HAMiltonian paths in vertex-symmetric graphs of order 5p

Dragan MaruŠi¢ and T.D. Parsons

The Pennsylvania State University, University Park, PA 16802, USA

Received 16 April 1980
Revised 15 December 1981

It is shown that every connected vertex-symmetric graph of order 5p (p a prime) has a Hamiltonian path.

1. Introduction

It was conjectured by L. Lovász in 1968 that every connected vertex-symmetric graph has a Hamiltonian path. (See [7, p. 249].) This conjecture has been verified for graphs of certain special orders, usually with the stronger conclusion that the graph has a Hamiltonian cycle (aside from a few notable exceptions). Every cvsg (connected vertex-symmetric graph) of prime order is a circulant graph (see [15]), and so has a Hamiltonian cycle. Recently, Alspach [1] has shown that every cvsg of order 2p (p always denotes a prime) has a Hamiltonian cycle, except for the Petersen graph—which has a Hamiltonian path.

Marušić [10] has shown that every cvsg of order p², p³, or 3p has a Hamiltonian cycle. It is also known [11, 14] that every (finite) Cayley graph for an abelian group or the semi-direct product of a prime order group by an odd order abelian group, has a Hamiltonian cycle.

L. Babai [2] has raised the question of constructing an infinite family of cvsg’s without Hamiltonian cycles. These are now only four such graphs known (if we disregard the trivial cases of the complete graphs on one or two points), namely the Petersen graph, the Coxeter graph, and two graphs obtained from these by replacing their vertices with triangles. C. Thomassen has conjectured that only finitely many such graphs exist (see [3, p. 163]).

The question of Hamiltonian cycles, or their absence, in cvsg’s is a difficult one. Our goal in this paper is more modest: we shall prove that every cvsg of order 5p has a Hamiltonian path. In a later paper, we shall consider the same question for graphs of order 4p.

In general, our method identifies certain families of graphs of order mp (m < p) which have a ‘homogeneous automorphism’, and whose structure does not make obvious the existence of a Hamiltonian path. Like the ‘generalized Petersen graphs’ [9], these graphs need not be vertex-symmetric; but those (if any) which
are vertex-symmetric may provide further examples of cvsg's without Hamiltonian cycles.

2. Preliminaries

In this section we introduce some definitions, notations, lemmas, and propositions used in proving our main theorem. It will be convenient to state these with greater generality than just for graphs of order $5p$.

An $(m, n)$-homogeneous permutation on a finite set $V$ is a permutation having exactly $m$ orbits, each of cardinality $n$. A graph $G$ with vertex set $V$ is $(m, n)$-galactic if its automorphism group $\text{Aut}(G)$ contains an $(m, n)$-homogeneous permutation on $V$. The proof of our main result is based on the following theorem of Marušič [12]:

Proposition 1. If $m \leq p$, then every vertex-symmetric graph $G$ of order $mp$ is $(m, p)$-galactic.

Henceforth, $G$ will denote a cvsg of order $mp$, where $2 \leq m < p$, and $\gamma$ will denote some (fixed) particular $(m, p)$-homogeneous automorphism of $G$. The term 'orbit' will always mean an orbit of $\gamma$, and we shall use letters $A, B, X, Y, Z, A_1, A_2, \ldots$ to denote orbits. The restriction $\gamma^A$ of $\gamma$ to an orbit $A$ is a cyclic permutation of order $p$. The factor graph $G/p$ is the graph whose vertices are the orbits of $\gamma$, and whose edges are those unordered pairs $\{A, B\}$ of distinct orbits $A, B$ for which at least one edge of $G$ joins a point of $A$ to a point of $B$. Our notation $G/p$ is justified because the factor graph is actually independent of the choice of $\gamma$. (This is because, from $m < p$, it follows that the orbits of any $(m, p)$-homogeneous automorphism of $G$ are actually the orbits of some Sylow $p$-subgroup of $\text{Aut}(G)$; and since any two Sylow $p$-subgroups are conjugate in $\text{Aut}(G)$, any two factor graphs arising from different choices of $\gamma$ are isomorphic.)

Our strategy will be to consider the possible structure of $G$ relative to the factor graph, and in each case to show that either this structure implies that $G$ has a Hamiltonian path, or else the structure is inconsistent with the vertex-symmetry of $G$. This approach actually proves a stronger result than the one claimed in our abstract: it shows that nearly all connected $(5, p)$-galactic graphs have a Hamiltonian path, and that those which do not cannot be vertex-symmetric.

If $S$ is a subset of the vertex set of $G$, then $\langle S \rangle$ denotes the subgraph of $G$ induced by $S$. We shall use the notation $\langle \cdot \rangle$ only for induced subgraphs of $G$. Therefore it should cause no confusion if, for any subgraph $F$ of the factor graph $G/p$, we let $\langle F \rangle$ denote the subgraph of $G$ induced by the union of those orbits which are the vertices of $F$.

If $A_1, A_2, \ldots, A_k$ are distinct orbits and $U$ is their union, then $\langle U \rangle$ is clearly $(k, p)$-galactic relative to the restriction $\gamma^U$ of $\gamma$ to $U$. In particular, if $A$ is any
orbit, then \( \langle A \rangle \) is \((1, p)\)-galactic, and so is a circulant graph of order \( p \). Since \( p \) is odd, we have:

(1) For any orbit \( A \), \( \langle A \rangle \) is regular of some even degree \( d(A) \). If \( d(A) > 0 \), then \( \langle A \rangle \) contains a Hamiltonian cycle.

If \( A, B \) are distinct orbits, then \([A, B] \) denotes the bipartite graph with bipartition \( A, B \) and whose edges are all those edges of \( G \) joining points of \( A \) to points of \( B \). Clearly, \([A, B] \) is \((2, p)\)-galactic relative to \( \gamma^A \gamma^B \), and so is regular of some degree \( d(A, B) \). If \( d(A, B) \geq 2 \), then some point \( b \in B \) is adjacent to two different points \( a, a' \) of \( A \), and \( a' = \gamma^t(a) \) for some \( t \neq 0 \) (mod \( p \)). Letting \( \beta = \gamma^t \), we have that \( ab \beta(a) \beta(b) \cdots \beta_p^{-1}(a) \beta_p^{-1}(b) a \) is a Hamiltonian cycle in \([A, B] \).

We summarize:

(2) If \( A, B \) are distinct orbits, then \([A, B] \) is regular of some degree \( d(A, B) > 0 \). If \( d(A, B) \geq 2 \), then \([A, B] \) contains a Hamiltonian cycle.

A path \( A_0 A_1 \cdots A_k \) in the factor graph \( G/p \) is called an orbit-path of length \( k \); similarly, a cycle \( A_0 A_1 \cdots A_k A_0 \) in \( G/p \) is an orbit-cycle of length \( k + 1 \). If \( P_1 = A_0 A_1 \cdots A_i \) and \( P_2 = B_0 B_1 \cdots B_i \) are orbit-paths with no common orbits, and if \( d(A_i, B_i) \geq 1 \), then we may concatenate them to form the orbit-path \( P_1 P_2 : = A_0 A_1 \cdots A_i B_0 B_1 \cdots B_i \).

A good orbit-path is an orbit-path of the form \( P_1 P_2 \cdots P_k (k \geq 1) \) where each \( P_i \) (\( 1 \leq i \leq k \)) is either an orbit \( A \) with \( d(A) \geq 2 \) or an orbit-path \( AB \) of length 1 with \( d(A, B) \geq 2 \). Often, to identify quickly the 'goodness' of an orbit-path, we shall use a notation like \( X - Y * Z - W \), where \( Y * Z \) means \( d(Y, Z) \geq 2 \), and single orbits like \( X \) have \( d(X) \geq 2 \). We shall refer to this as a 'decomposition' of a good orbit-path. We omit the easy proof of the following lemma.

**Lemma 2.** If \( P = A_0 A_1 \cdots A_k \) is a good orbit-path, then \( \langle P \rangle \) has a Hamiltonian path with one endpoint in \( A_0 \) and the other endpoint in \( A_k \).

**Proposition 3 (Alspach [1]).** If the orbit \( A \) has \( d(A) \geq 4 \), and if \( x, y \) are distinct vertices of \( \langle A \rangle \), then \( \langle A \rangle \) has a Hamiltonian path whose endpoints are \( x \) and \( y \).

**Lemma 4.** Let \( AB \) be a length 1 orbit-path with \( d(B) \geq 2 \) and \( d(A, B) \geq 2 \). Then \( \langle AB \rangle \) has a Hamiltonian path with both its endpoints in \( A \).

**Proof.** Let \( b \in B \), and let \( x, y \) be distinct vertices in \( A \) adjacent to \( b \). There is an integer \( r \neq 0 \) (mod \( p \)) such that \( \gamma'^r(x) = y \). Let \( \gamma' = \gamma^r \). There is an integer \( r \) (\( 1 \leq r \leq p - 1 \)) such that \( \beta^r(b) \) is adjacent to \( b \). Now \( x b \beta(x) \) is a path of length 2 in \( G \), so that for all integers \( i \), \( \beta^i(x) \beta^i(b) \beta^{i+1}(x) \) is also a path of length 2. Thus

\[
\beta^r(x) \beta^{-1}(b) \beta^{-1}(x) \beta(x) b \beta^r(b) \beta^{r+1}(x) \beta^{r+1}(b) \cdots \beta^{p-1}(b) x
\]

is a Hamiltonian path in \( \langle AB \rangle \) with both its endpoints in \( A \).
Lemma 5. Let \( k \geq 2 \) and let \( C = A_0A_1 \cdots A_kA_0 \) be an orbit-cycle in \( G/p \). If \( \langle C \rangle \) does not contain a Hamiltonian cycle, then \( d[A_k, A_0] = 1 = d[A_i, A_{i+1}] \) for \( i = 0, 1, \ldots, k-1 \), and the graph \( K \) induced by the edges of the graphs \( [A_k, A_0] \) and \( [A_i, A_{i+1}] \) for \( i = 0, 1, \ldots, k-1 \) is a disjoint union of \( p \) cycles of length \( k+1 \) in \( G \).

Proof. Since \( C \) is an orbit-cycle, we have that \( d[A_k, A_0] \geq 1 \) and \( d[A_i, A_{i+1}] \geq 1 \) for \( i = 0, 1, \ldots, k-1 \). Thus there is a path \( x_0x_1 \cdots x_k \) in \( G \) such that \( x_i \in A_i \) for \( i = 0, 1, \ldots, k \). Suppose that \( x_k \) is adjacent to some vertex \( x_0 \neq x_0 \), where \( x_0 \in A_0 \). There is an integer \( t \equiv 0 \pmod{p} \) such that \( \gamma'(x_0) = x_0 \). Let \( \beta = \gamma' \). Then

\[
\begin{align*}
x_0x_1 \cdots x_k \beta(x_0) \beta(x_1) \cdots \beta(x_k) \cdots \beta^{p-1}(x_0) \beta^{p-1}(x_1) \cdots \beta^{p-1}(x_k) x_0
\end{align*}
\]

is a Hamiltonian cycle in \( \langle C \rangle \).

If \( d[A, B] \geq 2 \) for two successive orbits \( A, B \) on the cycle \( C \), then we may assume without loss of generality that \( A = A_0 \) and \( B = A_0 \). Then the vertex \( x_k \) is certainly adjacent to some \( x_0 \) in \( A_0 \setminus \{x_0\} \), so by what was just shown, \( \langle C \rangle \) has a Hamiltonian cycle. It follows that, if \( \langle C \rangle \) has no Hamiltonian cycle, then \( d[A, B] = 1 \) for every pair \( A, B \) of successive orbits in \( C \), so \( [A, B] \) is the graph of a perfect matching of \( A \) to \( B \) in each case. Also, in the path \( x_0x_1 \cdots x_k \) above, we must have that \( x_k \) is adjacent to \( x_0 \), and to no other vertex of \( A_0 \). Then \( \beta'(x_0) \beta'(x_1) \cdots \beta'(x_k) \beta'(x_0) \) for \( i = 0, 1, \ldots, p-1 \) are disjoint cycles in \( G \) of length \( k+1 \), and they exhaust the edges of \( K \).

It will be convenient to introduce a sort of 'picture notation' which simultaneously gives information about the structures of the factor graph \( G/p \) and the graph \( G \) itself. The following notations (in which \( A, B \) are distinct orbits) have the meanings specified:

1. \( A \quad \cdots \quad B \) means \( d[A, B] \geq 1 \), (i.e. \( A \) adjacent to \( B \) in \( G/p \)),
2. \( A \quad \cdots \quad B \) means \( d[A, B] \geq 2 \),
3. \( A \quad \cdots \quad B \) means \( d[A, B] = k \), \( k \geq 1 \),
4. \( A \quad \cdots \quad B \) means \( d(A) = k \), \( k \geq 0 \),
5. \( A \quad \cdots \quad B \) means \( d(A) \geq k \), (used only for \( k > 0 \)).

We alert the reader that these notations are by no means mutually exclusive; in
particular, notation (3) makes no claim other than \( d[A, B] \geq 1 \) about the structure of \( \langle A \rangle, \langle B \rangle, \) or \([A, B]\). For example, both

\[
\begin{align*}
\text{(8)} & \\
\begin{array}{c}
\circ \quad A \\
\circ \quad B
\end{array}
\quad \text{and} \quad \\
\begin{array}{c}
\circ \quad A \\
\circ \quad B
\end{array}
\end{align*}
\]

are special cases of (3), and (4) is a special case of (3). Similarly, (7) with \( k = 4 \) is a special case of (7) with \( k = 2 \), etc.

3. Hamiltonian paths

By the results of Alspach and of Marušić mentioned in our Introduction, every cvsg \( G \) of order \( 5p \) for \( p = 2, 3, \) or \( 5 \) has a Hamiltonian path. We therefore let \( G \) be a cvsg of order \( 5p \) for \( p > 5 \). Then \( G \) is \((5, p)\)-galactic by Proposition 1; we let \( \gamma \) be some (fixed) \((5, p)\)-homogeneous automorphism of \( G \), and we use the notations, with \( m = 5 \), discussed in Section 2. The factor graph \( G/p \) is then some connected graph on 5 vertices; we shall successively discuss the cases where \( G/p \) is a tree, or in which the longest cycle in \( G/p \) is of length 3, 4, or 5.

Henceforth we assume that

\[
(9) \quad G \text{ is regular of even degree } d_0 \geq 4.
\]

(Clearly \( G \) is regular of positive even degree, but degree 2 would imply that \( G \) is a cycle.)

In proofs, we shall use the phrase ‘by regularity’ to justify claims which are easily deducible from (1), (2), and (9).

Case 1: The factor graph \( G/p \) is a tree.

Subcase 1(a): \( G/p \) is a path \( A_0A_1A_2A_3A_4 \) of length 4. By regularity, \( G/p \) must have the form

\[
\begin{align*}
\text{(10)} & \\
\circ \quad A_0 & \quad \circ \quad A_1 & \quad \circ \quad A_2 & \quad \circ \quad A_3 & \quad \circ \quad A_4
\end{align*}
\]

In this case, we will justify our claim at length, so that the reader will understand our phrase ‘by regularity’. First, \( d_0 = d(A_0) + d(A_0, A_1) \) and both \( d_0 \) and \( d(A_0) \) are even, so \( d(A_0, A_1) \) is even; similarly, \( d[A_3, A_4] \) is even. Now \( d_0 = d(A_1) + d[A_0, A_1] + d[A_1, A_2] \), thus \( d[A_1, A_2] \) is even, and similarly so is \( d[A_2, A_3] \). Since \( d[A_0, A_{i+1}] \geq 1 \) for \( i = 0, 1, 2, 3 \) and all these degrees are even, we have that \( d[A_i, A_{i+1}] \geq 2 \) for \( i = 0, 1, 2, 3 \). This justifies the ‘double bonds’ in (10). Next,

\[
d(A_0) + d[A_0, A_1] = d_0 = d[A_0, A_1] + d(A_1) + d[A_1, A_2],
\]

so \( d(A_0) = d(A_1) + d[A_1, A_2] \geq d[A_1, A_2] \geq 2 \). Similarly, \( d(A_4) \geq 2 \). We have now
justified the claim (10). (In subsequent arguments, we shall usually just make such claims 'by regularity', and leave the easy details to the reader.) From (10) we see that $G/p$ is itself a good orbit-path, with the decomposition $A_0 - A_1^* A_2 - A_3^* A_4$. Thus $G$ has a Hamiltonian path by Lemma 2.

**Remark.** We have *not* claimed that there exists a cvsg of order $5p$ for which the factor graph is a path of length 4. We have merely shown that if such a graph exists, then it has a Hamiltonian path. The question of the existence of cvsg's with factor graphs of certain specified structures is interesting, and we shall discuss this later.

**Subcase 1(b):** $G/p$ is the tree

\[ (11) \]

By regularity, it can be shown that $G/p$ has the form

\[ (12) \]

By Lemma 4, with $A = A_2$ and $B = A_3$, $(A_2 A_3)$ has a Hamiltonian path whose endpoints $x, y$ are in $A_2$. By (1), (2) each of $(A_4)$ and $[A_0, A_1]$ contains a Hamiltonian cycle. Since $x$ is adjacent to some vertex in $A_1$, and $y$ to some vertex in $A_4$, obviously $G$ has a Hamiltonian path.

**Subcase 1(c):** $G/p$ is the tree

\[ (13) \]
By regularity, $G/p$ must have the form

$$\begin{array}{c}
B = \{b_0, b_1, \ldots, b_{p-1}\} \\
A = \{a_0, a_1, \ldots, a_{p-1}\}.
\end{array}$$

Since $d(B) \geq 6$, there exist $i, j \in \{0, 1, \ldots, p-1\}$ such that $1 < i < j < p-1$ and $b_i$ is adjacent to both $b_j$ and $b_{p-1}$. Let $x_0, x_{i-1} \in X$ be adjacent, respectively, to $a_0$ and $a_{i-1}$. Let $y_{i-1}$ and $y_{j-1} \in Y$ be adjacent, respectively, to $a_{i-1}$ and $a_{j-1}$. Let $z_{j-1}, z_{p-1} \in Z$ be adjacent, respectively, to $a_{j-1}$ and $a_{p-1}$. By Proposition 3, there are Hamiltonian paths $P_x, P_y, P_z$ in $X, Y, Z$ whose initial and terminal points are $x_0$ and $x_{i-1}, y_{i-1}$ and $y_{j-1}, z_{j-1}$ and $z_{p-1}$, respectively. Then

$$b_{i-1}a_{i-2}b_{i-2} \cdots b_1a_0P_xa_{i-1}P_ya_{j-1}P_za_{p-1}b_{p-1}a_{p-2}b_{p-2}$$

is a Hamiltonian path in $G$.

Case 2: $G/p$ contains a triangle, but no 4-cycle.

Subcase 2(a): $G/p$ is the graph

$$\begin{array}{c}
A_4 \\
A_3 \\
A_2 \\
A_1
\end{array}$$

By regularity, $G/p$ must have one of the two forms

$$\begin{array}{c}
A_4 \\
A_3 \\
A_2
\end{array}$$

(a)

$$\begin{array}{c}
A_4 \\
A_3 \\
A_2
\end{array}$$

(b)

In (16), in either case (a) or (b), by Lemma 4 $\langle A_3, A_1 \rangle$ has a Hamiltonian path $P_1$ whose initial point $x$ and terminal point $y$ are both in $A_3$, and $\langle A_2 \rangle$ contains a Hamiltonian cycle, by (1). Let $x$ be adjacent to $w \in A_4$, and $y$ be adjacent to $z \in A_2$. Let $P_2$ be a Hamiltonian path in $A_2$ with $z$ as the initial point. Using (1),
and $d[A_4, A_5] = 1$, clearly $(A_4A_5)$ has a Hamiltonian path $P_3$ with $w$ as the terminal point, in either of the cases (a) or (b) of (16). Then $P_3P_1P_2$ is a Hamiltonian path in $G$.

**Subcase 2(b):** $G/p$ is the graph

![Graph](image)

By regularity, we must have $d(A_2)$, $d[A_4, A_5]$, and $d[A_3, A_4]$ all even and $\geq 2$. If $d(A_1, A_2) = 2$, then $A_1\cdot A_2 - A_3\cdot A_4 - A_5$ is a good orbit-path. If $d[A_1, A_2] = 1$, then by regularity, both $d(A_3) \geq 2$ and $d(A_4) \geq 2$, so that $A_1 - A_2 - A_3 \cdot A_4 - A_5$ is a good orbit-path. By Lemma 2, $G$ has a Hamiltonian path.

**Subcase 2(c):** $G/p$ is the graph

![Graph](image)

By regularity, all of $d(A_1)$, $d(A_3)$, $d[A_2, A_4]$, $d[A_4, A_5]$ are even and $\geq 2$. If $d(A_3) = 2$, then $A_1 \cdot A_2 - A_3 - A_4 \cdot A_5$ is a good orbit-path. If $d(A_3) = 0$, then since $d_n \geq 4$, either $d[A_2, A_4] \geq 2$ or $d[A_3, A_4] \geq 2$, giving a good orbit-path $A_1 - A_2 \cdot A_4 - A_5$ or $A_1 \cdot A_2 - A_3 \cdot A_4 - A_5$ respectively. Therefore, $G$ has a Hamiltonian path, by Lemma 2.

**Subcase 2(d):** $G/p$ is the graph

![Graph](image)

Suppose that $d(U) \geq 2$. Then by regularity, either $d[W, X] \geq 2$ or both $d(W)$, $d(X) \geq 2$, and similarly, either $d[Y, Z] \geq 2$ or both $d(Y)$ and $d(Z) \geq 2$. In every case, $WXUYZ$ is a good orbit-path.

Suppose that $d(U) = 0$. By Lemma 5, if at least one of $d[W, U]$, $d[U, X]$, $d[X, W]$ is $\geq 2$, then there is a Hamiltonian cycle $C$ in $(UWX)$. By regularity, $d(Y) + d[Y, Z]$ and $d(Z) + d[Y, Z]$ are both $\geq 3$; thus either $d[Y, Z] \geq 2$ or else $d[Y, Z] = 1$ and both $d(Y)$, $d(Z) \geq 2$. In either case, $YZ$ is a good orbit-path, and so $(YZ)$ contains a Hamiltonian path $P$. Since an endpoint of $P$ is adjacent to some vertex of $C$, and $P, C$ are vertex-disjoint, $G$ contains a Hamiltonian path. We conclude that, if $d(U) = 0$, then we may assume that $d[W, U] = 1 = d[U, X] =$. 


$d(X, W)$, and similarly, we may assume that $d(U, Y) = 1 = d(Y, Z) = d(Z, U)$. In this case, by regularity, $d(W) = 2 = d(X) = d(Y) = d(Z)$, and $G/p$ has the form

\[ (20) \]

Since both $WX$ and $YZ$ are good orbit-paths, clearly $G$ will have a Hamiltonian path if either $<UWX>$ or $<UYZ>$ has a Hamiltonian cycle; thus we assume that neither of these induced subgraphs of $G$ has a Hamiltonian cycle. By Lemma 5, we may label the vertices of $U, W, X, Y, Z$ as $u_i, w_i, x_i, y_i, z_i$ $(i = 1, \ldots, p)$ such that for each $i$, the subgraph $\langle \{u_i, w_i, x_i, y_i, z_i\} \rangle$ is the wedge of two triangles

\[ (21) \]

and if the edges of the four cycles $\langle W \rangle, \langle X \rangle, \langle Y \rangle, \langle Z \rangle$ are deleted from $G$, then the resulting graph is the disjoint union of the $p$ wedges for $i = 1, \ldots, p$. Then $u_i$ is in exactly two triangles of $G$, but $x_i$ is in only one triangle of $G$—contradicting the vertex-symmetry of $G$. Therefore this case cannot occur. We conclude that $G$ has a Hamiltonian path.

Case 3: $G/p$ has a 4-cycle, but no 5-cycles.

There are precisely six connected graphs with five vertices, having a 4-cycle but no 5-cycle. These graphs are:

\[ (22) \]
Now $G/p$ must be one of these six graphs. First, we consider the cases (a), (b), (c), (d). In each of these cases, by regularity both $d[W, U]$ and $d(U)$ must be even and $\geq 2$. By (1), $\langle U \rangle$ has a Hamiltonian cycle $C_1$. If $\langle WXYZ \rangle$ has a Hamiltonian cycle $C_2$, then since $C_1, C_2$ are disjoint and are joined by an edge of $[W, U]$, $G$ will have a Hamiltonian path. We may therefore assume that $\langle WXYZ \rangle$ has no Hamiltonian cycle. By Lemma 5, $d[W, X] = 1 = d[X, Y] = d[Y, Z] = d[Z, W]$. Also, in the case (d), considering the orbit-cycle $WXYZ$, we would have $d[X, Z] = 1 = d[W, Y]$, but this would imply that $d_0 = d(X) + 3$ is odd, which is impossible—so in case (d), $G$ has a Hamiltonian path. Since $d_0 \geq 4$, regularity now gives $d(X) = d(Y) = d(Z) = 2$ in case (a); $d(Y) = 2$ and $d[X, Z]$ is even and $\geq 2$ in case (b); and $d(X) = d(Z) = 2$ and $d(Y) = d(W) + d[W, U] = 2$ in case (c). The orbit-paths $U * W - X - Y - Z$, $U * W - X * Z - Y$, $U * W - X - Y - Z$ are then good respectively for cases (a), (b), and (c)—so by Lemma 2, $G$ has a Hamiltonian path in these cases.

In case (e), at least one of $d[W, X]$, $d[W, U]$, $d[W, Z]$ must be even and $\geq 2$. Without loss of generality, let $d[W, U]$ be even and $\geq 2$. By regularity, $d(U)$ must also be even and $\geq 2$. By Lemma 5, each of $\langle XWUY \rangle$ and $\langle ZWUY \rangle$ contains a Hamiltonian cycle. Clearly $G$ will have a Hamiltonian path unless both $d(X) = 0$ and $d(Z) = 0$. Thus let $d(X) = 0 = d(Z)$. Then by regularity, it can be shown that $d(U) \geq 2$. But

$$d[X, W] = d(X) + d[X, W] = d(Y) + d[Y, U] + d(Y, Z) = 2,$$

so by Lemma 5, $\langle XWZY \rangle$ has a Hamiltonian cycle $C$. Since $\langle U \rangle$ has a Hamiltonian cycle disjoint from $C$ and joined to $C$ by an edge of $[U, W]$, $G$ has a Hamiltonian path.

Suppose $G/p$ is the graph of case (f) of (22). By regularity, not all of $d(X)$, $d(U)$, $d(Z)$ are zero. Without loss of generality, we may assume that $d(U) \neq 0$, so $d(U) \geq 2$ and $\langle U \rangle$ contains a Hamiltonian cycle $C_0$ by (1). If $\langle XYZW \rangle$ has a Hamiltonian cycle $C_1$, then $C_0$ and $C_1$ easily give a Hamiltonian path in $G$. Thus we assume that $\langle XYZW \rangle$ has no Hamiltonian cycle; by Lemma 5, then


Since $d_0 \geq 4$, this gives $d(X) \geq 2$ and $d(Z) \geq 2$, so $\langle X \rangle$ and $\langle Z \rangle$ each have a Hamiltonian cycle. By the argument just used, we may assume that neither $\langle UYZW \rangle$ nor $\langle XYUW \rangle$ has a Hamiltonian cycle (else using the cycles in $\langle X \rangle$ and $\langle Z \rangle$ we would quickly find a Hamiltonian path in $G$). Thus $d[U, W] = 1 = d[U, Y]$ by Lemma 5. By regularity, we now have $d(Y) = d(W)$. If $d(Y) \geq 2$, then $X - Y - U - W - Z$ would be a good orbit-path, giving a Hamiltonian path in $G$. Therefore we assume that $d(Y) = 0 = d(W)$.

Suppose that $\langle XYW \rangle$ contains a Hamiltonian cycle $C$. Some edge of $C$ is of the form $\{y, w\}$, for $y \in Y$ and $w \in W$. The vertex $y$ is adjacent in $G$ to some $u \in U$, and $w$ is adjacent to some $z \in Z$, and $\langle U \rangle$, $\langle Z \rangle$ contain Hamiltonian cycles, so clearly $G$ has a Hamiltonian path. Therefore we may assume that $\langle XYW \rangle$ has no
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Hamiltonian cycle, and similarly that \( (UYW) \) and \( (YZW) \) have no Hamiltonian cycles. By Lemma 5, then \( d[W, Y] = 1 \).

We are now left with the case where \( G/p \) has the structure

\[
(23)
\]

and where, further, we have

\( (24) \) None of the induced subgraphs \( (XYW), (UYW), (YZW), (XYZW), (UYZW), (XYUW) \) of \( G \) has a Hamiltonian cycle.

Let \( H \) be the graph of order \( 5p \) obtained from \( G \) by deleting the edges of the cycles \( (X), (U), (Z) \). Applying Lemma 5 to (24), it follows that we may label the points of \( X, Y, Z, W, U \) as \( x_i, y_i, z_i, w_i, u_i \) \( (i = 1, \ldots, p) \) so that \( H \) is the disjoint union of \( p \) components of the form

\[
(25)
\]

But then \( w_i \) is in exactly three triangles of \( G \) and \( x_i \) is in only one triangle of \( G \)---which contradicts the vertex-symmetry of \( G \). Therefore no cvsg of order \( 5p \) has its factor graph \( G/p \) of the form (23) with also the property (24). We conclude that every order \( 5p \) cvsg \( G \) with \( G/p \) satisfying case 3 has a Hamiltonian path.

We remark that the 'odd graph' \( O_4 \) (see [4, p. 56]) is a cvsg of order 5-7 whose factor graph \( O_4/7 \) has the form (23) (but, of course, here the property (24) cannot hold). This graph \( O_4 \) has a Hamiltonian cycle (see [13]).

Case 4: \( G/p \) has a 5-cycle. Let \( A_1A_2A_3A_4A_5A_1 \) be a 5-cycle in \( G/p \).

Henceforth, assume that \( G \) has no Hamiltonian cycle. By Lemma 5, then \( d[A_5, A_1] = 1 = d[A_i, A_{i+1}] \) for \( i = 1, 2, 3, 4 \); and the subgraph of \( G \) spanned by all the edges of the graphs \( [A_5, A_1] \) and \( [A_i, A_{i+1}] \) for \( i = 1, 2, 3, 4 \) is a disjoint union of \( p \) cycles of length 5.

Suppose that \( G/p \) is a 5-cycle. Then the connectedness and regularity of \( G \) imply that \( d(A_i) = d_o - 2 \geq 2 \) for \( 1 \leq i \leq 5 \). In this case, \( A_1 - A_2 - A_3 - A_4 - A_5 \) is a good orbit-path, so \( G \) has a Hamiltonian path.
Suppose that $G$ is neither a 5-cycle nor a complete graph. If it has two nonadjacent vertices of degree 3—say $A_1$ and $A_2$—then both $d[A_1, A_2], d[A_3, A_4] > 0$. By regularity, now $d[A_1, A_2]$ is even and $\geq 2$. But then Lemma 5, applied to the orbit-cycle $A_1A_2A_3A_4A_5A_1$, would give that $G$ is Hamiltonian, contrary to hypothesis. Thus we may assume that no two nonadjacent vertices of $G/p$ are of degree 3. In the case under consideration, we may then assume that $G/p$ has one of the three forms (26)

In the first case, $d[A_2, A_4]$ must be even and $\geq 2$, and Lemma 5 applied to $A_1A_3A_2A_4A_5A_1$ would give that $G$ is Hamiltonian. In the other two cases, by regularity, $d[A_1, A_3], d(A_2), d(A_4), d(A_5) \geq 2$ so, $A_2 - A_1 - A_3 - A_4 - A_5$ is a good orbit-path, and $G$ has a Hamiltonian path.

We may now assume that $G/p$ is a complete graph. By Lemma 5, then $d[A_i, A_j] = 1$ for $1 \leq i < j \leq 5$. Therefore, by regularity, $d(A_i) = d(A_i)$ for $i = 1, 2, 3, 4, 5$. If $d(A_i) = 2$, then $A_1 - A_2 - A_3 - A_4 - A_5$ is a good orbit-path. Therefore by (1) we may assume that $d(A_i) = 0$, for $1 \leq i \leq 5$. Also, by Lemma 5, we may assume, since $G$ is non-Hamiltonian, that

(27) For every 5-cycle $X_1X_2X_3X_4X_5$ of $G/p$, the subgraph of $G$ spanned by the edges of the graphs $[X_i, X_j]$ and $[X_i, X_{i+1}, X_i]$, $1 \leq i \leq 4$, is a union of $p$ disjoint 5-cycles.

There are vertices $a \in A_1$, $b \in A_2$, $c \in A_3$, $d \in A_4$, $e \in A_5$ such that $abcdea$ is a 5-cycle in $G$, and there exists $r \in \{0, 1, \ldots, p-1\}$ such that

(28) $c$ is adjacent in $G$ to $\gamma'(r)$.

Since $A_1A_2A_3A_4A_5A_6A_1$, $A_2A_3A_4A_5A_6A_1A_2$, $A_3A_4A_5A_6A_1A_3$, $A_4A_5A_6A_1A_4$, and $A_5A_6A_1A_5A_5A_5$ are 5-cycles of $G/p$ and $\gamma \in \text{Aut}(G)$, (27) and (28) imply that $abc\gamma'(r)d, bcd\gamma'(r)a, cde\gamma'(r)b, dce\gamma'(r)c, ade\gamma'(r)c, bcd\gamma'(r)b, cde\gamma'(r)c, dce\gamma'(r)b$, and $eabc\gamma'(d)c$ are 5-cycles of $G$. In particular,

(29) $\gamma'(c)$ is adjacent in $G$ to $e$.

Now $\gamma \in \text{Aut}(G)$ and (28), (29) imply that both $e$ and $\gamma^{2r}(e)$ are adjacent to $\gamma'(c)$; since $d[A_3, A_5] = 1$, we must have $\gamma^{2r}(e) = e$, so that $2r = 0 \pmod{p}$. Since $p$ is odd, this gives $r = 0$. But then $\langle a, b, c, d, e \rangle$ is a complete graph on 5 vertices, and $G$ is the disjoint union of $p$ copies of this complete graph, and is discon-
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connected. This contradiction shows that $G$ has a Hamiltonian path. We have now proved:

**Theorem 6.** Every connected vertex-symmetric graph of order $5p$ has a Hamiltonian path.

4. Factor graphs

We now return to the notation of Section 2, in which $G$ is a cvsg of order $mp$, where $m \lt p$. As was previously remarked, the factor graph $G/p$ is independent of the choice $\gamma$ of an $(m, p)$-homogeneous automorphism of $G$. The question arises as to which graphs of order $m$ can be the factor graph of some such graph $G$, and what further structure is implied by the structure of $G$.

The fact that $G/p$ need not be vertex-symmetric can be seen from various examples.

(i) If $G$ is the line graph of the Petersen graph, then $G/5$ has the form

```
   2   0   2   2
```

(ii) If $G$ is the Coxeter graph (see [5], and [7, p. 241]), then $G/7$ has the form

```
   2
   1
   1
   1
   0

   2
```

(iii) The odd graph $O_4$ is of order $5\cdot7$ and $O_4/7$ has the form displayed earlier in (i).

(iv) There is a certain highly symmetric cvsg of order $102 = 6\cdot17$ (see [4, p. 153], [5], [6]) for which $G/17$ has the form

```
   2
   1
   1
   1
   0

   2
   1
   1
   1
```

Such examples suggest the following questions: Which non-vertex-symmetric graphs of order $m$ can be the factor graph $G/p$ of a cvsg $G$ of order $mp$, $m < p$? In particular, which trees can be $G/p$, and especially, which stars can be $G/p$? (The case of the stars seems to be one in which it is more difficult to find Hamiltonian paths in $G$.) The graph mentioned above in (33) actually has a Hamiltonian cycle (see [5]), and although the Coxeter graph (31) has no Hamiltonian cycle, it does have a Hamiltonian path; thus even in cases in which it is hard to find Hamiltonian paths or cycles by using the factor graph, such paths or cycles may exist.

Sometimes one can use different homogeneous automorphisms to help find Hamiltonian paths or cycles in $G$. For example, the graph $O_4$ not only has a $(5, 7)$-homogeneous automorphism, but it also has a $(7, 5)$-homogeneous automorphism; and, using the same definition of 'factor graph', any two such $(7, 5)$-homogeneous automorphisms give the same factor graph $O_4/5$, which has the structure

Now our Lemma 5 holds for this factor graph too, and the orbit-cycle $ABCDXYZA$ has $d[Z, A] = 2$, therefore $O_4$ has a Hamiltonian cycle, by Lemma 5. It seems that such use of homogeneous automorphisms will often be helpful in finding Hamiltonian cycles and paths in cvsg's. Although we have concentrated on the graphs of order $mp$ where $m < p$, it is clear that factor graphs may be used more generally.

Marušić [12] has asked whether every cvsg $G$ is $(m, n)$-galactic for some $m, n$ (where $n > 1$). If the answer to this question is affirmative, then for any cvsg $G$ and each $(m, n)$-homogeneous automorphism $\gamma$, we may consider the corresponding factor graph $G/\gamma$ (which will, in general, now depend upon the choice of $\gamma$). Hopefully, the factor graphs will be useful in studying Hamiltonian and other properties of cvsg's.

Sometimes the existence of a homogeneous automorphism of a regular graph can be deduced by considering the degree of the graph. This is intriguing, because the degrees of vertices have heretofore played a dominant role in results on Hamiltonian cycles.
Theorem 7. Let $G$ be a connected regular graph of degree $d_0$ and of order $n$. If $p > d_0$ and $p$ divides $|\text{Aut}(G)|$, then $p$ divides $n$ and $G$ is $(n/p, p)$-galactic.

Proof. Since $p$ divides the order of $\text{Aut}(G)$, there exists some $\gamma \in \text{Aut}(G)$ such that $\gamma$ is of order $p$. Let $F = \{x \in V: \gamma(x) = x\}$ and $S = V \setminus F$. Since $\gamma$ has order $p$, $S \neq \emptyset$. Suppose that $F \neq \emptyset$, and let $x \in F$. If $x$ were adjacent to some $y \in S$, then $x$ would be adjacent to $y, \gamma(y), \ldots, \gamma^{p-1}(y)$, and so would have degree $\geq p > d_0$. Thus no vertex in $F$ is adjacent to a vertex in $S$—but this implies that $G$ is not connected, contrary to hypothesis. Therefore $F = \emptyset$, and $\gamma$ has all its orbits of length $p$. This implies that the prime $p$ divides $n$ and $\gamma$ is $(n/p, p)$-homogeneous.

Now if $G$ is a cvsg of order $n$, then $n$ divides $|\text{Aut}(G)|$, so an immediate consequence of Theorem 7 is the following corollary, which shows that there are strong restrictions on the automorphism groups of cvsg's having small degree $d_0$.

Corollary 8. Let $G$ be a cvsg of order $n$ and degree $d_0$. If $p > d_0$, then $p$ divides $|\text{Aut}(G)|$ if and only if $p$ divides $n$, and in the latter case $G$ is $(n/p, p)$-galactic.

Although Theorem 7 and Corollary 8 are very simple observations, they are rather useful. As an application, the graph $O_4$ has degree 4 and order 57; since $5 > 4$, $O_4$ is $(7, 5)$-galactic, and this can be used to show that $O_4$ is Hamiltonian, as was done above—see (34). As another application, consider any cubic graph $G$ of order $4p$, where $p > 3$ (such as the Coxeter graph, where $p = 7$). Corollary 8 implies that $|\text{Aut}(G)| = 2^r 3^s p$, for some integers $r, s$—and that if $G$ is vertex-symmetric, then $G$ is $(4, p)$-galactic. (The fact that $G$ is $(4, p)$-galactic also follows from Proposition 1, but Corollary 8 is much simpler than Proposition 1.) Tutte [16] has shown that $s \leq 1$ must hold.

In a sequel paper, we shall prove that every cvsg of order $4p$ has a Hamiltonian path. The only 'difficult' case turns out to be the possible existence of 'generalized Coxeter graphs' whose factor graphs would be the three-valent star.

References