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The representation type of Segre varieties

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Abstract

The goal of this work is to prove that all Segre varieties $\Sigma_{n_1,\ldots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i + 1) - 1$, (unless the quadric surface in $\mathbb{P}^3$) support families of arbitrarily large dimension and rank of simple Ulrich (and hence ACM) vector bundles. Therefore, they are of wild representation type.

Keywords: Segre varieties; Ulrich bundles; Representation type

Contents

1. Introduction

The projective space $\mathbb{P}^n$ holds a very remarkable property: the only indecomposable vector bundle $\mathcal{E}$ without intermediate cohomology (i.e., $\mathrm{H}^i(\mathbb{P}^n, \mathcal{E}(t)) = 0$ for $t \in \mathbb{Z}$ and $1 < i < n$), up

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to twist, is the structural line bundle $\mathcal{O}_{\mathbb{P}^n}$. This is the famous Horrocks' Theorem, proven in [11]. Ever since this result was stated, the study of the category of indecomposable arithmetically Cohen–Macaulay bundles (i.e., bundles without intermediate cohomology) supported on a given projective variety X has raised a lot of interest since it is a natural way to understand the complexity of the underlying variety X. Mimicking an analogous trichotomy in Representation Theory, in [7] a classification of ACM projective varieties was proposed as *finite, tame or wild* (see Definition 2.7) according to the complexity of their associated category of ACM vector bundles and it was proved that this trichotomy is exhaustive for the case of ACM curves: rational curves are finite, elliptic curves are tame and curves of higher genus are wild.

Notwithstanding the fact that ACM varieties of finite representation type have been completely classified in a very short list (see [3, Theorem C] and [8, p. 348]), it remains a challenging problem to find out the representation type of the remaining ones. Up to now, del Pezzo surfaces and Fano blow-ups of points of $\mathbb{P}^r$ (cf. [14], the cases of the cubic surface in $\mathbb{P}^3$ and the cubic threefold in $\mathbb{P}^4$ have also been handled in [4]) and ACM rational surfaces on $\mathbb{P}^4$ (cf. [15]) have been shown to be of wild representation type. It is worthwhile to point out that in [5] an example of projective variety, namely the quadratic cone, that does not fall into this trichotomy has been found.

Among ACM vector bundles $\mathcal{E}$ on a given variety X, it is interesting to spot a very important subclass for which its associated module $\oplus_t \mathrm{H}^0(X, \mathcal{E}(t))$ has the maximal number of generators, which turns out to be $\deg(X)\mathrm{rk}(\mathcal{E})$. This property was isolated by Ulrich in [18], and ever since modules with this property have been called Ulrich modules and correspondingly Ulrich bundles in the geometric case (see [9] for more details on Ulrich bundles). It is therefore a meaningful question to find out if a given projective variety X is of wild representation type with respect to the much more restrictive category of its indecomposable Ulrich vector bundles.

In this paper, we are going to focus our attention on the case of *Segre varieties* $\Sigma_{n_1,\dots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i + 1) - 1$ for $1 \le n_1, \dots, n_s$. It is a classical result that the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$ only supports three indecomposable ACM vector bundles, up to shift: $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$, $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)$ and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)$. For the rest of Segre varieties we construct large families of simple (and, hence, indecomposable) Ulrich vector bundles on them and this will allow us to conclude that they are of wild representation type. Up to our knowledge, they will be the first family of examples of varieties of arbitrary dimension for which wild representation type is witnessed by means of Ulrich vector bundles. Part of the results of this paper have recently been generalized by the second author who in [13] has proved that non-singular rational normal scrolls $S(a_0, \dots, a_k) \subseteq \mathbb{P}^N$, $N = \sum_{i=0}^{k}(a_i) + k$, are all of wild representation type unless $\mathbb{P}^{k+1}$, $S(a)$, $S(1,1)$ and $S(1,2)$ which are of finite representation type.

The method for constructing large families of indecomposable Ulrich vector bundles on Segre varieties $\Sigma_{n_1,\dots,n_s} \subseteq \mathbb{P}^N$ will depend on the values $n_1, \dots, n_s$. More concretely, when $n_1 \ge 2$, the construction of Ulrich vector bundles on $\Sigma_{n_1,\dots,n_s}$ will start up by pulling back certain vector bundles (not even ACM, since they would be decomposable) from one of its factors. This will allow to prove the existence of families of large dimension of indecomposable Ulrich vector bundles on $\Sigma_{n_1,\dots,n_s}$ for certain scattered ranks. Then a construction of vector bundles through extensions will provide the existence for the remaining ranks. Summing up, we get (see Theorem 3.11):

Theorem. *Fix integers $2 \le n_1 \le \cdots \le n_s$ and let $\Sigma_{n_1,\dots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i + 1) - 1$ be a Segre variety. For any integer $r \ge n_2\binom{n_1}{2}$, set $r = an_2\binom{n_1}{2} + l$ with $a \ge 1$ and $0 \le l \le$*

$n_2\binom{n_1}{2}-1$. *Then there exists a family of dimension* $a^2(n_2^2+2n_2-4)+1+l\left(an_2\binom{n_1+1}{2}-l\right)$ *of simple (hence, indecomposable) initialized Ulrich vector bundles on* $\Sigma_{n_1,\dots,n_s}\subseteq\mathbb{P}^N$ *of rank* r.

In particular, $\Sigma_{n_1,\dots,n_s}\subseteq\mathbb{P}^N$ *is of wild representation type.*

On the other hand, when $n_1 = 1$ we are going to use a different approach. We are going first to use the Ulrich line bundles on $\Sigma_{n_1,\dots,n_s}$ obtained in Proposition 3.2 as building blocks through an iterated extension procedure to obtain indecomposable Ulrich vector bundles of rank up to $(\Sigma_{i=2}^s n_i - 1)\prod_{i=2}^s(n_i+1)$ (see Theorem 4.1). They will be used in turn to provide indecomposable Ulrich vector bundles of arbitrary large rank. Our main result in this case is (see Theorem 4.4 and Corollary 4.5):

Theorem. *Consider the Segre variety* $\Sigma_{1,n_2,\dots,n_s}\subseteq\mathbb{P}^N$ *for either* $s\geq 3$ *or* $n_2\geq 2$.

(i) *Then for any* $r=2t,\ t\geq 2$, *there exists a family of dimension*

$$(2t-1)(\Sigma_{i=2}^s n_i-1)\prod_{i=2}^s(n_i+1)-3(t-1)$$

of initialized simple Ulrich vector bundles of rank r.

(ii) *Let us suppose that* $s\geq 3$ *and* $n_2=1$. *Then for any* $r=2t+1,\ t\geq 2$, *there exists a family of dimension* $\geq (t-1)((\sum_{i=2}^s n_i-1)(n_3+2)\prod_{i=4}^s(n_i+1)-1)$ *of initialized simple Ulrich vector bundles of rank* r.

(iii) *Let us suppose that* $s\geq 3$ *and* $n_2>1$. *For any integer* $r=an_3\binom{n_2}{2}+l\geq n_3\binom{n_2}{2}$ *with* $a\geq 1$ *and* $0\leq l\leq n_3\binom{n_2}{2}-1$, *there exists a family of dimension* $a^2(n_3^2+2n_3-4)+1+l\left(an_3\binom{n_2+1}{2}-l\right)$ *of simple (hence, indecomposable) initialized Ulrich vector bundles of rank* r.

In particular, the Segre variety $\Sigma_{1,n_2,\dots,n_s}\subseteq\mathbb{P}^N$, $N=2\prod_{i=2}^s(n_i+1)-1$, *for* $s\geq 3$ *or* $s=2$ *and* $n_2\geq 2$ *is of wild representation type.*

Let us summarize here the structure of this paper. In Section 2, we introduce the definitions and main properties that will be used throughout the paper, mainly those related to ACM and Ulrich sheaves, representation type of varieties and Segre varieties. In Section 3, we pay attention to the case of Segre varieties $\Sigma_{n,m}\subseteq\mathbb{P}^N$, $N:=nm+n+m$, for $2\leq n,m$. We are going to construct families of arbitrarily large dimension of simple (and, hence, indecomposable) Ulrich vector bundles on them by pulling-back certain vector bundles on each factor. This will allow us to conclude that they are of wild representation type. In Section 4, we move forward to the case of Segre varieties of the form $\Sigma_{n_1,n_2,\dots,n_s}\subseteq\mathbb{P}^N$ for either $n_1=1$ and $s\geq 3$ or $n_1=1, s=2$ and $n_2\geq 2$. In this case the families of indecomposable Ulrich vector bundles of arbitrarily high rank will be obtained as iterated extensions of lower rank vector bundles but also in this case this will be enough to conclude that they are of wild representation type. Finally, we will conclude this paper with some remarks which naturally arise and the question concerning the existence of low rank indecomposable Ulrich vector bundles on $\Sigma_{n,m}\subseteq\mathbb{P}^N$, $N:=nm+n+m$, for $2\leq n,m$.

2. Preliminaries

In this section we are going to introduce the definitions and main properties of Segre varieties as well as those of ACM and Ulrich sheaves that are going to be used throughout the rest of the paper.

Let us start fixing some notation. We will work over an algebraically closed field k of characteristic zero. We set $R = k[x_0, x_1, \dots, x_n]$, $\mathfrak{m} = (x_0, \dots, x_n)$ and $\mathbb{P}^n = \mathrm{Proj}(R)$. Given a non-singular variety X equipped with an ample line bundle $\mathcal{O}_X(1)$, the line bundle $\mathcal{O}_X(1)^{\otimes l}$ will be denoted by $\mathcal{O}_X(l)$. For any coherent sheaf $\mathcal{E}$ on X we are going to denote the twisted sheaf $\mathcal{E} \otimes \mathcal{O}_X(l)$ by $\mathcal{E}(l)$. As usual, $\mathrm{H}^i(X, \mathcal{E})$ stands for the cohomology groups, $\mathrm{h}^i(X, \mathcal{E})$ for their dimension and $\mathrm{H}^i_*(X, \mathcal{E}) = \oplus_{l\in\mathbb{Z}} \mathrm{H}^i(X, \mathcal{E}(l))$. We also set $\mathrm{ext}^i(\mathcal{E}, \mathcal{F}) := \dim_k \mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$.

2.1. *Segre varieties*

The aim of this paper is to construct Arithmetically Cohen–Macaulay sheaves (i.e. sheaves without intermediate cohomology) with the maximal permitted number of global sections, the so-called Ulrich sheaves, on a class of Arithmetically Cohen–Macaulay varieties: the Segre varieties $\Sigma_{n_1,\dots,n_s}$. Let us recall here the definitions:

Definition 2.1. A subscheme $X \subseteq \mathbb{P}^n$ is said to be arithmetically Cohen–Macaulay (briefly, ACM) if its homogeneous coordinate ring R_X is a Cohen–Macaulay ring, i.e. $\mathrm{depth}(R_X) = \dim(R_X)$.

Thanks to the graded version of the Auslander-Buchsbaum formula (for any finitely generated R-module M):

$$\mathrm{pd}(M) = n + 1 - \mathrm{depth}(M),$$

we deduce that a subscheme $X \subseteq \mathbb{P}^n$ is ACM if and only if $\mathrm{pd}(R_X) = \mathrm{codim}\, X$. Hence, if $X \subseteq \mathbb{P}^n$ is a codimension c ACM subscheme, a graded minimal free R-resolution of I_X is of the form:

$$0 \to F_c \xrightarrow{\varphi_c} F_{c-1} \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} I_X \to 0$$

where $F_i = \oplus_{j\in\mathbb{Z}} R(-j)^{\beta_{i,j}}$, $i = 1, \dots, c$ (in this setting, minimal means that $\mathrm{im}\, \varphi_i \subseteq \mathfrak{m}F_{i-1}$).

Given integers $1 \le n_1, \dots, n_s$, we denote by

$$\sigma_{n_1,\dots,n_s} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \longrightarrow \mathbb{P}^N, \quad N = \prod_{i=1}^{s}(n_i + 1) - 1$$

the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$. The image of $\sigma_{n_1,\dots,n_s}$ is the Segre variety $\Sigma_{n_1,\dots,n_s} := \sigma_{n_1,\dots,n_s}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}) \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i + 1) - 1$. Notice that in terms of very ample line bundles, this embedding is defined by means of $\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_s}}(1, \dots, 1)$.

The equations of the Segre varieties are familiar to anyone who has studied algebraic geometry. Indeed, if we let T be the $(n_1 + 1) \times \cdots \times (n_s + 1)$ tensor whose entries are the homogeneous coordinates in $\mathbb{P}^N$, then it is well known that the ideal of $\Sigma_{n_1,\dots,n_s}$ is generated by the 2×2 minors of T and we have

Proposition 2.2. *Fix integers $1 \le n_1, \dots, n_s$ and denote by $\Sigma_{n_1,\dots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i + 1) - 1$, the Segre variety. It holds:*

(i) $\dim(\Sigma_{n_1,\dots,n_s}) = \sum_{i=1}^{s} n_i$ *and* $\deg(\Sigma_{n_1,\dots,n_s}) = \frac{(\sum_{i=1}^{s} n_i)!}{\prod_{i=1}^{s}(n_i)!}$.

(ii) $\Sigma_{n_1,\dots,n_s}$ *is ACM and* $I_{\Sigma_{n_1,\dots,n_s}}$ *is generated by* $\binom{N+2}{2} - \prod_{i=1}^{s}\binom{n_i+2}{2}$ *hyperquadrics.*

Let p_i denote the i-th projection of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ onto $\mathbb{P}^{n_i}$. There is a canonical isomorphism $\mathbb{Z}^s \longrightarrow \mathrm{Pic}(\Sigma_{n_1,\ldots,n_s})$, given by

$$(a_1, \ldots, a_s) \mapsto \mathcal{O}_{\Sigma_{n_1,\ldots,n_s}}(a_1, \ldots, a_s) := p_1^*(\mathcal{O}_{\mathbb{P}^{n_1}}(a_1)) \otimes \cdots \otimes p_s^*(\mathcal{O}_{\mathbb{P}^{n_s}}(a_s)).$$

For any coherent sheaves $\mathcal{E}_i$ on $\mathbb{P}^{n_i}$, we set $\mathcal{E}_1 \boxtimes \cdots \boxtimes \mathcal{E}_s := p_1^*(\mathcal{E}_1) \otimes \cdots \otimes p_s^*(\mathcal{E}_s)$. We will denote by $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \longrightarrow X_i := \mathbb{P}^{n_1} \times \cdots \times \widehat{\mathbb{P}^{n_i}} \times \cdots \times \mathbb{P}^{n_s}$ the natural projection and given sheaves $\mathcal{E}$ and $\mathcal{F}$ on X_i and $\mathbb{P}^{n_i}$, respectively, $\mathcal{E} \boxtimes \mathcal{F}$ stands for $\pi_i^*(\mathcal{E}) \otimes p_i^*(\mathcal{F})$. By the Künneth's formula, we have

$$\mathrm{H}^\ell(\Sigma_{n_1,\ldots,n_s}, \mathcal{E} \boxtimes \mathcal{F}) = \bigoplus_{p+q=\ell} \mathrm{H}^p(X_i, \mathcal{E}) \otimes \mathrm{H}^q(\mathbb{P}^{n_i}, \mathcal{F}).$$

While given a coherent sheaf $\mathcal{H}$ on $\Sigma_{n_1,\ldots,n_s}$, $\mathcal{H}(t)$ stands for $\mathcal{H} \otimes \mathcal{O}_{\Sigma_{n_1,\ldots,n_s}}(t, \ldots, t)$.

2.2. *ACM and Ulrich sheaves*

In this subsection we recall the definition of ACM and Ulrich sheaves and we summarize the properties that will be needed in the sequel.

Definition 2.3. Let $(X, \mathcal{O}_X(1))$ be a polarized variety. A coherent sheaf $\mathcal{E}$ on X is *Arithmetically Cohen Macaulay* (ACM for short) if it is locally Cohen–Macaulay (i.e., depth $\mathcal{E}_x = \dim \mathcal{O}_{X,x}$ for every point $x \in X$) and has no intermediate cohomology:

$$\mathrm{H}^i_*(X, \mathcal{E}) = 0 \quad \text{for all } i = 1, \ldots, \dim X - 1.$$

Notice that when X is a non-singular variety, which is going to be mainly our case, any coherent ACM sheaf on X is locally free. For this reason we are going to speak uniquely of ACM bundles. ACM sheaves are closely related to their algebraic counterpart, the maximal Cohen–Macaulay modules:

Definition 2.4. A graded R_X-module E is a *maximal Cohen–Macaulay* module (MCM for short) if depth $E = \dim E = \dim R_X$.

Indeed, it holds:

Proposition 2.5. *Let $X \subseteq \mathbb{P}^n$ be an ACM scheme. There exists a bijection between ACM sheaves $\mathcal{E}$ on X and MCM R_X-modules E given by the functors $E \to \widetilde{E}$ and $\mathcal{E} \to \mathrm{H}^0_*(X, \mathcal{E})$.*

Definition 2.6. Given a polarized variety $(X, \mathcal{O}_X(1))$, a coherent sheaf $\mathcal{E}$ on X is *initialized* if

$$\mathrm{H}^0(X, \mathcal{E}(-1)) = 0 \quad \text{but } \mathrm{H}^0(X, \mathcal{E}) \neq 0.$$

Notice that when $\mathcal{E}$ is a locally Cohen–Macaulay sheaf, there always exists an integer k such that $\mathcal{E}_{init} := \mathcal{E}(k)$ is initialized.

It is well known that ACM sheaves provide a criterium to determine the complexity of the underlying variety. Indeed, this complexity can be studied in terms of the dimension and number of families of indecomposable ACM sheaves that it supports. Recently, inspired by an analogous classification for quivers and for k-algebras of finite type, the classification of any ACM variety as being of *finite, tame or wild representation type* (cf. [7] for the case of curves and [6] for the higher dimensional case) has been proposed. Let us recall the definitions:

Definition 2.7. An ACM scheme $X \subseteq \mathbb{P}^n$ is of *finite representation type* if it has, up to twist and isomorphism, only a finite number of indecomposable ACM sheaves. An ACM scheme $X \subseteq \mathbb{P}^n$ is of *tame representation type* if for each rank r, the indecomposable ACM sheaves of rank r form a finite number of families of dimension at most one. On the other hand, X will be of *wild representation type* if there exist l-dimensional families of non-isomorphic indecomposable ACM sheaves for arbitrary large l.

Varieties of finite representation type have been completely classified into a short list in [3, Theorem C] and [8, p. 348]. They are three or less reduced points on $\mathbb{P}^2$, a projective space, a non-singular quadric hypersurface $X \subseteq \mathbb{P}^n$, a cubic scroll in $\mathbb{P}^4$, the Veronese surface in $\mathbb{P}^5$ or a rational normal curve. The only known example of a variety of tame representation type is the elliptic curve. On the other hand, so far only a few examples of varieties of wild representation type are known: curves of genus $g \geq 2$ (cf. [7]), del Pezzo surfaces and Fano blow-ups of points in $\mathbb{P}^n$ (cf. [14], the cases of the cubic surface and the cubic threefold have also been handled in [4]) and ACM rational surfaces on $\mathbb{P}^4$ (cf. [15]).

The ACM bundles that we are interested in share another stronger property, namely they have the maximal possible number of global sections.

Definition 2.8. Given a projective scheme $X \subseteq \mathbb{P}^n$ and a coherent sheaf $\mathcal{E}$ on X, we say that $\mathcal{E}$ is an *Ulrich sheaf* if $\mathcal{E}$ is an ACM sheaf and $\mathrm{h}^0(\mathcal{E}_{init}) = \deg(X)\mathrm{rk}(\mathcal{E})$.

The following result justifies the above definition:

Theorem 2.9. *Let $X \subseteq \mathbb{P}^n$ be an integral subscheme and let $\mathcal{E}$ be an ACM sheaf on X. Then the minimal number of generators $m(\mathcal{E})$ of the R_X-module $\mathrm{H}^0_*(\mathcal{E})$ is bounded by*

$$m(\mathcal{E}) \leq \deg(X)\mathrm{rk}(\mathcal{E}).$$

Therefore, since it is obvious that for an initialized sheaf $\mathcal{E}, \mathrm{h}^0(\mathcal{E}) \leq m(\mathcal{E})$, the minimal number of generators of Ulrich sheaves is as large as possible. Modules attaining this upper bound were studied by Ulrich in [18]. A complete account is provided in [9]. In particular we have:

Theorem 2.10. *Let $X \subseteq \mathbb{P}^N$ be an n-dimensional ACM variety and let $\mathcal{E}$ be an initialized ACM coherent sheaf on X. The following conditions are equivalent:*

(i) *$\mathcal{E}$ is Ulrich.*
(ii) *$\mathcal{E}$ admits a linear $\mathcal{O}_{\mathbb{P}^N}$-resolution of the form:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-N+n)^{a_{N-n}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1)^{a_1} \longrightarrow \mathcal{O}_{\mathbb{P}^N}^{a_0} \longrightarrow \mathcal{E} \longrightarrow 0.$$

(iii) $\mathrm{H}^i(\mathcal{E}(-i)) = 0$ *for* $i > 0$ *and* $\mathrm{H}^i(\mathcal{E}(-i-1)) = 0$ *for* $i < n$.
(iv) *For some (resp. all) finite linear projections $\pi : X \longrightarrow \mathbb{P}^n$, the sheaf $\pi_*\mathcal{E}$ is the trivial sheaf $\mathcal{O}_{\mathbb{P}^n}^t$ for some t.*

In particular, initialized Ulrich sheaves are 0*-regular and therefore they are globally generated.*

Proof. See [9, Proposition 2.1]. □

The search for Ulrich sheaves on a particular variety is a challenging problem. In fact, few examples of varieties supporting Ulrich sheaves are known, although in [9] it has been conjectured that any variety has an Ulrich sheaf. In this paper we are going to focus our attention on the existence of Ulrich bundles on Segre varieties, providing the first example of wild varieties of arbitrary dimension whose wildness is witnessed by means of the existence of families of simple Ulrich vector bundles of arbitrary high rank and dimension.

One of the properties that is shared by all Ulrich bundles is semistability. Let us briefly recall this notion.

Definition 2.11. A vector bundle $\mathcal{E}$ on a non-singular projective variety X is semistable if for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ we have the inequality

$$P(\mathcal{F})/\mathrm{rk}(\mathcal{F}) \leq P(\mathcal{E})/\mathrm{rk}(\mathcal{E}),$$

where $P(\mathcal{F})$ and $P(\mathcal{E})$ are the Hilbert polynomials of $\mathcal{F}$ and $\mathcal{E}$, respectively. It is stable if strict inequality holds.

$\mathcal{E}$ is said to be μ-semistable if for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$,

$$\mu(\mathcal{F}) := \deg(c_1(\mathcal{F}))/\mathrm{rk}(\mathcal{F}) \leq \mu(\mathcal{E}) := \deg(c_1(\mathcal{E}))/\mathrm{rk}(\mathcal{E}).$$

It is μ-stable if we have strict inequality.

Remark 2.12. The two definitions are related as follows

$$\mu\text{-stable} \;\Rightarrow\; \text{stable} \;\Rightarrow\; \text{semistable} \;\Rightarrow \mu\text{-semistable}.$$

We end this section with a proposition where we gather some properties of Ulrich bundles on projective varieties that turn out to be very useful.

Proposition 2.13. *Let $X \subseteq \mathbb{P}^n$ be a nonsingular projective variety of degree d and let $\mathcal{E}$ be a rank r Ulrich bundle on X. Then*

(1) *$\mathcal{E}$ is semistable.*
(2) *If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$ is an exact sequence of coherent sheaves with $\mathcal{G}$ torsion-free and $\mu(\mathcal{F}) = \mu(\mathcal{E})$, then $\mathcal{F}$ and $\mathcal{G}$ are both Ulrich bundles.*
(3) *If $\mathcal{E}$ is stable, then it is also μ-stable.*
(4) $\deg\mathcal{E} = r(d + g - 1)$ *where g is the sectional genus of X. In particular, all Ulrich bundles have the same slope.*

Proof. See [4, Theorem 2.9]. □

3. Representation type of $\Sigma_{n,m}$

The goal of this section is the construction of families of arbitrarily large dimension of simple (and, hence, indecomposable) Ulrich vector bundles on Segre varieties $\Sigma_{n,m} \subseteq \mathbb{P}^N$, $N := nm + n + m$, for $2 \leq n, m$. Let us start by determining the complete list of Ulrich line bundles on Segre varieties $\Sigma_{n_1,\ldots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i + 1) - 1$.

First of all, notice that it follows from Horrocks' Theorem [11] that

Lemma 3.1. *The only initialized Ulrich bundle on $\mathbb{P}^n$ is the structural sheaf $\mathcal{O}_{\mathbb{P}^n}$.*

The list of Ulrich line bundles on $\Sigma_{n_1,\ldots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i + 1) - 1$, is given by

Proposition 3.2. *Let $\Sigma_{n_1,\dots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i+1)-1$, be a Segre variety. Then there exist $s!$ initialized Ulrich line bundles on $\Sigma_{n_1,\dots,n_s}$. They are of the form*

$$\mathcal{L}_{X_i} \boxtimes \mathcal{O}_{\mathbb{P}^{n_i}}\left(\sum_{k\neq i} n_k\right),$$

where $\mathcal{L}_{X_i}$ is a rank one initialized Ulrich bundle on the Segre variety $X_i := \Sigma_{n_1,\dots,\widehat{n_i},\dots,n_s} \subseteq \mathbb{P}^{N'}$, $N' = \prod_{\substack{1\le j\le s\\ j\neq i}}(n_j+1)-1$. More explicitly, the initialized Ulrich line bundles on $\Sigma_{n_1,\dots,n_s}$ are of the form $\mathcal{O}_{\Sigma_{n_1,\dots,n_s}}(a_1,\dots,a_s)$ where, if we order the coefficients $0 = a_{i_1} \le \cdots \le a_{i_k} \le \cdots \le a_{i_s}$ then $a_{i_k} = \sum_{1\le j<k} n_{i_j}$.

Proof. The existence of this set of initialized Ulrich line bundles is a straightforward consequence of [9, Proposition 2.6]. In order to see that this list is exhaustive, let us consider an initialized Ulrich line bundle $\mathcal{L} := \mathcal{O}_{\Sigma_{n_1,\dots,n_s}}(a_1,\dots,a_s)$ with $a_{i_1} \le \cdots \le a_{i_k} \le \cdots \le a_{i_s}$. Given that $\mathcal{L}$ is initialized, it holds that $a_{i_1} = 0$. Since $\mathcal{L}$ is ACM, we have

$$\mathrm{H}^{\sum_{j=1}^{k} n_{i_j}}(\Sigma_{n_1,\dots,n_s}, \mathcal{L}(-\Sigma_{j=1}^{k} n_{i_j} - 1)) = 0$$

for $k = 1,\dots,s-1$. In particular, using Künneth's formula, it holds

$$\prod_{l=1}^{k} \mathrm{h}^{n_{i_l}}(\mathbb{P}^{n_{i_l}}, \mathcal{O}_{\mathbb{P}^{n_{i_l}}}(a_{i_l} - \Sigma_{j=1}^{k} n_{i_j} - 1)) \cdot \prod_{l=k+1}^{s} \mathrm{h}^{0}(\mathbb{P}^{n_{i_l}}, \mathcal{O}_{\mathbb{P}^{n_{i_l}}}(a_{i_l} - \Sigma_{j=1}^{k} n_{i_j} - 1)) = 0,$$

from where it follows that, by induction, $a_{i_{k+1}} \le b_{i_{k+1}} := \Sigma_{1\le j\le k} n_{i_j}$ for $k = 1,\dots,s-1$ (and $b_{i_1} := 0$). But, on the other hand, since an easy computation shows that

$$\mathrm{h}^0(\Sigma_{n_1,\dots,n_s}, \mathcal{O}_{\Sigma_{n_1,\dots,n_s}}(b_1,\dots,b_s)) = \frac{\left(\sum_{i=1}^{s} n_i\right)!}{\prod_{i=1}^{s}(n_i)!} = \deg(\Sigma_{n_1,\dots,n_s})$$

we are forced to have $a_{i_j} = b_{i_j}$ for $j = 1,\dots,s$. $\square$

Let us move forward to the construction of Ulrich bundles of higher rank on Segre varieties $\Sigma_{n_1,\dots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i+1)-1$. For the rest of this section we are going to assume that $n_1 \ge 2$ and postpone the remaining case ($n_1 = 1$) to the last section. Let us fix some notation as presented in [10]: let us consider k-vector spaces A and B of respective dimension a and b. Set $V = \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$ and let $M = \mathrm{Hom}(B, A \otimes V)$ be the space of $(a \times b)$-matrices of linear forms. It is well-known that there exists a bijection between the elements $\phi \in M$ and the morphisms $\phi : B \otimes \mathcal{O}_{\mathbb{P}^m} \longrightarrow A \otimes \mathcal{O}_{\mathbb{P}^m}(1)$. Taking the tensor with $\mathcal{O}_{\mathbb{P}^m}(1)$ and considering global sections, we have morphisms $\mathrm{H}^0(\phi(1)) : \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)^b) \longrightarrow \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(2)^a)$. The following result tells us under which conditions the aforementioned morphisms ϕ and $\mathrm{H}^0(\phi(1))$ are surjective:

Proposition 3.3. *For $a \ge 1$, $b \ge a + m$ and $2b \ge (m+2)a$, the set of elements $\phi \in M$ such that $\phi : B \otimes \mathcal{O}_{\mathbb{P}^m} \longrightarrow A \otimes \mathcal{O}_{\mathbb{P}^m}(1)$ and $\mathrm{H}^0(\phi(1)) : \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)^b) \longrightarrow \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(2)^a)$ are surjective forms a non-empty open dense subset that we will denote by V_m.*

Proof. See [10, Proposition 4.1]. $\square$

For any $2 \le m$ and any $1 \le a$, we denote by $\mathcal{E}_{m,a}$ any vector bundle on $\mathbb{P}^m$ given by the exact sequence

$$0 \longrightarrow \mathcal{E}_{m,a} \longrightarrow \mathcal{O}_{\mathbb{P}^m}(1)^{(m+2)a} \xrightarrow{\phi(1)} \mathcal{O}_{\mathbb{P}^m}(2)^{2a} \longrightarrow 0 \tag{3.1}$$

where $\phi \in V_m$. Note that $\mathcal{E}_{m,a}$ has rank ma.

We need to know the value of the cohomology groups of these vector bundles:

Lemma 3.4. *With the above notation we have:*

(i)

$$\mathrm{h}^0(\mathbb{P}^m, \mathcal{E}_{m,a}(t)) = \begin{cases} 0 & \text{for } t \le 0, \\ a\left((m+2)\binom{m+t+1}{m} - 2\binom{m+t+2}{m}\right) & \text{for } t > 0. \end{cases}$$

(ii)

$$\mathrm{h}^1(\mathbb{P}^m, \mathcal{E}_{m,a}(t)) = \begin{cases} 0 & \text{for } t < -2 \text{or } t \ge 0, \\ am & \text{for } t = -1 \\ 2a & \text{for } t = -2. \end{cases}$$

(iii) $\mathrm{h}^i(\mathbb{P}^m, \mathcal{E}_{m,a}(t)) = 0$ *for all* $t \in \mathbb{Z}$ *and* $2 \le i \le m-1$.
(iv) $\mathrm{h}^m(\mathbb{P}^m, \mathcal{E}_{m,a}(t)) = 0$ *for* $t \ge -m-1$.

Proof. Since $\phi \in V_m$, by Proposition 3.3, $\mathrm{H}^0(\phi(1))$ is surjective. But, since the k-vector spaces $\mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)^{(m+2)a})$ and $\mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(2)^{2a})$ have the same dimension, $\mathrm{H}^0(\phi(1))$ is an isomorphism and therefore $\mathrm{H}^0(\mathcal{E}_{m,a}) = 0$. *A fortiori*, $\mathrm{H}^0(\mathcal{E}_{m,a}(t)) = 0$ for $t \le 0$. On the other hand, again by the surjectivity of $\mathrm{H}^0(\phi(1)), \mathrm{H}^1(\mathcal{E}_{m,a}) = 0$. Since it is obvious that $\mathrm{H}^i(\mathcal{E}_{m,a}(1-i)) = 0$ for $i \ge 2$ it turns out that $\mathcal{E}_{m,a}$ is 1-regular and in particular, $\mathrm{H}^1(\mathcal{E}_{m,a}(t)) = 0$ for $t \ge 0$. The rest of cohomology groups can be easily deduced from the long exact cohomology sequence associated to the exact sequence (3.1). □

Proposition 3.5. (i) *The vector bundles $\mathcal{E}_{m,a}$ constructed in* (3.1) *are simple.*
(ii) *The vector bundles $\mathcal{E}_{m,1}$ constructed in* (3.1) *are μ-stable.*

Proof. (i) It follows from Kac's theorem (see [12, Theorem 4]) arguing as in [14, Proposition 3.4] that $\mathcal{E}_{m,a}$ is simple.

(ii) Since μ-stability is preserved by duality, it is enough to check that $\mathcal{E}_{m,1}^{\vee}$ is μ-stable. Because $\mathcal{E}_{m,1}^{\vee}$ is a rank m vector bundle on $\mathbb{P}^m$ sitting in an exact sequence of the following type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^m}(-2)^2 \longrightarrow \mathcal{O}_{\mathbb{P}^m}(-1)^{m+2} \longrightarrow \mathcal{E}_{m,1}^{\vee} \longrightarrow 0$$

its μ-stability follows from [2, Theorem 2.7]. □

We are now ready to construct families of simple (hence indecomposable) Ulrich bundles on the Segre variety $\Sigma_{n,m} \subseteq \mathbb{P}^{nm+n+m}$, $2 \le n, m$, of arbitrary high rank and dimension and conclude that Segre varieties $\Sigma_{n,m}$ are of wild representation type. The main ingredient on the construction of simple Ulrich bundles on $\Sigma_{n,m} \subseteq \mathbb{P}^{nm+n+m}$, $2 \le n \le m$, will be the family of simple vector bundles $\mathcal{E}_{m,a}$ on $\mathbb{P}^m$ given by the exact sequence (3.1) as well as the vector bundles of p-holomorphic forms of $\mathbb{P}^n$, $\Omega^p_{\mathbb{P}^n} := \wedge^p \Omega^1_{\mathbb{P}^n}$, where $\Omega^1_{\mathbb{P}^n}$ is the cotangent bundle. The values of $\mathrm{h}^i(\Omega^p_{\mathbb{P}^n}(t))$ are given by the *Bott's formula* (see, for instance, [16, p. 8]).

Theorem 3.6. *Fix integers $2 \leq n \leq m$ and let $\Sigma_{n,m} \subseteq \mathbb{P}^{nm+n+m}$ be the Segre variety. For any integer $a \geq 1$ there exists a family of dimension $a^2(m^2+2m-4)+1$ of initialized simple Ulrich vector bundles $\mathcal{F} := \Omega^{n-2}_{\mathbb{P}^n}(n-1) \boxtimes \mathcal{E}_{m,a}(n-1)$ of rank $am\binom{n}{2}$.*

Proof. Let $\mathcal{F}$ be the vector bundle $\Omega^{n-2}_{\mathbb{P}^n}(n-1) \boxtimes \mathcal{E}_{m,a}(n-1)$ for $\mathcal{E}_{m,a}$ a general vector bundle obtained on $\mathbb{P}^m$ from the exact sequence (3.1). The first goal is to prove that $\mathcal{F}$ is ACM, namely, we should show that $\mathrm{H}^i(\Sigma_{n,m}, \mathcal{F} \otimes \mathcal{O}_{\Sigma_{n,m}}(t,t)) = 0$ for $1 \leq i \leq n+m-1$ and $t \in \mathbb{Z}$. By Künneth's formula

$$\mathrm{H}^i(\Sigma_{n,m}, \mathcal{F} \otimes \mathcal{O}_{\Sigma_{n,m}}(t,t)) = \bigoplus_{p+q=i} \mathrm{H}^p(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1+t)) \otimes \mathrm{H}^q(\mathbb{P}^m, \mathcal{E}_{m,a}(n-1+t)). \tag{3.2}$$

According to Bott's formula the only non-zero cohomology groups of $\Omega^{n-2}_{\mathbb{P}^n}(n-1+t)$ are:

$$\begin{aligned}
&\mathrm{H}^0(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1+t)) \quad \text{for } t \geq 0 \text{ and } n \geq 3 \text{ or } t \geq -1 \text{ and } n = 2,\\
&\mathrm{H}^{n-2}(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1+t)) \quad \text{for } t = -n+1,\\
&\mathrm{H}^n(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1+t)) \quad \text{for } t \leq -n-2.
\end{aligned}$$

On the other hand, by Lemma 3.4, the only non-zero cohomology groups of $\mathcal{E}_{m,a}(n-1+t)$ are:

$$\begin{aligned}
&\mathrm{H}^0(\mathbb{P}^m, \mathcal{E}_{m,a}(n-1+t)) \quad \text{for } t \geq -n+2,\\
&\mathrm{H}^1(\mathbb{P}^m, \mathcal{E}_{m,a}(n-1+t)) \quad \text{for } -n-1 \leq t \leq -n,\\
&\mathrm{H}^m(\mathbb{P}^m, \mathcal{E}_{m,a}(n-1+t)) \quad \text{for } t \leq -n-m-1.
\end{aligned}$$

Hence, by (3.2), $\mathrm{H}^i(\Sigma_{n,m}, \mathcal{F} \otimes \mathcal{O}_{\Sigma_{n,m}}(t,t)) = 0$ for $1 \leq i \leq n+m-1$ and $t \in \mathbb{Z}$. Since for $n \geq 3\mathrm{H}^0(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-2)) = 0$ and for $n = 2\mathrm{H}^0(\mathbb{P}^m, \mathcal{E}_{m,a}) = 0$ (Lemma 3.4), $\mathcal{F}$ is an initialized ACM vector bundle on $\Sigma_{n,m}$. Let us compute the number of global sections. Recall that, by Bott's formula, $\mathrm{h}^0(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1)) = \binom{n+1}{2}$. Hence:

$$\begin{aligned}
\mathrm{h}^0(\mathcal{F}) &= \mathrm{h}^0(\Sigma_{n,m}, \Omega^{n-2}_{\mathbb{P}^n}(n-1) \boxtimes \mathcal{E}_{m,a}(n-1))\\
&= \mathrm{h}^0(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1))\mathrm{h}^0(\mathbb{P}^m, \mathcal{E}_{m,a}(n-1))\\
&= \binom{n+1}{2} a \left((m+2)\binom{m+n}{m} - 2\binom{m+n+1}{m}\right)\\
&= a\left(\frac{(m+2)(m+n)!(n+1)!}{m!n!(n-1)!2!} - \frac{2(m+n+1)!(n+1)!}{m!(n+1)!(n-1)!2!}\right)\\
&= a\left(\frac{n!(m+n)!}{2!(n-2)!m!n!} \cdot \frac{(n+1)(m+2) - 2(m+n+1)}{n-1}\right)\\
&= a\binom{n}{2}\binom{m+n}{m}\frac{m(n-1)}{n-1}\\
&= a\binom{n}{2}\binom{m+n}{m} m\\
&= \mathrm{rk}(\mathcal{F})\deg(\Sigma_{n,m})
\end{aligned}$$

where the last equality follows from the fact that $\deg(\Sigma_{n,m}) = \binom{m+n}{m}$ and $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{E}_{m,a})\,\mathrm{rk}(\Omega^{n-2}_{\mathbb{P}^n}) = am\binom{n}{2}$. Therefore, $\mathcal{F}$ is an initialized Ulrich vector bundle on $\Sigma_{n,m}$. With respect to simplicity, we need only to observe that

$$\begin{aligned}\mathrm{Hom}(\mathcal{F},\mathcal{F}) &\cong \mathrm{H}^0(\Sigma_{n,m},\mathcal{F}^\vee\otimes\mathcal{F})\\ &\cong \mathrm{H}^0(\mathbb{P}^n,\Omega^{n-2}_{\mathbb{P}^n}(n-1)^\vee\otimes\Omega^{n-2}_{\mathbb{P}^n}(n-1))\\ &\quad\otimes \mathrm{H}^0(\mathbb{P}^m,\mathcal{E}_{m,a}(n-1)^\vee\otimes\mathcal{E}_{m,a}(n-1))\end{aligned}$$

and use the fact that $\Omega^{n-2}_{\mathbb{P}^n}$ and $\mathcal{E}_{m,a}$ are both simple (Proposition 3.5(i)).

It only remains to compute the dimension of the family of simple Ulrich bundles $\mathcal{F} := \Omega^{n-2}_{\mathbb{P}^n}(n-1)\boxtimes\mathcal{E}_{m,a}(n-1)$ on $\Sigma_{n,m}$. Since they are completely determined by a general morphism $\phi\in M := \mathrm{Hom}_{\mathbb{P}^m}(\mathcal{O}^{(m+2)a}_{\mathbb{P}^m},\mathcal{O}_{\mathbb{P}^m}(1)^{2a})$, this dimension turns out to be:

$$\begin{aligned}&\dim M - \dim \mathrm{Aut}(\mathcal{O}^{(m+2)a}_{\mathbb{P}^m}) - \dim \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^m}(1)^{2a}) + 1\\ &\quad = 2a^2(m+2)(m+1) - a^2(m+2)^2 - 4a^2 + 1 = a^2(m^2+2m-4)+1\end{aligned}$$

which proves what we want. □

Corollary 3.7. *For any integers $2 \le n, m$, the Segre variety $\Sigma_{n,m} \subseteq \mathbb{P}^{nm+n+m}$ is of wild representation type.*

In the particular case of $n = 2$ the Ulrich vector bundles $\mathcal{F}$ on $\Sigma_{2,m}$ of the lowest possible rank (i.e., of rank m) constructed in Theorem 3.6 share a stronger property than simplicity, namely, they are even μ-stable and they define a family whose closure is a generically smooth component of the corresponding moduli space. To prove this, let us fix some notation.

Notation 3.8. (1) We will denote by $M_{\Sigma_{2,m}}(m; c_1,\dots,c_m)$ the moduli space of μ-stable rank m vector bundles $\mathcal{E}$ on $\Sigma_{2,m}$ with Chern classes $c_i(\mathcal{E}) = c_i$.

(2) We will denote by $\mathcal{D}_m$ the family of μ-stable rank m vector bundles $\mathcal{E}_m$ on $\mathbb{P}^m$ sitting in an exact sequence of the following type

$$0 \longrightarrow \mathcal{E}_m \longrightarrow \mathcal{O}_{\mathbb{P}^m}(1)^{m+2} \xrightarrow{\phi(1)} \mathcal{O}_{\mathbb{P}^m}(2)^2 \longrightarrow 0$$

where $\phi\in V_m$ ($\mathcal{D}_m$ is non-empty by Proposition 3.5). We define the family $\mathcal{M}_m$ of initialized simple Ulrich rank m vector bundles $\mathcal{F} := \mathcal{O}_{\mathbb{P}^2}(1)\boxtimes\mathcal{E}_m(1)$ on $\Sigma_{2,m}$ where $\mathcal{E}_m\in\mathcal{D}_m$. Notice that by Theorem 3.6, $\mathcal{M}_m$ is non-empty and any $\mathcal{O}_{\mathbb{P}^2}(1)\boxtimes\mathcal{E}_m(1)\in\mathcal{M}_m$ sits in the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)\boxtimes\mathcal{E}_m(1) \longrightarrow \mathcal{O}_{\Sigma_{2,m}}(1,2)^{(m+2)} \longrightarrow \mathcal{O}_{\Sigma_{2,m}}(1,3)^2 \longrightarrow 0. \tag{3.3}$$

In particular, any $\mathcal{F}\in\mathcal{M}_m$ is a rank m vector bundle with $c_1(\mathcal{F}) = (m, 2m-2)$.

Theorem 3.9. *Fix $2 \le m$ and let $\Sigma_{2,m} \subseteq \mathbb{P}^{3m+2}$ be the Segre variety. Then, we have:*

(i) *Any $\mathcal{F}\in\mathcal{M}_m$ is μ-stable.*

(ii) *The closure of $\mathcal{M}_m$ in $M_{\Sigma_{2,m}}(m; c_1(\mathcal{F}),\dots,c_m(\mathcal{F}))$ is a generically smooth component of dimension m^2+2m-3.*

Proof. (i) By Proposition 2.13 we know that the Ulrich vector bundle $\mathcal{F}$ is semistable. To abut to a contradiction, let us suppose that there exists a subsheaf $\mathcal{G}\subseteq\mathcal{F}$ of rank r, $1\le r\le m-1$, such that $\mu(\mathcal{G}) = \mu(\mathcal{F})$. Set $c_1(\mathcal{G}) = (a,b)$. Since $\mathcal{F}$ is a rank m vector bundle with $c_1(\mathcal{F}) = (m, 2m-2)$, the equality $\mu(\mathcal{G}) = \mu(\mathcal{F})$ is equivalent to

$$\frac{a(m+1) + b\binom{m+1}{2}}{r} = (m+1)m$$

or, equivalently,

$$a + \frac{bm}{2} = rm.$$

Since $\mathcal{F}$ sits into the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\Sigma_{2,m}}(1,2)^{(m+2)} \longrightarrow \mathcal{O}_{\Sigma_{2,m}}(1,3)^2 \longrightarrow 0 \tag{3.4}$$

and $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ we have the inclusion $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{\Sigma_{2,m}}(1,2)^{(m+2)}$ and applying the r-exterior power to it we obtain the injection

$$0 \longrightarrow \wedge^r \mathcal{G} \cong \mathcal{O}_{\Sigma_{2,m}}(a,b) \longrightarrow \wedge^r \mathcal{O}_{\Sigma_{2,m}}(1,2)^{(m+2)} \cong \mathcal{O}_{\Sigma_{2,m}}(r,2r)^{\binom{m+2}{r}}$$

which allows us to deduce that $a \le r$ and $b \le 2r$. Hence, we have

$$rm = a + bm/2 \le m - 1 + bm/2$$

from where we can deduce $2r - 1 \le b \le 2r$. On the other hand,

$$0 \longrightarrow \wedge^r \mathcal{G} \cong \mathcal{O}_{\Sigma_{2,m}}(a,b) \longrightarrow \wedge^r \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^2}(r) \boxtimes \wedge^r \mathcal{E}(1).$$

Hence $\mathrm{H}^0(\Sigma_{2,m}, \wedge^r(\mathcal{F}) \otimes \mathcal{O}_{\Sigma_{2,m}}(-a,-b)) \ne 0$ and by Künneth's formula $\mathrm{H}^0(\mathbb{P}^m, \wedge^r(\mathcal{E}(1))(-b)) \ne 0$. However, by Proposition 3.5, $\mathcal{E}^\vee(-1)$ is μ-stable and thus, by [2, Theorem 2.7] it should hold that

$$2m - 2 - b > -(m-r)\mu(\mathcal{E}^\vee(-1)) = -(m-r)(-2 + 2/m)$$

and hence $b < 2r - 2r/m$. Now, if $b = 2r - 1$ then $a = m/2 \le r$. But it would hold $b < 2r - 2r/m \le 2r - 1$ which is a contradiction. Since $b = 2r < 2r - 2r/m$ also provides with a contradiction we should conclude that $\mathcal{F}$ is μ-stable.

(ii) By deformation theory we know that $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$ is isomorphic to the tangent space of the moduli space $M_{\Sigma_{2,m}}(m; c_1(\mathcal{F}), \dots, c_m(\mathcal{F}))$ at $[\mathcal{F}]$. But since the dimension of this k-vector space

$$\begin{aligned}\dim\ \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) &= \mathrm{h}^1(\Sigma_{2,m}, \mathcal{F}^\vee \otimes \mathcal{F})\\ &= \mathrm{h}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \cdot \mathrm{h}^1(\mathbb{P}^m, \mathcal{E}^\vee \otimes \mathcal{E})\\ &= \dim\ \mathrm{Ext}^1(\mathcal{E},\mathcal{E})\\ &= m^2 + 2m - 3\end{aligned}$$

amounts to the dimension of the family $\mathcal{M}_m$, it turns out that $M_{\Sigma_{2,m}}(m; c_1(\mathcal{F}), \dots, c_m(\mathcal{F}))$ is smooth at $[\mathcal{F}]$ of dimension $m^2 + 2m - 3$ and, moreover, a general element of its irreducible component is obtained through our construction. □

Notice that in Theorem 3.6 we were able to construct simple Ulrich vector bundles on $\Sigma_{n,m} \subseteq \mathbb{P}^N$ for some scattered ranks, namely for ranks of the form $am\binom{n}{2}$, $a \ge 1$. The next goal will be to construct simple Ulrich bundles on $\Sigma_{n,m} \subseteq \mathbb{P}^{nm+n+m}$, $2 \le n \le m$, of the remaining ranks $r \ge m\binom{n}{2}$.

Theorem 3.10. *Fix integers $2 \le n \le m$ and let $\Sigma_{n,m} \subseteq \mathbb{P}^{nm+n+m}$ be the Segre variety. For any integer $r \ge m\binom{n}{2}$, set $r = am\binom{n}{2} + l$ with $a \ge 1$ and $0 \le l \le m\binom{n}{2} - 1$. Then, there exists a family of dimension $a^2(m^2+2m-4)+1+l\left(am\binom{n+1}{2} - l\right)$ of simple (hence, indecomposable) initialized Ulrich vector bundles $\mathcal{G}$ on $\Sigma_{n,m}$ of rank r.*

Proof. Note that for any $r \ge m\binom{n}{2}$, there exists $a \ge 1$ and $m\binom{n}{2} - 1 \ge l \ge 0$, such that $r = am\binom{n}{2} + l$. For such a, consider the family $\mathcal{P}_a$ of initialized Ulrich bundles of rank $am\binom{n}{2}$ given by Theorem 3.6. Notice that

$$\dim\ \mathcal{P}_a = a^2(m^2 + 2m - 4) + 1.$$

Hence it is enough to consider the case $l > 0$. To this end, for any $l > 0$ we construct the family $\mathcal{P}_{a,l}$ of vector bundles $\mathcal{G}$ given by a non-trivial extension

$$e : 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{\Sigma_{n,m}}(0, n)^l \longrightarrow 0 \tag{3.5}$$

where $\mathcal{F} \in \mathcal{P}_a$ and $e := (e_1, \dots, e_l) \in \mathrm{Ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0, n)^l, \mathcal{F}) \cong \mathrm{Ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0, n), \mathcal{F})^l$ with $e_1, \dots, e_l$ linearly independent.

Since

$$\begin{aligned}
\mathrm{ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0, n), \mathcal{F}) &= \mathrm{h}^1(\Sigma_{n,m}, \Omega^{n-2}_{\mathbb{P}^n}(n-1) \boxtimes \mathcal{E}(-1)) \\
&= \mathrm{h}^0(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(n-1)) \cdot \mathrm{h}^1(\mathbb{P}^m, \mathcal{E}(-1)) \\
&= \binom{n+1}{2} am \\
&> m\binom{n}{2}
\end{aligned}$$

such extension exists.

It is obvious that $\mathcal{G}$, being an extension of initialized Ulrich vector bundles, is also an initialized Ulrich vector bundle. Let us see that $\mathcal{G}$ is simple, i.e., $\mathrm{Hom}(\mathcal{G}, \mathcal{G}) \cong k$. If we apply the functor $\mathrm{Hom}(\cdot, \mathcal{G})$ to the exact sequence (3.5) we obtain:

$$0 \longrightarrow \mathrm{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0, n)^l, \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{G}, \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{G}).$$

On the other hand, if we apply $\mathrm{Hom}(\mathcal{F}, \cdot)$ to the same exact sequence we have

$$0 \longrightarrow k \cong \mathrm{Hom}(\mathcal{F}, \mathcal{F}) \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{O}_{\Sigma_{n,m}}(0, n)^l).$$

But

$$\begin{aligned}
\mathrm{Hom}(\mathcal{F}, \mathcal{O}_{\Sigma_{n,m}}(0, n)) &\cong \mathrm{Ext}^{n+m}(\mathcal{O}_{\Sigma_{n,m}}(0, n), \mathcal{F}(-n-1, -m-1)) \\
&\cong \mathrm{H}^{n+m}(\Sigma_{n,m}, \mathcal{F}(-n-1, -m-n-1)) \\
&= \mathrm{H}^n(\mathbb{P}^n, \Omega^{n-2}_{\mathbb{P}^n}(-2)) \otimes \mathrm{H}^m(\mathbb{P}^m, \mathcal{E}(-m-2)) = 0
\end{aligned} \tag{3.6}$$

by Serre's duality and Bott's formula. This implies that $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong k$.

Finally, using the fact that $\mathrm{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0, n), \mathcal{F}) \cong \mathrm{H}^0(\mathcal{F}(0, -n)) = 0$ and applying the functor $\mathrm{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0, n), \cdot)$ to the short exact sequence (3.5), we obtain

$$\begin{aligned}
&0 \longrightarrow \mathrm{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0, n), \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0, n), \mathcal{O}_{\Sigma_{n,m}}(0, n)^l) \\
&\quad \cong k^l \xrightarrow{\ \phi\ } \mathrm{Ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0, n), \mathcal{F}).
\end{aligned}$$

Since, by construction, the image of ϕ is the subvector space generated by $e_1, \ldots, e_l$ it turns out that ϕ is injective and in particular $\mathrm{Hom}(\mathcal{O}_{\Sigma_{n,m}}(0,n), \mathcal{G}) = 0$. Summing up, $\mathrm{Hom}(\mathcal{G}, \mathcal{G}) \cong k$, i.e., $\mathcal{G}$ is simple.

It only remains to compute the dimension of $\mathcal{P}_{a,l}$. Assume that there exist vector bundles $\mathcal{F}, \mathcal{F}' \in \mathcal{P}_a$ giving rise to isomorphic bundles, i.e.:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \xrightarrow{j_1} & \mathcal{G} & \xrightarrow{\alpha} & \mathcal{O}_{\Sigma_{n,m}}(0,n)^l & \longrightarrow & 0 \\ & & & & i \,\|\wr & & & & \\ 0 & \longrightarrow & \mathcal{F}' & \xrightarrow{j_2} & \mathcal{G}' & \xrightarrow{\beta} & \mathcal{O}_{\Sigma_{n,m}}(0,n)^l & \longrightarrow & 0. \end{array}$$

Since by (3.6), $\mathrm{Hom}(\mathcal{F}, \mathcal{O}_{\Sigma_{n,m}}(0,n)) = 0$, the isomorphism i between $\mathcal{G}$ and $\mathcal{G}'$ lifts to an automorphism f of $\mathcal{O}_{\Sigma_{n,m}}(0,n)^l$ such that $f\alpha = \beta i$ which allows us to conclude that the morphism $ij_1 : \mathcal{F} \longrightarrow \mathcal{G}'$ factorizes through $\mathcal{F}'$ showing up the required isomorphism from $\mathcal{F}$ to $\mathcal{F}'$.

Therefore, since $\dim\, \mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 1$, we have

$$\begin{aligned} \dim\, \mathcal{P}_{a,l} &= \dim\, \mathcal{P}_a + \dim\, Gr(l, \mathrm{Ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0,n), \mathcal{F})) \\ &= \dim\, \mathcal{P}_a + l \dim\, \mathrm{Ext}^1(\mathcal{O}_{\Sigma_{n,m}}(0,n), \mathcal{F}) - l^2 \\ &= a^2(m^2 + 2m - 4) + 1 + l\left(am\binom{n+1}{2} - l\right). \quad \square \end{aligned}$$

As a by-product of the previous results we can extend the construction of simple Ulrich bundles on $\Sigma_{n,m}$, $n \geq 2$, to the case of Segre embeddings of more than two factors and get:

Theorem 3.11. *Fix integers $2 \leq n_1 \leq \cdots \leq n_s$ and let $\Sigma_{n_1,\ldots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^s (n_i + 1) - 1$ be a Segre variety. For any integer $r \geq n_2\binom{n_1}{2}$, set $r = an_2\binom{n_1}{2} + l$ with $a \geq 1$ and $0 \leq l \leq n_2\binom{n_1}{2} - 1$. Then there exists a family of dimension $a^2(n_2^2 + 2n_2 - 4) + 1 + l\left(an_2\binom{n_1+1}{2} - l\right)$ of simple (hence, indecomposable) initialized Ulrich vector bundles on $\Sigma_{n_1,\ldots,n_s} \subseteq \mathbb{P}^N$ of rank r.*

Proof. By Theorem 3.6 we can suppose that $s \geq 3$. Therefore, by [9, Proposition 2.6], the vector bundle of the form $\mathcal{H} := \mathcal{G} \boxtimes \mathcal{L}(n_1 + n_2)$, for $\mathcal{G}$ belonging to the family constructed in Theorem 3.10 and $\mathcal{L}$ an Ulrich line bundle on $\mathbb{P}^{n_3} \times \cdots \times \mathbb{P}^{n_s}$ as constructed in Proposition 3.2, is an initialized simple Ulrich bundle. In order to show that in this way we obtain a family of the aforementioned dimension it only remains to show that whenever $\mathcal{G} \not\cong \mathcal{G}'$ then $\mathcal{H} \not\cong \mathcal{H}'$, or equivalently $\mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}} \not\cong \mathcal{G}' \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}}$. But if there exists an isomorphism

$$\phi : \mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}} \xrightarrow{\cong} \mathcal{G}' \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}}$$

$\pi_*\phi$ would also be an isomorphism between $\pi_*(\mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}}) \cong \mathcal{G}$ and $\pi_*(\mathcal{G}' \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}\times\cdots\times\mathbb{P}^{n_s}}) \cong \mathcal{G}'$ in contradiction with the hypothesis. $\square$

Corollary 3.12. *For any integers $2 \leq n_1, \ldots, n_s$, the Segre variety $\Sigma_{n_1,\ldots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^s (n_i + 1) - 1$ is of wild representation type.*

4. Representation type of $\Sigma_{n_1,n_2,\ldots,n_s}$

In this last section we are going to focus our attention on the construction of simple Ulrich bundles on Segre varieties of the form $\Sigma_{n_1,n_2,\ldots,n_s} \subseteq \mathbb{P}^N$ for either $n_1 = 1$ and $s \geq 3$ or

$n_1 = 1$ and $n_2 \geq 2$. We are going to show that they also are of wild representation type. Opposite to the Segre varieties that we studied in the previous section, the Ulrich bundles on $\Sigma_{1,n_2,\ldots,n_s} \subseteq \mathbb{P}^N$, $N = 2\prod_{i=2}^{s}(n_i+1)-1$, will not be obtained as products of vector bundles constructed on each factor, but they will be obtained directly as iterated extensions.

Theorem 4.1. *Let $X := \Sigma_{1,n_2,\ldots,n_s} \subseteq \mathbb{P}^N$ for either $s \geq 3$ or $n_2 \geq 2$. Let r be an integer, $2 \leq r \leq (\Sigma_{i=2}^{s} n_i - 1)\prod_{i=2}^{s}(n_i+1)$. Then:*

(i) *There exists a family Λ_r of rank r initialized simple Ulrich vector bundles $\mathcal{E}$ on X given by nontrivial extensions*

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_X(0,1,1+n_2,\ldots,1+\Sigma_{i=2}^{s-1}n_i) \longrightarrow \mathcal{E} \\ &\longrightarrow \mathcal{O}_X(\Sigma_{i=2}^{s}n_i,0,n_2,\ldots,\Sigma_{i=2}^{s-1}n_i)^{r-1} \longrightarrow 0 \end{aligned} \tag{4.1}$$

with first Chern class $c_1(\mathcal{E}) = ((r-1)\Sigma_{i=2}^{s}n_i, 1, 1+rn_2, \ldots, 1+r(\Sigma_{i=2}^{s-1}n_i))$.

(ii) *There exists a family Γ_r of rank r initialized simple Ulrich vector bundles $\mathcal{F}$ on X given by nontrivial extensions*

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_X(0,1+n_3,1,1+n_2+n_3,\ldots,1+\Sigma_{i=2}^{s-1}n_i) \longrightarrow \mathcal{F} \\ &\longrightarrow \mathcal{O}_X(\Sigma_{i=2}^{s}n_i,n_3,0,n_2+n_3,\ldots,\Sigma_{i=2}^{s-1}n_i)^{r-1} \longrightarrow 0 \end{aligned} \tag{4.2}$$

with first Chern class $c_1(\mathcal{F}) = ((r-1)\Sigma_{i=2}^{s}n_i, 1+rn_3, 1, \ldots, 1+r(\Sigma_{i=2}^{s-1}n_i))$.

Proof. To simplify we set $\mathcal{A} := \mathcal{O}_X(0,1,1+n_2,\ldots,1+\Sigma_{i=2}^{s-1}n_i)$, $\mathcal{B} := \mathcal{O}_X(\Sigma_{i=2}^{s}n_i,0,n_2,\ldots,\Sigma_{i=2}^{s-1}n_i)$, $\mathcal{C} := \mathcal{O}_X(0,1+n_3,1,1+n_2+n_3,\ldots,1+\Sigma_{i=2}^{s-1}n_i)$ and $\mathcal{D} := \mathcal{O}_X(\Sigma_{i=2}^{s}n_i,n_3,0,n_2+n_3,\ldots,\Sigma_{i=2}^{s-1}n_i)$. We are going to give the details of the proof of statement (i) since statement (ii) is proved analogously. Recall that by Proposition 3.2, $\mathcal{A}$ and $\mathcal{B}$ are initialized Ulrich line bundles on X. On the other hand, the dimension of $\mathrm{Ext}^1(\mathcal{B},\mathcal{A})$ can be computed as:

$$\begin{aligned} \dim \mathrm{Ext}^1(\mathcal{B},\mathcal{A}) &= \mathrm{h}^1(X,\mathcal{O}_X(-\Sigma_{i=2}^{s}n_i,1,\ldots,1)) \\ &= \mathrm{h}^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-\Sigma_{i=2}^{s}n_i))\prod_{i=2}^{s}\mathrm{h}^0(\mathbb{P}^{n_i},\mathcal{O}_{\mathbb{P}^{n_i}}(1)) \\ &= (\Sigma_{i=2}^{s}n_i-1)\prod_{i=2}^{s}(n_i+1). \end{aligned}$$

So, exactly as in the proof of Theorem 3.10, if we take l linearly independent elements $e_1,\ldots,e_l$ in $\mathrm{Ext}^1(\mathcal{B},\mathcal{A})$, $1 \leq l \leq (\Sigma_{i=2}^{s}n_i-1)\prod_{i=2}^{s}(n_i+1)-1$, these elements provide with an element $e := (e_1,\ldots,e_l)$ of $\mathrm{Ext}^1(\mathcal{B}^l,\mathcal{A}) \cong \mathrm{Ext}^1(\mathcal{B},\mathcal{A})^l$. Then the associated extension

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{E} \longrightarrow \mathcal{B}^l \longrightarrow 0 \tag{4.3}$$

gives a rank $l+1$ initialized simple Ulrich vector bundle. $\square$

Remark 4.2. (i) With the same technique, using other initialized Ulrich line bundles, it is possible to construct initialized simple Ulrich bundles of ranks covered by Theorem 4.1 with different first Chern class.

(ii) Notice that for $s = 2$, we have constructed rank r simple Ulrich vector bundles on $\Sigma_{1,m} \subseteq \mathbb{P}^{2m+1}$, $r \leq m^2$ as extensions of the form:

$$0 \longrightarrow \mathcal{O}_{\Sigma_{1,m}}(0,1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\Sigma_{1,m}}(m,0)^{r-1} \longrightarrow 0.$$

(iii) Observe that all vector bundles $\mathcal{E}$ constructed in Theorem 4.1 as extensions of Ulrich bundles are strictly μ-semistable.

Lemma 4.3. *Consider the Segre variety $\Sigma_{1,n_2,\dots,n_s} \subseteq \mathbb{P}^N$ for either $s \geq 3$ or $n_2 \geq 2$ and keep the notation introduced in Theorem 4.1. We have:*

(i) *For any two non-isomorphic rank 2 initialized Ulrich bundles $\mathcal{E}$ and $\mathcal{E}'$ from the family Λ_2 obtained from the exact sequence (4.1), it holds that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}') = 0$. Moreover, the set of non-isomorphic classes of elements of Λ_2 is parameterized by*

$$\mathbb{P}(\mathrm{Ext}^1(\mathcal{B}, \mathcal{A})) \cong \mathbb{P}\left(\mathrm{H}^1\left(\Sigma_{1,n_2,\dots,n_s}, \mathcal{O}_{\Sigma_{1,n_2,\dots,n_s}}\left(-\sum_{i=2}^{s} n_i, 1, \dots, 1\right)\right)\right)$$

and, in particular, it has dimension $(\Sigma_{i=2}^{s} n_i - 1)\prod_{i=2}^{s}(n_i + 1) - 1$.

(ii) *For any pair of bundles $\mathcal{E} \in \Lambda_2$ and $\mathcal{F} \in \Gamma_3$ obtained from the exact sequences (4.1) and (4.2), it holds that $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0$ and $\mathrm{Hom}(\mathcal{F}, \mathcal{E}) = 0$.*

Proof. The first statement is a direct consequence of Proposition [17, Proposition 5.1.3]. Regarding the second statement, it is a straightforward computation applying the functors $\mathrm{Hom}(\mathcal{F}, \cdot)$ and $\mathrm{Hom}(\mathcal{E}, \cdot)$ to the short exact sequences (4.1) and (4.2) respectively, and taking into account that there are no nontrivial morphisms among the vector bundles $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$. $\square$

In the next theorem we are going to construct families of increasing dimension of simple Ulrich bundles for arbitrary large rank on the Segre variety $\Sigma_{1,n_2,\dots,n_s}$. In case $s \geq 3$ we can use the two distinct families of rank 2 and rank 3 Ulrich bundles obtained in Theorem 4.1 to cover all the possible ranks. However, when $s = 2$, since there exists just a unique family, we will have to restraint ourselves to construct Ulrich bundles of arbitrary even rank. In any case, it will be enough to conclude that these Segre varieties are of wild representation type.

Theorem 4.4. *Consider the Segre variety $\Sigma_{1,n_2,\dots,n_s} \subseteq \mathbb{P}^N$ for either $s \geq 3$ or $n_2 \geq 2$.*

(i) *Then for any $r = 2t$, $t \geq 2$, there exists a family of dimension*

$$(2t-1)(\Sigma_{i=2}^{s} n_i - 1)\prod_{i=2}^{s}(n_i + 1) - 3(t-1)$$

of initialized simple Ulrich vector bundles of rank r.

(ii) *Let us suppose that $s \geq 3$ and $n_2 = 1$. Then for any $r = 2t + 1$, $t \geq 2$, there exists a family of dimension $\geq (t-1)((\sum_{i=2}^{s} n_i - 1)(n_3 + 2)\prod_{i=4}^{s}(n_i + 1) - 1)$ of initialized simple Ulrich vector bundles of rank r.*

(iii) *Let us suppose that $s \geq 3$ and $n_2 > 1$. For any integer $r = an_3\binom{n_2}{2} + l \geq n_3\binom{n_2}{2}$ with $a \geq 1$ and $0 \leq l \leq n_3\binom{n_2}{2} - 1$, there exists a family of dimension $a^2(n_3^2 + 2n_3 - 4) + 1 + l\left(an_3\binom{n_2+1}{2} - l\right)$ of simple (hence, indecomposable) initialized Ulrich vector bundles of rank r.*

Proof. (i) Let $r = 2t$ be an even integer and set $a := \mathrm{ext}^1(\mathcal{B}, \mathcal{A}) = (\Sigma_{i=2}^{s} n_i - 1)\prod_{i=2}^{s}(n_i + 1)$ with $\mathcal{A}$ and $\mathcal{B}$ defined as in the proof of Theorem 4.1. Denote by U the open subset of $\mathbb{P}^a \times \overset{t)}{\cdots} \times \mathbb{P}^a$, $\mathbb{P}^a \cong \mathbb{P}(\mathrm{Ext}^1(\mathcal{B}, \mathcal{A})) \cong \Lambda_2$, parameterizing closed points $[\mathcal{E}_1, \dots, \mathcal{E}_t] \in \mathbb{P}^a \times \overset{t)}{\cdots} \times \mathbb{P}^a$ such that $\mathcal{E}_i \not\cong \mathcal{E}_j$ for $i \neq j$ (i.e. U is $\mathbb{P}^a \times \overset{t)}{\cdots} \times \mathbb{P}^a$ minus the small diagonals). Given

$[\mathcal{E}_1, \dots, \mathcal{E}_t] \in U$, by Lemma 4.3, the set of vector bundles $\mathcal{E}_1, \dots, \mathcal{E}_t$ satisfy the hypothesis of Proposition [17, Proposition 5.1.3] and therefore, there exists a family of rank r simple Ulrich vector bundles $\mathcal{E}$ parameterized by

$$\mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_t, \mathcal{E}_1)) \times \cdots \times \mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_t, \mathcal{E}_{t-1}))$$

and given as extensions of the form

$$0 \longrightarrow \oplus_{i=1}^{t-1} \mathcal{E}_i \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_t \longrightarrow 0.$$

Next we observe that if we consider $[\mathcal{E}_1, \dots, \mathcal{E}_t] \neq [\mathcal{E}'_1, \dots, \mathcal{E}'_t] \in U$ and the corresponding extensions

$$0 \longrightarrow \oplus_{i=1}^{t-1} \mathcal{E}_i \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_t \longrightarrow 0,$$

and

$$0 \longrightarrow \oplus_{i=1}^{t-1} \mathcal{E}'_i \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E}'_t \longrightarrow 0$$

then $\mathrm{Hom}(\mathcal{E}, \mathcal{E}') = 0$ and in particular $\mathcal{E} \not\cong \mathcal{E}'$. Therefore, we have a family of non-isomorphic rank r simple Ulrich vector bundles $\mathcal{E}$ on $\Sigma_{1,n_2,\dots,n_s}$ parameterized by a projective bundle $\mathbb{P}$ over U of dimension

$$\dim\ \mathbb{P} = (t-1)\dim(\mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_t, \mathcal{E}_1))) + \dim\ U.$$

Applying the functor $\mathrm{Hom}(\cdot, \mathcal{E}_1)$ to the short exact sequence (4.1) we obtain:

$$0 \longrightarrow \mathrm{Hom}(\mathcal{A}, \mathcal{E}_1) \cong k \longrightarrow \mathrm{Ext}^1(\mathcal{B}, \mathcal{E}_1) \longrightarrow \mathrm{Ext}^1(\mathcal{E}_t, \mathcal{E}_1) \longrightarrow \mathrm{Ext}^1(\mathcal{A}, \mathcal{E}_1) = 0.$$

On the other hand, applying $\mathrm{Hom}(\mathcal{B}, \cdot)$ to the same exact sequence we have

$$\begin{aligned} 0 = \mathrm{Hom}(\mathcal{B}, \mathcal{E}_1) &\longrightarrow \mathrm{Hom}(\mathcal{B}, \mathcal{B}) \cong k \longrightarrow \mathrm{Ext}^1(\mathcal{B}, \mathcal{A}) \\ &\cong k^a \longrightarrow \mathrm{Ext}^1(\mathcal{B}, \mathcal{E}_1) \longrightarrow \mathrm{Ext}^1(\mathcal{B}, \mathcal{B}) = 0. \end{aligned}$$

Summing up, we obtain $\mathrm{ext}^1(\mathcal{E}_t, \mathcal{E}_1) = a - 2$ and so

$$\dim\ \mathbb{P} = (t-1)(a-3) + ta = (2t-1)a - 3(t-1).$$

(ii) Now, let us suppose that $s \geq 3$ and $n_2 = 1$ and take $r = 2t+1$, $t \geq 2$. Let $\mathcal{E}_1, \dots, \mathcal{E}_{t-1}$ be $t-1$ non-isomorphic rank 2 Ulrich vector bundles from the exact sequence (4.1) and let $\mathcal{F}$ be a rank 3 Ulrich bundle from the exact sequence (4.2). Again, by Lemma 4.3, this set of vector bundles satisfies the hypothesis of Proposition [17, Proposition 5.1.3] and therefore, there exists a family $\mathbb{G}$ of rank r simple Ulrich vector bundles $\mathcal{E}$ parameterized by

$$\mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{F})) \times \cdots \times \mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_{t-1}, \mathcal{F}))$$

and given as extensions of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \oplus_{i=1}^{t-1} \mathcal{E}_i \longrightarrow 0.$$

It only remains to compute the dimension of the family

$$\dim\ \mathbb{G} = (t-1)\dim(\mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{F}))).$$

Let us fix the notation

$$b := \mathrm{ext}^1(\mathcal{B}, \mathcal{C}) = \mathrm{h}^1\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(-\sum_{i=2}^{s} n_i\right)\right) \mathrm{h}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1+n_3)) \prod_{i=4}^{s} \mathrm{h}^0(\mathbb{P}^{n_i}, \mathcal{O}_{\mathbb{P}^{n_i}}(1))$$

$$= \left(\sum_{i=2}^{s} n_i - 1\right)(n_3 + 2)\prod_{i=4}^{s}(n_i + 1).$$

Applying the functor $\mathrm{Hom}(\cdot, \mathcal{F})$ to the short exact sequence (4.1) we obtain:

$$0 = \mathrm{Hom}(\mathcal{A}, \mathcal{F}) \longrightarrow \mathrm{Ext}^1(\mathcal{B}, \mathcal{F}) \longrightarrow \mathrm{Ext}^1(\mathcal{E}_1, \mathcal{F}) \longrightarrow \mathrm{Ext}^1(\mathcal{A}, \mathcal{F}).$$

On the other hand, applying $\mathrm{Hom}(\mathcal{B}, \cdot)$ to the short exact sequence (4.2) we have

$$\mathrm{Hom}(\mathcal{B}, \mathcal{D}) = 0 \longrightarrow \mathrm{Ext}^1(\mathcal{B}, \mathcal{C}) \cong k^b \longrightarrow \mathrm{Ext}^1(\mathcal{B}, \mathcal{F}) \longrightarrow \mathrm{Ext}^1(\mathcal{B}, \mathcal{D}) = 0.$$

Summing up, we obtain $\mathrm{ext}^1(\mathcal{E}_1, \mathcal{F}) \geq b$ and therefore $\dim \mathbb{G} \geq (t-1)(b-1)$.

(iii) It follows from Theorem 3.10 and [9, Proposition 2.6]. $\square$

Corollary 4.5. *The Segre variety $\Sigma_{1,n_2,\ldots,n_s} \subseteq \mathbb{P}^N$, $N = 2\prod_{i=2}^{s}(n_i+1) - 1$, for $s \geq 3$ or $s = 2$ and $n_2 \geq 2$ is of wild representation type.*

Putting together Corollaries 3.7, 3.12 and 4.5, we get

Theorem 4.6. *All Segre varieties $\Sigma_{n_1,n_2,\ldots,n_s} \subseteq \mathbb{P}^N$, $N = \prod_{i=1}^{s}(n_i+1) - 1$, are of wild representation type unless the quadric surface in $\mathbb{P}^3$ (which is of finite representation type).*

We would like to finish this paper with some related comments and open problems.

First of all, regarding the existence of Ulrich vector bundles of lower rank on the Segre varieties $\Sigma_{n,m} \subseteq \mathbb{P}^N$, $N := nm + n + m$, for $2 \leq n, m$, notice that in this paper we have shown their existence in the cases of rank one and rank higher than $m\binom{n}{2}$. So, it remains to prove their existence for the intermediate ranks:

Open Problem 4.7. Construct indecomposable Ulrich vector bundles of rank r, for $1 < r < m\binom{n}{2}$, on the Segre variety $\Sigma_{n,m} \subseteq \mathbb{P}^N$, $N := nm + n + m$, for $2 \leq n, m$.

Finally, we want to pay attention to the case of the Segre variety $\Sigma_{1,2} \subseteq \mathbb{P}^5$. It is interesting to relate the known ACM vector bundles on $\Sigma_{1,2}$ with their restriction to the general hyperplane section $X \cap \mathbb{P}^4$ which turns out to be the cubic scroll $S(1,2) \subseteq \mathbb{P}^4$. As it has been mentioned, $S(1,2)$ is a variety of finite representation type. Moreover, in [1] there has been given a complete list of non-isomorphic irreducible ACM vector bundles on $S(1,2)$, up to shift. If we write $\mathrm{Pic}(S(1,2)) \cong \langle C_0, f\rangle$ for C_0 a section and f a fiber of the structural morphism $\pi : S(1,2) \longrightarrow \mathbb{P}^1$, they turn out to be: $\mathcal{O}_{S(1,2)}$, $\mathcal{O}_{S(1,2)}(f)$, $\mathcal{O}_{S(1,2)}(2f)$, $\mathcal{O}_{S(1,2)}(f + C_0)$ and the first sygyzy bundle $\mathcal{E}$ of the free resolution as $\mathcal{O}_{S(1,2)}$-module of $\mathcal{O}_{S(1,2)}(3f + 2C_0)$, which has rank two. Notice that the last three vector bundles on the list are Ulrich.

Lemma 4.8. *Let $S(1,2) \subseteq \mathbb{P}^4$ be the cubic scroll seen as a general hyperplane section of the Segre variety $\Sigma_{1,2}$. Then the five non-isomorphic irreducible ACM vector bundles on $S(1,2)$ can be obtained as restrictions to $S(1,2)$ of ACM irreducible vector bundles on $\Sigma_{1,2}$.*

Proof. First of all, we have $\mathcal{O}_{\Sigma_{1,2}|S(1,2)} \cong \mathcal{O}_{S(1,2)}$. On the other hand, since the restriction to $S(1,2)$ of $\mathcal{O}_{\Sigma_{1,2}}(1,1)$ is the very ample line bundle $\mathcal{O}_{S(1,2)} := \mathcal{O}_{S(1,2)}(2f + C_0)$ that provides the embedding $S(1,2) \subseteq \mathbb{P}^4$, it is easy to see that $\mathcal{O}_{\Sigma_{1,2}}(0,1)_{|S(1,2)} \cong \mathcal{O}_{S(1,2)}(f + C_0)$, $\mathcal{O}_{\Sigma_{1,2}}(2,0)_{|S(1,2)} \cong \mathcal{O}_{S(1,2)}(2f)$ and therefore $\mathcal{O}_{\Sigma_{1,2}}(1,0)_{|S(1,2)} \cong \mathcal{O}_{S(1,2)}(f)$. Finally, we can consider the rank two Ulrich vector bundle $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \Omega^1_{\mathbb{P}^2}(2)$ on $\Sigma_{1,2}$ that fits in the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \Omega^1_{\mathbb{P}^2}(2) \longrightarrow \mathcal{O}_{\Sigma_{1,2}}(1,1)^3 \longrightarrow \mathcal{O}_{\Sigma_{1,2}}(1,2) \longrightarrow 0.$$

If we restrict this exact sequence to $S(1,2)$,

$$0 \longrightarrow \mathcal{E} := \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \Omega^1_{\mathbb{P}^2}(2)_{|S(1,2)} \longrightarrow \mathcal{O}_{S(1,2)}(1)^3 \longrightarrow \mathcal{O}_{S(1,2)}(3f + 2C_0) \longrightarrow 0$$

we obtain the rank two Ulrich sygyzy bundle $\mathcal{E}$, completing the list. □

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