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Idempotent analysis and continuous semilattices

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Abstract

In this survey article we give a brief overview of various aspects of the recently emerging field of idempotent analysis and suggest potential connections with domain theory.

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1. Introduction

Classical linear algebra and functional analysis have been based on (topological) vector spaces over fields, particular the real and complex fields. In idempotent analysis (the formal term was introduced in 1989) vector spaces are replaced by modules (of functions) over idempotent semirings, the most common and important idempotent semiring being $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with idempotent addition and multiplication given by

$$a \oplus b = \max\{a, b\} \quad \text{and} \quad a \odot b = a + b,$$

or isomorphic reformulations thereof.

The structure of idempotent semirings is sufficiently rigid that significant analogs of many basic results in functional analysis can be derived. Indeed certain practitioners of idempotent analysis advocate a "correspondence principle" for idempotent analysis, a wide-ranging assertion (somewhere in the realm of metamathematics and supposedly akin in spirit to N. Bohr's correspondence principle in quantum mechanics) that there is a correspondence between important, useful, and interesting constructions and results in analysis over the field of real (or complex) numbers and appropriately analogous

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constructions and results in the idempotent analysis over idempotent semirings, particularly $\mathbb{R}^{-\infty}$. The correspondence principle frequently manifests itself in practice in settings where one considers certain "limiting" or "asymptotic" cases arising in classical analysis (see Section 12 of this survey). Ref. [9] gives a variety of illustrations and applications of the correspondence principle.

Despite its recent origins, idempotent analysis already boasts an extensive literature and widespread applications (although in many cases the idempotent structure appears more implicitly than explicitly). It seems particularly suited to a variety of optimization problems, for example optimal organization of parallel data processing. This makes sense when one realizes that in scheduling parallel processes, a process dependent on two earlier parallel processes must wait for the *maximum* run time of the two earlier processes before it can be initiated. The finite dimensional linear algebra is typically the suitable tool for discrete optimization problems, while the functional analysis aspect applies to continuous problems.

Two good sources to learn about idempotent analysis are [7] and [9]. The second reference begins with a helpful and well-written survey article. Both books also contain extensive references. Another useful reference is [10]. The discrete theory, which is sometimes referred to as max-plus algebra, is treated in [3] and [4]. For the theory of continuous lattices and domains we refer the reader to [1] or the more extensive treatment of [6].

2. Idempotent semirings

A *semiring* is a quintuple $(S, \oplus, \odot, \mathbf{0}, \mathbf{1})$ satisfying
(i) $(S, \oplus, \mathbf{0})$ is a commutative monoid with identity $\mathbf{0}$.
(ii) $(S, \odot, \mathbf{1})$ is a monoid with identity $\mathbf{1}$.
(iii) $a \odot \mathbf{0} = \mathbf{0} = \mathbf{0} \odot a$ for all $a \in S$.
(iv) $a \odot (b \oplus c) = a \odot b \oplus a \odot c$, $(b \oplus c) \odot a = b \odot a \oplus c \odot a$ for all $a, b, c \in S$.
An *idempotent semiring* or *dioid* is a semiring for which the addition is idempotent:
(v) $a \oplus a = a$ for all $a \in S$.
Note that $(S, \oplus)$ is commutative idempotent monoid. If we define an order by $a \leqslant b$ iff $a \oplus b = b$, then the order is a partial order and any two elements a, b have a least upper bound or join, namely $a \oplus b$. Such structures are called (*join*) *semilattices*. Thus the additive structure of an idempotent semiring is that of a semilattice with least element $\mathbf{0}$. Conversely given a join semilattice with smallest element $\mathbf{0}$, the operation of taking join converts it to a commutative idempotent semigroup with identity $\mathbf{0}$. We thus may pass freely back and forth between the order-theoretic notion of a semilattice and the algebraic notion of a commutative idempotent semigroup.

In some situations we may prefer to define the order in the reverse manner: $a \leqslant b$ iff $a \oplus b = a$. In this case every two elements of the commutative idempotent semigroup have a greatest lower bound or meet and we obtain a meet semilattice with largest element $\mathbf{0}$. For any specific example or setting of an idempotent semiring, we typically fix the order by one means or the other so that the additive structure is either thought of as a meet semilattice or a join semilattice. But unless the meet semilattice structure is specified or clearly intended from context, we typically assume that S is a join semilattice under $\oplus$ with smallest element $\mathbf{0}$.

There is a corresponding category of dioids whose objects are dioids and whose morphisms are functions between dioids that preserve addition, multiplication, $\mathbf{0}$, and $\mathbf{1}$.

We frequently impose further algebraic conditions on a dioid that provide a richer algebraic theory. We say that a dioid is *commutative* if the multiplicative operation $\odot$ is commutative and *cancellative* if the multiplicative operation is cancellative for non-zero elements. It is *algebraically complete* if the equation $x^n = a$ always has a unique solution.

An important case where the cancellation law is satisfied is requiring the nonzero elements to form a group under multiplication. If it is additionally commutative, then we call the dioid a *semifield.*

Proposition 2.1. *If S is a dioid for which $\odot$ is a group operation, then inversion is order-reversing on $S\backslash\{\mathbf{0}\}$, and hence $(S, \leqslant)$ is a lattice.*

3. Topological idempotent semirings

A commutative idempotent semigroup (or semilattice) S is called *metric* if it is equipped with a metric ρ that satisfies the *minimax* axiom

- $\rho(a\oplus b, c\oplus d) \leqslant \max(\rho(a,c), \rho(b,d))$ for all $a,b,c,d \in S$; and the *monotonicity axiom*
- $a \leqslant b \leqslant c$ implies $\rho(b,c) \leqslant \rho(a,c)$ and $\rho(a,b) \leqslant \rho(a,c)$.

An idempotent semiring is *metric* if the additive part is metric in the above sense and the multiplication $\odot : S\times S \to S$ is uniformly continuous on any order bounded subset of $S\times S$, where $S\times S$ is given the coordinate-sup metric.

The minimax axiom implies the uniform continuity of $\oplus$ and the *minimax inequality*

$$\rho\left(\bigoplus_{i=1}^{n} a_i, \bigoplus_{j=1}^{n} b_j\right) \leqslant \min_{\pi} \max_{i} \rho(a_i, b_{\pi(i)}),$$

where the infimum is taken over all permutations π of the set $\{1, \ldots, n\}$.

4. Idempotent semiring examples

1. *The max-plus reals* $\mathbb{R}_{\max}$: on the set $\mathbb{R}\cup\{-\infty\}$ define $a\oplus b := \max\{a,b\}$, $a\odot b = a+b$, $\mathbf{0} = -\infty$, and $\mathbf{1} = 0$. This example is actually an idempotent semifield $((\mathbb{R},\odot)$ is a commutative group) that is algebraically complete. It is also an example of a metric idempotent semiring with metric $\rho(x,y) = |e^x - e^y|$; this metric gives rise to the usual topology on $\mathbb{R}$ and the open left rays as a basis of neighborhoods for $-\infty$.
2. *The min-plus reals* $\mathbb{R}_{\min}$: on $\mathbb{R}^{\infty} = \mathbb{R}\cup\{\infty\}$ define $a\oplus b := \min\{a,b\}$, $a\odot b = a+b$, $\mathbf{0} = \infty$, and $\mathbf{1} = 0$. This is an isomorphic form of $\mathbb{R}_{\max}$, the isomorphism sending

$x \to -x$. We also write this example as $\mathbb{R}^{\infty}$, particularly when our emphasis is more on the topological space than on the semiring structure. The semiring $\mathbb{R}_{\min}$ is often convenient for applications to analysis and continuous optimization.

3. *The max-plus extended reals*: Set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ with $a \oplus b := \max\{a, b\}$ and $a \odot b = a + b$, where $\infty + (-\infty) = -\infty + \infty = -\infty$. While one loses the metric and semifield properties with this example, it has the advantage of order completeness. Since it is also a continuous lattice, it is often a good choice in developing connections with domain theory. There is also an order dual version of this example, the min-plus extended reals, where ∞ becomes the $\mathbf{0}$-element.
4. Let $\mathbb{R}^n_+$ be the nonnegative octant in $\mathbb{R}^n$ with $\oplus$ the coordinatewise maximum operation. We can take for $\odot$ either coordinatewise addition or coordinatewise multiplication. In the first case we obtain an algebraically complete cancellative commutative dioid. Note that in the second case for $n = 1$ we obtain an isomorphic copy of $\mathbb{R}_{\max}$ (the isomorphism being exponentiation).
5. Let M be a monoid, a semigroup with identity 1. Let $\mathscr{P}(M)$ be the power set and $\mathscr{P}^{\mathrm{Fin}}(M)$ be the collection of finite subsets with addition $\oplus$ given by union and multiplication $\odot$ given by set multiplication: $A \odot B := \{ab : a \in A, b \in B\}$. Then $\mathbf{0} = \emptyset$ and $\mathbf{1} = \{1\}$. If $M = (\mathbb{R}^n, +)$, then multiplication in $\mathscr{P}(\mathbb{R}^n)$ corresponds to (Minkowski) addition of subsets. If A is an alphabet and $M = A^*$ is the free monoid over A, then $\mathcal{P}(A)$ is the dioid of (formal) languages over A.

5. Semimodules

A *semimodule* over an idempotent semiring R consists of a commutative monoid $(M, +)$ together with a left R action $R \times M \to M$ (scalar multiplication) satisfying for $r, s \in R$ and $a, b \in M$

- $r \cdot (a + b) = r \cdot a + r \cdot b$, $r \cdot 0_M = 0_M$,
- $(r \oplus s) \cdot a = r \cdot a + s \cdot a$, $\mathbf{0} \cdot a = 0_M$,
- $r \cdot (s \cdot a) = (r \cdot s) \cdot a$, $\mathbf{1} \cdot a = a$.

Note that $a + a = \mathbf{1} \cdot a + \mathbf{1} \cdot a = (\mathbf{1} + \mathbf{1}) \cdot a = \mathbf{1} \cdot a = a$, so that $(M, +)$ must also be idempotent.

If M and N are R-semimodules, a homomorphism from M to N is a homomorphism of monoids that also respects scalar multiplication. The notions of submodules and product modules should be clear.

A semimodule M over a metric idempotent semiring R is *metric* if it is a metric idempotent semigroup for which the scalar multiplication from $R \times M$ to M is uniformly continuous on order bounded subsets of $R \times M$.

We give some basic examples of semimodules.

1. Let X be a set and let S be an commutative idempotent (metric) semigroup. The set $B(X, S)$ of bounded mappings (mappings with order-bounded range) is a commutative idempotent (metric) semigroup with respect to the pointwise operation. (In the metric case, the metric on $B(X, S)$ is given by the uniform metric $\rho(f, g) = \sup_x \{\rho(f(x), g(x)) : x \in X\}$. Since f, g are bounded, it follows from the monotonicity axiom that the supremum exists and is finite.) If R is a (metric)

semiring, then $B(X,R)$ is a (metric) R-semimodule with respect to pointwise addition and scalar multiplication.

2. If $X=\{x_1,\dots,x_n\}$ is a finite set, then every function from X to a dioid R is bounded and $B(X,R)$ is isomorphic to R^n.
3. If X is a topological space and R is a metric idempotent semiring, then one can consider the subsemimodule $C(X,R)$ of all continuous bounded functions. Further, if X is locally compact, then one can consider the subsemimodules $C_0(X,R)$ of continuous functions with compact support and $C_0^\infty(X,R)$ of continuous functions vanishing at ∞.
4. Let R be a metric idempotent semiring, and let $B(X,R)$ be the metric semimodule of bounded functions. A linear operator on $B(X,R)$ is a continuous mapping $H: B(X,R) \to B(X,R)$ such that

$$H(g\oplus h)=H(g)\oplus H(h),\quad H(r\odot g)=r\odot H(g)$$

for any bounded functions g and h and any constant r. With respect to the operations $\oplus$ of pointwise addition and $\odot$ of composition and with the metric

$$\rho(H_1,H_2)=\sup\{\rho(H_1(f),H_2(f)): \forall x, f(x)\in[\mathbf{0},\mathbf{1}]\},$$

the linear operators become an idempotent semiring, and the pointwise scalar multiplication gives them the structure of a *semialgebra*.

6. Domain theory connections

The additive structure of an idempotent semiring and of a semimodule over an idempotent semiring is that of a semilattice (where the operation is sometimes viewed as join and sometimes as meet, according to the context). The theory of modules over idempotent semirings has both discrete and continuous or analytic components. Typically, in the analytic setting the semilattices have been assumed to be the metric semilattices, as defined in Section 3. However domain theory suggests a modified approach.

We recall that in a partially ordered set, the *approximation relation* is defined $q\ll p$ if whenever D is a directed set for which the supremum $\sup D$ exists and satisfies $p\leqslant \sup D$, then $q\leqslant d$ for some $d\in D$. A partially ordered set P is a *continuous poset* if every element in P is the directed supremum of the elements that approximate it. A poset P is a *dcpo* (directed complete partially ordered set) if every directed set has a supremum and a *continuous domain* if it both a continuous poset and a dcpo. A *continuous lattice* is a complete lattice that is also a continuous poset. A *dually continuous lattice* is one that is a continuous lattice in its order dual; other dual notions of continuity are expressed similarly.

Following the lead of [2], we suggest a modified version of a continuous lattice as a suitable structure for replacing (and generalizing) the notion of a metric semilattice in idempotent analysis, or at least in substantial portions of it. Our proposal is the following: *replace the assumption that S is metric by the assumption that each order* *interval* $[a,b]=\{x: a\leqslant x\leqslant b\}$ *is a continuous or dually continuous lattice, the choice depending on context*. Such a hypothesis typically generalizes assumptions made in the metric case, may often pinpoint more closely the assumptions needed, and allows the theory of continuous lattices and domains to come into play.

The next propositions establish close connections between continuity and metric and provide motivation for our proposal.

Proposition 6.1. *Let S be metric meet (resp. join) semilattice. Then it is a topological semilattice with a basis of subsemilattices. If an order interval* $[a,b]$ *is a complete lattice, then it is a continuous (resp. dually continuous) lattice and the relative metric topology agrees with the Lawson topology.*

Proof. In metric semilattices the open ε-balls are all subsemilattices, so the first assertion is true. The remaining assertions can be readily deduced from the theory presented in Chapters III and VI of [6]. □

Proposition 6.2. *Let S be a meet semilattice that is a continuous domain with a countable basis. Then S admits the structure of a metric semilattice such that the metric and Lawson topologies agree.*

Proof. The proof follows along the lines sketched in Exercise VI-3.17 of [6]. □

Since domain connections to idempotent analysis have hitherto been drawn only to a limited extent, one would like to examine the theory from this perspective. This will necessitate some extensions of the standard theory of domains, since one will be working with ordered objects that are not necessarily domains, but have ordered intervals that are continuous lattices. Conversely, idempotent analysis may provide an opportunity to import useful and rather natural algebraic and analytic ideas into domain theory that will make it a much more powerful tool in certain contexts and provide new areas of application in theoretical computer science.

Problem. Find general conditions that yield equivalences between
(i) metric semilattices and continuous semilattices and
(ii) the metric and Lawson topologies.

We point out further connections and potential connections between domain theory and idempotent analysis in the following sections.

7. Max-plus algebra

For the semimodule R^n over a semiring R, the second example of Section 5, one has a satisfactory linear algebra, typically referred to as *max-plus* algebra, that shares many features in common with classical linear algebra. For example any vector $\mathbf{v}\in R^n$ can be uniquely represented in the form $\mathbf{v}=\bigoplus_{i=1}^n r_i\odot \mathbf{e_i}$, where $\mathbf{e_i}$ is the "unit" vector with $\mathbf{1}$ in the ith coordinate and $\mathbf{0}$ in the remaining coordinates. The collection $\{\mathbf{e_1},\dots,\mathbf{e_n}\}$

is called the *standard basis*. As in conventional linear algebra, one readily proves that any *linear functional* that is, semimodule homomorphism $m: R^n \to R$, has the form

$$m(\mathbf{v}) = \bigoplus_{i=1}^{n} m^i \odot a_i, \quad \mathbf{v} = (a_1, \ldots, a_n) \in R^n,$$

where $m^i \in R$. Therefore, the semimodule of linear functionals on R^n is isomorphic to R^n.

By analogy with Euclidean space, we define an inner product on R^n by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \bigoplus_{i=1}^{n} a_i \odot b_i.$$

The inner product is bilinear with respect to $\oplus$ and $\odot$, and the standard basis is orthonormal with respect to this inner product. Furthermore, each linear functional has a unique representation of the form $m(\mathbf{a}) = \langle \mathbf{a}, \mathbf{b} \rangle$ for some $\mathbf{b} \in R^n$.

Any semimodule endomorphism $H: R^n \to R^n$ (a linear operator on R^n) has a matrix representation with respect to the standard basis. Thus the endomorphism ring has a representation as the $n \times n$-matrices (with matrix operations like the usual ones, except with respect to $\oplus$ and $\odot$). It is a special case of the semialgebra given in the fourth example of Section 5.

Unlike the classical theory, the class of invertible matrices in max-plus algebra is rather thin. However, there is another matrix in this context that plays an important role in the computations of max-plus algebra. Let us consider max-plus algebra over the max-plus extended reals $\overline{\mathbb{R}}$. Let $A: \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^m$ be a (max-plus) linear map. Then $x \mapsto Ax$ preserves arbitrary suprema, and thus is residuated, or equivalently, the lower adjoint of a Galois adjoint pair. The *upper adjoint* $A^\dagger$ preserves arbitrary infs, and one verifies that it is also homogeneous with respect to the $\odot$-scalar multiplication. Thus $A^\dagger$ is linear in the context of semimodules over the min-plus extended reals. The fact that $\overline{\mathbb{R}}^n$ is a continuous lattice and the prevalence of Galois connections provide ties with the theory of continuous lattices and domains.

Max-plus algebra has been applied to a variety of discrete optimization problems such as graph optimization problems, parallel computation and production scheduling, queueing systems with finite capacity, and timed Petri nets, timed event graphs and stochastic versions thereof. We refer to [3,4]; further references may be found in [7,9].

8. A discrete-time Bellman equation

In this section we consider one important discrete optimization problem that can be effectively studied from the view-point of max-plus algebra. Let $X = \{x_1, \ldots, x_n\}$ and consider $A := B(X, R) \cong R^n$, where R is a metric idempotent semiring. Let $H = [h_{ij}]$ be a linear operator on A, and let $b \in A$. We consider the discrete-time equation

$$x_{t+1} = Hx_t \oplus b,$$

called the *generalized evolution Bellman equation*. The equation

$$x = Hx \oplus b$$

is called the *generalized stationary* (*steady state*) *Bellman equation*. There equations are *homogeneous* if $b = \mathbf{0}$ and *nonhomogeneous* otherwise. They are formulations in the context of idempotent analysis of important classes of the discrete Hamilton–Jacobi–Bellman systems in control theory (see, for example [11]) or of systems that arise when optimization problems on graphs are solved by dynamic programming methods.

We consider first the stationary steady state Bellman equation

$$x = Hx \oplus b.$$

If we define the order so that $\oplus$ is sup and $\mathbf{0} = (\mathbf{0}, \ldots, \mathbf{0})$ is the bottom element of R^n, then we observe that the right-hand side is a continuous order-preserving map, and that we are looking for fixed points, in particular, a least fixed point for this "affine" equation. However, we are not assuming the dcpo property in this context, so fixed points are not guaranteed, indeed need not exist in general, even under the hypothesis of conditional completeness. Thus we get into interesting questions about existence of fixed points. Therefore a somewhat different fixed point theory arises from that of domain theory; nevertheless, one observes certain kinships between the two versions.

Proposition 8.1. *If principal ideals in R are dcpo's and if fixed points do exist, then there is a smallest solution to the stationary Bellman equation, namely the least fixed point given by*

$$x = \bigvee\nolimits^{\uparrow} \{H^{(n)} b : n \in \mathbb{N}\},$$

where

$$H^{(n)} := \sup\{H^k : 0 \leqslant k \leqslant n\} = \bigoplus_{k=0}^{n} H^k = (H \oplus I)^n.$$

The solutions are frequently derived from the asymptotic limit of the appropriate powers of H, provided this limit exists.

Proposition 8.2. *If the sequence $H^{(t)} = \bigoplus_{k=0}^{t} H^k = (H \oplus I)^t$ converges to H^* as $t \to \infty$, then $H^* b$ is a solution to the stationary Bellman equation, and is the smallest of all the solutions. Every solution y is of the form $y = w \oplus H^* b$, where w is a solution of the homogeneous equation $Hw = w$.*

By linearity it suffices to solve the stationary equation for $b = e_i$ for each i. If ε_i is the solution of the stationary equation for $b = e_i$ (such solutions are called *sources*), then

$$x = \bigoplus_{i=1}^{n} b_i \odot \varepsilon_i$$

is the solution for arbitrary $b = \bigoplus_{i=1}^{n} b_i \odot e_i$. Thus the general solution of the stationary Bellman equation has a source representation.

We turn now to the evolution equation. The solution x_t of the evolution equation for $x_0 = \mathbf{0}$ has the form

$$x_t = H^{(t-1)}b := \bigoplus_{k=0}^{t-1} H^k b.$$

The solution of the homogeneous evolution equation with arbitrary initial value $x_0 = \oplus_{i=1}^{n} x_0^i \odot e_i$ is given by

$$x_t = \bigoplus_{i=1}^{n} x_0^i \odot H^t e_i,$$

that is, a linear combination of the source functions (Green's functions) $H^t e_i$, which are solutions of the homogeneous evolution equation with initial value e_i. Thus the evolution Bellman equation also has a source representation.

We remark that in the context of $\mathbb{R}_{\max}$-semimodules, one could work instead with the max-plus extended reals and always be guaranteed a fixed point or solution. There are two reasons for not doing so: first of all one is frequently interested in a *finite* solution, and so the general existence of an infinite solution is no help. Secondly, the max-plus extended reals do not form a metric idempotent semialgebra nor a semifield, and it is sometimes useful to have such tools as the minimax inequality to show existence of finite solutions.

There is a spectral theory for operators in idempotent analysis that diverges rather dramatically from the classical theory. It is sometimes useful for determining existence of asymptotic limits of operators.

A scalar $\lambda \in R$ is an *eigenvalue* of the linear operator H on R^n if

$$Hv = \lambda \odot v$$

for some $v \in R^n \backslash \{\mathbf{0}\}$. Any such v satisfying the equation is called an eigenvector for λ.

Proposition 8.3. *For $R = \mathbb{R}_{\max}$, the max-plus reals, every linear operator H on R^n has at least one eigenvalue. If all matrix entries of H are not equal to $\mathbf{0}$, then $\lambda \neq \mathbf{0}$, and is unique. Furthermore the vector v is unique up to $\odot$-multiplication by a constant.*

The operator H is said to be *nilpotent* if $H^{(t)} = H^{(t+1)}$ for some $t \geqslant 1$. Note that in this case $H^* = H^{(t)}$.

Proposition 8.4. *Suppose that the linear operator H on $\mathbb{R}^n_{\max}$ has all entries not equal to $\mathbf{0}$. Then H is nilpotent iff $\lambda \geqslant \mathbf{1} = 0$, where λ is the eigenvalue.*

9. Quantales

The Bellman equations of the previous section always have solutions in the context of quantales. A *quantale* is a complete sup-semilattice (thus $\oplus$ admits infinite sums) with an associative multiplication $\odot$ that distributes over arbitrary sups. Quantales are often not required to have a multiplicative **1**, but in order to be a dioid, this must be the case. The earlier example of the power set of a free semigroup A^* under union and set product is a quantale, actually the free one over the set A.

In a quantale the simple Bellman system

$$x = (a \odot x) \oplus b.$$

has a least solution $a^* \odot b$, where a^* is the Kleene star of a, defined by

$$a^* = \mathbf{1} \oplus a \oplus a^2 + \cdots = \sup_{0 \leqslant i} a^i.$$

The star operation satisfies many interesting identities, for example:

$$(a \oplus b)^* = (a^* \odot b)^* \odot a^*.$$

For a quantale Q, the semialgebra of $n \times n$ matrices over Q is again a quantale. This applies in particular to the quantale of the max-plus extended reals $\overline{\mathbb{R}}$; moreover, the semialgebra is a completely distributive, hence dually continuous, lattice. There are recursion formulas for reducing the star operation for matrices back to the computation of the star operation in Q (see Section 2.7 of [7]).

10. Function spaces

The functional analysis aspects of idempotent analysis arise in the study of certain important classes of function spaces. The analysis has for the most part been worked out for the case of $\mathbb{R}_{\min}$, so our presentation is given in this framework. It is in this context that some of the major theorems of the theory have arisen. One important function space is the $\mathbb{R}_{\min}$-semimodule $C_0(X)$ of continuous functions with compact support from a locally compact Hausdorff space X into $\mathbb{R}^\infty$, where $\mathbb{R}^\infty$ is the underlying space of $\mathbb{R}_{\min}$. Note that in this context compact support means that the function takes only the value $\mathbf{0} = \infty$ outside a compact set.

The first theorem connects continuous linear operators and kernel functions. Note that in idempotent analysis infinite infima (or suprema) typically play the role of the integral.

Theorem 10.1. *Let X, Y be locally compact Hausdorff spaces. If $B : C_0(Y) \to C_0(X)$ is a continuous ($\mathbb{R}_{\min}$-)linear operator, then there exists a unique Scott continuous $k : X \times Y \to \mathbb{R}^\infty$ such that*

$$(Bf)(x) = \int^{\oplus} k(x,y) \odot f(y)\, \mathrm{d}y = \inf\{k(x,y) + f(y) : y \in Y\}.$$

The theorem specializes to the following characterization of functionals.

Corollary 10.2. *If $\phi : C_0(Y) \to \mathbb{R}_{\min}$ is a continuous linear functional on the space of continuous functions of compact support on a locally compact Hausdorff space Y, then there exists a unique lower semicontinuous map $k = k_\phi : Y \to \mathbb{R}^\infty$*

such that

$$\phi(f) = \int^{\oplus} k(y) \odot f(y)\,\mathrm{d}y$$

Thus the map $\phi \mapsto k_\phi$ *is an isomorphism of the semimodule* $C_0^*(Y)$ *onto the semimodule of lower semicontinuous functions.*

See Chapter 1 of [9] for proofs of the preceding results. We remark that lower semicontinuous functions on locally compact spaces have received considerable attention in domain theory. However, there is again a twist here similar to that we encountered earlier in looking at fixed point results. In this case the absence of a bottom element in $\mathbb{R}_{\min}$ means that the semimodule of lower semicontinuous functions isomorphic to the dual space $C_0^*(Y)$ will generally not be a domain. Thus idempotent analysis motivates the study of structures that are more general than domains, for example, dcpos in which every order interval is a domain.

Let us specialize to the case that X is an open subset of $\mathbb{R}^n$. In the study of partial differential equations on X, one sometimes considers generalized solutions that are distributions, functionals on a linear space of test functions. For an $\mathbb{R}_{\min}$-linear partial differential equation, one takes for the test functions $C_0(X)$. As we have seen, the dual space is then the semimodule of lower semicontinuous functions into $\mathbb{R}^{\infty}$. These "idempotent distributions" provide an alternative version of the viscosity solutions and thus provide an important impetus for functional idempotent analysis.

11. Measures in idempotent analysis

We further develop some of the ideas of the preceding section, but now in the context of $\mathbb{R}_{\max}$ instead of $\mathbb{R}_{\min}$ (to connect with the literature that we cite in this section). Let X be a topological space. An idempotent analog of the usual integration can be defined by the formula

$$\int_X^{\oplus} \phi(x)\,\mathrm{d}x = \sup_{x\in X} \phi(x)$$

if ϕ is a continuous or upper semicontinuous function from X into $\mathbb{R}_{\max}$. The set function

$$\mu_\phi(B) = \sup_{x\in B} \phi(x),$$

where $B \subseteq X$ is called an *idempotent measure* on X. Since $\mu_\phi(\bigcup A_\alpha) = \oplus_\alpha\ \mu_\phi(A_\alpha) = \sup\ \mu_\phi(A_\alpha)$, the measure is *completely additive*. The function ϕ is called a *density function* for the idempotent measure μ_ϕ. More generally one can define measures as completely additive functions on the lattice of open sets, or more general lattices closed under arbitrary unions, and investigate their properties, particularly the existence of a density function and the extent to which it is unique. Such investigations have been carried out in [8] and in quite some detail in [2]. Both treatments use notions from the theory of continuous lattices quite heavily in their development. The second reference generalizes to measures defined into more general idempotent semirings and makes substantial use of the hypotheses suggested in Section 6 that intervals be either dually continuous or continuous.

An idempotent measure gives rise to an idempotent integral defined by

$$\int_X^{\oplus} \psi(x)\,\mathrm{d}\mu_\phi = \int_X^{\oplus} \psi(x) \odot \phi(x)\,\mathrm{d}x = \sup_{x \in X} \psi(x) \odot \phi(x).$$

This integration is max-plus linear and defines a continuous linear form on the space of bounded continuous functions. In [2] a Riesz representation theorem for idempotent analysis is established, which shows that under quite general conditions a continuous linear functional on the space of continuous bounded functions can be realized as idempotent integration against an idempotent measure. The theorem is proved in the more general setting of idempotent semirings in which each order interval is a continuous lattice.

12. Deformations, asymptotics, and superposition

Let $\mathbb{R}^\infty$ the underlying space of $\mathbb{R}_{\min}$ consider the inverse mappings $u = e^{-w/h} : \mathbb{R}^\infty \to \mathbb{R}^+ = [0, \infty)$ and $w = -h \ln u : \mathbb{R}^+ \to \mathbb{R}^\infty$. These bijections induce isomorphisms between the following operations and systems:

$$a \oplus_h b := -h \ln(\mathrm{e}^{-a/h} + \mathrm{e}^{-b/h}) \overset{\approx}{\longleftrightarrow} a + b,$$

$$a \odot b = a + b \overset{\approx}{\longleftrightarrow} ab,$$

$$(\mathbb{R}^\infty, \oplus_h, \odot) \overset{\approx}{\longleftrightarrow} (\mathbb{R}^+, +, \cdot).$$

Such constructions in Lie theory and quantum group theory are called *deformations*:

$$(\mathbb{R}^+, +, \cdot) \overset{\text{deforms}}{\longrightarrow} (\mathbb{R}^\infty, \oplus_h, \odot)$$

We note that $\lim_{h \to 0} a \oplus_h b = \min\{a, b\} = a \oplus b$. Thus $\mathbb{R}_{\min}$ appears as the asymptotic structure, and in this fashion idempotent analysis appears in the asymptotics of the study of what are sometimes called "large deviation limits."

Let us consider a rather simple illustration of such asymptotics. If u_1, u_2 are solutions of the one-dimensional heat equation $\partial_t u = h \partial_x^2 u$ for a small parameter $h > 0$, then so is $a_1 u_1 + a_2 u_2$. This linearity of solutions is a special case of what is sometimes called a *superposition principle*. We transform the heat equation and linearity of solutions via the transforms of the preceding paragraphs.

$$\partial_t u = h \partial_x^2 u \overset{w = -h \ln u}{\longrightarrow} \partial_t w + (\partial_x w)^2 - h \partial_x^2 w = 0 \qquad (\bigstar)$$

$$w_1, w_2 \text{ solutions} \overset{\text{superposition}}{\longrightarrow} -h \ln(\mathrm{e}^{-(a_1 + w_1)/h} + \mathrm{e}^{-(a_2 + w_2)/h})$$

As $h \to 0$, we obtain

$$(\bigstar) \to \partial_t w + (\partial_x w)^2 = 0 \qquad \text{(HJ)}$$

$$-h \ln(e^{-a/h} + e^{-b/h}) \to \min\{a, b\}$$

$$w_1, w_2 \text{ solutions} \overset{\text{superposition}}{\longrightarrow} \min\{a_1 + w_1, a_2 + w_2\}$$
$$= (a_1 \odot w_1) \oplus (a_2 \odot w_2)$$

The transformed asymptotic equation (HJ) is the Hamilton–Jacobi equation, a basic equation in continuous optimization. Its superposition principle has a natural formulation in the language of idempotent analysis. In this manner one observes that idempotent analysis has close connections with the Hamilton–Jacobi equation and its viscosity solutions.

13. Conclusion

The preceding discussion has been an attempt to give some of the flavor of idempotent analysis, with a view toward pointing out how many important aspects of the theory resonate with aspects of domain theory. It appears to the author that indeed there should be points of fruitful interaction between the two theories, and a major goal of this article is to point out possibilities for and encourage such developments.

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