1 \cap \Omega_2 \), and let the pair \( \{ \Omega_1, \Omega_2 \} \) satisfy the fuzzy qualification condition (3.2) around this point. Then there exists a number \( \eta > 0 \) such that for every \( \varepsilon > 0 \) and every \( x^* \in \bar{N}(\bar{x}; \Omega_1 \cap \Omega_2) \) there are elements \( x_i \in \Omega_i \cap (\bar{x} + \varepsilon \mathbb{B}) \) and \( x_i^* \in \bar{N}(x_i; \Omega_i) \) as \( i = 1, 2 \) satisfying the relations
\[
\|x^* - x_i^* - x_i^*\| \leq \varepsilon \quad \text{and} \quad \|x_i^*\| \leq \eta(1 + \|x^*\| + \varepsilon),
\]

which imply the fuzzy intersection rule for Fréchet normals:
\[
\bar{N}(\bar{x}; \Omega_1 \cap \Omega_2) \subset \bigcup_{\|x_i - \bar{x}\| \leq \varepsilon, i=1,2} \left[ \bar{N}(x_i; \Omega_i) + \bar{N}(x_i; \Omega_i) \right] + \varepsilon \mathbb{B}^*.
\]

**Proof.** Note that the validity of (3.5) automatically requires that \( x_i \in \Omega_i \) as \( i = 1, 2 \) under the union sign. The estimates in (3.4) follow from [18, Theorem 3.7] with a rather complicated proof based on the extremal principle. Let us now show that the fuzzy intersection rule (3.5) can be directly derived from Lemma 3.1. To proceed, we assume the opposite and thus find a number \( \varepsilon > 0 \) and a Fréchet normal \( x^* \in \bar{N}(\bar{x}; \Omega_1 \cap \Omega_2) \) such that
\[
\|x^* - x_i^* - x_i^*\| > \varepsilon \quad \text{whenever} \quad x_i \in \Omega_i \cap (\bar{x} + \varepsilon \mathbb{B}) \quad \text{and} \quad x_i^* \in \bar{N}(x_i; \Omega_i), \ i = 1, 2.
\]

Taking this normal \( x^* \) and a sequence \( \varepsilon_k \downarrow 0 \) as \( k \to \infty \), apply Lemma 3.1 to the pair \( (x^*, \varepsilon_k) \) for each \( k \in \mathbb{N} \) and find sequences \( x_{i_k} \to \bar{x}, \ x_{i_k}^* \in \bar{N}(x_{i_k}; \Omega_i) \) as \( i = 1, 2 \), and \( \lambda_k \geq 0 \) satisfying the relations
\[
\|\lambda_k x^* - x_{i_k}^* - x_{i_k}^*\| \to 0 \quad \text{and} \quad \max\{\lambda_k, \|x_{i_k}^*\|\} \to 1 \quad \text{as} \quad k \to \infty.
\]

By the second relation in (3.7), suppose without loss of generality that \( \lambda_k \to \lambda \in [0, 1] \) as \( k \to \infty \). If \( \lambda > 0 \), we get from the first relation in (3.7) that \( x^* - x_{i_k}^*/\lambda - x_{i_k}^*/\lambda \to 0 \) as \( k \to \infty \), which obviously contradicts (3.6) for large \( k \in \mathbb{N} \). In the remaining case of \( \lambda = 0 \) we conclude from (3.7) that
\[
\|x_{i_k}^* + x_{i_k}^*\| \to 0 \quad \text{and} \quad \|x_{i_k}^*\| \to 1 \quad \text{as} \quad k \to \infty,
\]

which contradicts the fuzzy qualification condition (3.2) and thus justifies (3.5). \( \square \)

Employing Lemma 3.3 together with the qualification and SNC conditions discussed above, we establish next the new fuzzy coderivative rule estimating the Fréchet coderivative (2.3) of the intersection mapping
\[
F(x) := F_1(x) \cap F_2(x), \quad x \in X,
\]

at \((\bar{x}, \bar{y}) \in gph F\) via the Fréchet coderivative for each \( F_i \) around the reference point.
Theorem 3.4 (Fuzzy intersection rule for the Fréchet coderivative). Let $F : X \rightrightarrows Y$ be the intersection mapping \((3.8)\) generated by set-valued mappings $F_1$ and $F_2$ between Asplund spaces whose graphs are locally closed around $(\bar{x}, \bar{y}) \in \text{gph} \, F_1 \cap \text{gph} \, F_2$. Then there exists a number $\eta > 0$ such that for every $\varepsilon > 0$ and every $x^* \in \hat{D}^* \, F(\bar{x}, \bar{y})(y^*)$ there are
\[(x_i, y_i) \in \text{gph} \, F_i \cap \{(\bar{x}, \bar{y}) + \varepsilon \mathbb{B}\} \text{ and } x_i^* \in \hat{D}^* \, F_i(x_i, y_i)(y_i^*), \quad i = 1, 2,
\]
satisfying the relations
\[
\|x^* - x_1^* - x_2^*\| \leq \varepsilon, \quad \|y^* - y_1^* - y_2^*\| \leq \varepsilon, \quad \text{and} \quad \|(x_1^*, y_1^*)\| \leq \eta(1 + \|(x^*, y^*)\| + \varepsilon)
\]
in each of the following cases:
(a) The pair $\{F_1, F_2\}$ satisfies the fuzzy qualification condition around $(\bar{x}, \bar{y})$.
(b) The pair $\{F_1, F_2\}$ satisfies the limiting qualification condition at $(\bar{x}, \bar{y})$, and either $F_1$ is PSNC at $(\bar{y}, \bar{x})$ and $F_2^{-1}$ is strongly PSNC at $(\bar{y}, \bar{x})$, or $F_1^{-1}$ is PSNC at $(\bar{y}, \bar{x})$ and $F_2$ is strongly PSNC at $(\bar{x}, \bar{y})$.
(c) The pair $\{F_1, F_2\}$ satisfies the limiting qualification condition at $(\bar{x}, \bar{y})$, and one of the mappings $F_1$ and $F_2$ is SNC at $(\bar{x}, \bar{y})$.

In particular, the above assumptions ensure the fuzzy coderivative intersection rule
\[
\hat{D}^* \, F(\bar{x}, \bar{y})(y^*) \subseteq \bigcup_{\max\{\|(x_i, y_i) - (\bar{x}, \bar{y})\| \leq \varepsilon, i = 1, 2\}, \max\{\|x^* - x_1^* - x_2^*\| \leq \varepsilon\}} \left[\hat{D}^* \, F_1(x_1, y_1)(y_1^*) + \hat{D}^* \, F_2(x_2, y_2)(y_2^*)\right] + \varepsilon \mathbb{B}^*.
\]

Proof. To justify the results of the theorem in case (a), we simply observe that
\[
\text{gph} \, F = \text{gph} \, F_1 \cap \text{gph} \, F_2 \subset X \times Y
\]
in the Asplund space $X \times Y$ and apply to these sets the corresponding results for Fréchet normals from Lemma 3.3 combined with definition (2.3) of the Fréchet coderivative.

To proceed in case (b) under the limiting qualification condition, suppose for definiteness (the other combination in (b) can be treated similarly) that $F_1$ is PSNC at $(\bar{x}, \bar{y})$ and $F_2^{-1}$ is strongly PSNC at $(\bar{y}, \bar{x})$. This means, according to the definitions of these properties, that the graph of $F_1$ is PSNC at $(\bar{x}, \bar{y})$ with respect to $X$ while the graph of $F_2$ is strongly PSNC at this point with respect to $Y$. Let us show that in this case the limiting qualification condition for $\{F_1, F_2\}$ implies the fulfillment of the fuzzy qualification condition for these mappings. Assuming by contradiction that the latter condition does not hold for $\{F_1, F_2\}$, we find sequences $(x_{ik}, y_{ik})$ \(\text{gph} \, F_i\) $(\bar{x}, \bar{y})$ as $k \to \infty$ and $(x_{ik}^*, y_{ik}^*) \in \tilde{N}(x_{ik}, y_{ik}; \text{gph} \, F_i), i = 1, 2$, such that
\[
\|(x_{ik}^*, y_{ik}^*) + (x_{2k}^*, y_{2k}^*)\| \to 0 \text{ as } k \to \infty \quad \text{and} \quad \frac{1}{3} \leq \|(x_{ik}^*, y_{ik}^*)\| \leq 2, \quad k \in \mathbb{N}.
\]
(3.9)

Since $X$ and $Y$ are Asplund and thus the unit ball in $X^* \times Y^*$ is weak* sequentially compact, both sequences $\{(x_{ik}^*, y_{ik}^*)\}$ contain subsequences that weak* converge to some $(x_i^*, y_i^*) \in X^* \times Y^*$, $i = 1, 2$. Employing the first relation in (3.9) and the limiting qualification condition for $\{F_1, F_2\}$, we get
\[
(x_{ik}^*, y_{ik}^*) \rightharpoonup^* (0, 0) \quad \text{and} \quad (x_{2k}^*, y_{2k}^*) \rightharpoonup^* (0, 0) \quad \text{as } k \to \infty.
\]
(3.10)

Further, by the assumed strong PSNC property of $F_2^{-1}$ at $(\bar{y}, \bar{x})$, we have from the second relation in (3.10) that $\|y_{2k}^*\| \to 0$ and hence, by the first one in (3.9), $\|y_{ik}^*\| \to 0$ as well. In turn, the PSNC property of $F_1$ at $(\bar{x}, \bar{y})$ and the first relation in (3.10) yield that $\|x_{ik}^*\| \to 0$ as $k \to \infty$. The latter contradicts the second relation in (3.9) and justifies the results of the theorem in case (b).

The proof given in case (b) surely applies to case (c), with a simplification. Indeed, the SNC assumption on either $F_1$ or $F_2$ imposed in case (c) implies that the weak* convergence of the corresponding pair in (3.10) ensures its norm convergence to $(0, 0)$. The latter immediately contradicts (3.9) and thus completes the proof of the theorem. \(\Box\)

It turns out that the assumptions imposed in Theorem 3.4 ensure the validity of the exact/point-based intersection rule for the normal coderivative (2.4).
Theorem 3.5 (Exact intersection rule for the normal coderivative). Let \( F : X \rightrightarrows Y \) be the intersection mapping defined in (3.8) under the assumptions of Theorem 3.4. Then we have the coderivative intersection rule
\[
D_N^F \tilde{x}(\tilde{y})(y^*) \subset \bigcup_{y_1^* + y_2^* = y^*} \left[ D_N^{F_1} \tilde{x}(\tilde{y})(y_1^*) + D_N^{F_2} \tilde{x}(\tilde{y})(y_2^*) \right], \quad y^* \in Y^*.
\] (3.11)

**Proof.** To justify (3.11) in case (a) of Theorem 3.4, pick any \((x^*, y^*) \in X^* \times Y^*\) satisfying \(x^* \in D_N^F \tilde{x}(\tilde{y})(y^*)\). Using the normal coderivative definition (2.4), find sequences \((x_k, y_k) \to (\tilde{x}, \tilde{y})\) and \((x_k^*, y_k^*) \to (x^*, y^*)\) as \(k \to \infty\) such that
\[
(x_k, y_k) \in \text{gph } F \quad \text{and} \quad x_k^* \in D_N^F \tilde{x}(x_k, y_k)(y_k^*), \quad k \in \mathbb{N}.
\] (3.12)

Observe that the fuzzy qualification condition imposed on \(\{F_1, F_2\}\) around \((\tilde{x}, \tilde{y})\) allows us to employ the fuzzy coderivative intersection rule of Theorem 3.4 at \((x_k, y_k)\) in (3.12) for all \(k \in \mathbb{N}\) sufficiently large. In this way we find a number \(\eta > 0\) such that for any selected sequence \(\varepsilon_k \downarrow 0\) as \(k \to \infty\) there are elements
\[
(x_{ik}, y_{ik}) \in \text{gph } F_i \quad \text{and} \quad x_{ik}^* \in \hat{D}^F_{\tilde{x}}(x_{ik}, y_{ik})(y_{ik}^*), \quad i = 1, 2,
\] (3.13)
satisfying the relations \(\|x_{ik} - (x_k, y_k)\| \leq \varepsilon_k\) for \(i = 1, 2\) and
\[
\begin{align*}
\|x_k^* - x_{1k}^* - x_{2k}^*\| &\leq 2\varepsilon_k, \\
\|y_k^* - y_{1k}^* - y_{2k}^*\| &\leq 2\varepsilon_k, \\
\|x_{ik}^* - y_{ik}^*\| &\leq \eta(1 + \|x_{ik}^* - y_{ik}^*\| + \varepsilon_k) \quad \text{for all } k \in \mathbb{N}.
\end{align*}
\] (3.14)

Since the sequence \(\{(x_{ik}^*, y_{ik}^*)\}\) weak* converges, it is bounded in \(X^* \times Y^*\) by the classical uniform boundedness principle. By (3.14) this implies the boundedness of both sequences \(\{(x_{ik}^*, y_{ik}^*)\}\) in \(X^* \times Y^*\), \(i = 1, 2\). As in the proof of Theorem 3.4, we suppose without loss of generality (by the Asplund property of \(X\) and \(Y\)) that there are \((x_i^*, y_i^*) \in X^* \times Y^*\) with
\[
(x_{ik}^*, y_{ik}^*) \rightharpoonup (x_i^*, y_i^*) \quad \text{as } k \to \infty, \quad i = 1, 2.
\]
It follows directly from (3.13) and the normal coderivative construction (2.4) that the inclusions \(x_i^* \in D_N^F \tilde{x}(\tilde{y})(y_i^*)\) hold for \(i = 1, 2\). By passing to the limit as \(k \to \infty\) in the first line of (3.14) and using the lower semicontinuity of the dual norm in the weak* topology, we arrive at the equalities
\[
x^* = x_1^* + x_2^* \quad \text{and} \quad y^* = y_1^* + y_2^*,
\]
which justify (3.11) and complete the proof of the theorem in case (a). To justify the theorem in cases (b) and (c) listed in Theorem 3.4, we employ the relations between the qualification conditions under consideration established in the proof of Theorem 3.4. This completes the proof of this theorem. \(\Box\)

As a simple particular case of the above theorem, we have the validity of the coderivative intersection rule (3.11) if either \(F_1\) or \(F_2\) is SNC at \((\tilde{x}, \tilde{y})\) and the pair \(\{F_1, F_2\}\) satisfies the normal qualification condition
\[
N((\tilde{x}, \tilde{y}); \text{gph } F_1) \cap [ -N((\tilde{x}, \tilde{y}); \text{gph } F_2) ] = \{0\}.
\]
In this case the coderivative rule (3.11) was derived in [14, Proposition 3.20] by reducing it to the intersection rule for limiting normals.

Let us present useful consequences of the intersections rules obtained in Theorems 3.4 and 3.5 to estimating the Fréchet and normal coderivatives for the so-called restrictive range mappings \(G : X \rightrightarrows Y\) given by
\[
G(x) := F(x) \cap \Theta \quad \text{with } F : X \rightrightarrows Y \text{ and } \Theta \subset Y,
\] (3.15)
which are important in many applications; see, e.g., [14,15,23]. In particular, the domain of \(G\) from (3.15) is the inverse image (or preimage) of the set \(\Theta\) under the mapping \(F\):
\[
\text{dom } G = F^{-1}(\Theta) := \{x \in X \mid F(x) \cap \Theta \neq \emptyset\}.
\]
The following new results of fuzzy and exact coderivative calculus are, from one viewpoint, specifications of those obtained in Theorems 3.4 and 3.5 for the mappings
F_1(x) := F(x) \quad \text{and} \quad F_2(x) \equiv \Theta \quad (3.16)

while, on the other hand, the specific structures of F_1 and F_2 in (3.16) allow us to express the limiting qualification condition for \{F_1, F_2\} = \{\Theta, \Theta\} in the fully pointbased form via the so-called reversed mixed coderivative of F defined by

\[ \tilde{D}_M^* F(\bar{x}, \bar{y})(y^*) := -D_M^* F^{-1}(\bar{y}, \bar{x})(-y^*), \quad y^* \in Y^*. \quad (3.17) \]

Recall also that for any mapping H : X \rightrightarrows Y the kernel of H is denoted by

\[ \ker H := \{ x \in X \mid 0 \in H(x) \}. \]

**Corollary 3.6 (Coderivatives of restrictive range mappings).** Let \( \bar{y} \in F(\bar{x}) \cap \Theta \), where F : X \rightrightarrows Y is locally closed-graph around (\bar{x}, \bar{y}), and where \( \Theta \subset Y \) is locally closed around \( \bar{x} \). Suppose also that one of the following groups of assumptions holds:

(a) The pair \( \{F, \Theta\} \) satisfies the fuzzy qualification condition around (\( \bar{x}, \bar{y} \)): there is \( \gamma > 0 \) such that

\[ (\tilde{N}(x, y); gph F) + yB^* + \gamma B^* + \tilde{N}(v; \Theta) \subseteq \frac{1}{2}B^* \]

for all \( (x, y) \in gph F \cap ((\bar{x}, \bar{y}) + \gamma B) \) and \( v \in \Theta \cap (\bar{y} + B) \).

(b) The pair \( \{F, \Theta\} \) satisfies the pointbased qualification condition at \( (\bar{x}, \bar{y}) \):

\[ \ker \tilde{D}_M^* F(\bar{x}, \bar{y}) \subseteq \tilde{N}(\bar{y}; \Theta) = \{0\} \]

and either \( \Theta \) is SNC at \( \bar{y} \) or \( F^{-1} \) is PSNC at \( \bar{y} \).

Then, considering the restrictive range mapping \( G(x) := F(x) \cap \Theta \) and taking any \( \varepsilon > 0 \) and \( y^* \in Y^* \), we have the fuzzy upper estimate

\[ \tilde{D}^* G(\bar{x}, \bar{y})(y^*) \subseteq \bigcup_{\|(x, y)-(\bar{x}, \bar{y})\| \leq \varepsilon} \left[ \tilde{D}^* F(x, y)(y^* + \tilde{N}(v; \Theta) + \varepsilon B^*) \right] + \varepsilon B^* \quad (3.19) \]

of its Fréchet coderivative and the pointbased upper estimate of the normal coderivative

\[ D_N^* G(\bar{x}, \bar{y})(y^*) \subseteq D_N^* F(\bar{x}, \bar{y})(y^* + \tilde{N}(\bar{y}; \Theta)). \quad (3.20) \]

**Proof.** Representing the restrictive range mapping G from (3.15) in the intersection form \( F_1(x) \cap F_2(x), x \in X \), with \( F_1 \) defined in (3.16) and taking into account that the Fréchet coderivative of \( F_2 \) is computed by

\[ \tilde{D}^* F_2(u, v)(v^*) = \begin{cases} 0 & \text{if } -v^* \in \tilde{N}(v; \Theta), \ u \in X, \ v \in \Theta, \\ \emptyset & \text{otherwise}, \end{cases} \quad (3.21) \]

we can easily observe that the fuzzy qualification condition from Definition 3.2(i) reduces to the one formulated in (a) of this corollary and that the fuzzy intersection formula from Theorem 3.4 reduces to (3.19) for the mapping G under consideration.

Furthermore, since the formula for computing the normal coderivative \( D_N^* F_2 \) of the mapping \( F_2(x) \equiv \Theta \) is similar to (3.21) with just replacing \( \tilde{N}(v; \Theta) \) by \( N(v; \Theta) \), we deduce the normal coderivative estimate (3.20) from the intersection formula (3.11) of Theorem 3.5. To complete the proof of this corollary, it remains to specify in case (3.16) the limiting qualification condition and the PSNC/SNC properties required in assumptions (b) and (c) of Theorem 3.4.

It is easy to check that the limiting qualification condition for \( \{F, \Theta\} \) reads: for any sequences \( (x_k, y_k) \rightrightarrows (\bar{x}, \bar{y}), \|x_k^*\| \to 0, y_k^* \rightharpoonup y^*, v_k \rightharpoonup \bar{y}, \) and \( v_k \rightharpoonup v^* \) as \( k \to \infty \) with \( (x_k^*, y_k^*) \in \tilde{N}((x_k, y_k); gph F) \) and \( v_k^* \in \tilde{N}(v_k; \Theta), k \in \mathbb{N} \), one has

\[ \|y_k^* + v_k^*\| \to 0 \quad \Rightarrow \quad y^* = v^* = 0. \quad (3.22) \]
It follows from the above constructions and definitions (2.2) and (3.17) that \( v^* \in N(\bar{y}; \Theta) \) and \( 0 \in \tilde{D}_M^* F(\bar{x}, \bar{y})(-y^*) \). Passing to the limit in (3.22) as \( k \to \infty \), we thus conclude that the limiting qualification condition for \( \{F, \Theta\} \) is equivalent to the pointbased one (3.18) imposed in the corollary. By similar arguments we can check that all the possible PSNC/SNC alternatives listed in cases (b) and (c) of Theorem 3.4 for the mappings \( F_1 \) and \( F_2 \) from (3.16) reduce to the best two formulated in case (b) of the corollary. \( \square \)

4. Coderivatives of solution maps and more calculus

In this section we employ the coderivative intersection rule derived in the previous section for estimating the normal coderivative of solution maps to extended generalized equations and establish also some related calculus results of independent interest. In fact, this approach allows us to establish a variety of other calculus rules for normals to sets and coderivatives of set-valued mappings; see the discussions below.

Let us start with a simple but useful proposition that does not seem to be mentioned in the literature. It is partly employed in what follows while being of certain interest for its own. Recall that, given a mapping \( F : X \rightrightarrows Y \), there is a straightforward relation

\[
\text{rge } F = \text{dom } F^{-1} := \{ y \in Y \mid F^{-1}(y) \neq \emptyset \}
\]

(4.1)

between its range \( \text{rge } F := F(X) \) and the domain of the inverse mapping \( F^{-1} \). Recall also that \( F \) is inner semicontinuous at \((\bar{x}, \bar{y}) \in gph F \) if for every sequence \( x_k \to \bar{x} \) with \( x_k \in \text{dom } F \) there is a sequence \( y_k \in F(x_k) \) that converges to \( \bar{y} \) as \( k \to \infty \). This mapping is inner semicompact at \( \bar{x} \) if for every sequence \( x_k \to \bar{x} \) there is a sequence \( y_k \in F(x_k) \) that contains a convergent subsequence.

**Proposition 4.1 (Basic normals to domains and ranges of mappings).** Let \( F : X \rightrightarrows Y \) be an arbitrary set-valued mapping, and let \( \bar{x} \in \text{dom } F \). The following assertions hold:

(i) Assume that for some \( \bar{y} \in F(\bar{x}) \) the mapping \( F \) is inner semicontinuous at \((\bar{x}, \bar{y}) \). Then we have the inclusion

\[
N(\bar{x}; \text{dom } F) \subset D_M^* F(\bar{x}, \bar{y})(0),
\]

(4.2)

and furthermore the domain set \( \text{dom } F \) is SNC at \( \bar{x} \) provided that \( F \) is PSNC at \((\bar{x}, \bar{y}) \). If \( F \) is merely semicompact at \( \bar{x} \), then

\[
N(\bar{x}; \text{dom } F) \subset \bigcup_{\bar{y} \in F(\bar{x})} D_M^* F(\bar{x}, \bar{y})(0).
\]

(ii) Given \( \bar{y} \in \text{rge } F \) and \( \bar{x} \in F^{-1}(\bar{y}) \), we have the inclusions

\[
N(\bar{y}; \text{rge } F) \subset \text{ker } \tilde{D}_M^* F(\bar{x}, \bar{y}), \text{ respectively } N(\bar{y}; \text{rge } F) \subset \bigcup_{\bar{x} \in F^{-1}(\bar{y})} \text{ker } \tilde{D}_M^* F(\bar{x}, \bar{y})
\]

if \( F^{-1} \) is inner semicontinuous at \((\bar{y}, \bar{x}) \), respectively inner semicompact at \( \bar{y} \).

**Proof.** We justify inclusion (4.2) under the inner semicontinuity assumption on \( F \); similar arguments work in the case of inner semicompactness. Take \( x^* \in N(\bar{x}; \text{dom } F) \) and by (2.2) find sequences \( x_k \to \bar{x} \) and \( x_k^* \rightharpoonup x^* \) as \( k \to \infty \) satisfying the inclusions

\[
x_k \in \text{dom } F \quad \text{and} \quad x_k^* \in \bar{N}(x_k; \text{dom } F) \quad \text{for all } k \in \mathbb{N}.
\]

Since \( F \) is inner semicontinuous at \((\bar{x}, \bar{y}) \), there are \( y_k \in F(x_k) \) such that \( y_k \to \bar{y} \) as \( k \to \infty \). It is easy to observe from the definition of Fréchet normals (2.1) that

\[
(x_k^*, 0) \in \bar{N}(x_k; \text{rge } F) \quad \text{for all } k \in \mathbb{N}.
\]

(4.3)

Thus, setting \( y_k^* \equiv 0 \) in definition (2.5) of the mixed coderivative, we get \( x^* \in D_M^* F(\bar{x}, \bar{y})(0) \) and hence justify (4.2). It also follows directly from (4.3) and the PSNC/SNC definitions that the PSNC property of \( F \) at \((\bar{x}, \bar{y}) \) implies the
SNC property of the domain of $F$ at $\bar{x}$, which completes the proof of assertion (i). Assertion (ii) is a consequence of (i) due to formula (4.1) and the obvious relation

$$D^*_M F^{-1}(\bar{y}, \bar{x})(0) = \ker \tilde{D}^*_M F(\bar{x}, \bar{y})$$

between the mixed coderivative (2.5) of the inverse $F^{-1}$ and the kernel of the reversed mixed coderivative (3.17) of the mapping $F$.  \(\square\)

The next theorem is certainly of independent interest while implying upper estimates for the normal coderivative of solution maps to extended generalized equations of type (1.5). Its proof is based on using the intersection rule for set-valued mappings from Theorem 3.5 and Proposition 4.1 established above.

**Theorem 4.2 (Basic normals to kernels of sums for set-valued mappings).** Consider the kernel set

$$\Omega := \ker (F_1 + F_2) = \{ x \in X \mid 0 \in F_1(x) + F_2(x) \}$$

(4.4)

generated by two set-valued mappings $F_1$, $F_2$: $X \rightrightarrows Y$, and let $\bar{x} \in \Omega$. The following assertions hold:

(i) Suppose that there is $\bar{y} \in F_1(\bar{x}) \cap (-F_2(\bar{x}))$ such that $F_1$ and $F_2$ are locally closed-graph around $(\bar{x}, \bar{y})$ and $(\bar{x}, -\bar{y})$, respectively, that the intersection mapping $F_1 \cap (-F_2)$ is inner semicontinuous at $(\bar{x}, \bar{y})$, and that one of the assumption groups (a)–(c) from Theorem 3.4 is satisfied for $\{F_1, -F_2\}$. Then we have the inclusion

$$N(\bar{x}; \Omega) \subset \bigcup_{y^* \in Y^*} \left[ D^*_N F_1(\bar{x}, \bar{y})(y^*) + D^*_N F_2(\bar{x}, -\bar{y})(y^*) \right].$$

(4.5)

(ii) Suppose that $F_1 \cap (-F_2)$ is inner semicompact at $\bar{x}$ and that the assumptions of (i) are fulfilled at for all $\bar{y} \in F_1(\bar{x}) \cap (-F_2(\bar{x}))$. Then we have

$$N(\bar{x}; \Omega) \subset \bigcup_{\bar{y} \in F_1(\bar{x}) \cap (-F_2(\bar{x}))} \bigcup_{y^* \in Y^*} \left[ D^*_N F_1(\bar{x}, \bar{y})(y^*) + D^*_N F_2(\bar{x}, -\bar{y})(y^*) \right].$$

(4.6)

**Proof.** First observe that the set $\Omega$ in (4.4) can be represented in the domain form

$$\Omega = \text{dom}(F_1 \cap (-F_2)) = \{ x \in X \mid F_1(x) \cap (-F_2)(x) \neq \emptyset \},$$

and thus we can apply Proposition 4.1(i) to estimate the normal cone to this set. We proceed with justifying inclusion (4.5) in case (i) of the theorem, which is based on the application of Proposition 4.1(i) in the case of inner semicontinuity; the proof of (4.6) is similar. It follows from (4.2) and the relation between the mixed and normal coderivatives that

$$N(\bar{x}; \Omega) \subset D^*_M (F_1 \cap (-F_2))(\bar{x}, \bar{y})(0) \subset D^*_N (F_1 \cap (-F_2))(\bar{x}, \bar{y})(0).$$

(4.7)

Applying now inclusion (3.11) of Theorem 3.5 with $y^*_1 + y^*_2 = 0$ to the intersection mapping in (4.7), we have the estimate

$$N(\bar{x}; \Omega) \subset \bigcup_{y^* \in Y^*} \left[ D^*_N F_1(\bar{x}, \bar{y})(y^*) + D^*_N (-F_2)(\bar{x}, \bar{y})(-y^*) \right]$$

under the assumptions made in assertion (i) of the theorem. The latter inclusion obviously reduces to (4.5), since

$$D^*_N (-F)(\bar{x}, \bar{y})(-y^*) = D^*_N F(\bar{x}, -\bar{y})(y^*)$$

for any mapping $F$. This completes the proof of the theorem.  \(\square\)

The results obtained in Theorem 4.2 easily imply upper estimates for the normal coderivative of solution maps (1.7) to general parametric variational systems.
Corollary 4.3 (Normal coderivative estimates for solution maps to generalized equations). Let \( S : X \rightrightarrows Y \) be the solution map (1.7) to the extended generalized equation with both set-valued base mapping \( F : X \times Y \rightrightarrows Z \) and field mapping \( Q : X \times Y \rightrightarrows Z \). Given \((\bar{x}, \bar{y}) \in \text{gph} S\), we have the following estimates of the normal coderivative of \( S \) at \((\bar{x}, \bar{y})\):

(i) Suppose that there is \( \bar{z} \in F(\bar{x}, \bar{y}) \cap (-Q(\bar{x}, \bar{y})) \) such that \( F \) and \( Q \) are locally closed-graph around \((\bar{x}, \bar{y}, \bar{z})\) and \((\bar{x}, \bar{y}, -\bar{z})\), respectively, that the intersection mapping \( F \cap (-Q) \) is inner semicontinuous at \((\bar{x}, \bar{y}, \bar{z})\), and that one of the assumption groups (a)–(c) from Theorem 3.4 is satisfied for \{\( F, -Q \)\} at \((\bar{x}, \bar{y}, \bar{z})\). Then for all \( y^* \in Y^* \) we have the inclusion

\[
D^*_N S(\bar{x}, \bar{y})(y^*) \subset \{x^* \in X^* \mid (x^* - y^*) \in D^*_N F(\bar{x}, \bar{y}, \bar{z})(z^*) + D^*_N Q(\bar{x}, \bar{y}, -\bar{z})(\bar{z}^*), \; \bar{z}^* \in Z^* \}.
\]

(ii) Suppose that \( F \cap (-Q) \) is inner semicompact at \((\bar{x}, \bar{y})\) and that the assumptions of (i) are fulfilled whenever \( \bar{z} \in F(\bar{x}, \bar{y}) \cap (-Q(\bar{x}, \bar{y})) \). Then for all \( y^* \in Y^* \) we have

\[
D^*_N S(\bar{x}, \bar{y})(y^*) \subset \{x^* \in X^* \mid \exists \bar{z}^* \in Z^* \text{ and } \bar{z} \in F(\bar{x}, \bar{y}) \cap (-Q(\bar{x}, \bar{y})) \text{ with } (x^* - y^*) \in D^*_N F(\bar{x}, \bar{y}, \bar{z})(\bar{z}^*) + D^*_N Q(\bar{x}, \bar{y}, -\bar{z})(\bar{z}^*) \}.
\]

Proof. Both assertions of the corollary follow directly from the corresponding assertions of Theorem 4.2 with \( F_1 = F : X \times Y \rightrightarrows Z \) and \( F_2 = Q : X \times Y \rightrightarrows Z \) due to definition (2.4) of the normal coderivative and the fact that

\[
\text{gph} S = \{(x, y) \in X \times Y \mid 0 \in F_1(x, y) + F_2(x, y)\}
\]

for the solution map \( S \) in (1.7) and the mappings \( F_i, \; i = 1, 2 \), under consideration.

Remark 4.4 (Comparison with known results). The coderivative inclusions for solution maps obtained in Corollary 4.3 significantly improve the known results in this directions derived by a different way and summarized in [14, Theorem 4.46]. Indeed, only the case of single-valued bases \( F = f : X \times Y \rightrightarrows Z \) is considered in [14], which automatically ensures the fulfillment of the inner semicontinuity property of \( F \cap (-Q) \) imposed in (i), assuming also that the normal qualification condition (3.3) holds for \{\( F, -Q \)\} and that either \( Q \) is SNC at \((\bar{x}, \bar{y}, -\bar{z})\), or \( \dim Z < \infty \) and \( f \) is locally Lipschitzian around \((\bar{x}, \bar{y})\)—the latter assumptions imply the SNC property of \( f \) at \((\bar{x}, \bar{y})\). We can easily see that the qualification conditions and PSNC/SNC properties formulated in (a)–(c) of Theorem 3.4 and imposed on \{\( F, -Q \)\} in this corollary offer a much larger variety of efficient conditions for the validity of the coderivative estimate in (i) even for single-valued base mappings \( f \). In particular, the PSNC (vs. SNC) property of \( f \) holds for any locally Lipschitzian base mapping \( f \) with no restriction on \( \dim Z < \infty \); the same is true for the Lipschitz-like property in the case of set-valued base mappings \( F : X \times Y \rightrightarrows Z \).

Remark 4.5 (More calculus rules for coderivatives). Besides the above applications of Theorem 4.2 to coderivative estimates of solution maps to extended generalized equations, this theorem provides various opportunities to derive coderivatives calculus rules for set-valued mappings. Observe, in particular, that the sum of mappings \( F_i : X \rightrightarrows Y_i \), \( i = 1, 2 \), can be represented in the kernel form (4.4) as

\[
F_1(x) + F_2(x) = \{y \in Y \mid 0 \in F_1(x) - y + F_2(x)\}
\]

and the composition \( F \circ G \) of mappings \( G : X \rightrightarrows Y \) and \( F : Y \rightrightarrows Z \) can be written as

\[
(F \circ G)(x) = \{z \in Z \mid 0 \in F(x) + (-G^{-1}(z))\}.
\]

In this way, applying Theorem 4.2 to the above representations, we recover various sum and chain rules for set-valued mappings derived in [14] by using different approaches and also establish new calculus results in this direction under general fuzzy qualification conditions induced by (a) of Theorem 3.4, which are not considered in [14].

On the other hand, we can apply the coderivative sum rule from [14, Theorem 3.10] (the strongest known result in this direction) to the summation mapping \( F + Q \) in (1.7) and derive in this way a version of Corollary 4.3 from coderivative estimates of solution maps to nonstructural inclusions \{\( y \in Y \mid 0 \in G(x, y) \)\}. Such a device leads us to some particular cases of Corollary 4.3 under more restrictive assumptions in comparison with those imposed in the above corollary, which is derived with no use of any sum rule.
Remark 4.6 (Composite subdifferential forms of generalized equations). Many variational systems that are most interesting for applications correspond to generalized equations of type (1.5), where either field mappings $Q$ or/and base mappings $F$ admit certain subdifferential representations. In particular, this is the case of stationary point multifunctions defined in (1.4). In [14, Section 4.4], the reader can find several specifications of normal coderivative estimates for solution maps to conventional generalized equations (1.1), including also some cases of those with $Q = Q(x,y)$, when field mappings are given in one of the composite subdifferential forms

$$Q(x,y) = \partial(\varphi \circ g)(x,y) \quad \text{and} \quad Q(x,y) = (\partial \varphi \circ g)(x,y)$$

with $\varphi : X \times Y \rightarrow W$ and $\varphi : W \rightarrow \mathbb{R}$. The proofs of these coderivative estimates obtained in [14] for solution maps to generalized equations with composite subdifferential fields of types (4.8) are based on calculus rules for second-order subdifferentials of extended-real-valued functions defined by the scheme

$$\partial^2 \psi := D^*(\partial \psi)$$

via coderivatives of first-order subdifferentials; see [14] for more details. Using these developments and the new results from Corollary 4.3, we can obtain efficient estimates of solutions maps to extended generalized equations given by (1.5), where not only field mappings $Q$ but also base mappings $F$ admit the subdifferential representations of types (4.8). The latter case is particularly important for handling stationary point multifunctions (1.4) and corresponding KKT systems arising from first-order optimality conditions in problems of nonsmooth composite optimization whose costs are described by the so-called amenable functions playing a crucial role in many aspects of variational analysis and optimization; see, e.g., [23]. In subsequent research we intend to thoroughly investigate these and related issues in optimization theory with applications to specific classes of variational systems.

5. Lipschitzian stability of restrictive range mappings and extended generalized equations

In this section we employ the calculus results obtained above and the coderivative characterization of the Lipschitz-like property for arbitrary set-valued mappings from Theorem 2.1 to derive verifiable conditions for robust Lipschitzian stability of extended generalized equations and restrictive range mappings. Our approach is based on applying the explicit coderivative formulas established in Sections 3 and 4 for the classes of set-valued mappings under consideration and also—in the case of mappings between infinite-dimensional spaces—on deriving efficient conditions that ensure the fulfillment of the PSNC property for the structural mappings of our study expressed via their initial data. The latter requires yet another mixed qualification condition for sets in product spaces, which is situated between the limiting and normal qualification conditions discussed in Section 3; see [14, Section 3.3].

Definition 5.1 (Mixed qualification condition for set systems in product spaces). Given sets $\Omega_1, \Omega_2 \subset X \times Y$, we say that the set system $\{\Omega_1, \Omega_2\}$ satisfies the mixed qualification condition relative to $Y$ at $(\bar{x}, \bar{y}) \in \Omega_1 \cap \Omega_2$ if for any sequences $(x_{1k}, y_{1k}) \xrightarrow{\Omega_1} (\bar{x}, \bar{y})$ and $(x_{ik}^*, y_{ik}^*) \xrightarrow{w^*} (x_{i}^*, y_{i}^*)$ as $k \rightarrow \infty$ with

$$(x_{ik}^*, y_{ik}^*) \in \hat{N}_i((x_{ik}, y_{ik}); \Omega_i) \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad k \in \mathbb{N},$$

we have the implication

$$\left[x_{1k}^* + x_{2k}^* \xrightarrow{w^*} 0, \|y_{1k}^* + y_{2k}^*\| \rightarrow 0\right] \Rightarrow (x_1^*, y_1^*) = (x_2^*, y_2^*) = (0,0).$$

(5.1)

It is easy to see that, being considered in the product space $X \times Y$, the mixed qualification condition from Definition 5.1 is implied by the normal qualification condition (3.3) in the space $X \times Y$ and implies in turn the limiting qualification condition from Definition 3.2(ii) in this space; the latter conditions do not take into account the product structure of the space in question. Indeed, the normal qualification condition for $\{\Omega_1, \Omega_2\}$ corresponds to the weak* convergence of both components in (5.1), while the limiting one involves the norm convergence of both terms therein. Note that the product structure of spaces is intrinsic in the case of graphical sets for mappings $F : X \rightrightarrows Y$.

We start with establishing pointbased conditions for robust Lipschitzian stability, with exact bound estimates, for restrictive range mappings $F(x) \cap \Theta$ defined in (3.15). To the best of our knowledge, these questions have not been considered in the literature, except some specific situations.
Theorem 5.2 (Lipschitzian stability of restrictive range mappings). Let \( G : X \rightrightarrows Y \) be given by \( G(x) := F(x) \cap \Theta \), where \( \Theta \subset Y \) is locally closed around \( \bar{y} \in \Theta \), and where \( F : X \rightrightarrows Y \) is locally closed-graph around \((\bar{x}, \bar{y}) \in \text{gph} \, F\). Assume that

\[
\ker D_N^* F(\bar{x}, \bar{y}) \cap \text{N}(\bar{y}; \Theta) = \{0\},
\]

(5.2)

\[
D_N^* F(\bar{x}, \bar{y}) \circ \text{N}(\bar{y}; \Theta) := \bigcup_{y^* \in \text{N}(\bar{y}; \Theta)} D_N^* F(\bar{x}, \bar{y})(y^*) = \{0\},
\]

(5.3)

and that either \( F \) is SNC at \((\bar{x}, \bar{y})\), or \( F \) is PSNC at this point and \( \Theta \) is SNC at \( \bar{y} \). Then \( G \) is Lipschitz-like around \((\bar{x}, \bar{y})\). If in addition \( \dim X < \infty \), we have the upper estimate

\[
\text{lip} \, G(\bar{x}, \bar{y}) \leq \sup \{ \| x^* \| \mid x^* \in D_N^* F(\bar{x}, \bar{y})(y^* + \text{N}(\bar{y}; \Theta)), \| y^* \| \leq 1 \}
\]

(5.4)

for the exact Lipschitzian bound of the restrictive range mapping.

Proof. It is easy to see that the mapping \( G(x) = F(x) \cap \Theta \) is locally closed-graph around \((\bar{x}, \bar{y})\) under the closedness assumptions of the theorem. Provided the fulfillment of the assumptions in Corollary 3.6, condition (5.3) immediately comes from formula (3.20) for upper estimating the normal coderivative of the restrictive range mapping obtained therein and the coderivative criterion

\[
D^*_M G(\bar{x}, \bar{y})(0) \subset D^*_N G(\bar{x}, \bar{y})(0) = \{0\}
\]

for the Lipschitz-like property given in (2.9) of Theorem 2.1. Furthermore, the upper estimate (5.4) for the exact Lipschitzian bound of \( G \) follows from that in (2.10) and the coderivative inclusion (3.20) when \( \dim X < \infty \). We obviously get the validity of all the assumptions in group (b) of Corollary 3.6 under the fulfillment of the qualification condition (5.2) and the SNC/PSNC properties imposed in this theorem. To complete the proof of the theorem, it remains to show that the latter assumptions ensure the PSNC property of \( G \) at \((\bar{x}, \bar{y})\), which allows us to meet the other characterization requirement of Theorem 2.1.

Since \( \text{gph} \, G = \text{gph} \, F \cap (X \times \Theta) \), we have by the PSNC calculus rule from [14, Corollary 3.80] that \( \text{gph} \, G \) is PSNC at \((\bar{x}, \bar{y})\) if either \( \text{gph} \, F \) is SNC and \( X \times \Theta \) is PSNC, or \( \text{gph} \, F \) is PSNC and \( X \times \Theta \) is SNC at this point provided that the mixed qualification condition from Definition 5.1 holds for the set system \( \{ \text{gph} \, F, X \times \Theta \} \). By the structure of \( X \times \Theta \), this set is obviously PSNC with respect to \( X \) being SNC at \((\bar{x}, \bar{y})\) if and only if \( \Theta \) is SNC at \((\bar{x}, \bar{y})\). Let us finally check the mixed qualification condition for \( \{ \text{gph} \, F, X \times \Theta \} \). It reads in the case under consideration as

\[
[x_k^* \xrightarrow{w} 0, \| y_{1k}^* + y_{2k}^* \| \xrightarrow{w} 0] \quad \Rightarrow \quad y_1^* = y_2^* = 0
\]

(5.5)

for any sequences \((x_k^*, y_{1k}^*) \in \hat{N}(x_k, y_{1k}; \text{gph} \, F) \) and \( y_{2k}^* \in \hat{N}(y_{2k}; \Theta), k \in N, \) with \( (y_{1k}^*, y_{2k}^*) \xrightarrow{w} (y_1^*, y_2^*), (x_k, y_{1k}) \xrightarrow{\text{gph} \, F} (\bar{x}, \bar{y}), \) and \( y_{2k} \xrightarrow{\Theta} \bar{y} \) as \( k \to \infty \). Thus we have \((0, y_1^*) \in N((\bar{x}, \bar{y}); \text{gph} \, F) \) and \( y_2^* \in N(\bar{y}; \Theta) \); so (5.5) reduces to the assumed qualification condition (5.2). This completes the proof of the theorem. \( \square \)

Our final result in this paper establishes verifiable conditions ensuring the validity of Lipschitz-like property of solution maps (1.7) to extended generalized equations. As in Theorem 5.2, these conditions are expressed in terms of the robust coderivative constructions for the initial data computed at the reference points. Applying the coderivative estimates of Corollary 4.3, we consider for brevity only the case of inner semicontinuity in assertion (i) therein. The case of inner semicompactness can be considered similarly.

Theorem 5.3 (Lipschitzian stability of solution maps to extended generalized equations). Let \( S : X \rightrightarrows Y \) be the solution map to the extended generalized equations defined in (1.7) with \( F : X \times X \rightrightarrows Z \) and \( Q : X \times Y \rightrightarrows Z \), and let \((\bar{x}, \bar{y}) \in \text{gph} \, S \). Given \( \bar{z} \in F(\bar{x}, \bar{y}) \cap (-Q(\bar{x}, \bar{y})) \), assume that the mapping \( F \cap (-Q) \) is inner semicontinuous at \((\bar{x}, \bar{y}, \bar{z})\), that \( F \) and \( Q \) are locally closed-graph around \((\bar{x}, \bar{y}, \bar{z})\) and \((\bar{x}, \bar{y}, -\bar{z})\), respectively, that the qualification conditions

\[
[(x^*, y^* , z^*) \in D^*_N F(\bar{x}, \bar{y}, \bar{z})(z^*) \cap (-D^*_N Q(\bar{x}, \bar{y}, -\bar{z})(z^*))] \quad \Rightarrow \quad (x^*, y^*, z^*) = (0, 0, 0),
\]

(5.6)

\[
[(x^*, 0) \in D^*_N F(\bar{x}, \bar{y}, \bar{z})(z^*) + D^*_N Q(\bar{x}, \bar{y}, -\bar{z})(z^*)] \quad \Rightarrow \quad x^* = 0
\]

(5.7)
are satisfied, and that either $F$ is PSNC at $(\bar{x}, \bar{y}, \bar{z})$ and $Q$ is SNC at $(\bar{x}, \bar{y}, -\bar{z})$ or vice versa. Then $S$ is Lipschitz-like around $(\bar{x}, \bar{y})$. If in addition $\dim X < \infty$, we have the upper estimate for the exact Lipschitzian bound of $S$:

$$\lip S(\bar{x}, \bar{y}) \leq \sup \{ \| x^* \| \mid \text{there exists} \ z^* \in Z^* \ \text{with} \ (x^*, -y^*) \in D_S^\nu F(\bar{x}, \bar{y}, \bar{z})(z^*) + D_S^\nu Q(\bar{x}, \bar{y}, -\bar{z})(z^*), \ \| y^* \| \leq 1 \}. \quad (5.8)$$

**Proof.** Having the coderivative upper estimate for the solution map $S$ in Corollary 4.3(i) and using relations (2.9) and (2.10) of Theorem 2.1, we get condition (5.7) for the Lipschitz-like property of $S$ and the Lipschitzian bound estimate (5.8) provided that the qualification and SNC/PSNC assumptions of Corollary 4.3(i) are satisfied and that $S$ is PSNC at $(\bar{x}, \bar{y})$. It is easy to see that the qualification condition (5.6) and the SNC/PSNC properties of $F$ and $Q$ imposed in this theorem imply all the assumptions from group (b) of Theorem 3.4 imposed on $\{F, -Q\}$ in Corollary 4.3(i). Let us show that they also ensure the fulfillment of the SNC (and hence PSNC) property of $S$ at $(\bar{x}, \bar{y})$, which is needed to apply the characterization of Lipschitzian stability in Theorem 2.1.

To justify this statement, we employ the result of Proposition 4.1(i) that guarantees the SNC property of the set in interest

$$\gph S = \ker(F + Q) = \text{dom}(F \cap (-Q))$$

provided that the intersection mapping $[F \cap (-Q)]: X \times Y \Rightarrow Z$ is PSNC at $(\bar{x}, \bar{y}, \bar{z})$. The latter means that the intersection set

$$\gph(F \cap (-Q)) = (\gph F) \cap \gph(-Q) \subset X \times Y \times Z \quad (5.9)$$

is PSNC at $(\bar{x}, \bar{y}, \bar{z})$ with respect to $X \times Y$. By the result of [14, Corollary 3.80] on the PSNC property of set intersections in product spaces, we have this property of the graphical set (5.9) if one of the sets $\gph F$, $\gph(-Q)$ is PSNC at $(\bar{x}, \bar{y}, \bar{z})$ with respect to $X \times Y$ while the other set is SNC at this point, and if the pair $\{\gph F, \gph(-Q)\}$ satisfies the mixed qualification condition at $(\bar{x}, \bar{y}, \bar{z})$ relative to $Z$. The required SNC/PSNC properties of the sets $\gph F$ and $\gph(-Q)$ obviously reduce to those assumed in the theorem. Furthermore, it is easy to derive directly from the definitions that the mixed qualification condition for the set system $\{\gph F, \gph(-Q)\}$ is implied by the pointbased coderivative condition (5.6). This completes the proof of the theorem. \qed

The results obtained in Theorem 5.3 extend those from [14, Theorem 4.59] established by another approach for standard generalized equations with $F = f : X \times Y \rightarrow Z$ in (1.7) under the PSNC assumption on $f$ and the SNC assumption on $Q$. Similarly to the discussions in Remark 4.6 we can apply the new results to deriving verifiable conditions for Lipschitzian stability for variational systems with composite subdifferential forms of either base or field mappings, which particular cover stationary point multifunctions of type (1.4). The general assumptions imposed in Theorem 5.3 can be significantly simplified if either $F$ or $Q$ is Lipschitz-like around the reference point.

**Corollary 5.4** (Simplified conditions for Lipschitzian stability of extended generalized equations). Let either $F$ be Lipschitz-like around $(\bar{x}, \bar{y}, \bar{z})$ or $Q$ be Lipschitz-like around $(\bar{x}, \bar{y}, -\bar{z})$ in the framework of Theorem 5.3. Then $S$ is Lipschitz-like around $(\bar{x}, \bar{y})$ under the qualification conditions (5.6) and (5.7) and the SNC assumption on the other mapping at the corresponding point. If in addition to its Lipschitz-like property either $F$ or $Q$ is strongly coderivative normal at the reference point, then the qualification conditions (5.6) and (5.7) can be unified and equivalently replaced by the following one:

$$[(x^*, 0) \in D_S^\nu F(\bar{x}, \bar{y}, \bar{z})(z^*) + D_S^\nu Q(\bar{x}, \bar{y}, -\bar{z})(z^*)] \Rightarrow x^* = 0, \ z^* = 0. \quad (5.10)$$

**Proof.** It follows from Theorem 2.1 that the Lipschitz-like property of a set-valued mapping ensures its PSNC property at the reference point. Furthermore, from the coderivative criterion (2.9) in Theorem 2.1 and the assumed strong coderivative normality in (2.6) we easily derive that the fulfillment of both qualification conditions (5.6) and (5.7) in this case is equivalent to the validity of (5.10). \qed

**Acknowledgment**

The authors are grateful to the referee for his/her careful reading the paper and valuable remarks, which allowed us to improve the original presentation.
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