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Graphs Where Every Maximal Path Is Maximum

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We give a complete characterization of all graphs in which every simple path is contained in a maximum path. Among Hamiltonian graphs, such graphs were previously characterized by C. Thomassen.

The characterization of certain families of graphs, where some greedy schemes are guaranteed to produce a Hamiltonian path, or a Hamiltonian circuit, has received considerable attention in the literature. Such families include the set of all graphs where every simple path is contained in a Hamiltonian path (circuit) and the one consisting of all graphs, on a vertex set $V=\{v_1, ..., v_n\}$, where every simple path $(v_{i_1}, v_{i_2}, ..., v_{i_k})$ can be completed to a Hamiltonian path (circuit) $(v_{i_1}, v_{i_2}, ..., v_{i_k}, v_{i_{k+1}}, ..., v_{i_n})$. Similar families can be defined for directed graphs, where the paths involved are required to be directed. These sets and similar ones are fully characterized in [2, 3, 4, 7, 8] and [5].

Such characterization problems can be viewed within the general framework of detecting greedily solvable instances of combinatorial problems, which are hard in general (such as the *NP*-complete Hamiltonian path problem). A thorough discussion of this topic can be found in [1].

The goal of this article is to obtain the corresponding characterization for maximum paths. We give a complete characterization of all simple undirected graphs in which every simple path can be extended to a simple path of maximum length. Although the characterization we present is easily stated, our proof is rather long and involved. Similar phenomena occur as well with the results we have retrieved.

Since the case, where the graph under discussion has a Hamiltonian path, has already been treated (in [8]), we excluded it from the current discussion.

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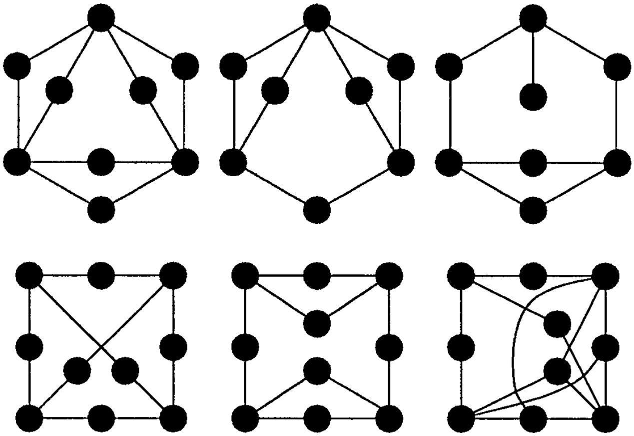

FIGURE 1

THEOREM 1. *Let G be a connected simple graph which admits no Hamiltonian path. Every simple path of G is contained in a simple path of maximum length, if and only if G is one of the following*:

The union of simple paths, all of the same length, which share one common endvertex and are otherwise vertex disjoint, or

a bipartite graph on vertex set $V = X \cup Y$, where $|Y| \geqslant |X| + 2$ and every vertex of X is adjacent to all, but at most one, vertex of Y, or one of the six graphs, presented in Fig. 1.

Our proof of Theorem 1 is organized into several lemmas. Throughout the proof G is assumed to satisfy the conditions of Theorem 1, namely, it is connected, admits no Hamiltonian path and any of its paths is contained in a maximum length path.

LEMMA 1. *Theorem* 1 *holds for the case where G is a tree.*

Proof. Let us assume that G is a tree. If it is a path then we are done, so let x be a vertex of degree 3 or more. The paths from x to all leaves are of the same length; otherwise, by the concatenations of pairs of such paths, maximal paths of different length are obtained. Other than x, there is no vertex y of degree 3 or more, because, considering x as the root of G, a maximal path between two descendents of y would obviously be shorter than maximum. It turns out that G is the union of equally sized paths with a common endvertex x and otherwise vertex disjoint. ∎

LEMMA 2. *If G is not a tree then it is bipartite on vertex set $V = X \cup Y$, (X and Y are independent), where $|Y| \geqslant |X| + 2$. The graph G has a circuit of length $2|X|$ and all maximal paths of G are of length $2|X|$.*

Proof. Let $(0, 1, ..., m-1)$ (modulo m) be the vertices, in that order, of a maximum length circuit C of G. In the sequel we use addition modulo m to indicate the vertices of C. For $i = 0, 1, ..., m-1$, let the connected component of the subgraph spanned by $(V \backslash C) \cup \{i\}$, which includes the vertex i, be denoted by B_i. Let k_i denote the maximum length of a path in B_i, which starts at the vertex i.

Claim 1. For every $i = 0, 1, ..., m-1$, the components B_i and B_{i+1} are disjoint.

Otherwise, the edge $(i, i+1)$ in C could be replaced by a path of length two or more, contradicting the maximality of C.

Claim 2. $k_i = k_{i+2}$ for all $i = 0, 1, ..., m-1$.

The path obtained from C by deleting an edge $(i, i+1)$ is extended to a maximum path by appending two end segments, one in B_i and another in B_{i+1}. By Claim 1, these two components are disjoint and hence the length of a maximum path is $m - 1 + k_i + k_{i+1}$. This value is the same for every $i = 0, 1, ..., m-1$. The assertion of Claim 2 immediately follows.

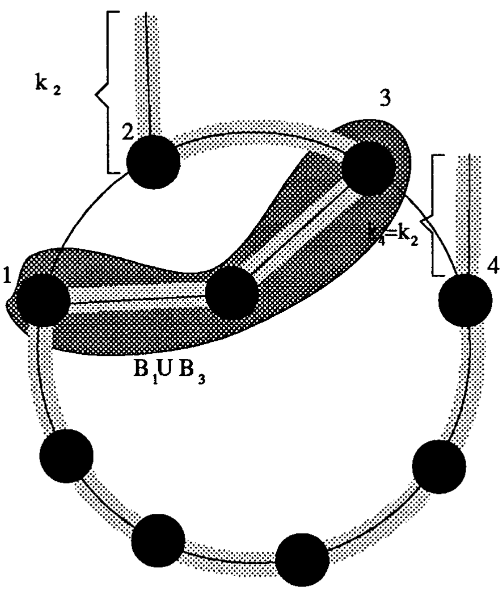

FIGURE 2

Let us select the vertex labelling such that $k_1 \leqslant k_2$.

Claim 3. $k_1 = 0$.

Assume to the contrary $0 < k_1 \leqslant k_2$. Consider the path obtained from C by deleting the two edges $(1, 2)$ and $(2, 3)$. If B_1 is disjoint from B_3 then the path can be extended in these two components to a maximal path of length $m - 2 + 2k_1$, which is less than maximum. Hence B_1 and B_3 intersect. By the maximality of C and $k_1 > 0$, a maximum path from 1 to 3 in $B_1 \cup B_3$ has length 2, but then, by the maximality of C, the components B_2 and B_4 are disjoint and there exists a path of length $m + 2k_2$, as shown in Fig. 2. Since this exceeds the length of a maximum path, a contradiction results. For the last argument we assumed the existence of four distinct vertices on C. Indeed, if C is a triangle then $k_0 = k_1 = k_2 = k$ and there exist a maximal path of length $k + 1$ and another one of length $k + 2$.

Claim 4. The length m of the maximum circuit C is even and the length of a maximum (every maximal) path is m.

If m is odd then by Claims 2 and 3, $k_i = 0$ for $i = 0, 1, ..., m - 1$. Since G is connected and not Hamiltonian, there does exist a vertex in $V \backslash C$ adjacent to a vertex in C and hence m is even. We have already observed that a maximum path in G is of length $m - 1 + k_1 + k_2$, which equals $m - 1 + k_2$ by Claim 3.

If B_0 is disjoint from B_2 then there exists a path of length $m - 2 + k_2$, (Fig. 3), which implies that $k_2 = 1$.

If B_0 and B_2 intersect then, by the maximality of C, a maximum path from 0 to 2 in $B_0 \cup B_2$ has length 2. Another maximum circuit C' is obtained from C when $(0, 1, 2)$ is replaced by such a path $(0, t, 2)$ (See Fig. 4).

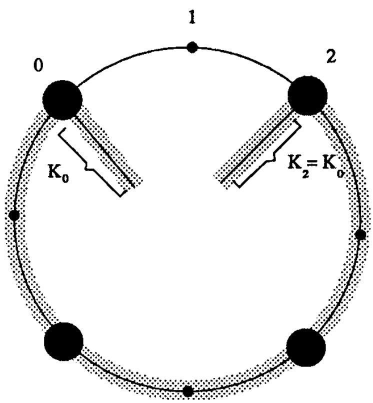

FIGURE 3

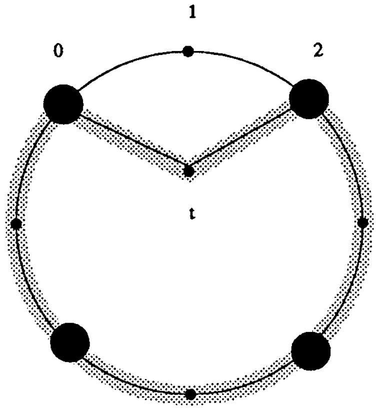

FIGURE 4

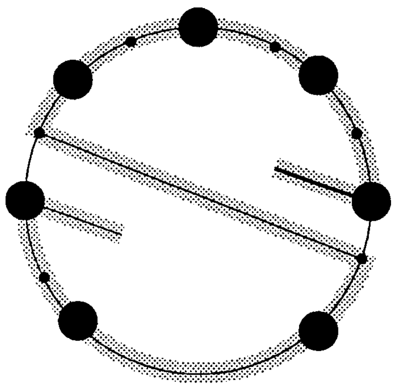

FIGURE 5

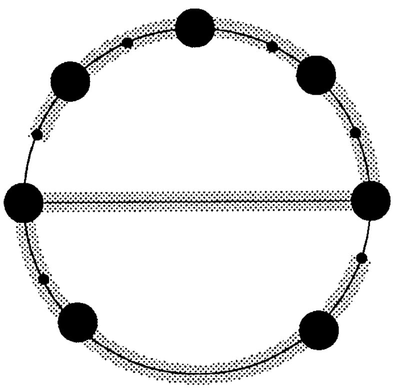

FIGURE 6

By Claim 3, only one of two consecutive vertices on a maximum circuit is adjacent to a vertex not on the circuit. The vertex 0 of C' is adjacent to 1 which is not on C'; thus t is not adjacent to any vertex not on C' (or C) and thus $k_2 = 1$, which completes the proof of Claim 4.

In the sequel we consider the partition of V into three sets: *even* vertices, *odd* vertices and *external vertices*. The first two sets include the vertices of C and the third is $V \setminus C$.

Claim 5. G is bipartite. Its independent sets are the even vertices on one side and the odd and external vertices on the other.

By Claim 4, no external path, starting at an even vertex, has length exceeding 1 and hence the external vertices form an independent set. By Claim 3, no odd vertex is adjacent to an external one. To verify the claim we should also eliminate the possibility of *even chords* (edges joining two even vertices, and that of *odd chords* (joining two odd vertices). Indeed if an odd chord exists then it is contained in a path of length $m+1$ (see Fig. 5), or a circuit of this size, if both ends of that path coincide. An even chord, on the other hand, implies a maximal path of length $m-1$ (see Fig. 6). Any of these yields a contradiction to Claim 4.

Let now X stand for the set of $m/2$ even vertices and Y for that of odd and external ones. There must be at least two external vertices, otherwise G would have a Hamiltonian path, hence $|Y| \geqslant |X| + 2$. The assertion of Lemma 2 is the conclusion of claims 4 and 5. ∎

Notice that the lemma implies that every maximum circuit, as well as every maximal path, includes all the vertices of X.

Let x and y be two vertices of the circuit C. We define the *open interval* $(\cdot x, y \cdot)$ to be the path on C, which starts at $x+1$ and ends at $y-1$, following the positive (clockwise) direction modulo m.

LEMMA 3. *Any pair of external vertices dominates all even vertices.*

Proof. We make the following two claims.

Claim 1. Let x and y be two non-consecutive even vertices, $y \geqslant x+4$, such that each of x and y is adjacent to at least one of two external vertices a and b, while no even vertex in the open interval $(\cdot x, y \cdot)$ is adjacent to either a or b, then both x and y are adjacent to the same external vertex, say a, and neither of them is adjacent to b.

Indeed if this is not the case then there exists a maximal path not containing the even vertices on the interval $(\cdot x, y \cdot)$ (See Fig. 7), contradicting Lemma 2.

Claim 2. Every set of three external vertices dominates all even vertices.

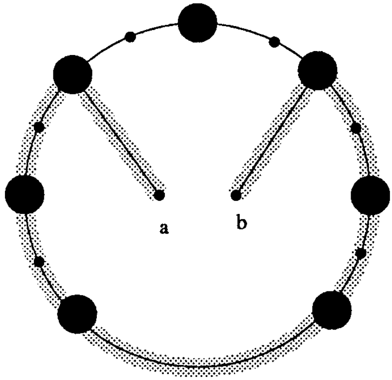

FIGURE 7

Assume to the contrary that there exist three external vertices a, b and c which do not dominate all even vertices. Let x and y, where $y \geqslant x+4$, be two even vertices, dominated by $\{a, b, c\}$, where no even vertex in $(\cdot x, y \cdot)$ is dominated by that set. By Claim 1, both x and y are adjacent to the same one of the 3 vertices a, b, c say to a, and both are nonadjacent to the other two vertices b and c. All even vertices adjacent to either b or c are in the complementary interval $(\cdot y, x \cdot)$. By Claim 1 there are two *consecutive* even vertices z and $z+2$, one adjacent to b and the other adjacent to c (it does not matter if either of z and $z+2$ is adjacent to both b and c and also to a). Such a configuration, however, admits a maximal path which misses the even vertices in $(\cdot x, y \cdot)$ (See Fig. 8)—a contradiction to Lemma 2.

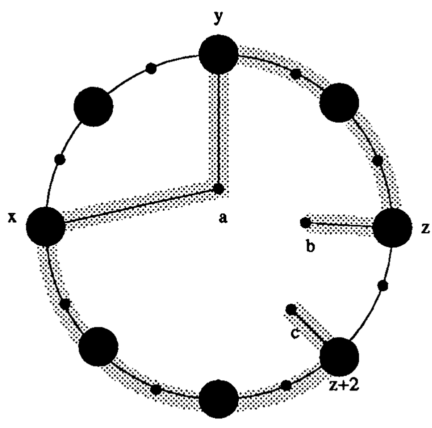

FIGURE 8

Assume now, contrary to Lemma 3, that an even vertex t is adjacent to only one vertex a among the three external vertices a, b and c. Let us select b and c such that an even vertex adjacent to b is encountered in the sequence $t-2$, $t-4$, $t-6$, ... before one which is adjacent to c. Notice that, by Claim 1, the first even vertex adjacent to b cannot also be adjacent to c and even vertices adjacent to b and to c are encountered in the same order in the sequence t, $t+2$, $t+4$, ... Let x be the greatest even vertex which is smaller than t and adjacent to c. Let us also select t to be the smallest even vertex, greater than x, which is adjacent only to one of a and b. It turns out that all even vertices in $(\cdot x, t\cdot)$ are adjacent to a and b, but not to c and by Claim 1 there exists at least one such vertex. Let s be the smallest even vertex, greater than t, which is adjacent to either b or c. By Claim 1 s is nonadjacent to c and hence it is adjacent to b. Let y be the smallest even vertex, greater than s, adjacent to c (We use "greater" and "smaller" to indicate clockwise and anticlockwise direction on C. The argument stays valid also if y and x coincide). By Claim 1, $y-2$ is adjacent to a (and also to b, but this is irrelevant to our needs). As shown in Fig. 9, such configuration contains a maximal path which misses the even vertices in the interval $(\cdot t-2, s\cdot)$—a contradiction to Lemma 2. ▮

LEMMA 4. *Theorem* 1 *holds for the case where there are at least* 4 *external vertices.*

Proof. Let a, b, c, d be four distinct external vertices. Assume that an even vertex x is nonadjacent to external vertex a. Let s be any odd vertex. By Lemma 3, each of the even vertices $s+1$ and $s-1$ is adjacent to at least three of a, b, c, d. It implies that s and $s+1$ have two common neighbors in $\{a, b, c, d\}$, at least one of which is not a, say d. Replacing the path

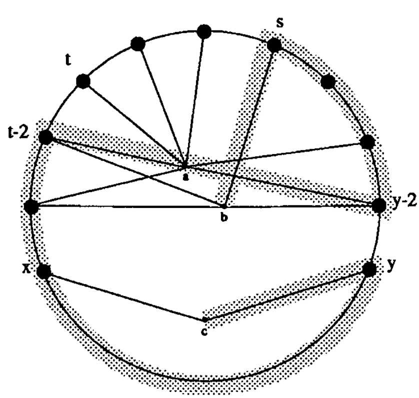

FIGURE 9

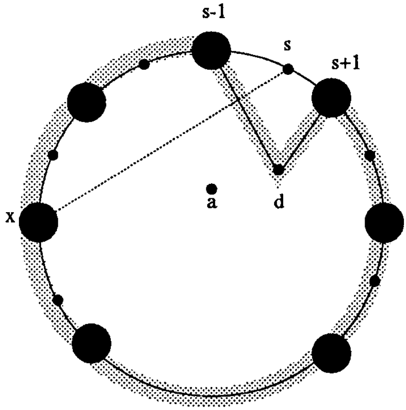

FIGURE 10

$(s-1, s, s+1)$ in C by $(s-1, d, s+1)$, a new maximum circuit is obtained, for which a and s are external (See Fig. 10). By Lemma 3, even vertex x is adjacent to at least one of them, that is, to s.

Let now y be an even vertex nonadjacent to an odd vertex t. By an argument similar to the above, we can construct a new maximal circuit where t is external and hence, again, y is adjacent to all other odd vertices. We have shown that every even vertex is indeed adjacent to all odd and external vertices, with at most one exception. ∎

LEMMA 5. *Theorem* 1 *holds for the case where there are at least five even vertices.*

Proof. Throughout this proof r and g (for *red* and *green*) are two vertices, external to a maximum circuit C with at least ten vertices (of which at least five are even). An even vertex adjacent to r is called *red* and one adjacent to g is *green*. By Lemma 3, every even vertex is either red or green or both. We call a red and green even vertex *bicolored*, otherwise it is *monochromatic*. A monochromatic vertex x is *deprived* if there exists an odd vertex nonadjacent to x (we shall prove that no deprived vertex exists). An edge joining an odd vertex to an even one is a *chord*. The length of a chord is the difference modulo m (odd and at most $\frac{m}{2}$) between its endvertices. A *short* chord is a chord of length 3. The short chords which *bypass* even vertex x are $(x-1, x+2)$ and $(x-2, x+1)$.

Claim 1. There is no short chord bypassing a deprived vertex.

Let x be a deprived vertex, say red, which is not adjacent to some odd vertex s. If both $s-1$ and $s+1$ are red then $(s-1, r, s+1)$ can replace

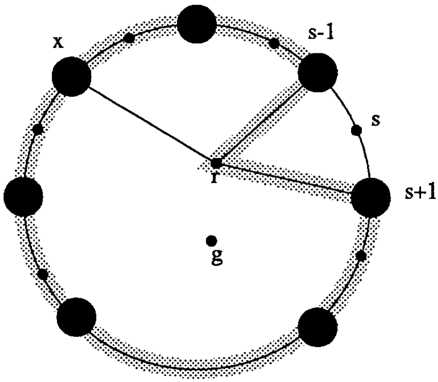

FIGURE 11

$(s-1, s, s+1)$ in C to obtain a maximum circuit where x is nonadjacent to two external vertices g and s, in contradiction to Lemma 3 (Fig. 11). Thus, either $s-1$ or $s+1$ is monochromatic green. In that case, a short chord which bypasses x completes a maximal path which does not include x (Fig. 12), contrary to Lemma 2.

Claim 2. If x is deprived then at least one of $x-2$ and $x+2$ is deprived and its color differs from that of x. Recall, in the proof of Claim 1, that if both $x-2$ and $x+2$ are red then x can be bypassed by $(x-2, r, x+2)$ (Fig. 13). Hence one of $x-2$ and $x+2$ is monochromatic green. Since it is also an end vertex of a nonexisting short chord which bypasses x (Claim 1), then it is indeed deprived.

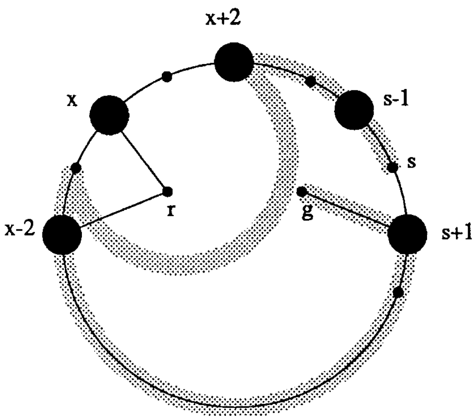

FIGURE 12

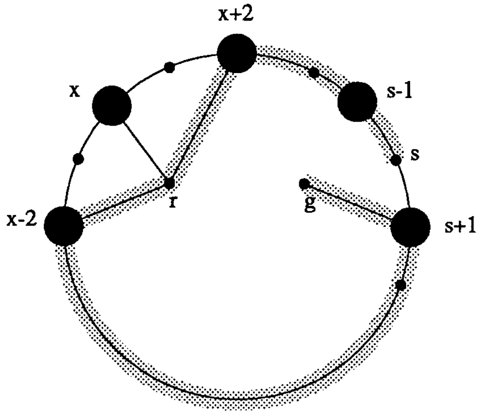

FIGURE 13

Claim 3. A monochromatic even neighbor ($x-2$ or $x+2$) of a deprived vertex x is deprived.

This is an immediate consequence of Claim 1.

Claim 4. If there exists a deprived vertex, then both neighbors of any bicolored vertex are deprived.

Let us start from a deprived vertex and move clockwise on C until the first bicolored vertex x is found. By Claim 3, x is also the first non-deprived and thus $x-2$ is deprived, say green. By Claim 2, $x-4$ is deprived red. Assume to the contrary of Claim 4 that $x+2$ is bicolored. In that configuration there exists a maximum circuit for which $x-1$ and $x+1$ are

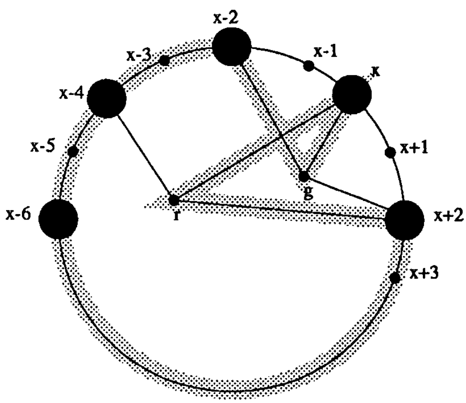

FIGURE 14

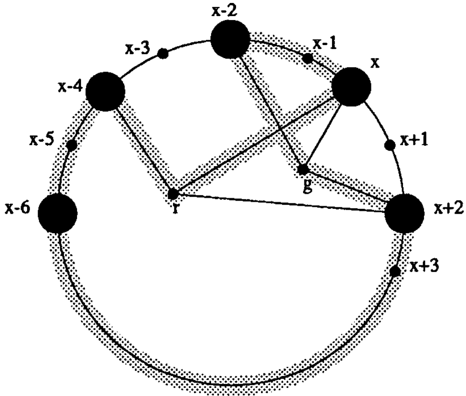

FIGURE 15

external (Fig. 14). By Claim 1, $x-4$ is nonadjacent to $x-1$ and thus, by Lemma 3, $x-4$ is adjacent to $x+1$. There also exists a maximum circuit for which $x-3$ and $x+1$ are external (Fig. 15) and thus by a similar argument also $x-6$ is adjacent to $x+1$. Now we can draw a maximum circuit for which r and $x-5$ are external (Fig. 16), but neither of these two is adjacent to $x-2$ and that is a contradiction to Lemma 3.

The assumption that $x+2$ is bicolored is proven wrong and hence it is monochromatic. Our goal is to show that this vertex is deprived. Assume to the contrary that it is adjacent to all odd vertices. An edge $(x-1, x+2)$ provides a maximum circuit where r and $x+1$ are external (Fig. 17) and an edge $(x-3, x+2)$ gives another, where $x-1$ and $x+1$ are external (Fig. 18).

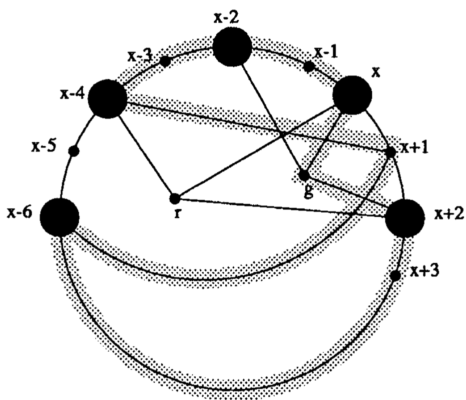

FIGURE 16

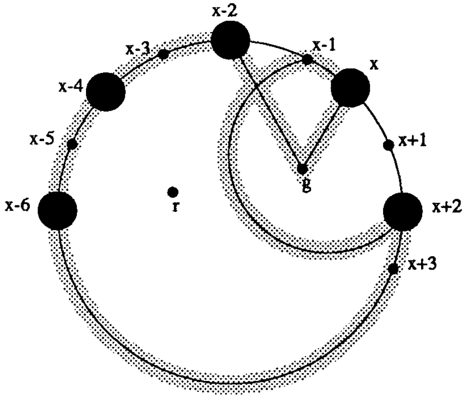

FIGURE 17

By Lemma 3 and Claim 1, $x+1$ is adjacent to $x-2$ (nonadjacent to r) and to $x-4$ (nonadjacent to $x-1$). Let us assume that $x-6$ is red. In that case there exists a maximum path without $x-4$ (Fig. 19), contradicting Lemma 2.

It turns out that $x-6$ is monochromatic green. By Claim 1, $x-6$ is nonadjacent to $x-3$. This leads to a maximum circuit with external vertices $x-3$ and r, both nonadjacent to $x-6$ (Fig. 20)—a contradiction, which shows that $x+2$ is indeed deprived. Claim 4 is now proven by keeping going forward on C.

Claim 5. If there exists a deprived vertex then all even vertices are deprived.

Let $x-4$, $x-2$ and x be as assumed while proving the previous claim. $x+2$ is now known to be deprived. Assume first that it is green. By

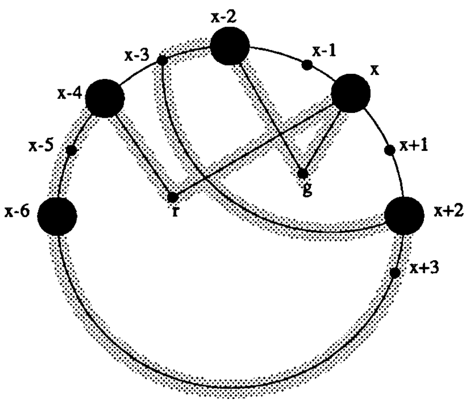

FIGURE 18

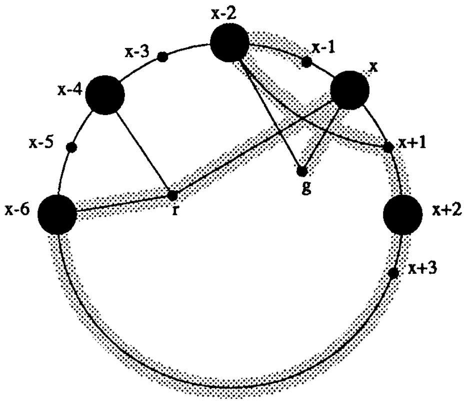

FIGURE 19

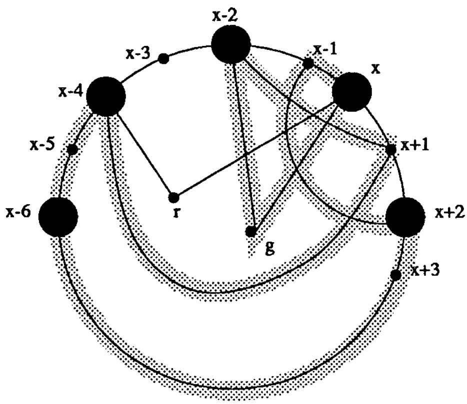

FIGURE 20

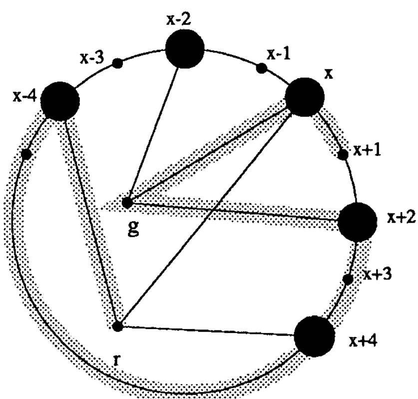

FIGURE 21

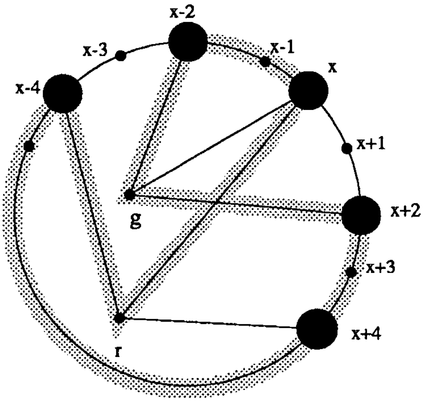

FIGURE 22

Claim 2, $x+4$ is deprived red. The path from $x+1$ to r, as shown in Fig. 21, cannot be maximal. Thus, since $x-2$ is not red, it is adjacent to $x+1$.

The vertices $x-3$ and $x+1$ are external to a maximum circuit, as shown in Fig. 22. Hence $x+4$, which, by Claim 1, is nonadjacent to $x+1$, is adjacent to $x-3$.

If there is no even vertex in the interval $(\cdot x+4, x-4\cdot)$, then $(x+4, x-3)$ is a short chord which bypasses $x-4$, contrary to Claim 1, so there must be an additional even vertex $x-6$. $x-6$ is nonadjacent to $x-3$ and hence (refer, again, to Fig. 22), it is adjacent to $x+1$.

Consider now the path from $x-5$ to $x+3$, sketched in Fig. 23. It is too short for maximal and since there is no short chord $(x, x+3)$, there exists an edge $(x-5, x)$.

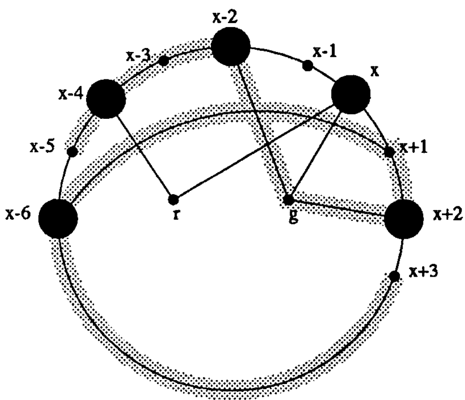

FIGURE 23

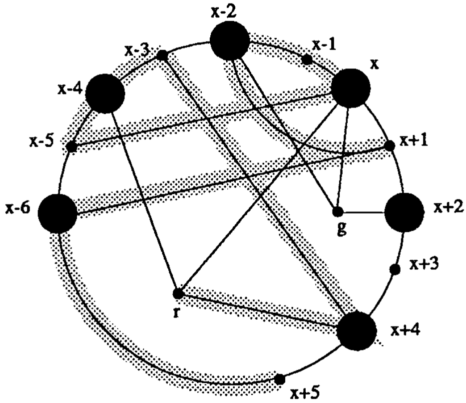

FIGURE 24

We have shown so far the existence of edges: $(x-2, x+1)$, $(x-3, x+4)$, $(x-5, x)$ and $(x-6, x+1)$. All these combine to a maximal path which does not include $x+2$ (Fig. 24) and hence contradicting Lemma 2.

We should assume then, that $x+2$ is deprived red. In that case, however, there exists a maximum circuit, where $x-3$ and $x+3$ are external (Fig. 25). By Claim 1, neither of these two vertices is adjacent to x, in contradiction to Lemma 3.

It remains to eliminate the possibility that all even vertices are deprived, which is the situation we assume in what follows. Notice that if this is the case then, by Claim 1, no short chord exists.

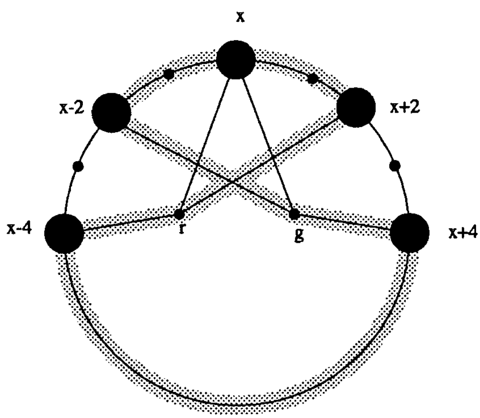

FIGURE 25

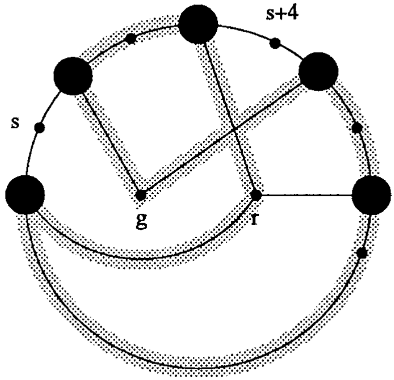

FIGURE 26

Claim 6. There is no sequence of five consecutive deprived vertices with colors red, green, x, red, green (red and green can be switched for that matter), in that order, where x is either green or red.

In fact, this was already proven in the last argument dealing with Claim 5 (recall Fig. 25).

Claim 7. The "color sequence" of the (deprived) even vertices on C is either one of:

(i) red, green, red, green,... with an even number of even vertices, or

(ii) red, red, green, green, red, red, green, green,... with $4k$ even vertices.

It is straightforward to verify that these are indeed the only patterns allowed by Claims 6 and 2.

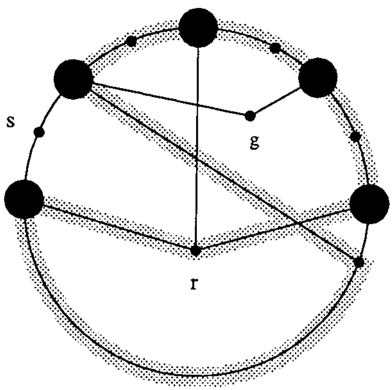

FIGURE 27

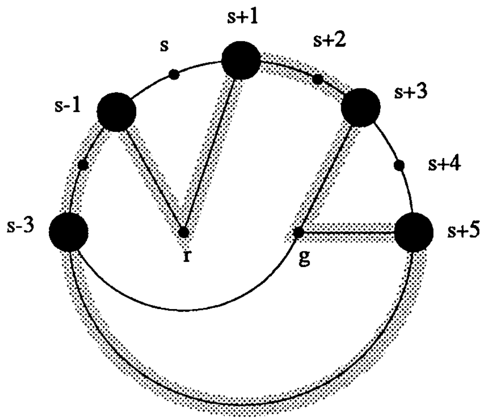

FIGURE 28

Claim 8. The pattern of Claim 7(i) yields a contradiction.

The pattern at hand provides a maximum circuit, for which two external vertices are odd vertices of C, with distance 4 between them (Fig. 26).

Since there are no short chords of length 3, Lemma 3 implies the existence of all chords of length 7. A chord of length 7 provides a maximum circuit for which g and an odd vertex s of C are external (Fig. 27).

Due to the full symmetry of the pattern, s, for that matter, can be any odd vertex and g can be replaced by r. Since every even vertex is nonadjacent either to r or to g, it is, by Lemma 3, adjacent to every odd vertex s, but then it is not deprived, in contradiction to our working assumption.

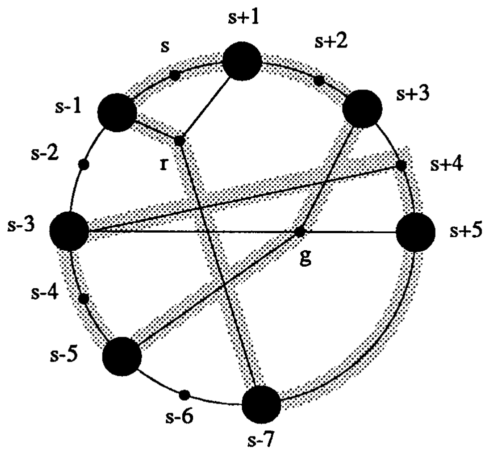

FIGURE 29

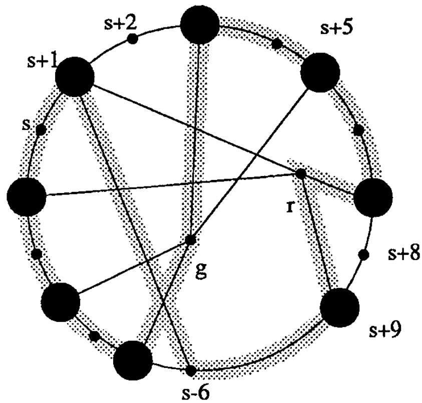

FIGURE 30

Claim 9. The pattern of Claim 7(ii) yields a contradiction.

Let s be an odd vertex such that the color sequence of the even vertices $s-1$, $s+1$, $s+3$, $s+5$ is red, red, green, green. There exists a maximum circuit for which s and $s+4$ are external (Fig. 28).

Hence, $s-3$, which is nonadjacent to s, is adjacent to $s+4$. The edge $(s-3, s+4)$ completes a maximum circuit for which $s-6$ and $s-2$ are external (Fig. 29) and hence $s+1$ is adjacent to $s-6$.

The edge $(s-6, s+1)$ completes a maximum circuit for which $s+2$ and $s+8$ are external (Fig. 30), but neither of these two is adjacent to $s+5$, in contradiction to Lemma 3. Notice that Lemma 5 relates to a maximum circuit C with at least five even vertices. Since we deal here with $4k$ even vertices, it means at least 8, as required for the last argument.

Claims 1 through 9 show that no deprived vertex exists. To complete the proof of Lemma 5 we should prove:

Claim 10. Every pair of odd vertices dominates the set of even vertices. Assume to the contrary that an even vertex x is adjacent to neither of two odd vertices s and t. If either $s-1$ and $s+1$, or $t-1$ and $t+1$, share a common color than there exists a maximum circuit, with respect to which x is deprived (nonadjacent to s and t, an external vertex and an odd one, Fig. 31). Since no deprived vertex exists, then $s+1$ and $t-1$ are monochromatic and hence each of them is adjacent to all odd vertices. In that situation there exists a maximal path which does not include x (See Fig. 32. The argument is also valid if $s+1$ and $t-1$ coincide), in contradiction to Lemma 2. The proof of Lemma 5 is now completed. ▮

Combining Lemmas 4 and 5, there are only finitely many graphs for which the validity of Theorem 1 should still be checked. Lemmas 2 and 3,

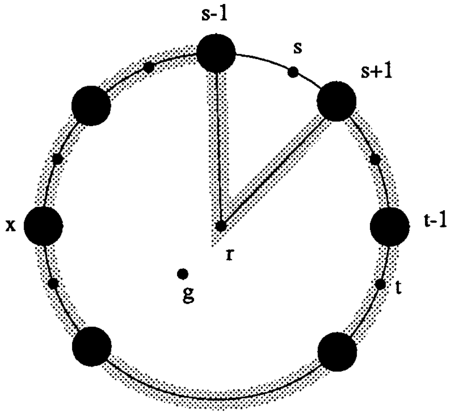

FIGURE 31

as well as most of the Claims stated along the proofs of Lemmas 4 and 5, are applicable also to small graphs (at most three external vertices and four even ones). We can also provide a general argument (omitted) to show that the theorem holds for the case where there exist three external vertices and at least four even ones. Considering all that, the graphs remained to be tested are indeed few and easy to check. Out of those, the only ones which satisfy the required property, are the six graphs drawn in Fig. 1. The details of the straightforward verification are omitted, as well as the easy proof of the sufficiency claim in Theorem 1.

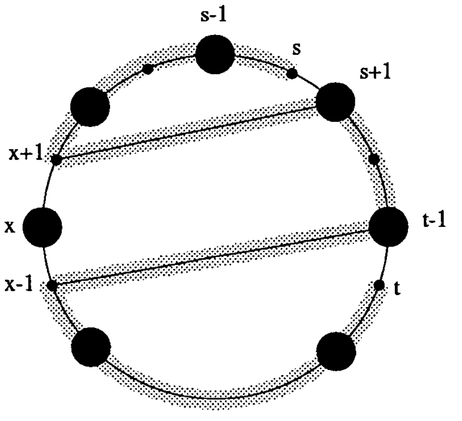

FIGURE 32

Acknowledgment

While preparing the final version of this article for publication, it was brought to my attention that M. S. Jacobson, A. E. Kézdy, and J. Lehel [6] have recently independently proved Theorem 1. Furthermore, they pointed out an error in my original work, where only five of the six exceptional graphs, listed in Fig. 1, were detected. I wish to thank them for this very important contribution.

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