Completely Monotonic and Related Functions

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We study completely monotonic and related functions whose first \( N \) derivatives are definite on an interval. Let \( \mathcal{L}^N \) denote the class of functions defined by

\[
f \in \mathcal{L}^N \iff (-1)^k f^{(k)}(t) \geq 0, \quad \forall t > 0, \forall k, 0 \leq k \leq N.
\]

For \( N \to \infty \) we write \( f \in \mathcal{L} \); such functions are called completely monotonic on \( (0, \infty) \). We prove in particular that the implication

\[
f \in \mathcal{L}^N \implies \left[ \forall \alpha > 1: f^\alpha \in \mathcal{L}^N \right],
\]

is true for \( 0 \leq N \leq 5 \), but false for \( N \geq 6 \). In contrast, L. Lorch and D. J. Newman (J. London Math. Soc. (2), 28, 1983, 31–45, Theorem 8) claimed this implication to be false for \( N = 5 \). Further we prove that the implication

\[
f \in \mathcal{L} \implies \left[ \forall \alpha > 1: f^\alpha \in \mathcal{L}^N \right],
\]

is true for \( 0 \leq N \leq 6 \), but false for \( N \geq 20 \).

1. INTRODUCTION AND DEFINITIONS

A function \( f \) is called completely monotonic (c.m.) on \( (0, \infty) \) if

\[
(-1)^k f^{(k)}(t) \geq 0, \quad \forall t > 0, \forall k \in \mathbb{N} := \{0, 1, \ldots\}, \tag{1.1}
\]

see Widder [1, Chap. 4; 2, Chap. 5]. We shall denote the class of functions satisfying (1.1) by \( \mathcal{L} \). The famous Bernstein–Widder theorem states that \( f \in \mathcal{L} \) if and only if (iff) \( f \) can be represented by a Laplace–Stieltjes integral with non-decreasing integrator function \( \chi \), see Theorem C in Sect. 3. Schoenberg [3] studied c.m. functions in connection with metric spaces.
For c.m. functions in relation to c.m. sequences and the moment problem of Hausdorff see, e.g., Widder [1] and Lorch and Newman [4]. For c.m. functions in relation to differential equations, see Lorch, Muldoon, and Szego [5] and the references quoted therein, and Mahajan and Ross [6].

If (1.1) holds only for \( k = 0, 1, \ldots, N \), we shall write \( f \in \mathcal{L}^N \). Functions in \( \mathcal{L}^N \) are called “monotonic of order \( N \)” in [4, 5], whereas in Fink [7] they are called “\((N - 1)\)-alternating monotonic functions” and abbreviated as \( \text{AM}(N - 1) \). Williamson [8] defined an “\( n \)-times monotonic function” \( f \) by

\[
( -1)^k f^{(k)}(t) \text{ is non-negative, non-increasing, and convex,}
\]

for \( t > 0 \), and for \( k = 0, 1, \ldots, n - 2 \ (n \geq 2) \),

(1.2)

see also Schoenberg [9], Royall [10], and Muldoon [11]. The class of such functions is slightly more general than the class \( \mathcal{L}^n \). For example, the function

\[
f(t) := 0, t \geq 1, \quad f(t) := (1 - t)^{n - 1}, 0 < t < 1 \ (n \geq 2),
\]

is “\( n \)-times monotonic,” but it does not belong to \( \mathcal{L}^n \). Williamson [8, Theorems 1 and 2] proved that \( f \) is “\( n \)-times monotonic” on \((0, \infty)\) iff it is representable in the form

\[
f(t) = \int_0^{1/t} (1 - ut)^{n - 1} d\gamma(u), \quad t > 0,
\]

where \( \gamma(u) \) is non-decreasing and bounded below, and that \( \gamma(u) \) is determined at its points of continuity if \( \gamma(0) = 0 \).

In this paper we shall study real, finite-valued functions that are defined and monotonic on an open interval \( I := (a, b) \), where \( -\infty < a < b \leq \infty \). The notation \( f > 0 \) will mean \( 0 < f(t) < \infty \), \( \forall t \in I \). It will be understood that all differentiations are with respect to the variable \( t \) (or \( x \)), where \( t \in I \), unless there is evidence to the contrary. We shall mostly suppress the dependence upon this underlying variable \( t \); the precise meaning will be clear from the context. Let us now first define the classes of functions \( \mathcal{L}^N(I) \) and \( \mathcal{R}^N(I) \).

**Definitions.** Let \( I \) be an open interval \((a, b)\), where \(-\infty < a < b \leq \infty\), and let us define the class of functions \( \mathcal{L}^N(I) \) for any \( N \in \mathbb{N}^\# := \mathbb{N} \cup \{\infty\} \) as (the letter \( \mathcal{L} \) stems from “Laplace,” z see below)

\[
f \in \mathcal{L}^N(I) \iff 0 \leq ( -1 )^k f^{(k)}(t) < \infty, \quad \forall t \in I, \forall k = 0, 1, \ldots, N.
\]
Further we define $\mathcal{L}_\alpha^N(I), \forall \alpha \in \mathbb{R}$, as follows: $\forall \alpha > 0$,

\[
f \in \mathcal{L}_\alpha^N(I) \iff [f^n \in \mathcal{L}^N(I), \text{ and } f \geq 0 \text{ on } I], \quad N \geq 0,
\]

\[
f \in \mathcal{L}_\alpha^{-N}(I) \iff [(f^{-a})' \in \mathcal{L}^{N-1}(I), \text{ and } f > 0 \text{ on } I], \quad N \geq 1;
\]

for $\alpha = 0$,

\[
f \in \mathcal{L}_0^N(I) \iff [- (f') \in \mathcal{L}^{N-1}(I), \text{ and } f > 0 \text{ on } I], \quad N \geq 1;
\]

and $\forall \alpha \geq 0$,

\[
f \in \mathcal{L}_\alpha^0(I) \iff f > 0 \quad \text{on } I.
\]

Hence, in particular, $\mathcal{L}_\alpha^N(I) = \mathcal{L}_\alpha^0(I)$. We shall assume that $N \in \mathbb{N}^*$ unless there is evidence to the contrary. As the value of $\alpha$ (finite) is unimportant in the following we may choose $\alpha = 0$, for example. However, there exists an essential difference between the cases $b < \infty$ and $b = \infty$.

We shall suppress the dependence upon $I$ when $I = (0, \infty)$ and write $\mathcal{L}_\alpha^N$ for $\mathcal{L}_\alpha^N((0, \infty))$. Further, we shall omit $N$ when $N = \infty$.

Further we define the class $\mathcal{K}^N(I)$ for $N \geq 1$ by

\[
f \in \mathcal{K}^N(I) \iff [f' \in \mathcal{L}^{N-1}(I), \text{ and } f \text{ is defined on } I].
\]

Again we shall suppress the dependence upon $I$ when $I = (0, \infty)$, and omit $N$ when $N = \infty$. Then the class $\mathcal{K}$ is similar to the class $T$ defined by Schoenberg [3, p. 825]: $\varphi \in T$ iff $\varphi$ is defined on $[0, \infty)$, $\varphi(0) = 0$, and

\[
\varphi(t) = \int_0^t \psi(x) \, dx, \quad t > 0,
\]

where the integral is convergent and $\psi$ is completely monotonic on $(0, \infty)$, so $\psi \in \mathcal{L}$. Note that $T \neq \mathcal{K}$. For example, if $f(t) = -t^{-1}, t > 0$, we have $f \in \mathcal{K}, f \notin T$. Obviously it is possible that $f(0+) = -\infty$ when $f \in \mathcal{K}$. If we require in addition that $f(0+) = 0$, then the associated subclass of $\mathcal{K}$ may actually be identified with $T$.

In Sections 2 and 3 we shall discuss relations, integral representations, and inequalities in connection with the classes $\mathcal{L}_\alpha^N(I)$ and $\mathcal{K}^N(I)$. Several of the theorems given here, in particular, Theorems A–F, are modifications of known theorems, or are (partly) known in the literature. In such cases we shall mostly give a different proof or omit the proof. In Theorem 2 we shall prove an interesting monotonicity property of the class $\mathcal{L}_\alpha^N(I)$, and in Theorem 4 we shall list inclusion relations for $\mathcal{L}_\alpha^N(I)$.

The main results of this paper are given in Theorems 2 and 4, and Theorems 8 and 9 in Section 4. The proofs of the latter are lengthy and...
not straightforward. In particular, Theorem 8, for the case $N = 5$, corrects a statement made in [4, Theorem 8].

2. SOME RELATIONS BETWEEN THE CLASSES $\mathcal{L}_N^N(I)$ AND $\mathcal{A}_N^N(I)$

The following theorem is given by [5, Lemma 2.1]. It is similar to [3, Theorem 8; 4, Theorem E*].

**Theorem A.** For $N \in \mathbb{N}^*$ we have

$$g \in \mathcal{A}_N^N(I), \quad f \in \mathcal{L}_N^N(g(I)) \Rightarrow f(g) \in \mathcal{L}_N^N(I),$$

and, in particular,

$$g > 0, \quad g \in \mathcal{A}_N, \quad f \in \mathcal{L}_N \Rightarrow f(g) \in \mathcal{L}_N.$$

**Proof.** This follows by induction, by applying Leibniz's rule, on writing $(f(g))^{n+1} = (f'(g)g^n)^n$, cf. [4].

The following theorem is an extension of [3, p. 833, Corollary 3].

**Theorem B.** For $N \in \mathbb{N}^*$ we have

$$g \in \mathcal{A}_N^N(I), \quad f \in \mathcal{L}_N^N(g(I)) \Rightarrow f(g) \in \mathcal{A}_N^N(I),$$

and, in particular,

$$g > 0, \quad g \in \mathcal{A}_N, \quad f \in \mathcal{L}_N \Rightarrow f(g) \in \mathcal{A}_N.$$

**Proof.** The proof is virtually identical to that of Theorem A. The only difference is that in this case the sign of $f^{(n)}$ is opposite to the sign of $f^{(n)}$ in Theorem A, for all $n \geq 1$.

The following theorem is similar to [3, Theorem 9], and to one direction of [4, Theorem E]. We shall give a different proof.

**Theorem 1.** For $N \in \mathbb{N}^*$ we have

$$f \in \mathcal{L}_N^N(I) \iff \left[ f > 0 \text{ on } I, \text{ and } \forall \alpha > 0: f^n \in \mathcal{L}_N^N(I) \right].$$

**Proof.** For $N = 0$ the proof is obvious, so let $N \geq 1$. Let us first assume that $f > 0$ and $\forall \alpha > 0: f^n \in \mathcal{L}_N^N(I)$. Then

$$\lim_{\alpha \downarrow 0} \frac{(-1)^k}{\alpha} \left[ f^n(t) \right]^{(k)} \geq 0, \quad \text{for } k = 1, 2, \ldots, N, \text{ and } t \in I,$$

hence $-(\ln f) \in \mathcal{L}_N^{N-1}(I)$, which yields $f \in \mathcal{L}_0^N(I)$. 

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392 H. VAN HAERINGEN
Second, let \( f \in \mathcal{L}_0^N(I) \). Then \( f > 0 \) and \( f(t) = e^{-g(t)} \), where
\[
g' \in \mathcal{L}_{N-1}^N(I), \quad \text{hence} \quad g \in \mathcal{N}^N(I).
\]
Now let \( f_1(x) := e^{-\alpha x}, \alpha > 0 \). Then \( f_1 \) is completely monotonic on \( \mathbb{R} \), so \( f_1 \in \mathcal{L}^N(g(I)) \), and thus we find from Theorem A that \( f_1(g) \in \mathcal{N}^N(I) \), which yields \( f^\alpha \in \mathcal{L}^N(I), \forall \alpha > 0 \), as desired.

**Theorem 2.** Let \( \alpha \geq 0 \), \( N \in \mathbb{N}^* \), then
\[
f \in \mathcal{L}_{-\alpha}^N(I) \Rightarrow f \in \mathcal{L}_{-\beta}^N(I), \quad \forall \beta \leq \alpha.
\]

**Proof.** If \( \alpha = 0 \), the proof follows immediately from Theorem 1. For \( N = 0 \) the proof is obvious. So let \( N > 0 \) and \( \alpha > 0 \), then we have \( f > 0 \), \( f' \leq 0 \), and
\[
(f^{-\alpha})' \in \mathcal{L}^{N-1}(I), \quad N \geq 1.
\]
Hence
\[
(-1)^k (f^{-\alpha})^{(k+1)} \geq 0, \quad 0 \leq k \leq N - 1, \text{ so } f^{-\alpha} \in \mathcal{N}^N(I).
\]
Let us now in Theorem B replace \( f \) by \( h \) and choose \( h(x) := x^\gamma, x > 0, 0 \leq \gamma \leq 1. \) Then clearly \( h(x) \in \mathcal{N}((0, \infty)) = \mathcal{N} \). Hence
\[
h(g) = g^\gamma \in \mathcal{N}^N(I), \quad \text{if } g > 0 \text{ and if } g \in \mathcal{N}^N(I).
\]
Choosing \( g := f^{-\alpha} \) we get \( f^{-\alpha \gamma} \in \mathcal{N}^N(I) \), which completes the proof for the case \( 0 \leq \beta = \alpha \gamma \leq \alpha \). Subsequently the proof for \( \beta < 0 \) follows from the case \( \beta = 0 \), by Theorem 1.

From the preceding theorem it follows that \( \mathcal{L}_a^N(I) \), where \( N \in \mathbb{N}^* \), is monotonic in \( \alpha \) on \( -\infty < \alpha \leq 0 \). Such a monotonicity does not hold on \( 0 < \alpha < \infty \), at least not for all \( N \). It turns out that there exists an important and interesting difference between the case \( b < \infty \) and the case \( b = \infty \) (assuming that \( I = (a, b) \)); see Theorems 4 and C below.

One easily verifies that the class \( \mathcal{L}_a^N(I) \) is invariant under addition and multiplication of its elements. For convenience we shall collect some invariance properties of \( \mathcal{L}_a^N(I) \) in Theorem 3, and some monotonicity properties in Theorem 4. Note that \( \mathcal{L}_a^{N+1}(I) \subset \mathcal{L}_a^N(I) \), and \( \mathcal{N}_a^{N+1}(I) \subset \mathcal{N}_a^N(I) \).

**Theorem 3.** We have, for \( N \in \mathbb{N}^* \),
(i) \( f, g \in \mathcal{L}^N(I) \Rightarrow fg, f + g \in \mathcal{L}^N(I) \),
(ii) \( f, g \in \mathcal{L}_0^N(I) \Rightarrow fg \in \mathcal{L}_0^N(I) \), if \( \alpha \geq 0 \).

On the other hand, the implications
(iii) \( f, g \in \mathcal{L}_a^N(I) \Rightarrow fg \geq \in \mathcal{L}_a^N(I) \), if \( \alpha < 0 \), and
(iv) \( f, g \in \mathcal{L}_0^N(I) \Rightarrow f + g \in \mathcal{L}_0^N(I) \),
are not valid, in general. Further,

(v) \( f, g \in \mathcal{A}^N(I) \Rightarrow f + g \in \mathcal{A}^N(I) \),
but, in general, \( f, g \in \mathcal{A}^N(I) \) does not imply that \( fg \in \mathcal{A}^N(I) \), and \( f \in \mathcal{L}^N(I) \),
\( g \in \mathcal{A}^N(I) \) does not imply that either \( fg \in \mathcal{L}^N(I) \) or \( fg \in \mathcal{A}^N(I) \).

Proof. Relations (i) and (v) are obvious, (ii) follows from Leibniz’s rule,
and the invalidness of (iii) from the counterexamples \( f(t) = g(t) = t^{-1} \),
and \( f(t) = g(t) = (1 - e^{-t})^{1/2} \), \( t > 0 \), \( \alpha < 0 \). In (iv), take \( f(t) = 1 \), \( g(t) = e^{-t} \), \( t > 0 \), and \( N = \infty \). Then \( 1 + e^{-t} \notin \mathcal{L}^0_0 \) because \( e^{-t}(1 + e^{-t})^{-1} \notin \mathcal{L} \).
Further, for \( N \geq 1 \), \( fg \notin \mathcal{A}^N \) if \( f(t) = g(t) = t \), and \( fg \notin \mathcal{L}^N \cap \mathcal{A}^N \) if \( f(t) = t^{-1/2} \), \( g(t) = 1 + t \).

Theorem 4. We have

(i) \( \mathcal{L}^N(I) \subset \mathcal{A}^N(I) \), if \( \alpha > 0 \), \( N \in \mathbb{N}^\# \), and \( n - 1 \in \mathbb{N} \),
whereas this relation does not hold, in general, if \( \alpha < 0 \), see (iii). Further,
(ii) \( f \in \mathcal{L}^N(I) \Rightarrow f^n \in \mathcal{L}^N(I) \), if \( N \in \mathbb{N}^\# \), \( n \in \mathbb{N} \),
(iii) \( \mathcal{L}^N(I) \subset \mathcal{L}^N(I) \), if \( \alpha \leq 0 \), \( \alpha \leq \beta \), and \( N \in \mathbb{N}^\# \).

Next,

(iv) \( \mathcal{L}^N(I) \subset \mathcal{L}^N(I) \), if \( 0 < \alpha \leq \beta \), and \( N = 0, 1, 2 \),
but this relation does not hold, in general, if \( N \geq 3 \), and
(v) \( \mathcal{L}^N(I) \subset \mathcal{A}^N(I) \), if \( 0 < \alpha \leq \beta \), and \( N = 1, \ldots, 5 \),
but this relation does not hold, in general, if \( N \geq 6 \). Finally,
(vi) \( \mathcal{L}^N(I) \subset \mathcal{A}^N(I) \), if \( 0 < \alpha \leq \beta \), and \( N = 1, \ldots, 6 \),
whereas this relation does not hold, in general, if \( N \geq 20 \).

Proof. Relations (i) and (ii) are obvious, (iii) follows from Theorem 2,
(iv) from Theorem 7, (v) from Theorem 8, and (vi) from Theorem 9.

Remark. Obviously, the class \( \mathcal{L}^N(I) \) plays a special role. From (iii) (or from Theorem 2) we get

\[
\mathcal{L}^N(I) \subset \mathcal{L}^N(I) \subset \mathcal{L}^N(I), \quad \forall \alpha > 0, N \in \mathbb{N}^\#.
\]

3. INTEGRAL REPRESENTATIONS AND INEQUALITIES

One of the most intriguing facts about functions in \( \mathcal{A} \) is that they can
be represented as a Laplace transform (this fact justifies the use of the
symbol \( \mathcal{L} \)), according to the famous Bernstein–Widder theorem [1, 2]:

Theorem C.

\[
f \in \mathcal{A} \Leftrightarrow f(t) = \int_{[0, \infty)} e^{-\lambda t} d\chi(\lambda), \quad t > 0,
\]

(3.1)
where the integral is convergent \( \forall t > 0 \) and where \( \chi(\lambda) \) is non-decreasing on \([0, \infty)\).

From Eq. (3.1) it follows that \( f(t) \) can be extended to a function \( f(t) \)
that is analytic in the half-plane \( \text{Re} \ t > 0 \), see Widder [1, 2]. Hence, as pointed out by Dubourdieu [12], one easily verifies that \( f \in \mathcal{D} \) implies that either \( f \) is identically constant on \([0, \infty)\), or

\[
(-1)^k f^{(k)}(t) > 0, \quad \forall k \geq 0, \forall t > 0. \quad (3.2)
\]

A simple proof may run as follows: Let \( g(t) := (-1)^n f^{(n)}(t) \), for any \( n \in \mathbb{N} \). Then clearly \( g \in \mathcal{D} \). Now if \( g \) has a zero at \( t_0 \) then \( g'(t) = 0, \forall t > t_0 \), because \( g \geq 0 \). But \( g(t) \) is analytic, so it must vanish identically. Hence \( f(t) \) must be a constant.

Alternatively, using Eq. (3.1) directly, if, for some \( k \in \mathbb{N} \) and for some \( t_0 > 0 \),

\[
(-1)^k f^{(k)}(t_0) = \int_{[0, \infty)} \lambda^k e^{-\lambda t_0} d\chi(\lambda) = 0,
\]

it follows that \( \chi(\lambda) \) is constant on \((0, \infty)\), which implies that \( f(t) \) is constant on \((0, \infty)\).

**Theorem 5.** Let \( h(t) \) be non-negative and non-increasing for \( t > a, a > 0 \), and assume that, for some real \( \nu \),

\[
\int_a^\infty t^\nu h(t) \, dt < \infty.
\]

Then

\[
\lim_{t \to \infty} t^{\nu+1} h(t) = 0.
\]

**Remark.** When \( t^\nu \) is replaced by a non-decreasing function \( g(t) \), \( g(t) \geq 0 \), the analogous statement is not necessarily true. Even for \( h(t) = e^{-t} \), there exists a non-decreasing function \( g(t) \geq 0 \) such that

\[
\int_a^\infty g(t) e^{-t} \, dt < \infty, \quad \limsup_{t \to \infty} t g(t) e^{-t} > 0.
\]

**Proof.** If \( \nu \leq 0 \), the function \( t^\nu h(t) \) is non-increasing, so that we have to prove the \( \nu = 0 \) case only. Therefore it will suffice to prove the theorem for the case \( \nu \geq 0 \). Let us suppose that \( \limsup_{t \to \infty} t^{\nu+1} h(t) \geq 2 \delta > 0 \). Then there exists a strictly increasing sequence \( \{t_k\} \), with \( \lim_{k \to \infty} t_k = \infty \), and such that, for some \( k_0 \),

\[
t_k^{\nu+1} h(t_k) \geq \delta, \quad \forall k \geq k_0.
\]
Then for \( k \geq k_0 \),

\[
\int_{t_k}^{t_{k+1}} t^\nu h(t) \, dt \geq h(t_{k+1}) \int_{t_k}^{t_{k+1}} t^\nu \, dt
\]

\[
\geq \frac{\delta}{t_{k+1}^{\nu+1}} \frac{1}{\nu + 1} (t_{k+1}^{\nu+1} - t_k^{\nu+1}) = \frac{\delta}{\nu + 1} (1 - y_k/y_{k+1}),
\]

where we put \( y_k := (t_k)^{\nu+1} \). Clearly the sequence \( (y_k) \) is strictly increasing and \( \lim_{k \to \infty} y_k = \infty \). Now we observe that

\[
\sum_{k=k_0}^{\infty} \frac{(y_{k+1}/y_k - 1)}{n} = \infty, \quad \therefore \sum_{k=k_0}^{\infty} \frac{1 - y_k/y_{k+1}}{n} = \infty,
\]

and hence \( \int_0^\infty t^\nu h(t) \, dt = \infty \), contrary to hypothesis, which completes the proof.

The representation (3.1) turns out to be very useful for deriving inequalities. Similarly, for functions in \( L^{N+1} \) there exists an integral representation from which inequalities can be derived. Part (i) of the following theorem is similar to [8, Lemma 1], whereas part (ii) is in fact contained in [7, Eq. (11)]. We shall give a different proof.

**Theorem 6.** Let \( f \in L^{N+1} \) and \( \lim_{t \to -\infty} f(t) = 0 \), then

\[
(i) \quad \lim_{t \to -\infty} t^k f^{(k)}(t) = 0, \quad \forall k, 0 \leq k \leq N, \quad (3.3)
\]

\[
(ii) \quad f(t) = \frac{1}{N!} \int_t^\infty (\lambda - t)^N d\varphi(\lambda), \quad \forall t > 0, \quad (3.4)
\]

where

\[
d\varphi(\lambda) := (-1)^{N+1} f^{(N+1)}(\lambda) \, d\lambda. \quad (3.5)
\]

**Proof.** Since \( 0 \leq (-1)^{N+1} f^{(N+1)}(t) < \infty, \forall t > 0 \), it follows that \( f^{(N)}(t) \) is monotonic and absolutely continuous on \((0, \infty)\), that \( f^{(N+1)}(t) \) is integrable on \([a, b]\), for \( 0 < a < b < \infty \), and that \( f^{(N)}(t) = -\int_t^b f^{(N+1)}(\lambda) \, d\lambda \), provided that \( \lim_{t \to \infty} f^{(N)}(t) = 0 \). Now the relation

\[
f(t) = f(b) + \sum_{k=1}^{N} \frac{(t - b)^k}{k!} f^{(k)}(b) + \int_b^t (\lambda - t)^N \frac{1}{N!} \, d\varphi(\lambda), \quad (3.6)
\]

which may be recognized as a Taylor expansion, follows easily by integration by parts. Clearly, (ii) follows immediately from (i), according to Eq.
(3.6) with \( b \to \infty \). The theorem is now easily proved by induction: Assume that both (i) and (ii) hold with \( N \) replaced by \( n - 1 \) (\( 1 \leq n \leq N \)). Then

\[
\int_t^\infty (\lambda - t)^{n-1} f^{(n)}(\lambda) \, d\lambda
\]

is convergent, so \( \lim_{t \to \infty} t^n f^{(n)}(t) = 0 \), according to the preceding Theorem 5. Consequently (i) and hence (ii), holds with \( N \) replaced by \( n \), which completes the proof by induction.

Fink [7, Theorems 1 and 2] derived inequalities for products of higher derivatives of \( f(t) \) by using the integral representations in Eqs. (3.1) and (3.4) and Muirhead's theorem, see Theorem F. We shall need these inequalities in particular for the product of \( n \) higher derivatives of \( f(t) \). For this special case we shall formulate these inequalities in Theorems D and E, and give direct proofs, without using Muirhead's theorem.

**Theorem D.** Let \( f \in \mathcal{L} \), \( n \geq k \geq m, k \geq n - k, \) and \( m \geq n - m \). Then

\[
(-1)^n f^{(k)} f^{(n-k)} \geq (-1)^n f^{(m)} f^{(n-m)}. \tag{3.7}
\]

**Proof.** Inserting Eq. (3.1) we get

\[
(-1)^n f^{(k)} f^{(n-k)}(t) - (-1)^n f^{(m)} f^{(n-m)}(t)
= \int \int \mu^k(x^k - x^m) e^{-(\lambda + \mu)x} \, d\lambda d\mu
= \frac{1}{2} \int \int \mu^k(x^k - x^m)(1 - x^{n-k-m}) e^{-(\lambda + \mu)x} \, d\lambda d\mu \geq 0,
\]

where \( x := \lambda/\mu \) and the integrals are over \([0, \infty)\). The last inequality follows because \( k \geq m \) and \( n - k - m \leq 0 \), and because the integrand of the second integral is symmetric in \( \lambda \) and \( \mu \).

**Theorem E.** Let \( f \in \mathcal{L}^{N+1} \), \( n \geq k \geq m, \ N \geq k \geq n - k, \) and \( m \geq n - m \). Then

\[
(-1)^n f^{(k)} f^{(n-k)} \geq A(-1)^n f^{(m)} f^{(n-m)}, \tag{3.8}
\]

where

\[
A := \frac{(N - m)!(N - n + m)!}{(N - k)!(N - n + k)!} \leq 1. \tag{3.9}
\]
Proof. Inserting Eq. (3.4) we get

\[
(-1)^n(N-k)!(N-n+k)! \left[ f^{(k)}(t)f^{(n-k)}(t) - Af^{(m)}(t)f^{(n-m)}(t) \right]
\]

\[
= \int_t e^\infty \int_t e^\infty (yz)^{-N} z^n(x^k - x^m) \phi(\lambda) \phi(\mu) dz d\lambda
\]

\[
= \frac{1}{2} \int_t e^\infty \int_t e^\infty (yz)^{-N} z^n(x^k - x^m)(1 - x^{n-k-m}) \phi(\lambda) \phi(\mu) \geq 0,
\]

(3.10)

where \( y := (\lambda - t)^{-1}, \ z := (\mu - t)^{-1}, \) and \( x := y/z. \) The last inequality follows because \( k \geq m \) and \( n - k - m \leq 0, \) and because the integrand of the second integral is symmetric in \( \lambda \) and \( \mu. \) Further, \( A \leq 1 \) follows from the convexity of \( \ln \Gamma(x), \ x > 0. \)

Remark 1. If \( f(\infty) > 0, \) one may apply the proof to \( f_1(t) := f(t) - f(\infty). \) Putting \( k = n, \) we get for \( N \geq n > m \geq n - m > 0, \)

\[
(-1)^n f^{(n)} \geq (-1)^n f^{(n)} f_1 \geq (-1)^n Af^{(m)} f^{(n-m)}.
\]

Therefore (3.8) is valid also when \( f(\infty) > 0, \) although in the proof we have assumed that \( f(\infty) = 0 \) (see Theorem 6).

Remark 2. Equality holds in Eq. (3.7) for \( f(t) := e^{-ct}, \) and in Eq. (3.8) for

\[
f(t) := (c - t)^N, \ 0 < t < c, \quad f(t) := 0, \ t \geq c, \quad (3.11)
\]

where \( c > 0. \) Strictly speaking, the latter function does not belong even to \( \mathcal{L}^N, \) because \( f^{(N)}(c) \) is not well-defined, since the left derivative differs from the right derivative at \( t = c. \) However, by applying a slight, arbitrarily small, deformation at \( t = c, \) one can construct a function \( g, \) "arbitrarily close to" \( f, \) with the desired property that \( g \in \mathcal{L}^{N+1}. \) It is not possible to find such a function \( g \) satisfying \( (1)^{N+2} g^{(N+2)}(t) \geq 0, \ \forall t. \)

One easily verifies that the function \( f \) defined by Eq. (3.11) is "(N + 1)-times monotonic" (see Eq. (1.2)) in the sense of Williamson [8].

Definition. Let the \( r \)-dimensional vectors \( (\alpha) \) and \( (\beta) \) satisfy

\[
\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 0, \quad \beta_1 \geq \beta_2 \geq \cdots \geq \beta_r \geq 0,
\]

\[
\sum_{i=1}^k \alpha_i \geq \sum_{i=1}^k \beta_i, \quad \text{for all} \ k = 1, 2, \ldots, r - 1,
\]
and
\[
\sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r} \beta_i. \tag{3.12}
\]
Then \((\beta)\) is majorized by \((\alpha)\), and we write \((\alpha) \succ (\beta)\), or \((\beta) \prec (\alpha)\). For example, \((2, 0) \succ (1, 1)\). The formulation of this definition in [7, p. 252] is not accurate, as it would yield incorrectly that \((0, 2) \prec (1, 1)\).

The following theorem is proved by Fink [7, Theorem 2].

**Theorem F.** Let \(\alpha, \beta\) be integers such that \(0 \leq \alpha_i, \beta_i \leq N\) for \(1 \leq i \leq r\), and suppose that \((\alpha) \succ (\beta)\). Let \(f \in \mathcal{L}^{N+1}\), then
\[
\left| \prod_{i=1}^{r} f^{(\alpha_i)}(t) \right| \geq K_N \left| \prod_{i=1}^{r} f^{(\beta_i)}(t) \right|, \quad \forall t > 0, \tag{3.13}
\]
where
\[
K_N := \prod_{i=1}^{r} \left( \frac{(N - \beta_i)!}{(N - \alpha_i)!} \right).
\]
Further
\[
0 < K_N \leq 1 = \lim_{N \to \infty} K_N. \tag{3.14}
\]

This theorem remains valid for \(N \to \infty\). In the case \(r = 2\) we retrieve Theorem E.

**4. MORE INCLUSION RELATIONS**

**Theorem 7.** Let \(I := (a, b)\) where \(-\infty < a < b < \infty\). Then the statement
\[
f \in \mathcal{L}^{N}(I) \Rightarrow \left[ \forall \alpha > 1: f^\alpha \in \mathcal{L}^{N}(I) \right], \tag{4.1}
\]
is true for \(N = 0, 1,\) and 2, but false for \(N \geq 3\).

**Proof.** (i) Suppose first that \(f(t) > 0, \forall t \in I\). Since
\[
(f^\alpha)^\prime = \alpha f^{\alpha-2} [f^{\prime 2} + (\alpha - 1)(f^\prime)^2],
\]
the proof for \(N \leq 2\) is easy. A counterexample suffices to give the proof for \(N \geq 3\). Let \(f(t) := 1 - t, 0 < t < 1\). Then \(f \in \mathcal{L}^{N}((0, 1)), \forall N \in \mathbb{N}\). But
\[
-[f^\alpha(t)]^\prime = \alpha(\alpha - 1)(\alpha - 2)(1 - t)^{\alpha-3} < 0, \quad \text{if } 1 < \alpha < 2,
\]
so
\[ f \notin \mathcal{L}_N^\alpha((0,1)), \quad \text{if } 1 < \alpha < 2 \text{ and } N \geq 3. \] (4.2)

(ii) Let us next suppose that \( f(t) > 0 \), for \( t < c \), and \( f(c) = 0 \), for some \( c \in (a, b) \). Let \( N = 2 \), then we have to prove that \( h'(c) = h''(c) = 0 \), where we put for notational convenience \( h(t) := [f(t)]^\alpha, \ \alpha > 1 \). Clearly, \( f(t) = f'(t) = f''(t), \ \forall t \in [c, b) \). Let \( f'_L \) and \( f'_R \) denote the left and right derivative of \( f \), respectively. Then
\[ 0 = f''_L(c) = f''_R(c) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} [f'(c) - f'(c - \varepsilon)], \]
so
\[ \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} f'(c - \varepsilon) = 0, \quad \text{and similarly} \quad \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} f(c - \varepsilon) = 0. \]
Now \( h'(t) = \alpha f'(t)[f(t)]^{\alpha - 1} \), if \( f(t) > 0 \), so we get
\[ h'_L(c) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} [h(c) - h(c - \varepsilon)] = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} [f(c - \varepsilon)]^\alpha = 0, \]
and similarly,
\[ h'_R(c) = \lim_{\varepsilon \downarrow 0} \frac{-\alpha}{\varepsilon} f'(c - \varepsilon)[f(c - \varepsilon)]^{\alpha - 1} = 0. \]
Hence, \( \forall \alpha > 1: h'(c) = h''(c) = 0 \), which completes the proof.

**Theorem 8.** The statement
\[ f \in \mathcal{L}^N \Rightarrow \exists \alpha > 1: f^\alpha \in \mathcal{L}^N, \] (4.3)
is true for \( N = 0, 1, \ldots, 5 \), but false for \( N \geq 6 \).

**Remark 1.** For \( N \leq 4 \), this theorem is given in [4, Theorem 8], for \( N = 4 \) without proof. Moreover, the above statement is claimed to be false for \( N = 5 \) in [4], by means of the following alleged counterexample:
\[ f(t) := 1 + 2(1 - t)^4, \ 0 < t < 1, \quad f(t) := 1, \ t \geq 1, \] (4.4a)
and it is claimed that \( F^{(5)}(0+) > 0 \), where \( F \) is defined by \( F(t) := [f(t)]^{3/2} \). However, we find that
\[ F^{(5)}(0+) = -2^5 \cdot 3^{-3/2} \cdot 5 \cdot 67 < 0. \] (4.4b)
More generally, if
\[ f(t) := 1 + b(1 - t)^4, \quad 0 < t < 1, \quad f(t) := 1, \quad t \geq 1, \] (4.5a)
then
\[ F^{(b)}(0+) = -360b^3(2b^3 + 7b^2 + 8b + 7)(1 + b)^{-7/2}. \] (4.5b)

**Remark 2.** The implication (4.3) means that \( T^N \subset T^N \), \( \forall N \geq 1, N \leq 5 \) (see Section 1). On the other hand, if \( \alpha < 1 \) we have \( T^N \not\subset T^N \), even for \( N = 1 \). To prove this, let \( \beta > 0 \),
\[ f(t) := (1 - t)\beta, \quad \text{for} \quad 0 < t < 1, \quad \text{and} \quad f(t) := 0, \quad \text{for} \quad t \geq 1. \] (4.6)
Then \( f(1) \) exists (and equals 0) and \( f \in L^1 \), iff \( \beta > 1 \). Hence the derivative of \( [f(t)]^\alpha \) at \( t = 1 \) exists iff \( \alpha \beta > 1 \). So if \( 0 < \alpha < 1 \), take \( \beta = 1/\alpha \), then \( f^\alpha \in L^1 \).

**Proof of Theorem 8.** The proof is trivial if \( N = 0 \) or if \( f \) vanishes identically. Let \( N \geq 1 \) and let us suppose that \( f(c) = 0 \), where \( c > 0 \), and that \( I := (0, c) \) is the support of \( f \) where \( f(t) > 0 \). The proof on this support will be given below. Naturally, that proof includes the case when \( I = (0, \infty) \).

We begin by considering the interval \( t \in [c, \infty) \). In particular, the derivatives at \( t = c \) have to be investigated separately. (This point is not mentioned in [4].) Since \( f'(t) \leq 0, \ \forall t > 0 \), we get \( f(t) = 0, \ \forall t > c \). By equating the right derivative \( f'_r \) with the left derivative \( f'_l \) at \( t = c \), we obtain
\[ f'(c) = f'_r(c) = 0 = f'_l(c) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}[f(c) - f(c - \varepsilon)]. \]

Since \( f^{(n)}(x) \) (finite) exists \( \forall x > 0, n \leq N \), we find by induction
\[ f^{(n)}(c) = 0 = -\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}f^{(n-1)}(c - \varepsilon), \quad n = 1, 2, \ldots, N. \] (4.7)

Let us put \( h(t) := [f(t)]^\alpha \) for notational convenience. Clearly \( h^{(n)}(t) = 0, \ \forall t > c, n \leq N \), even for \( \alpha > 0 \). Hence, by induction
\[ h^{(n)}(c) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left[h^{(n-1)}(c + \varepsilon) - h^{(n-1)}(c)\right] = 0, \quad n = 1, 2, \ldots, N. \]

Therefore we have to prove that, for any \( \alpha > 1 \),
\[ h^{(n)}(c) = 0, \quad n = 1, 2, \ldots, N. \]
For the case $N = 2$, this has been proved in an elementary way in the proof of Theorem 7. Somewhat surprisingly, this proof cannot be easily extended to the case $N > 2$.

We shall use the formula (cf. Eq. (4.14))

$$h^{(n)} = \alpha f^{n-1} \sum_{k=1}^{n} (-1)^{k-1} (1 - \alpha)^{k-1} f^{1-k} g_{n,k}, \quad n \geq 1, \quad (4.8)$$

which holds when $f(t) > 0$. Here $g_{n,k}$ consists of a finite sum of terms, each of which contains the product of precisely $k$ derivatives of $f$:

$$\prod_{i=1}^{k} f^{(n_i)}, \quad \text{where } n_i \in \{1, 2, \ldots, n\} \text{ and } \sum_{i=1}^{k} n_i = n.$$ 

Now let the $n_i$ be arranged in descending order of magnitude: $n_i \geq n_{i+1}$. Then we obtain $(n_1, n_2, \ldots, n_k) < (n, 0, \ldots, 0)$, and from Theorem H,

$$(-1)^n \prod_{i=1}^{k} f^{(n_i)} \leq (-1)^n f^{(n)} f^{(n-1)} \prod_{i=1}^{k} \frac{1}{(n - n_i)!}, \quad (4.9)$$

which holds for all $n \leq N - 1$, since $f \in \mathcal{C}^N$. Now let $t = c - \varepsilon$, then we get from (4.9), $\forall k, 1 \leq k \leq n$, and for $n \leq N - 1$, 

$$f^{1-k} |g_{n,k}| \leq \text{const.} |f^{(n)}(c) - \varepsilon|. \quad (4.10)$$

The right-hand side is $o(\varepsilon)$ for $\varepsilon \downarrow 0$, according to Eq. (4.7). Since $\lim_{\varepsilon \downarrow 0} f^{n-1} = 0$, for any $\alpha > 1$, we obtain from (4.7), (4.8), and (4.10),

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} h^{(n)}(c - \varepsilon) = 0, \quad \forall n \leq N - 1.$$ 

Hence, by induction,

$$h^{(n)}(c) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} [h^{(n-1)}(c) - h^{(n-1)}(c - \varepsilon)] = 0, \quad n = 1, 2, \ldots, N, \quad (4.11)$$

which completes the proof for the derivatives at $t = c$.

In connection with this proof, we note that $h^{(n)}(c) = 0$, for $\alpha > 1$, does not follow from $f^{(n)}(c) = 0$, $\forall n \leq N$, if $f \notin \mathcal{C}^N$. For example, consider the function $f(t) := t^\beta \sin(1/t)^2$, $t > 0$, $f(0) := 0$. 

In the following we shall assume that $f(t) > 0$, $\forall t \in I$, and we shall restrict $t$ to the open interval $I$ in most formulas. Let us define the function $D_n(\alpha)$ for $\alpha \in \mathbb{R}$ (for $\alpha = 0$ by continuity) by

$$D_n(\alpha) := \frac{-1}{\alpha} (f^{-\alpha})^{(n)}, \quad n = 0, 1, \ldots, N, \quad (4.12)$$

where the differentiations are with respect to the underlying variable $t$, which we shall mostly suppress. Then $f^\alpha \in \mathbb{R}^N$, $\forall \alpha > 1$, comes down to

$$( -1 )^n D_n(\alpha) \geq 0, \quad \forall \alpha < -1, \forall n, 0 \leq n \leq N. \quad (4.13)$$

It turns out that

$$D_n(\alpha) = \sum_{k=1}^{n} (-1)^{k-1}(\alpha + 1)_{k-1} f^{-\alpha-k} g_{n,k}, \quad n \geq 1, \quad (4.14)$$

where the $g_{n,k}$ are functions of the derivatives of $f$, but are independent of $\alpha$. In particular,

$$g_{1,1} = f', \quad g_{2,1} = f'', \quad \text{and} \quad g_{2,2} = (f')^2.$$ 

Defining

$$g_{n,0} := g_{n,n+1} := 0, \quad n \geq 1,$$

and using $(d/dt)D_n(\alpha) = D_{n+1}(\alpha)$, we obtain from Eq. (4.14) the recurrence relation

$$g_{n+1,k} = g'_{n,k} + f' g_{n,k-1}, \quad 1 \leq k \leq n + 1, n \geq 1. \quad (4.15)$$

Further we find that

$$f^{\alpha+n} D_n(\alpha) = f^{n-1} f^{(n)} - (\alpha + 1) U_n(\alpha), \quad n \geq 2, \quad (4.16a)$$

where $U_n(\alpha)$ is a polynomial in $\alpha$, of degree $n - 2$,

$$U_n(\alpha) = \sum_{k=2}^{n} (-1)^{k}(\alpha + 2)_{k-2} f^{-\alpha-k} g_{n,k}, \quad n \geq 2. \quad (4.16b)$$

Now we are going to prove that (4.3) is false for $N = 6$, by means of a counterexample. To this end, we define

$$f(t) := 1 + b(1 - t)^5, \quad \text{for} \ 0 < t < 1, b > 0,$$

$$f(t) := 1, \quad \text{for} \ t \geq 1. \quad (4.17)$$
After performing a slight deformation at \( t = 1 \) (see Remark 2 after Theorem E, Eq. (3.11)), we get \( f \in \mathcal{Z} \). Further we find in a straightforward way

\[
\lim_{t \downarrow 0} \lim_{\alpha \to -1} -\frac{1}{\alpha + 1} f^{\alpha + 6} D_6(\alpha) = 600b^2(4b^3 + 131b^2 - 371b + 126). \tag{4.18}
\]

This expression is negative for \( b = 1 \). Hence \( D_6(\alpha) \) is negative for \( \alpha < -1 \), \( \alpha \) close to \(-1\), \( t \approx 0 \). In view of (4.13), this proves that (4.3) is false for \( N = 6 \).

The complexity of the proof of (4.3) for \( N = 0, 1, \ldots, 5 \), is strongly increasing with \( N \). For \( N \leq 2 \) the proof is trivial since \( f^{n+2}D_2(\alpha) = ff'' - (\alpha + 1)(f')^2 \). For \( N = 3 \) it comes down to showing that \( 3ff'' \geq (\alpha + 2)(f')^2 \), \( \alpha < -1 \), which follows because \( ff'' \geq (1/2)(f')^2 \), according to (3.8). Now let \( N = 4 \), then it will suffice to prove that

\[
f^2\left[4ff'' + 3(ff'')^2\right] - 6(\alpha + 2)ff''(f')^2 + (\alpha + 2)ff''(f')^4 \geq 0, \\
\alpha < -1, \tag{4.19}
\]

where \((a)_b := \Gamma(a + b)/\Gamma(a)\). Taking \( N = 3 \) (note in (3.8) we get

\[
ff'' \geq \frac{3}{2}(f')^2, \quad ff'' \geq \frac{1}{2}(f')^2, \tag{4.20}
\]

so it will suffice to prove that

\[
5x^2 - 6(\alpha + 2)x + (\alpha + 2)(\alpha + 3) \geq 0, \quad \text{for } \alpha < -1 \text{ and } x \geq \frac{3}{2},
\]

where \( x := ff''(f')^{-2} \) (if \( f' \neq 0 \)). This inequality can be rewritten as

\[
\frac{1}{9}\left[5(3x - 2)^2 + 2(3x - 2) + 2\right] + (\alpha + 1)(\alpha + 4 - 6x) \geq 0, \tag{4.21}
\]

from which the proof follows.

For \( N = 5 \) we have to prove that \( U_5(\alpha) \leq 0, \forall \alpha < -1 \), where

\[
U_5(\alpha) = 5f^2\left[f^{(3)f'} + 2ff''f'ight] - 5(\alpha + 2)f^2\left[2ff''(f')^2 + 3(f'f'')^2\right] \\
+ 10(\alpha + 2)ff''(f')^3 - (\alpha + 2)(f')^5. \tag{4.22}
\]

Now

\[
U_5(-1) - U_5(\alpha) = 5(\alpha + 1)fV_5(\alpha), \tag{4.23}
\]
where
\[ V_\varepsilon(\alpha) := f^2 \left[ 2 f'' f' + 3 (f')^2 \right] - 2 (\alpha + 4) f'' (f')^2 + \frac{1}{2} (\alpha^2 + 8 \alpha + 18) (f')^4. \]

We shall first prove that \( V_\varepsilon(\alpha) \geq 0 \), \( \forall \alpha \leq -1 \), and second that \( U_\varepsilon(-1) \leq 0 \). From (3.8) we get \( f'^n \geq (3/4) (f')^2 \) and \( f'' f' \geq (2/3) (f')^2 \). Inserting the latter, and putting \( x := f'' (f')^{-2} \), we find that \( V_\varepsilon(\alpha) \geq 0 \) will follow from the inequality
\[ \frac{13}{2} x^2 - 2 (\alpha + 4) x + \frac{1}{2} (\alpha^2 + 8 \alpha + 18) \geq 0, \quad \forall \alpha \leq -1, \forall x \geq \frac{2}{3}. \]  
(4.24)

The left-hand side can be rewritten as
\[ \frac{13}{2} (x - \frac{2}{3})^2 - \frac{1}{2} (4 \alpha + 3) (x - \frac{2}{3}) + \frac{1}{2} (\alpha + \frac{2}{3})^2 + \frac{1}{27}, \]
which is clearly positive.

Now we come to the more complicated part of the proof of Theorem 8, i.e., the proof that \( U_\varepsilon(-1) \leq 0 \). Let us put \( P_5 := f'' - (3/4) (f')^2 \). From (3.8) we obtain \( P_5 \geq 0 \) and further
\[ f'' f' \geq \frac{1}{2} f'' f', \quad f'' f' \geq \frac{3}{2} (f')^2, \]
\[ f'' f' \leq \frac{1}{2} f'' f', \quad f'' f' \leq \frac{1}{2} f'' f'. \]  
(4.25)

In order to be able to use these inequalities adequately, we find that it is convenient to multiply \( U_\varepsilon(-1) \) by \( f' \). After some tedious manipulations we derive from Eq. (4.22) the identity
\[ \varepsilon f' U_\varepsilon(-1) = f^2 f' Y + \frac{1}{2} f^2 P_5 \left[ f'' f' - \frac{3}{2} (f')^2 \right] \]
\[ + \frac{1}{2} P_5 \left[ P_5 - \frac{3}{2} (f')^2 \right]^2 + \frac{1}{2} (P_5)^3 + \frac{3}{16} (f')^6, \]  
(4.26)

where \( Y \) is defined by
\[ Y := f \left[ f^{(4)} f' - \frac{1}{2} f'' f' \right] - \frac{1}{2} (f') \left[ f'' f' - \frac{3}{2} (f')^2 \right]. \]  
(4.27)

In view of (4.25) and (4.26) it will suffice to prove that \( Y \leq 0 \). The first term on the right-hand side of Eq. (4.27) is non-positive because of (4.25) and the fact that \( f(t) \geq 0, \forall t \). Clearly we may assume that \( \lim_{t \to -\infty} f(t) = 0 \).
Then we may apply Theorem 6 with $N = 4$ (note). Inserting Eq. (3.4), where $N = 4$ and $d\varphi(\lambda) = -f^{(5)}(\lambda)\,d\lambda$, into Eq. (4.27), we obtain

$$-144Y = \int_0^\infty \int_0^\infty \int_0^\infty d\varphi(\lambda_1)\,d\varphi(\lambda_2)\,d\varphi(\lambda_3)$$

$$\times \left[ (\lambda_1 - t)^4(\lambda_2 - t)^3 - (\lambda_1 - t)^4(\lambda_3 - t)^2 \right]$$

$$-(\lambda_1 - t)^3(\lambda_2 - t)^3(\lambda_3 - t) + (\lambda_1 - t)^3(\lambda_2 - t)^2(\lambda_3 - t)^2 \right]$$

$$= \int_0^\infty \int_0^\infty Q(q, r, s)\,d\varphi(q + t)\,d\varphi(r + t)\,d\varphi(s + t), \quad (4.28)$$

where $q := \lambda_1 - t$, $r := \lambda_2 - t$, $s := \lambda_3 - t$, and

$$Q(q, r, s) := q^4r^3 - q^4r^2s - q^3r^3s + q^3r^2s^2.$$  

For convenience we put $\lambda := s^{-1}$, $\mu := r^{-1}$, $\nu := q^{-1}$, and get

$$Q(q, r, s) = (\lambda \mu \nu)^{-4}R(\lambda, \mu, \nu),$$

where

$$R(\lambda, \mu, \nu) := \lambda \mu^2 - \lambda^2 \mu \nu + \lambda^2 \mu^2 \nu. \quad (4.29)$$

Then

$$R(\lambda, \mu, \nu) + R(\mu, \lambda, \nu) = \lambda \mu (\lambda - \mu)^2 (\lambda + \mu - \nu). \quad (4.30)$$

Now it will suffice to prove that the analog of $R(\lambda, \mu, \nu)$ that is completely symmetrized in the triple $(\lambda, \mu, \nu)$, is non-negative, $\forall \lambda, \mu, \nu \geq 0$, which is relatively easy; details may be found in [13]. This completes the proof of Theorem 8.

**Theorem 9.** The implication

$$f \in \mathcal{L} \Rightarrow \big[ \forall \alpha > 1 : f^\alpha \in \mathcal{L}^N \big], \quad (4.31)$$

is true for $N = 0, 1, \ldots, 6$, but false for $N \geq 20$.

**Remark.** This implication, for $N = 5$, is given in [4, Theorem 9], without proof.
Proof. In view of Theorem 8, it will suffice to prove that (4.31) is true for \( N = 6 \) and false for \( N = 20 \). Nevertheless we first give a short proof of (4.31) for \( N = 5 \). From Eq. (4.22) we obtain

\[
\frac{1}{2}U_5(-1) = f(\int f'' - f'f'')\left[3ff'' - 2(f')^2\right]
+ \frac{1}{6}(f^{(4)}f' - f'''f') + 2ff''(f')^3 - \frac{2}{5}(f')^5 \leq 0, \tag{4.32}
\]

where the inequality follows from (3.7). Thus the proof is completed by using (4.23) and (4.24).

The proof of (4.31) for \( N = 6 \) is considerably more complicated. Following essentially the same approach as in the proof of Theorem 8, we are able to complete this proof using inequalities obtained in this paper. For details the reader is referred to [13]. One easily verifies (by hand) that (4.31) is false for \( N = 20 \), choosing \( f(t) = 1 + 10^{-2}e^{-t} \) and \( \alpha = 3/2 \).

In [14] we introduce new techniques and derive some new inequalities involving c.m. functions which enable us to prove (4.31) for \( N = 7 \); computer calculations show that (4.31) is false for \( N = 13 \).

**References**

