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Criteria for the strong regularity of J-inner functions and γ-generating matrices

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Abstract

The class of left and right strongly regular J-inner mvf's plays an important role in bitangential interpolation problems and in bitangential direct and inverse problems for canonical systems of integral and differential equations. A new criterion for membership in this class is presented in terms of the matricial Muckenhoupt condition (A_2) that was introduced for other purposes by Treil and Volberg. Analogous results are also obtained for the class of γ-generating functions that intervene in the Nehari problem. The new criterion is simpler than the criterion that we presented earlier. A determinental criterion is also presented.

Keywords: J-inner matrix valued functions; Matricial Nehari problem; γ-generating matrices; Strong regularity; Bitangential interpolation problems; Matricial Muckenhoupt condition

1. Introduction

Let J be an $m \times m$ signature matrix, let Ω_+ denote either the open unit disk $\mathbb{D}$ or the open upper half plane $\mathbb{C}_+$ and let $\mathcal{U}(J, \Omega_+)$ denote the class of $m \times m$ J-inner mvf's (matrix valued functions) with respect to Ω_+.

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We recall that an $m \times m$ mvf $U(\lambda)$ that is meromorphic in Ω_+ is said to be J-inner with respect to Ω_+ if

(1) $U(\lambda)^* J U(\lambda) \leqslant J$ for every point $\lambda \in \Omega_+$ at which U is holomorphic;
(2) $U(\mu)^* J U(\mu) = J$ for a.e. point μ on the boundary Ω_0 of Ω_+.

We remark that condition (1) insures that every entry in U is the ratio of two functions that are holomorphic and bounded in Ω_+ and hence, by Fatou's lemma, that nontangential boundary limits $U(\mu)$ exist at a.e. point $\mu \in \Omega_0$.

It is well known that if J is equal to

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad p \geqslant 1,\ q \geqslant 1,\ p+q=m, \tag{1.1}$$

and if the $m \times m$ mvf

$$W(\lambda) = \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix} \tag{1.2}$$

with diagonal blocks of sizes $p \times p$ and $q \times q$, respectively, belongs to the class $\mathcal{U}(j_{pq}, \Omega_+)$, then the linear fractional transformation

$$T_W[\varepsilon] = (w_{11}\varepsilon + w_{12})(w_{21}\varepsilon + w_{22})^{-1} \tag{1.3}$$

maps the Schur class

$$\mathcal{S}^{p\times q}(\Omega_+) = \{p \times q \text{ mvf's } \varepsilon(\lambda)\colon\ \varepsilon(\lambda) \text{ is holomorphic and contractive in } \Omega_+\} \tag{1.4}$$

into itself. A mvf $W \in \mathcal{U}(j_{pq}, \Omega_+)$ is said to belong to the class $\mathcal{U}_{rsR}(j_{pq}, \Omega_+)$ of right strongly regular j_{pq}-inner mvf's if there exists at least one mvf $\varepsilon \in \mathcal{S}^{p\times q}(\Omega_+)$ such that

$$\|T_W[\varepsilon]\|_\infty < 1. \tag{1.5}$$

There are many bitangential interpolation problems in the class $\mathcal{S}^{p\times q}(\Omega_+)$ for which the set of solutions is equal to

$$T_W[\mathcal{S}^{p\times q}(\Omega_+)] = \{T_W[\varepsilon]\colon\ \varepsilon \in \mathcal{S}^{p\times q}(\Omega_+)\}$$

for an appropriately chosen $W \in \mathcal{U}(j_{pq}, \Omega_+)$; see, e.g., [11,12]. An interpolation problem in the class $\mathcal{S}^{p\times q}(\Omega_+)$ is said to be strictly completely indeterminate if there exists at least one solution $s(\lambda)$ such that $\|s\|_\infty < 1$. There exists a two sided correspondence between the class $\mathcal{U}_{rsR}(j_{pq}, \Omega_+)$ and the class of strictly completely indeterminate generalized bitangential interpolation problems in $\mathcal{S}^{p\times q}(\Omega_+)$:

(1) If $W \in \mathcal{U}_{rsR}(j_{pq}, \Omega_+)$, then

$$T_W[\mathcal{S}^{p\times q}(\Omega_+)] = \{s \in \mathcal{S}^{p\times q}(\Omega_+)\colon\ b_1^{-1}(s - s^\circ)b_2^{-1} \in H_\infty^{p\times q}(\Omega_+)\}, \tag{1.6}$$

for some mvf $s^\circ \in \mathcal{S}^{p\times q}(\Omega_+)$ and some pair of mvf's $b_1(\lambda)$ and $b_2(\lambda)$ of sizes $p \times p$ and $q \times q$, respectively, that are inner with respect to Ω_+.

(2) To every set of mvf's defined by the right hand-side of formula (1.6) that contains a mvf $s(\lambda)$ such that $\|s\|_\infty < 1$, there corresponds an essentially unique $W \in \mathcal{U}_{rsR}(j_{pq}, \Omega_+)$ such that formula (1.6) holds.

Additional information on this correspondence may be found, e.g., in [2,9]. Identification (1.6) implies that every problem in the class $\mathcal{S}^{p\times q}(\Omega_+)$ for which the set of solutions can be expressed as $T_W[\mathcal{S}^{p\times q}(\Omega_+)]$ for some mvf $W \in \mathcal{U}_{rsR}(j_{pq}, \Omega_+)$ is equivalent to a strictly completely indeterminate generalized bitangential interpolation problem.

The class $\mathcal{U}_{rsR}(j_{pq}, \mathbb{C}_+)$ was introduced in [3] because of the central role that it played in our study of direct and inverse problems for canonical integral and differential systems [3–8].

If $U \in \mathcal{U}(J, \Omega_+)$ and

$$V \text{ is a unitary matrix such that } V^*JV = j_{pq}, \tag{1.7}$$

then $W(\lambda) = V^*U(\lambda)V$ belongs to the class $\mathcal{U}(j_{pq}, \Omega_+)$ and we say that $U \in \mathcal{U}_{rsR}(J, \Omega_+)$ if $W \in \mathcal{U}_{rsR}(j_{pq}, \Omega_+)$. In [3] it was shown that $U \in \mathcal{U}_{rsR}(J, \Omega_+)$ if and only if the $m \times 1$ vvf's (vector valued functions) in the associated RKHS (reproducing kernel Hilbert space) $\mathcal{H}(U)$ all belong to $L_2^m(\Omega_0)$ (with respect to Lebesgue measure). This criterion leads easily to the following inclusion:

$$\mathcal{U}(J, \Omega_+) \cap L_\infty^{m\times m}(\Omega_0) \subset \mathcal{U}_{rsR}(J, \Omega_+). \tag{1.8}$$

An example that shows that the inclusion (1.8) is proper if $J \neq \pm I_m$ is presented in [9, Section 7.6].

To be more precise, in a number of our papers, the class $\mathcal{U}_{rsR}(J, \Omega_+)$ is referred to as $\mathcal{U}_{sR}(J, \Omega_+)$. The class $\mathcal{U}_{\ell sR}(J, \Omega_+)$ of left strongly regular J-inner mvf's was introduced later in [9]. The definition can be formulated most simply in terms of the mvf

$$U^\sim(\lambda) = U(-\bar{\lambda})^* \quad \text{if } \Omega_+ = \mathbb{C}_+ \quad \text{and} \quad U^\sim(\lambda) = U(-1/\bar{\lambda})^* \quad \text{if } \Omega_+ = \mathbb{D} \tag{1.9}$$

as follows:

$$U \in \mathcal{U}_{\ell sR}(J, \Omega_+) \quad \Leftrightarrow \quad U^\sim \in \mathcal{U}_{rsR}(J, \Omega_+). \tag{1.10}$$

A mvf $W \in \mathcal{U}_{\ell sR}(j_{pq}, \Omega_+)$ if and only if $W \in \mathcal{U}(j_{pq}, \Omega_+)$ and

$$\begin{aligned} &\big\{(w_{22} + \varepsilon w_{12})^{-1}(w_{21} + \varepsilon w_{11}) \colon \varepsilon \in \mathcal{S}^{q\times p}\big\} \quad \text{contains at least one} \\ &\quad \text{mvf } s \in \mathcal{S}^{q\times p}(\Omega_+) \quad \text{with } \|s\|_\infty < 1. \end{aligned} \tag{1.11}$$

This fact and additional discussion of the class $\mathcal{U}_{\ell sR}(J, \Omega_+)$ may be found in [9, Section 6].

A number of other characterizations of the classes $\mathcal{U}_{\ell sR}(J, \Omega_+)$ and $\mathcal{U}_{rsR}(J, \Omega_+)$ were obtained in [9] in terms of the matricial Muckenhoupt condition (A_2) of Treil and Volberg [13]. To formulate their condition and our results for both domains $\mathbb{D}$ and $\mathbb{C}_+$, it is convenient to use a flexible notation that is spelled out in Table 1.

In the last column of Table 1 the average $A_I(\Delta)$ of a mvf Δ is always computed with respect to a finite subinterval I of Ω_0 with length $|I| > 0$.

The matricial Muckenhoupt condition (A_2) may be written as

$$\sup_I \big\| \big(A_I(\Delta)\big)^{1/2}\big(A_I(\Delta^{-1})\big)^{1/2} \big\| < \infty \tag{1.12}$$

Table 1

Ω_+	$\rho_\omega(\lambda)$	Ω_0	$\mathrm{Int}_{\Omega_0}(f)$	$A_I(\Delta)$
$\mathbb{D}$	$1-\lambda\bar{\omega}$	$\mathbb{T}$	$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,d\theta$	$\frac{1}{\lvert I\rvert}\int_I \Delta(e^{i\theta})\,d\theta$
$\mathbb{C}_+$	$-2\pi i(\lambda-\bar{\omega})$	$\mathbb{R}$	$\frac{1}{\pi}\int_{-\infty}^{\infty} f(\mu)\,\frac{d\mu}{1+\mu^2}$	$\frac{1}{\lvert I\rvert}\int_I \Delta(\mu)\,d\mu$

for matrix valued weight functions $\Delta(\mu) \geqslant 0$. In [9] we obtained the following characterization of the classes $\mathcal{U}_{rsR}(j_p, \Omega_+)$ and $\mathcal{U}_{\ell sR}(j_p, \Omega_+)$, where $j_p = j_{pp}$, that will serve both as a good illustration and a useful tool for the developments in this paper.

Theorem 1. *Let $W \in \mathcal{U}(j_p, \Omega_+)$. Then $W \in \mathcal{U}_{rsR}(j_p, \Omega_+)$ if and only if the following two conditions are met*:

(1) $\mathrm{Int}_{\Omega_0}(W^*W)$ *is finite*; (1.13)

(2) *The $p \times p$ mvf*

$$\Delta(\mu) = \{w_{21}(\mu) + w_{22}(\mu)\}^*\{w_{21}(\mu) + w_{22}(\mu)\} \tag{1.14}$$

that is defined in terms of the bottom entries in the block decomposition (1.2) *of $W(\lambda)$ meets condition* (1.12).

The mvf $W \in \mathcal{U}_{\ell sR}(j_p, \Omega_+)$ if and only if (1) *and* (2) *hold, but with*

$$\Delta(\mu) = \{w_{12}(\mu) + w_{22}(\mu)\}\{w_{12}(\mu) + w_{22}(\mu)\}^* \tag{1.15}$$

in (2).

In this article we shall first present another condition that is equivalent to the matricial Muckenhoupt condition (A_2), but is formulated in terms of determinants rather than norms and has the potential advantage of dispensing with square roots. We shall then present a new characterization of the classes $\mathcal{U}_{rsR}(J, \Omega_+)$ and $\mathcal{U}_{\ell rR}(J, \Omega_+)$. In particular, if $W \in \mathcal{U}(j_{pq}, \Omega_+)$, then this criterion is most easily formulated in terms of the off diagonal blocks of the Potapov–Ginzburg transform

$$\begin{bmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{bmatrix} = \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}^{-1} \tag{1.16}$$

as follows:

Theorem 2. *Let $W \in \mathcal{U}(j_{pq}, \Omega_+)$. Then*:

(1) *$W \in \mathcal{U}_{rsR}(j_{pq}, \Omega_+) \Leftrightarrow$ the $m \times m$ weight*

$$\Delta(\mu) = \begin{bmatrix} I_p & s_{21}(\mu)^* \\ s_{21}(\mu) & I_q \end{bmatrix}$$

meets the matricial Muckenhoupt condition (1.12);

(2) *$W \in \mathcal{U}_{\ell s R}(j_{pq}, \Omega_+) \Leftrightarrow$ the $m \times m$ weight*

$$\Delta(\mu) = \begin{bmatrix} I_p & s_{12}(\mu) \\ s_{12}(\mu)^* & I_q \end{bmatrix}$$

meets the matricial Muckenhoupt condition (1.12).

Notice that this new criterion replaces two conditions (1.13) and (1.14) (respectively, (1.15)), by a single matricial Muckenhoupt condition.

Finally, in the last section, we shall briefly discuss some analogues for the class $\mathfrak{M}(p, q)$ of γ-generating functions that play an important role in the study of the Nehari problem.

We have already noted that class of strongly regular J-inner mvf's play an important role in the study of bitangential direct and inverse problems of canonical integral and differential systems and in bitangential interpolation problems. They also play a useful role in the study of operator nodes. Every J-inner mvf $U(\lambda)$ that is holomorphic at zero can be expressed as the characteristic function of a simple operator node with main operator A equal to the backward shift $R_0 : f \to \{f(\lambda) - f(0)\}/\lambda$ acting on the RKHS $\mathcal{H}(U)$. If $U \in \mathcal{U}_{rsR}(J, \Omega_+)$, then A and $\mathcal{H}(U)$ decompose in a nice way. This and a number of related results have been obtained by Arova in her Ph.D. thesis [10].

2. Preliminaries

Lemma 3. *Let Δ be a measurable positive semidefinite $p \times p$ mvf on Ω_0 such that Δ and (Δ^{-1}) are both summable on some interval I. Then the matrix $(A_I(\Delta))^{1/2}(A_I(\Delta^{-1}))^{1/2}$ is expansive*:

$$\left\| \left(A_I(\Delta)\right)^{1/2} \left(A_I(\Delta^{-1})\right)^{1/2} \xi \right\| \geqslant \|\xi\|$$

for every vector $\xi \in \mathbb{C}^p$.

Proof. This fact is established in Corollary 3.3 of Treil and Volberg [13]. □

Lemma 4. *Let X be a $p \times p$ expansive matrix. Then*

$$\|X\| \leqslant |\det X| \leqslant \|X\|^p. \tag{2.1}$$

Proof. This is immediate from the singular value decomposition

$$X = U \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s_p \end{bmatrix} V$$

of X, since U and V are unitary, $s_1 \geqslant \cdots \geqslant s_p$ and, under the given assumptions, $s_p \geqslant 1$. □

Theorem 5. *Let $\Delta(\mu)$ be a measurable positive semidefinite $p \times p$ mvf Δ on Ω_0 such that Δ and Δ^{-1} are both summable on each subinterval I of Ω_0 with $|I| < \infty$. Then $\Delta(\mu)$ will meet the matricial Muckenhoupt condition* (1.12) *if and only if*

$$\sup_I \bigl\{\det\bigl(A_I(\Delta)\bigr)\det\bigl(A_I(\Delta^{-1})\bigr)\bigr\} < \infty. \tag{2.2}$$

Proof. In view of the last two lemmas, inequality (2.1) is directly applicable to the matrix

$$X = \bigl(A_I(\Delta)\bigr)^{1/2}\bigl(A_I(\Delta^{-1})\bigr)^{1/2}$$

and also yields the bound

$$\|X\|^2 \leqslant \bigl\{\det(X)\bigr\}^2 \leqslant \|X\|^{2p}.$$

However, this does the trick, since

$$\bigl\{\det(X)\bigr\}^2 = \det(X^2) = \det\bigl(A_I(\Delta)\bigr)\det\bigl(A_I(\Delta^{-1})\bigr). \qquad \Box$$

Let J be any $m \times m$ signature matrix, and let

$$P = (I_m + J)/2 \quad \text{and} \quad Q = (I_m - J)/2. \tag{2.3}$$

Then, since P and Q are complementary orthogonal projectors on $\mathbb{C}^m$, i.e.,

$$P = P^2 = P^*, \qquad Q = Q^2 = Q^*, \quad \text{and} \quad P + Q = I_m, \tag{2.4}$$

it is readily checked that the $2m \times 2m$ matrix

$$\tilde{V} = \begin{bmatrix} P & Q \\ Q & P \end{bmatrix} \tag{2.5}$$

is also a signature matrix and that

$$\tilde{V}\begin{bmatrix} J & 0 \\ 0 & -J \end{bmatrix}\tilde{V} = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix} = j_m. \tag{2.6}$$

Lemma 6. *Let $U \in \mathcal{U}(J, \Omega_+)$ and let*

$$\tilde{W}(\lambda) = \tilde{V}\begin{bmatrix} U(\lambda) & 0 \\ 0 & I_m \end{bmatrix}\tilde{V}. \tag{2.7}$$

Then $\tilde{W} \in \mathcal{U}(j_m, \Omega_+)$ and

(1) $\tilde{W} \in \mathcal{U}_{rsR}(j_m, \Omega_+) \Leftrightarrow U \in \mathcal{U}_{rsR}(J, \Omega_+)$;
(2) $\tilde{W} \in \mathcal{U}_{\ell sR}(j_m, \Omega_+) \Leftrightarrow U \in \mathcal{U}_{\ell sR}(J, \Omega_+)$.

Proof. It is readily checked that

$$j_m - \tilde{W}(\lambda) j_m \tilde{W}(\omega)^* = \tilde{V}\begin{bmatrix} J - U(\lambda)JU(\omega)^* & 0 \\ 0 & 0 \end{bmatrix}\tilde{V}.$$

Thus, $\tilde{W} \in \mathcal{U}(j_m, \Omega_+)$ and the RK (reproducing kernel)

$$K_\omega^{\tilde{W}}(\lambda) = \frac{j_m - \tilde{W}(\lambda) j_m \tilde{W}(\omega)^*}{\rho_\omega(\lambda)}$$

of the RKHS $\mathcal{H}(\tilde{W})$ is related to the RK

$$K_\omega^U(\lambda) = \frac{J - U(\lambda)JU(\omega)^*}{\rho_\omega(\lambda)}$$

of the RKHS $\mathcal{H}(U)$ by the formula

$$K_\omega^{\tilde{W}}(\lambda) = \tilde{V}\begin{bmatrix} K_\omega^U(\lambda) & 0 \\ 0 & 0 \end{bmatrix}\tilde{V}.$$

Therefore,

$$\mathcal{H}(\tilde{W}) = \tilde{V}\begin{bmatrix} \mathcal{H}(U) \\ \oplus \\ \{0\} \end{bmatrix}$$

and hence

$$\mathcal{H}(\tilde{W}) \subset L_2^{2m}(\Omega_0) \quad \Leftrightarrow \quad \mathcal{H}(U) \subset L_2^m(\Omega_0).$$

Consequently assertion (1) follows from the criterion for right strong regularity that was established in Theorem 6.7 of [3] and was discussed earlier. Assertion (2) then follows by applying assertion (1) to $(\tilde{W})^\sim(\lambda)$ and $U^\sim(\lambda)$. □

The proof of the preceding lemma clearly exhibits the fact that the mvf $\tilde{W}(\lambda)$ has special structure. Another consequence of this special structure is revealed in the next lemma.

Lemma 7. *Let $\tilde{w}_{ij}(\lambda)$, $i, j = 1, 2$, denote the $m \times m$ mvf's in the standard four block decompositions of the $2m \times 2m$ mvf $\tilde{W}(\lambda)$ that is defined by formula (2.7). Then*:

(1) $\operatorname{Int}_{\Omega_0}\{(\tilde{w}_{21} + \tilde{w}_{22})^*(\tilde{w}_{21} + \tilde{w}_{22})\}$ *finite* $\Rightarrow \operatorname{Int}_{\Omega_0}\{(\tilde{w}_{22}^*\tilde{w}_{22})$ *finite*;
(2) $\operatorname{Int}_{\Omega_0}\{(\tilde{w}_{12} + \tilde{w}_{22})(\tilde{w}_{12} + \tilde{w}_{22})^*\}$ *finite* $\Rightarrow \operatorname{Int}_{\Omega_0}\{(\tilde{w}_{22}\tilde{w}_{22}^*)$ *finite.*

Proof. In view of formulas (2.3)–(2.7), it is readily checked that

$$\big(\tilde{w}_{21}(\mu) + \tilde{w}_{22}(\mu)\big)^*\big(\tilde{w}_{21}(\mu) + \tilde{w}_{22}(\mu)\big) = P + U(\mu)^*QU(\mu), \tag{2.8}$$

$$\tilde{w}_{22}(\mu)^*\tilde{w}_{22}(\mu) = P + QU(\mu)^*QU(\mu)Q, \tag{2.9}$$

$$\big(\tilde{w}_{12}(\mu) + \tilde{w}_{22}(\mu)\big)\big(\tilde{w}_{12}(\mu) + \tilde{w}_{22}(\mu)\big)^* = P + U(\mu)QU(\mu)^*, \tag{2.10}$$

$$\tilde{w}_{22}(\mu)\tilde{w}_{22}(\mu)^* = P + QU(\mu)QU(\mu)^*Q. \tag{2.11}$$

Moreover,

$$\operatorname{Int}_{\Omega_0}(U^*QU) \text{ is finite} \quad \Leftrightarrow \quad \operatorname{Int}_{\Omega_0}\big(\operatorname{tr}\{U^*QU\}\big) < \infty$$

and

$$\operatorname{Int}_{\Omega_0}\big(\operatorname{tr}\{U^*QU\}\big) = \operatorname{Int}_{\Omega_0}\left(\sum_{i=1}^m \|QUu_i\|^2\right) \geqslant \operatorname{Int}_{\Omega_0}\left(\sum_{i=1}^q \|QUu_i\|^2\right)$$

for every orthonormal basis $\{u_1, \ldots, u_m\}$ of $\mathbb{C}^m$. But if the basis is chosen so that $\{u_1, \ldots, u_q\}$ is an orthonormal basis for the q-dimensional subspace $Q\mathbb{C}^m$, then the last sum on the right is equal to

$$\mathrm{Int}_{\Omega_0}\left(\sum_{i=1}^{q} \|QUQu_i\|^2\right) = \mathrm{Int}_{\Omega_0}\bigl(\mathrm{tr}\{QU^*QUQ\}\bigr).$$

This serves to justify assertion (1) and also assertion (2), since the two are equivalent. □

Lemma 8. *Let $W \in \mathcal{U}(j_{pq}, \Omega_+)$. Then the following statements are equivalent*:

(1) $\mathrm{Int}_{\Omega_0}(W^*W)$ *is finite*;
(2) $\mathrm{Int}_{\Omega_0}(w_{22}^* w_{22})$ *is finite*;
(3) $\mathrm{Int}_{\Omega_0}(WW^*)$ *is finite*;
(4) $\mathrm{Int}_{\Omega_0}(w_{22} w_{22}^*)$ *is finite.*

Proof. The proof exploits the fact that

$$W(\mu)^* j_{pq} W(\mu) = j_{pq} = W(\mu) j_{pq} W(\mu)^*$$

for a.e. point $\mu \in \Omega_0$ and that

$$\mathrm{Int}_{\Omega_0}(j_{pq}) \text{ is finite.}$$

Thus,

$$\begin{aligned}
\text{(1) holds} \quad &\Leftrightarrow \quad \mathrm{Int}_{\Omega_0}(W^*W - W^* j_{pq} W) \text{ is finite} \\
&\Leftrightarrow \quad \mathrm{Int}_{\Omega_0}\left(\begin{bmatrix} w_{21}^* \\ w_{22}^* \end{bmatrix} [w_{21} \quad w_{22}]\right) \text{ is finite} \\
&\Leftrightarrow \quad \mathrm{Int}_{\Omega_0}\bigl(\mathrm{tr}\{w_{21}^* w_{21} + w_{22}^* w_{22}\}\bigr) < \infty \\
&\Leftrightarrow \quad \mathrm{Int}_{\Omega_0}\bigl(\mathrm{tr}\{s_{21}^* w_{22}^* w_{22} s_{21} + w_{22}^* w_{22}\}\bigr) < \infty \\
&\Leftrightarrow \quad \mathrm{Int}_{\Omega_0}\bigl(\mathrm{tr}\bigl\{w_{22}^* w_{22}(I_q + s_{21} s_{21}^*)\bigr\}\bigr) < \infty \\
&\Leftrightarrow \quad \mathrm{Int}_{\Omega_0}\bigl(\mathrm{tr}\{w_{22}^* w_{22}\}\bigr) < \infty \quad \Leftrightarrow \quad \text{(2) holds.}
\end{aligned}$$

Much the same sort of argument serves to justify the equivalence of (3) and (4). Therefore, since $\mathrm{tr}\{w_{22}^* w_{22}\} = \mathrm{tr}\{w_{22} w_{22}^*\}$, all four statements are equivalent. □

3. A new characterization of strongly regular J-inner mvf's

In this section we shall formulate and establish a new characterization of each of the subclasses $\mathcal{U}_{rsR}(J, \Omega_+)$ and $\mathcal{U}_{\ell sR}(J, \Omega_+)$ of $\mathcal{U}(J, \Omega_+)$ in terms of the $m \times m$ mvf's

$$G_r(\mu) = P + U(\mu)^* QU(\mu) \quad \text{and} \quad G_\ell(\mu) = P + U(\mu) QU(\mu)^*. \tag{3.1}$$

The orthogonal projections P and Q in formula (3.1) are defined in formula (2.3). The first step is to verify the invertibility of the mvf's defined in (3.1).

Lemma 9. *Let $U \in \mathcal{U}(J, \Omega_+)$. Then the $m \times m$ mvf's $G_r(\mu)$ and $G_\ell(\mu)$ that are defined in formula* (3.1) *are invertible for a.e. point $\mu \in \Omega_0$. Moreover, if V is a unitary matrix such that $V^*JV = j_{pq}$, then*

$$V^*G_r(\mu)^{-1}V = \begin{bmatrix} I_p & s_{21}(\mu)^* \\ s_{21}(\mu) & I_q \end{bmatrix} \tag{3.2}$$

and

$$V^*G_\ell(\mu)^{-1}V = \begin{bmatrix} I_p & -s_{12}(\mu) \\ -s_{12}(\mu)^* & I_q \end{bmatrix} \tag{3.3}$$

for a.e. point $\mu \in \Omega_0$, where $s_{12}(\mu)$ and $s_{21}(\mu)$ are the off-diagonal blocks in the Potapov–Ginzburg transform of $W(\lambda) = V^*U(\lambda)V$.

Proof. It is readily checked that

$$\begin{aligned} V^*G_r(\mu)V &= \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & w_{21}(\mu)^* \\ 0 & w_{22}(\mu)^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ w_{21}(\mu) & w_{22}(\mu) \end{bmatrix} \\ &= \begin{bmatrix} I_p + w_{21}(\mu)^*w_{21}(\mu) & w_{21}(\mu)^*w_{22}(\mu) \\ w_{22}(\mu)^*w_{21}(\mu) & w_{22}(\mu)^*w_{22}(\mu) \end{bmatrix} \\ &= \begin{bmatrix} I_p & -s_{21}(\mu)^* \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & w_{22}(\mu)^*w_{22}(\mu) \end{bmatrix} \begin{bmatrix} I_p & 0 \\ -s_{21}(\mu) & I_q \end{bmatrix} \end{aligned}$$

for a.e. point $\mu \in \Omega_0$. Therefore, $G_r(\mu)$ is invertible and

$$V^*G_r(\mu)^{-1}V = \begin{bmatrix} I_p & s_{21}(\mu)^* \\ s_{21}(\mu) & I_q \end{bmatrix}.$$

Similar considerations lead easily to the formula

$$V^*G_\ell(\mu)V = \begin{bmatrix} I_p & s_{12}(\mu) \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & w_{22}(\mu)w_{22}(\mu)^* \end{bmatrix} \begin{bmatrix} I_p & 0 \\ s_{12}(\mu)^* & I_q \end{bmatrix}$$

and hence to the invertibility of $G_\ell(\mu)$ for a.e. point $\mu \in \Omega_0$ and formula (3.3). □

Theorem 10. *Let $U \in \mathcal{U}(J, \Omega_+)$. Then the following conditions are equivalent*:

(1) $U \in \mathcal{U}_{rsR}(J, \Omega_+)$ (*respectively*, $U \in \mathcal{U}_{\ell sR}(J, \Omega_+)$);
(2) *The $m \times m$ mvf $G_r(\mu)$* (*respectively*, $G_\ell(\mu)$) *meets the matricial Muckenhoupt condition* (1.12);
(3) *The $m \times m$ mvf $G_r(\mu)$* (*respectively*, $G_\ell(\mu)$) *meets the determinental Muckenhoupt condition* (2.2).

Proof. By Lemma 6, (1) holds if and only if $\tilde{W} \in \mathcal{U}_{rsR}(j_m, \Omega_+)$ (respectively, $\tilde{W} \in \mathcal{U}_{\ell sR}(j_m, \Omega_+)$). Therefore, in view of Theorem 1, Lemmas 6–8, and formulas (2.8) and (2.10), (1) holds if and only if $G_r(\mu)$ (respectively, $G_\ell(\mu)$) meets the matricial Muckenhoupt condition (1.12). (These last conditions guarantee that the left hand side of (1) (respectively, (2)) in Lemma 7 is finite via formula (2.8) (respectively, (2.10)) and the discussion in Section 2 of [13] combined with Theorem 7.1 of that article.) Thus, we have verified (1) $\Leftrightarrow$ (2). The equivalence with (3) is now immediate from Theorem 5. $\square$

4. Strongly regular γ-generating matrices

Let

$$f^{\#}(\lambda)=\begin{cases} f(\bar{\lambda})^* & \text{if } \Omega_+=\mathbb{C}_+,\\ f(1/\bar{\lambda})^* & \text{if } \Omega_+=\mathbb{D}.\end{cases}$$

Let $\mathfrak{M}_r(p,q)$ denote the class of measurable $m\times m$ mvf's $\mathfrak{A}(\mu)$ on Ω_0 of the form

$$\mathfrak{A}(\mu)=\begin{bmatrix} \mathfrak{a}_{11}(\mu) & \mathfrak{a}_{12}(\mu)\\ \mathfrak{a}_{21}(\mu) & \mathfrak{a}_{22}(\mu)\end{bmatrix} \tag{4.1}$$

such that

(1) $\mathfrak{A}(\mu)$ is j_{pq}-unitary for a.e. point $\mu\in\Omega_0$;
(2) $\mathfrak{a}_{22}(\mu)$ and $\mathfrak{a}_{11}(\mu)^*$ are the boundary values of mvf's $\mathfrak{a}_{22}(\lambda)$ and $\mathfrak{a}_{11}^{\#}(\lambda)$ that are holomorphic in Ω_+ and, in addition, $(\mathfrak{a}_{22})^{-1}$ and $(\mathfrak{a}_{11}^{\#})^{-1}$ are outer mvf's of class $\mathcal{S}^{q\times q}(\Omega_+)$ and $\mathcal{S}^{p\times p}(\Omega_+)$, respectively;
(3_r) The mvf

$$s_{21}(\mu)=-\mathfrak{a}_{22}(\mu)^{-1}\mathfrak{a}_{21}(\mu)=-\mathfrak{a}_{12}(\mu)^*\big[\mathfrak{a}_{11}(\mu)^*\big]^{-1} \tag{4.2}$$

is the boundary value of a mvf $s_{21}(\lambda)$ that belongs to the class $\mathcal{S}^{q\times p}(\Omega_+)$.

This class of mvf's was introduced and investigated in [1]. It plays a fundamental role in the study of the matrix Nehari problem. The mvf's in this class are called γ-generating matrices.

Let $\mathfrak{M}_\ell(p,q)$ denote the class of measurable $m\times m$ mvf's $\mathfrak{A}(\mu)$ on Ω_0 of the form (4.1) that meet conditions (1) and (2) that are stated above for $\mathfrak{M}_r(p,q)$ and (in place of (3_r))

(3_ℓ) The mvf

$$s_{12}(\mu)=\mathfrak{a}_{12}(\mu)\mathfrak{a}_{22}(\mu)^{-1}=\big[\mathfrak{a}_{11}(\mu)^*\big]^{-1}\mathfrak{a}_{21}(\mu)^* \tag{4.3}$$

is the boundary value of a mvf $s_{12}(\lambda)$ that belongs to the class $\mathcal{S}^{p\times q}(\Omega_+)$. This class of functions was introduced and briefly discussed in [9, Section 7.3].

A mvf $\mathfrak{A}\in\mathfrak{M}_r(p,q)$ is said to be right strongly regular if there exists a mvf $\varepsilon\in\mathcal{S}^{p\times q}$ such that

$$\big\|(\mathfrak{a}_{11}\varepsilon+\mathfrak{a}_{12})(\mathfrak{a}_{21}\varepsilon+\mathfrak{a}_{22})^{-1}\big\|_\infty<1. \tag{4.4}$$

A mvf $\mathfrak{A}\in\mathfrak{M}_\ell(p,q)$ is said to be left strongly regular if there exists a mvf $\varepsilon\in\mathcal{S}^{q\times p}(\Omega_+)$ such that

$$\big\|(\mathfrak{a}_{22}+\varepsilon\mathfrak{a}_{12})^{-1}(\mathfrak{a}_{21}+\varepsilon\mathfrak{a}_{11})\big\|_\infty<1. \tag{4.5}$$

These two classes will be designated $\mathfrak{M}_{rsR}(p,q)$ and $\mathfrak{M}_{\ell sR}(p,q)$, respectively. There exists a two sided correspondence between the class $\mathfrak{M}_{rsR}(p,q)$ and the class of strictly completely indeterminate Nehari problems for mvf's $f \in \mathcal{B}^{p\times q}$, the unit ball in $L_\infty^{p\times q}(\Omega_0)$, that is expressed by the formula

$$T_{\mathfrak{A}}\big[\mathcal{S}^{p\times q}(\Omega_+)\big] = \big\{f \in \mathcal{B}^{p\times q}\colon\ (f - f^\circ) \in H_\infty^{p\times q}(\Omega_+)\big\} \tag{4.6}$$

for some mvf $f^\circ \in \mathcal{B}^{p\times q}$.

The classes $\mathfrak{M}_{rsR}(p,q)$ and $\mathfrak{M}_{\ell sR}(p,q)$ were introduced and characterized in terms of a matricial Muckenhoupt condition in [9]; see, e.g., Theorems 4.5, 4.8, and Section 7.3. In the special case that $q = p$, Theorem 4.5 of that paper yields the following result:

Theorem 11. *Let $\mathfrak{A} \in \mathfrak{M}_r(p,p)$. Then $\mathfrak{A} \in \mathfrak{M}_{rsR}(p,p)$ if and only if*

(1) $\mathrm{Int}_{\Omega_0}(\mathfrak{A}^*\mathfrak{A})$ *is finite*;
(2) *The $p \times p$ mvf*

$$\Delta(\mu) = \big\{\mathfrak{a}_{21}(\mu) + \mathfrak{a}_{22}(\mu)\big\}^*\big\{\mathfrak{a}_{21}(\mu) + \mathfrak{a}_{22}(\mu)\big\}$$

satisfies the matricial Muckenhoupt condition (1.12).

There is an analogous characterization of the class $\mathfrak{M}_{\ell sR}(p,p)$ that follows from the discussion in [9, Section 7.3]:

Theorem 12. *Let $\mathfrak{A} \in \mathfrak{M}_\ell(p,p)$. Then $\mathfrak{A} \in \mathfrak{M}_{\ell sR}(p,p)$ if and only if*

(1) $\mathrm{Int}_{\Omega_0}(\mathfrak{A}\mathfrak{A}^*)$ *is finite*;
(2) *The $p \times p$ mvf*

$$\Delta(\mu) = \big\{\mathfrak{a}_{12}(\mu) + \mathfrak{a}_{22}(\mu)\big\}\big\{\mathfrak{a}_{12}(\mu) + \mathfrak{a}_{22}(\mu)\big\}^*$$

satisfies the matricial Muckenhoupt condition (1.12).

The next step is to introduce the $2m \times 2m$ mvf

$$\tilde{\mathfrak{A}}(\mu) = \begin{bmatrix} \tilde{\mathfrak{a}}_{11}(\mu) & \tilde{\mathfrak{a}}_{12}(\mu) \\ \tilde{\mathfrak{a}}_{21}(\mu) & \tilde{\mathfrak{a}}_{22}(\mu) \end{bmatrix} = \begin{bmatrix} \mathfrak{a}_{11}(\mu) & 0 & 0 & \mathfrak{a}_{12}(\mu) \\ 0 & I_q & 0 & 0 \\ 0 & 0 & I_p & 0 \\ \mathfrak{a}_{21}(\mu) & 0 & 0 & \mathfrak{a}_{22}(\mu) \end{bmatrix}$$

with blocks $\tilde{\mathfrak{a}}_{ij}$ of size $m \times m$ and to check that

$$\mathfrak{A} \in \mathfrak{M}_r(p,q) \quad \Leftrightarrow \quad \tilde{\mathfrak{A}} \in \mathfrak{M}_r(m,m)$$

and

$$\mathfrak{A} \in \mathfrak{M}_\ell(p,q) \quad \Leftrightarrow \quad \tilde{\mathfrak{A}} \in \mathfrak{M}_\ell(m,m).$$

Lemma 13. $\mathfrak{A} \in \mathfrak{M}_{rsR}(p,q) \Leftrightarrow \tilde{\mathfrak{A}} \in \mathfrak{M}_{rsR}$.

Proof. If $\mathfrak{A} \in \mathfrak{M}_{rsR}(p,q)$, then there exists a mvf $\varepsilon \in \mathcal{S}^{p\times q}$ such that (4.4) holds. Thus, upon setting

$$\tilde{\varepsilon} = \begin{bmatrix} 0_{p\times p} & \varepsilon \\ 0_{q\times p} & 0_{q\times q} \end{bmatrix} \quad \text{and} \quad \tilde{s} = T_{\tilde{\mathfrak{A}}}[\tilde{\varepsilon}],$$

it is readily checked that

$$\tilde{s} = (\tilde{\mathfrak{a}}_{11}\tilde{\varepsilon} + \tilde{\mathfrak{a}}_{12})(\tilde{\mathfrak{a}}_{21}\tilde{\varepsilon} + \tilde{\mathfrak{a}}_{22})^{-1} = \begin{bmatrix} 0_{p\times p} & s \\ 0_{q\times p} & 0_{q\times q} \end{bmatrix}$$

meets the condition $\|\tilde{s}\|_\infty < 1$, since

$$s = (\mathfrak{a}_{11}\varepsilon + \mathfrak{a}_{12})(\mathfrak{a}_{21}\varepsilon + \mathfrak{a}_{22})^{-1}.$$

Conversely, if

$$\tilde{s} = (\tilde{\mathfrak{a}}_{11}\tilde{\varepsilon} + \tilde{\mathfrak{a}}_{12})(\tilde{\mathfrak{a}}_{21}\tilde{\varepsilon} + \tilde{\mathfrak{a}}_{22})^{-1}$$

meets the condition $\|\tilde{s}\|_\infty < 1$ for some choice of

$$\tilde{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} \in \mathcal{S}^{m\times m}(\Omega_+)$$

(with diagonal blocks ε_{11} of size $p \times p$ and ε_{22} of size $q \times q$), then there exists a positive constant $\gamma < 1$ such that

$$\tilde{s}(\mu)^*\tilde{s}(\mu) \leqslant \gamma I_m \quad \text{for a.e. } \mu \in \Omega_0.$$

Thus, the factors

$$\tilde{\alpha} = \tilde{\mathfrak{a}}_{11}\tilde{\varepsilon} + \tilde{\mathfrak{a}}_{12} \quad \text{and} \quad \tilde{\beta} = \tilde{\mathfrak{a}}_{21}\tilde{\varepsilon} + \tilde{\mathfrak{a}}_{22}$$

are subject to the inequality

$$\tilde{\alpha}(\mu)^*\tilde{\alpha}(\mu) \leqslant \gamma\tilde{\beta}(\mu)^*\tilde{\beta}(\mu) \quad \text{for a.e. } \mu \in \Omega_0.$$

Consequently, the inequality

$$[0 \quad I_q]\tilde{\alpha}(\mu)^*\tilde{\alpha}(\mu)\begin{bmatrix} 0 \\ I_q \end{bmatrix} \leqslant \gamma[0 \quad I_q]\tilde{\beta}(\mu)^*\tilde{\beta}(\mu)\begin{bmatrix} 0 \\ I_q \end{bmatrix}$$

must also hold for a.e. $\mu \in \Omega_0$. But this turn leads easily to the conclusion that

$$(\mathfrak{a}_{11}\varepsilon_{12} + \mathfrak{a}_{12})^*(\mathfrak{a}_{11}\varepsilon_{12} + \mathfrak{a}_{12}) \leqslant \gamma(\mathfrak{a}_{21}\varepsilon_{12} + \mathfrak{a}_{22})^*(\mathfrak{a}_{21}\varepsilon_{12} + \mathfrak{a}_{22})$$

for a.e. point $\mu \in \Omega_0$, and hence that $\mathfrak{A} \in \mathfrak{M}_{rsR}(p,q)$. □

Lemma 14. $\mathfrak{A} \in \mathfrak{M}_{\ell sR}(p,q) \Leftrightarrow \tilde{\mathfrak{A}} \in \mathfrak{M}_{\ell sR}(m,m)$.

Proof. The proof is much the same as the proof of the previous lemma, but with (4.5) in place of (4.4). □

Now, having these two lemmas available, the analysis of the preceding section can be applied directly to obtain the following conclusions:

Theorem 15. *Let* $\mathfrak{A} \in \mathfrak{M}_r(p,q)$. *Then* $\mathfrak{A} \in \mathfrak{M}_{rsR}(p,q)$ *if and only if the* $m \times m$ *matrix weight*

$$\Delta(\mu) = \begin{bmatrix} I_p & s_{21}(\mu)^* \\ s_{21}(\mu) & I_q \end{bmatrix} \tag{4.7}$$

meets the matricial Muckenhoupt condition (1.12).

Theorem 16. *Let* $\mathfrak{A} \in \mathfrak{M}_\ell(p,q)$. *Then* $\mathfrak{A} \in \mathfrak{M}_{\ell sR}(p,q)$ *if and only if the* $m \times m$ *matrix weight*

$$\Delta(\mu) = \begin{bmatrix} I_p & s_{12}(\mu) \\ s_{12}(\mu)^* & I_q \end{bmatrix} \tag{4.8}$$

meets the matricial Muckenhoupt condition (1.12).

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