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Buckling of a Nonlinear Elastic Rod

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The buckling of a pin-ended slender rod subjected to a horizontal end load is formulated as a nonlinear boundary value problem. The rod material is taken to be governed by constitutive laws which are nonlinear with respect to both bending and compression. The nonlinear boundary value problem is converted to a suitable integral equation to allow the application of bounded operator methods. By treating the integral equation as a bifurcation problem, the branch points (critical values of load) are determined and the existence and form of nontrivial solutions (buckled states) in the neighborhood of the branch points is established. The integral equation also affords a direct attack upon the question of uniqueness of the trivial solution (unbuckled state). It is shown that, under certain conditions on the material properties, only the trivial solution is possible for restricted values of the load. One set of conditions gives uniqueness up to the first branch point.

1. INTRODUCTION

In this paper we consider the buckling of a pin-ended slender rod which is subjected to an axial end load. The rod is taken to be originally straight, and its material behavior is characterized by constitutive laws which are nonlinear with respect to both bending and compression. Two aspects of this problem will be our principal concern. First, we shall establish the existence of buckled configurations in some neighborhood of the unbuckled state for values of the load near certain critical values. Our proof is constructive and yields some explicit information about the character of these configurations. Second, we shall examine some conditions involving the constitutive laws, which will provide that only the unbuckled state can exist for certain restricted

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values of the load. One set of conditions will even preclude buckling up to the first critical value of the load.

The mathematical formulation of the physical problem is obtained by considering the equilibrium conditions, constitutive laws, and geometric relations from the theory of slender rods. Under our assumed form of the constitutive laws, the problem can be posed as a nonlinear Sturm–Liouville problem for the tangent angle of the rod. Since there are advantages in dealing with bounded operators, we devote some effort toward converting this boundary value problem into a suitable integral equation. We then analyze the integral equation as a bifurcation problem and determine the branch-points (critical values of the load), as well as the nature of the nontrivial solutions (buckled states) near these branch points. Our integral equation also affords a direct attack upon the question of uniqueness of the trivial solution (unbuckled state). This leads to our results on sufficient conditions to preclude buckling.

The problem of interest here was considered in some detail by Greenberg [4]. However, his governing equations are developed from certain assumptions about the constitutive laws which are different from ours. For his equations, he has located the branchpoints and proven the existence of buckled states. Shortly after the publication of that work, there appeared a paper by Antman [1] which covered a general statement of the constitutive equations for plane-extensible elasticae. This, as well as related work by Antman [2] on global existence of equilibrium states of rods, has made it clear that his characterization of the problem is the appropriate one. More recently, Stakgold [13] has pointed out that the constitutive assumptions of Greenberg are not in accord with those of Antman. Moreover, he has proposed a different form of the constitutive laws which is compatible with the theory of Antman. It is our intention to explore the model suggested by Stakgold in some depth, so to obtain a correction and extension of the work initiated by Greenberg. Some preliminary results about the uniqueness problem have been reported by Olmstead [7].

2. Formulation of the Problem

We shall summarize the essential arguments from [1, 2, 4, 13] which are required for the proper formulation of our problem. Some compromise on notation is unavoidable, and we will strive to point out the important differences.

Consider a pin-ended slender rod subjected to a horizontal end load of magnitude P. The rod has an undeformed length ℓ. A given material point of the rod with coordinates $x = X$ and $y = 0$ for $0 \leqslant X \leqslant \ell$ in the undeformed state is located in the deformed configuration by $x = U(X) + X$ and $y = V(X)$, where $U(0) = V(0) = V(\ell) = 0$. The displacements $U(X)$ and $V(X)$ are geometrically related to the arc length $s(X)$ and the tangent angle $\varphi(X)$ of the deformed rod by the equations

$$U'(X) = s'(X) \cos \varphi(X) - 1 \tag{2.1}$$

and

$$V'(X) = s'(X) \sin \varphi(X). \tag{2.2}$$

A prime will always denote differentiation with respect to the independent variable indicated.

The strain measures δ (extension) and μ (bending) are defined as[1]

$$\delta = s'(X) \tag{2.3}$$

and

$$\mu = -\varphi'(X). \tag{2.4}$$

We follow Antman [1] in introducing a strain energy function

$$W = W(\mu, \delta, X).$$

He shows that the bending moment M and the axial force N, taken as positive in compression, depend upon W through the equations

$$M = \frac{\partial W}{\partial \mu} \tag{2.5}$$

and

$$-N = \frac{\partial W}{\partial \delta}. \tag{2.6}$$

Moreover he indicates in [2] the importance of requiring that the Hessian matrix

$$\begin{pmatrix} \partial^2 W/\partial \mu^2 & \partial^2 W/\partial \mu \, \partial \delta \\ \partial^2 W/\partial \delta \, \partial \mu & \partial^2 W/\partial \delta^2 \end{pmatrix} \tag{2.7}$$

be positive definite.

From the equilibrium equations for the moments and forces in the rod, we have that[2]

$$M = PV \tag{2.8}$$

and

$$N = P \cos \varphi. \tag{2.9}$$

[1] This definition of δ from [13] has the property of always being positive; it differs by unity from that used in [1]. Our δ corresponds to γ in [4].

[2] Here, the positive sense of N is the same as in [13] except that (2.6) is not given correctly there. Our positive sense is opposite to that used in [1]. Also, P corresponds to $-P_0$ in [4].

In order to obtain some explicit information about the buckling problem, it becomes necessary to make some specific constitutive assumptions. We follow Stakgold [13] in assuming that

$$M = \hat{M}(\mu) \tag{2.10}$$

and

$$N = \hat{N}(\delta). \tag{2.11}$$

These functions are smooth and invertible, so that

$$\mu = \hat{\mu}(M) \tag{2.12}$$

and

$$\delta = \hat{\delta}(N). \tag{2.13}$$

Moreover, these functions are endowed with some additional physically reasonable properties. We take $\hat{M}(\mu) \in C^3[-\mu_0, \mu_0]$, for some appropriate $\mu_0 > 0$, and

$$\hat{M}(0) = 0; \qquad \hat{M}(-\mu) = -\hat{M}(\mu); \qquad \hat{M}'(\mu) > 0; \qquad -\mu_0 \leqslant \mu \leqslant \mu_0 . \tag{2.14}$$

Furthermore we take $\hat{\delta}(N) \in C^2[-P, P]$ and

$$\hat{\delta}(0) = 1; \qquad \hat{\delta}(N) > 0, \qquad \hat{\delta}'(N) \leqslant 0, \qquad -P \leqslant N \leqslant P, \qquad P > 0. \tag{2.15}$$

The classical problem of the elastica corresponds to a material where $\hat{\delta}(N) \equiv 1$ and $\hat{M}(\mu) \equiv \hat{M}'(0)\,\mu$. A thorough treatment of that problem has been given by Love [6].

We hasten to add that these constitutive assumptions do satisfy the requirement that (2.7) be positive definite. This is readily seen by considering

$$\begin{gathered} \frac{\partial^2 W}{\partial \mu^2} = \hat{M}'(\mu) > 0, \\ \frac{\partial^2 W}{\partial \delta^2} = -[\hat{\delta}'(N)]^{-1} \geqslant 0, \qquad \text{and} \qquad \frac{\partial^2 W}{\partial \delta\, \partial \mu} = \frac{\partial^2 W}{\partial \mu\, \partial \delta} = 0. \end{gathered} \tag{2.16}$$

At the point we can make clear the difference between the constitutive assumptions of Greenberg and those used here. In [4], the assumptions are that $M = \hat{M}(\mu/\delta)$ and $N = \hat{N}(\delta)$ in place of (2.10) and (2.11). The consequence of this is that it precludes a variational formulation of the problem in terms of W; in particular, we find that $\partial^2 W/(\partial \mu\, \partial \delta) \neq \partial^2 W/(\partial \delta\, \partial \mu) = 0$, so that (2.7) is not positive definite.

To proceed with the study of the buckling phenomena, it is desirable to combine some of the given relations in order to obtain an appropriate boundary value problem. It is easily seen that φ and V must satisfy a pair of first-order differential equations:

$$-\varphi'(X) = \hat{\mu}[PV(X)] \tag{2.17}$$

and

$$V'(X) = \hat{\delta}[P \cos \varphi(X)] \sin \varphi(X) \tag{2.18}$$

for $0 \leqslant X \leqslant \ell$, with boundary conditions

$$\varphi'(0) = \varphi'(\ell) = V(0) = V(\ell) = 0. \tag{2.19}$$

Stakgold [13] explains how the symmetry of (2.17)–(2.19) is such that any physical configuration of the deflected rod can be found among the cases where

$$P > 0; \qquad \varphi(0) = \varphi_0\,, \qquad 0 \leqslant \varphi_0 < \pi. \tag{2.20}$$

A combination of (2.8) and (2.18) provides the differential equality

$$\hat{M}'(\mu)\,\mu\,d\mu = -P\hat{\delta}(P\cos\varphi)\sin\varphi\,d\varphi, \tag{2.21}$$

which upon integration gives

$$I(\mu) \equiv \int_0^{\mu} \hat{M}'(\tilde{\mu})\,\tilde{\mu}\,d\tilde{\mu} = \int_{P\cos\varphi_0}^{P\cos\varphi} \hat{\delta}(\tilde{N})\,d\tilde{N} \equiv \iota(\varphi). \tag{2.22}$$

Since $\hat{M}'(\mu) > 0$ and even, then $I(\mu) \geqslant 0$. Furthermore, $\hat{\delta}(N) > 0$ so that (2.22) implies $\cos\varphi \geqslant \cos\varphi_0$ and

$$|\varphi(X)| \leqslant \varphi_0 < \pi. \tag{2.23}$$

By elimination of V from (2.17) and (2.18), one finds a second-order nonlinear Sturm–Liouville problem for the tangent angle φ—namely,

$$\begin{gathered}\varphi''(X) + P\frac{\hat{\delta}[P\cos\varphi(X)]}{\hat{M}'[\varphi'(X)]}\sin\varphi(X) = 0, \qquad 0 < X < \ell, \quad P > 0, \\ \varphi'(0) = \varphi'(\ell) = 0; \qquad \varphi(0) = \varphi_0\,, \qquad 0 \leqslant \varphi_0 < \pi.\end{gathered} \tag{2.24}$$

It is this nonlinear boundary value problem, together with the properties of $\hat{\delta}$ and $\hat{M}$, that will yield our results about the buckling problem. Clearly, $\varphi(X) \equiv 0$, the unbuckled state, is a solution of (2.24) for all $P > 0$. By means of bifurcation analysis, we shall be able to show the existence and character of nontrivial solutions (buckled states) in the neighborhood of the trivial solution for values of P near the branch points (critical values of P). Also of interest to us is the uniqueness problem for (2.24). We will determine sufficient conditions involving $\hat{\delta}$, $\hat{M}$, and P, such that the trivial solution is the only solution of (2.24).

3. Main Results

In this section we state and discuss our principal theorems while postponing the proofs until Section 4. In this way we can present a concise picture of what our results reflect about the buckling problem.

As a preliminary task, we wish to recast the boundary value problem (2.24) into a nonlinear integral equation so that we can treat a bounded operator problem. Ordinarily, this might be accomplished in a straightforward manner by the use of an appropriate Green's function. However, this case is an exceptional one which requires some additional information about possible solutions of (2.24) before the desired integral equation can be derived. The essential properties are given by

Theorem 1. *If* $\varphi(X) \in C^2(0, \ell)$ *is a solution of* (2.24), *then*

$$\int_0^{\ell} \hat{\delta}[P \cos \varphi(X)] \{\hat{M}'[\varphi'(X)]\}^{-1} \sin \varphi(X)\, dX = 0 \tag{3.1}$$

and

$$\int_0^{\ell} \varphi(X)\, dX = 0. \tag{3.2}$$

These properties enable us to derive a suitable integral equation to replace (2.24). We obtain the following.

Theorem 2. *If* $\varphi(X) \in C^2(0, \ell)$ *is a solution of* (2.24), *then* $\varphi(X)$ *is also a solution of*

$$\varphi(X) = P \int_0^{\ell} g(X \mid \xi) \frac{\hat{\delta}[P \cos \varphi(\xi)]}{\hat{M}'[\varphi'(\xi)]} \sin \varphi(\xi)\, d\xi, \qquad 0 \leqslant X \leqslant \ell, \quad P > 0, \tag{3.3}$$

where

$$g(X \mid \xi) = \frac{\ell}{3} + \frac{X^2 + \xi^2}{2\ell} - \xi + (\xi - X)\, H(X - \xi). \tag{3.4}$$

We use $H(X - \xi)$ to denote the Heaviside function.

The kernel $g(X \mid \xi)$ admits a certain degree of arbitrariness, because if any constant is added to it then any solution of (3.3) will still satisfy (2.24). This is of no consequence in view of (3.1). In fact, we could omit the terms

$(\ell/3)+X^2/2\ell$. We choose to retain these terms because they facilitate our analysis of other results to follow.

Hereafter, we will refer to solutions of the buckling problem as those which satisfy (3.3) in some appropriate Banach space. The choice of the space of functions is rather crucial to the analysis. We will point out, in the course of our proofs in Section 4, the difficulties encountered by a choice other than the one we make here.

In particular, we will look for solutions $\varphi(X)$ which are Hölder continuous and have derivatives that are Hölder continuous. A function $\Psi(X)$ is Hölder continuous on $[0, \ell]$ with exponent α $(0 < \alpha < 1)$ if

$$\sup\{| X - Y |^{-\alpha} | \Psi(X) - \Psi(Y)|\} < \infty \qquad \text{for all } X, Y \in [0, \ell].$$

We say $\Psi(X) \in C^{1+\alpha}[0, \ell]$ if both $\Psi(X)$ and $\Psi'(X)$ are Hölder continuous on $[0, \ell]$ with exponent α, $0 < \alpha < 1$. This function space is a Banach space under the norm

$$\| \Psi \|_{1+\alpha} = \sup_{X\in[0,\ell]} | \Psi(X)| + \sup_{X,Y\in[0,\ell]} \frac{| \Psi(X) - \Psi(Y)|}{| X - Y |^{\alpha}}$$

$$+ \sup_{X\in[0,\ell]} | \Psi'(X)| + \sup_{X,Y\in[0,\ell]} \frac{| \Psi'(X) - \Psi'(Y)|}{| X - Y |^{\alpha}}, \qquad 0 < \alpha < 1.$$

The use of such spaces in the analysis of bifurcation problems has been considered by others (cf. Sattinger [10, 11] and Rabinowitz [8]).

Our results about the existence and nature of the buckled states are obtained through a bifurcation analysis of (3.3). That is, we look for nontrivial solutions which branch from the trivial one at certain critical values of P. The existence of such critical values is vitally dependent upon the decay properties of $\hat{\delta}$. To deal with this, it is convenient to introduce a hypothesis about $P\hat{\delta}(P)$. We first consider a mild assumption which will provide for one branchpoint.

Hypothesis 1. Let $\hat{\delta}(N)$ satisfy (2.15) and be such that there exists a $P_1 > 0$ which is the smallest value of P satisfying

$$P_1\hat{\delta}(P_1) = \hat{M}'(0)\, \pi^2/\ell^2, \tag{3.5}$$

and

$$\hat{\delta}(P_1) + P_1\hat{\delta}'(P_1) > 0. \tag{3.6}$$

Our results about bifurcation from the first branchpoint are given by

THEOREM 3. *Let Hypothesis* 1 *hold. Then, for P in some sufficiently small* *neighborhood of P_1, there exists a nontrivial solution $\varphi(X) \in C^{1+\alpha}[0, \ell]$ of* (3.3) *such that*

$$\begin{aligned} \varphi(X) &= 2\sqrt{2}\left[\left(1 - \frac{P_1}{P}\right) h_1\right]^{1/2} \cos\frac{\pi X}{\ell} + w(X); \\ \| w \|_{1+\alpha} &\leqslant C\left[\left(1 - \frac{P_1}{P}\right) h_1\right]^{3/2}, \end{aligned} \tag{3.7}$$

where

$$h_1 = \frac{\hat{\delta}(P_1) + P_1\hat{\delta}'(P_1)}{\hat{\delta}(P_1) + 3P_1\hat{\delta}'(P_1) + \pi^2\ell^{-2}\hat{\delta}(P_1)\,\hat{M}'''(0)\,[\hat{M}'(0)]^{-1}}. \tag{3.8}$$

Remark. Without Hypothesis 1, we have no branchpoint and hence no bifurcation, as has been discussed by Stakgold [13].

Remark. Here, as well as in Theorem 4, it easily follows that, if $\varphi(X) \in C^{1+\alpha}[0, \ell]$ and satisfies (3.3), then in fact $\varphi(X) \in C^2[0, \ell]$ and satisfies (2.24). This is so because to differentiate (3.3) twice and obtain (2.24) requires only that $\hat{\delta}[P\cos\varphi]\,[\hat{M}'(\varphi')]^{-1}\sin\varphi$ be continuous on $[0, \ell]$.

To provide for more branchpoints and to be able to order them, we introduce

Hypothesis 2. Let $\hat{\delta}(N)$ satisfy (2.15) and be such that

$$\hat{\delta}(N) + N\hat{\delta}'(N) > 0, \qquad 0 \leqslant N \leqslant \bar{N} \tag{3.9}$$

for some given $\bar{N} > 0$.

The results of our bifurcation analysis are then given by

THEOREM 4. *Let Hypothesis 2 hold for $\bar{N}\hat{\delta}(\bar{N}) \geqslant \hat{M}'(0)\, n^2\pi^2/\ell^2$, $n = 1, 2,\ldots, k$. Then there exist at least k branchpoints P_n for* (3.3) *that satisfy*

$$P_n\hat{\delta}(P_n) = \hat{M}'(0)\, n^2\pi^2/\ell^2, \qquad n = 1, 2,\ldots, k, \tag{3.10}$$

and

$$\hat{M}'(0)\,\pi^2/\ell^2 \leqslant P_1 < P_2 < \cdots < P_k. \tag{3.11}$$

Moreover, for P in some sufficiently small neighborhood of each P_n, there exists a nontrivial solution $\varphi(X) \in C^{1+\alpha}[0, \ell]$ of (3.3) *such that*

$$\begin{aligned} \varphi(X) &= 2\sqrt{2}\left[\left(1 - \frac{P_n}{P}\right) h_n\right]^{1/2} \cos\frac{n\pi X}{\ell} + w(X), \\ \| w \|_{1+\alpha} &\leqslant C\left[\left(1 - \frac{P_n}{P}\right) h_n\right]^{3/2}, \end{aligned} \tag{3.12}$$

where

$$h_n = \frac{\hat{\delta}(P_n) + P_n\hat{\delta}'(P_n)}{\hat{\delta}(P_n) + 3P_n\hat{\delta}'(P_n) + n^2\pi^2\ell^{-2}\hat{\delta}(P_n)\,\hat{M}'''(0)\,[\hat{M}'(0)]^{-1}}. \tag{3.13}$$

To illustrate the results of Theorem 3 and 4, we refer to Fig. 1, a bifurcation diagram depicting $\| \varphi \|$ versus P. The branchpoints (critical values) P_n are indicated, and a nontrivial solution emanates from each P_n. We have shown (solid lines) the two types of bifurcation at each P_n, depending upon the sign of h_n. Since $(P - P_n) h_n > 0$, we take the right branch if $h_n > 0$ and the left one if $h_n < 0$.

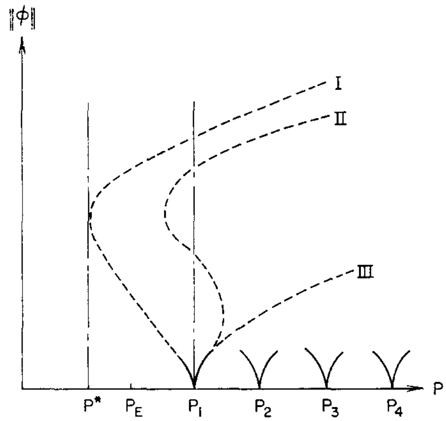

FIGURE 1

In Fig. 1 we have labeled

$$P_E = \hat{M}'(0)\, \pi^2/\ell^2, \tag{3.14}$$

and we call it the first Euler buckling load. It represents the first critical value of load for inextensible materials ($\hat{\delta} \equiv 1$). The solution of the classical problem of the elastica (cf. Love [6]) would give a branch emanating from P_E and going to the right.

Of course our results in Theorems 3 and 4 are valid only for sufficiently small values of $| P - P_n |$. Nevertheless, we can speculate about how extensions of these branches might appear in the bifurcation diagram. In Fig. 1, we have indicated (with dotted lines) three possible extensions of the branch emanating from P_1. Case I is an extension of the left branch ($h_1 < 0$), which ranges far to the left of P_1. Cases II and III both are extensions of the right branch ($h_1 > 0$), where case II bends back to the left of P_1 while case III does not.

One explicit example of case III is the solution of the classical problem of the elasticae with $\hat{\delta} \equiv 1$ and $\hat{M}(\mu) \equiv \hat{M}'(0)\, \mu$. It may be of some interest to the reader to see another explicit example of case III which has nonlinear constitutive dependence. Take $\ell = \pi$ and consider the constitutive relations

$$\hat{\delta}(N) = (1 + \gamma N)^{-1}, \qquad \gamma \geqslant 0, \qquad -\frac{1}{2\gamma} \leqslant -P \leqslant N \leqslant P, \tag{3.15}$$

and

$$\hat{M}(\mu) = \beta \int_0^\mu \{(1 - \eta^2)[\beta\gamma + (1 - \eta^2)^{1/2}]\}^{-1} \, d\eta, \qquad 0 < \beta < \frac{1}{2\gamma}, \qquad -1 < \mu < 1, \tag{3.16}$$

where β and γ are assumed to be given material constants. Now (2.24) takes the form

$$\varphi''(X) + \frac{P\{1 - (\varphi'(X))^2\}\{\beta\gamma + [1 - (\varphi'(X))^2]^{1/2}\}}{\beta[1 + \gamma P \cos \varphi(X)]} \sin \varphi = 0, \quad 0 < X < \pi, \qquad \varphi'(0) = \varphi'(\pi) = 0. \tag{3.17}$$

It is a straightforward matter to verify that (3.17) is satisfied by

$$\varphi(X) = \sin^{-1}\{[1 - (P_1/P)^2]^{1/2} \cos X\}, \qquad P_1 = \beta, \qquad P_1 \leqslant P \leqslant 1/2\gamma. \tag{3.18}$$

Next, we consider the matter of uniqueness of the trivial solution. Whenever $\varphi(X) \equiv 0$ is the only solution of (3.3), there cannot exist any buckled states and the rod remains straight. Theorem 3 gives the existence of buckled states near the smallest critical value P_1, so that we know not to expect uniqueness for $P > P_1$. Of course we could have buckled states which exist far to the left of P_1, as is suggested in Fig. 1 by cases I and II. Just how far case I or II might range to the left can be partially answered by our uniqueness theorems. Under some mild restrictions on $\hat{\delta}$ and $\hat{M}$, we will show that no branch can extend to the left of some $P^* < P_1$. Under stronger restrictions, we will show that no branch can extend to the left of P_1.

Our analysis of the uniqueness problem does not have to be confined to the function space $C^{1+\alpha}[0, \ell]$, which was needed for the existence problem. We find it more convenient to consider the space $C^1[0, \ell]$ of continuously differentiable functions on $[0, \ell]$, which is a Banach space under the norm

$$\| \Psi \|_1 = \sup_{X \in [0,\ell]} | \Psi(X)| + \sup_{X \in [0,\ell]} | \Psi'(X)| .$$

To obtain our first uniqueness theorem, we introduce some mild restrictions on $\hat{M}$ and $\hat{\delta}$ through

Hypothesis 3. Let $\hat{M}(\mu)$ and $\hat{\delta}(N)$ satisfy (2.14) and (2.15), respectively, and let there exist a constant m $(0 < m < \infty)$ such that

$$\hat{M}'(\mu) \geqslant \frac{1}{m}, \qquad -\mu_0 \leqslant \mu \leqslant \mu_0, \tag{3.19}$$

and

$$|\hat{\delta}'(N)| \leqslant \frac{m\ell^2}{6}, \qquad N \leqslant 0. \tag{3.20}$$

We then have the following.

THEOREM 5. *Let Hypothesis* 3 *hold. If* $P < P^* = 6/m\ell^2$, *then the only solution* $\varphi(X) \in C^1[0, \ell]$ *of* (3.3) *is* $\varphi(X) \equiv 0$.

Since (3.19) clearly implies $m \geqslant [\hat{M}'(0)]^{-1}$, we have that

$$P^* < P_E \leqslant P_1. \tag{3.21}$$

Some other uniqueness results similar to Theorem 5 have been derived by Olmstead [7]. While those results give uniqueness to the right of P^*, they depend upon a condition on $\hat{\delta}''$ which we have not imposed.

To extend uniqueness up to the first branchpoint P_1 requires more stringent restrictions on $\hat{M}$ and $\hat{\delta}$. We consider

Hypothesis 4. Let $\hat{M}(\mu)$ and $\hat{\delta}(N)$ satisfy (2.14) and (2.15), respectively, and be such that

$$\hat{M}'''(\mu) \geqslant 0, \qquad -\mu_0 \leqslant \mu \leqslant \mu_0 \tag{3.22}$$

and

$$\hat{\delta}(N) + 3N\hat{\delta}'(N) - (P^2 - N^2)\,\hat{\delta}''(N) > 0, \qquad -P \leqslant N \leqslant P. \tag{3.23}$$

Remark. The differential inequality (3.23) is not easily interpreted. We can characterize it to this extent: If $\hat{\delta}(N) \in C^2(-P, P)$ and satisfies (3.23), then it also satisfies

$$\hat{\delta}(N) < P\left[1 + \hat{\delta}'(0)\, P \sin^{-1}\left(\frac{N}{P}\right)\right](P^2 - N^2)^{-1/2}, \qquad 0 < |N| < P. \tag{3.24}$$

This inequality follows from well-known comparison theorems (cf. [3, Theorem 5.1]) and the fact that the bounding function would satisfy (3.23) with equality.

Our stronger result about uniqueness is given by

THEOREM 6. *Let Hypotheses* 1 *and* 4 *hold. If* $P < P_1$, *then the only solution* $\varphi(X) \in C^1[0, \ell]$ *of* (3.3) *is* $\varphi(X) \equiv 0$.

Thus we are able to show that, under appropriate restrictions on the constitutive relations, no buckled state can exist for $P < P_1$. Of course, this is the best we could expect since Theorem 3 provides for the existence of a nontrivial solution for $P > P_1$. Referring to Fig. 1, we see that Theorem 6 not only makes case I impossible but also implies that the branch in case II could not bend back to the left of P_1.

4. Proofs of Theorems

Out task remains to prove the results of Sec. 3. We begin with the following.

Proof of Theorem 1. The integral property (3.1) follows immediately from the boundary value problem (2.24). Integration of the differential equation over the interval $(0, \ell)$, while invoking the boundary conditions, yields the desired result.

To establish the integral condition (3.2), we will use a symmetry property of solutions of (2.24) along with some phase plane arguments. The desired property of symmetry is obtained by noting that, if $\varphi(X)$ is a solution of (2.24), then both $\varphi(\ell - X)$ and $-\varphi(\ell - X)$ will satisfy the differential equation and end-slope conditions. Only one of these cases can satisfy our specification of initial value $\varphi(0) = \varphi_0 > 0$. Thus, we are only able to conclude that either

$$\varphi(X) = \varphi(\ell - X) \tag{4.1}$$

or

$$\varphi(X) = -\varphi(\ell - X). \tag{4.2}$$

Knowing this symmetry condition, we turn to a phase plane analysis of (2.24). It is easily found that the phase plane equation is simply (2.21) or (2.22) with $\mu = -\varphi'$. That is,

$$\hat{M}'(\varphi')\, \varphi'\, d\varphi' = -P\hat{\delta}(P \cos \varphi) \sin \varphi\, d\varphi, \tag{4.3}$$

which can be integrated to give

$$I(\varphi') = \iota(\varphi). \tag{4.4}$$

Furthermore, we see that

$$0 \leqslant I(\varphi') = I(-\varphi'), \qquad \iota(\varphi) = \iota(-\varphi), \qquad \text{and} \qquad I(0) = \iota(\pm\varphi_0). \tag{4.5}$$

These relationships imply a phase plane diagram which is a closed curve symmetric with respect to both axes. Furthermore, this closed curve has negative slope in the first (and third) quadrant, while it has positive slope in the second (and fourth) quadrant.

The independent variable X is increasing along the closed curve in the clockwise direction with $X = 0$ located at $(\varphi, \varphi') = (\varphi_0, 0)$. From the symmetry property (4.1) and (4.2) the point $X = l$ is located at either $(\varphi, \varphi') = (\varphi_0, 0)$ or $(-\varphi_0, 0)$.

It follows that an evaluation of $\int_0^\ell \varphi(X)\, dX$ in the phase plane involves integration along the closed curve in a clockwise direction over either the whole curve or its lower half, depending upon the location of $X = \ell$. The symmetry of the closed curve is such that the integration over its lower half as half as well as over its upper half is zero.

Next, we consider the following.

Proof of Theorem 2. It is easily verified that the boundary value problem (2.24) is satisfied by the expression

$$\varphi(X) = \frac{1}{\ell} \int_0^\ell \varphi(\xi)\, d\xi + P \int_0^\ell g(X \mid \xi) \frac{\hat{\delta}[P \cos \varphi(\xi)]}{\hat{M}'[\varphi'(\xi)]} \sin \varphi(\xi)\, d\xi, \tag{4.6}$$

with $g(X \mid \xi)$ given by (3.4). That verification requires the use of (3.1). The term $\ell^{-1} \int_0^\ell \varphi(\xi)\, d\xi$ is needed for consistency; that is, integration of both sides of (4.6) over $(0, \ell)$ must yield an identity. We are justified in dropping this term only because of (3.2). Thus (3.3) is obtained.

This difficulty in converting the boundary value problem (2.24) into a suitable integral equation stems from the nature of the linear problem obtained by setting $P = 0$ in (2.24). A Green's function for that problem does not exist. The remedy for such situations is to construct a modified Green's function (cf. Stakgold [12]) like $g(X \mid \xi)$. The modified Green's function is typically nonunique, as is evident from our comments following Theorem 2.

Our next two results in Section 3 involve similar proofs. Once the possible branchpoints have been established, the method of construction of the solutions is the same. Some preliminary definitions and results will facilitate the discussion.

For convenience, we introduce an operator notation so that (3.3) can be expressed as

$$\varphi = PA(\varphi, \varphi', P) \equiv P \int_0^\ell g(X \mid \xi) \frac{\hat{\delta}[P \cos \varphi(\xi)]}{\hat{M}'[\varphi'(\xi)]} \sin \varphi(\xi)\, d\xi. \tag{4.7}$$

Our existence proof will be a constructive one known as the Lyapunov–Schmidt method. We will follow this method essentially as it has been outlined by Stakgold [13]. The intricacies of our proof will require the use of four Banach spaces. We consider $C^0[0, \ell]$, the space of continuous functions on $[0, \ell]$ with norm

$$\| \Psi \|_0 = \sup_{X \in [0, \ell]} | \Psi(X) | .$$

The space $C^1[0, \ell]$ of continuously differentiable functions has the norm

$$\| \Psi \|_1 = \| \Psi \|_0 + \| \Psi' \|_0 .$$

In anticipation of the use of Hölder continuous functions on $[0, \ell]$, we first note that such functions form a Banach space $C^{0+\alpha}[0, \ell]$ under the norm

$$\| \Psi \|_{0+\alpha} = \| \Psi \|_0 + \sup_{X, Y \in [0,\ell]} \frac{| \Psi(X) - \Psi(Y) |}{| X - Y |^\alpha}, \qquad 0 < \alpha < 1.$$

Then $C^{1+\alpha}[0, \ell]$ is the Banach space of functions which are Hölder continuous and have Hölder continuous derivatives. Its norm can be expressed as

$$\| \Psi \|_{1+\alpha} = \| \Psi \|_{0+\alpha} + \| \Psi' \|_{0+\alpha}, \qquad 0 < \alpha < 1.$$

It is easily seen that

$$C^1[0, \ell] \subset C^0[0, \ell], \qquad C^{1+\alpha}[0, \ell] \subset C^{0+\alpha}[0, \ell],$$
$$C^{0+\alpha}[0, \ell] \subset C^0[0, \ell], \qquad \text{and} \qquad C^{1+\alpha}[0, \ell] \subset C^1[0, \ell].$$

The need for these Hölder spaces will become clear as our proof of existence unfolds.

Some information about the linearized form of (4.7) will be needed. Linearization about the trivial solution yields the eigenvalue problem

$$\theta = \nu L \theta \equiv \nu \int_0^\ell g(X \mid \xi)\, \theta(\xi)\, d\xi, \qquad \nu = \hat{\nu}(P) \equiv P \hat{\mathcal{S}}(P) / \hat{M}'(0). \tag{4.8}$$

Moreover, $(\nu/P)L$ is the Fréchet derivative of A at $\varphi \equiv 0$. This eigenvalue problem is satisfied by

$$\theta_n(X) = \left(\frac{2}{\ell}\right)^{1/2} \cos \frac{n\pi X}{\ell}, \qquad \nu_n = \frac{n^2\pi^2}{\ell^2}, \qquad n = 1, 2, \ldots. \tag{4.9}$$

To gain insight about possible branchpoints, we consider

LEMMA 1. *Nontrivial solutions of* (4.7) *can branch from the trivial solution only at values of* $P = P_c$ *which satisfy any one of the equations*

$$\hat{\nu}(P_c) = \frac{n^2\pi^2}{\ell^2}, \qquad n = 1, 2, \ldots. \tag{4.10}$$

Proof. This follows immediately from a well-known result of bifurcation theory which states that branchpoint of the nonlinear operator A must be in the spectrum of the linearized operator L (cf. Stakgold [13]).

Of course Lemma 1 does not tell us if there is any branching. It just gives us the candidates for the branchpoints. To show that there is branching, we must prove the existence of nontrivial solutions for values of P in some neighborhood of a given P_c which satisfies (4.10).

One of the key parts of the Lyapunov–Schmidt method involves a rearrangement of (4.7) into a form more appropriate for considering P near P_c or, equivalently, ν near ν_n. We express (4.7) as

$$L\varphi - \frac{1}{\nu_n}\varphi = \epsilon\varphi - R(\varphi, \varphi', P), \qquad \epsilon = \frac{1}{\nu} - \frac{1}{\nu_n}, \tag{4.11}$$

where the remainder operator R is defined as

$$\begin{aligned} R(\varphi, \varphi', P) &= \frac{P}{\nu} A(\varphi, \varphi', P) - L\varphi = \int_0^\ell g(X \mid \xi) F[\varphi(\xi), \varphi'(\xi), P]\, d\xi, \\ \text{for} & \\ F(\varphi, \varphi', P) &= \frac{\hat{\delta}(P\cos\varphi)\, \hat{M}'(0)}{\hat{\delta}(P)\, \hat{M}'(\varphi')} \sin\varphi - \varphi. \end{aligned} \tag{4.12}$$

To obtain a suitable representation of $\varphi(X)$, we utilize the set $\{\theta_n\}$ of orthonormal functions. Since $\int_0^\ell \varphi(X)\, dX = 0$, this set is complete, and we can form the series expansion

$$\varphi(X) = \sum_{i=1}^{\infty} c_i\theta_i(X), \qquad c_i = \langle\varphi, \theta_i\rangle, \qquad i = 1, 2, \ldots. \tag{4.13}$$

Here we have introduced an inner product notation only for some conciseness of presentation. We denote

$$\langle\chi, \Psi\rangle = \int_0^\ell \chi(X)\, \Psi(X)\, dX.$$

In bifurcation analysis, a special decomposition of the expansion (4.13) is used. For a given branch point P_n, we single out the eigenfunction θ_n associated with the corresponding ν_n. This gives

$$\begin{aligned} \varphi(X) = v(X) + w(X); \qquad v(X) &= a(\epsilon)\, \theta_n(X), \\ w(X) &= \sum_{\substack{i=1 \\ i\neq n}}^{\infty} c_i\theta_i(X), \end{aligned} \tag{4.14}$$

where we have set $c_n = a(\epsilon)$. In the course of our analysis, we will determine $a(\epsilon)$ and establish a bound for $w(X)$ in terms of $a(\epsilon)$.

Upon substituting (4.14) into (4.11) we find

$$Lw - \frac{1}{\nu_n} w = \epsilon(v + w) - R(v + w, v' + w', P). \tag{4.15}$$

The inversion of the operator $(L - (1/\nu_n) I)$ in (4.15) involves the concept of the pseudoinverse. General statements about the character of the pseudoinverse are available (cf. Stakgold [13]). However, in our case it is a simple matter to derive it explicitly. From (4.8) and (4.14) we have that

$$Lw - \frac{1}{\nu_n} w = \sum_{\substack{i=1 \\ i \neq n}}^{\infty} c_i \left(L\theta_i - \frac{1}{\nu_n} \theta_i \right) = \sum_{\substack{i=1 \\ i \neq n}}^{\infty} \left(\frac{1}{\nu_i} - \frac{1}{\nu_n} \right) c_i \theta_i , \tag{4.16}$$

so that

$$\left(\frac{1}{\nu_i} - \frac{1}{\nu_n} \right) c_i = \left\langle \theta_i , Lw - \frac{1}{\nu_n} w \right\rangle , \qquad i \neq n. \tag{4.17}$$

Solving for c_i, we then have

$$w(X) = \sum_{\substack{i=1 \\ i \neq n}}^{\infty} \frac{\nu_i \nu_n}{\nu_n - \nu_i} \langle \theta_i , Lw - \frac{1}{\nu_n} w \rangle \theta_i(X), \tag{4.18}$$

which provides the desired inversion for (4.15). We denote the pseudoinverse operator by T, where

$$Tu = \sum_{\substack{i=1 \\ i \neq n}}^{\infty} \frac{\nu_i \nu_n}{\nu_n - \nu_i} \langle \theta_i , u \rangle \theta_i(X). \tag{4.19}$$

By substituting (4.15) into (4.18), while noting that $Tv = 0$, we obtain

$$w = Bw \equiv T[\epsilon w - R(v + w, v' + w', P)]. \tag{4.20}$$

In this equation we regard P and, hence, ϵ as given. We think of $a = a(\epsilon)$ as fixed but unknown. Thus $v = a\theta_n$ is also fixed but unknown.

To obtain an independent equation for $a = a(\epsilon)$, we take the inner product of (4.15) with θ_n. Since

$$\langle \theta_n , v \rangle = a, \qquad \langle \theta_n , w \rangle = 0,$$

and

$$\langle \theta_n , Lw \rangle = \langle L\theta_n , w \rangle = \frac{1}{\nu_n} \langle \theta_n , w \rangle = 0,$$

we find that

$$\epsilon a(\epsilon) = \langle \theta_n , R(v + w, v' + w', P) \rangle. \tag{4.21}$$

Once we have established that a solution of (4.20) exists for fixed $a = a(\epsilon)$, then we can use (4.21) to determine $a = a(\epsilon)$.

It is Eqs. (4.20) and (4.21) that will be analyzed to establish the existence of nontrivial solutions of (4.7). Our choice of function space is motivated by the nature of these two equations. To prove a fixed point theorem for (4.20) necessitates that the operator T be bounded. The space $C^1[0, \ell]$ is convenient for many of the estimates, but T is not bounded on it (as we will indicate). While T is bounded on some Sobolev spaces, there are certain operators used in the analysis of (4.21) which are not well defined on these spaces. Undoubtedly, there are several possible remedies to this situation. We have chosen to use the Hölder space $C^{1+\alpha}[0, \ell]$ because T is bounded on it, and we will have the convenience of continuous differentiability for other estimates.

We prove the boundedness of T in the following.

LEMMA 2. *There exists a constant* $\| T \| < \infty$ *such that*

$$\| Tu \|_{1+\alpha} \leqslant \| T \| \, \| u \|_1 \leqslant \| T \| \, \| u \|_{1+\alpha} \tag{4.22}$$

for all $u \in C^{1+\alpha}[0, \ell]$.

Proof. First consider $u \in C^{0+\alpha}[0, \ell]$. It suffices to take

$$\langle u, 1 \rangle = \langle u, \theta_n \rangle = 0$$

since these terms do not appear in (4.19). It is well known that the series $\sum_{i=1}^{j} \langle u, \theta_i \rangle \, \theta_i(X)$ converges pointwise to $u(X)$ as $j \to \infty$.[3] Thus we consider

$$S_j(X) = \sum_{\substack{i=1 \\ i \neq n}}^{j} \frac{\langle u, \theta_i \rangle}{\lambda_i - \lambda_n} \theta_i(X) = \sum_{\substack{i=1 \\ i \neq n}}^{j} \frac{\lambda_i \langle u, \theta_i \rangle}{\lambda_n(\lambda_i - \lambda_n)} \theta_i(X) - \frac{1}{\lambda_n} \sum_{i=1}^{j} \langle u, \theta_i \rangle_j \theta_i(X),$$

where $\lambda_i = 1/\nu_i = \ell^2/\pi^2 i^2$. Clearly, $S_j(X)$ converges as $j \to \infty$ since the first series converges absolutely while the second converges pointwise to $u(X)$. Thus $S_j(X) \to Tu$ and

$$Tu(X) = G(X) - \frac{1}{\lambda_n} u(X), \qquad G(X) = \frac{1}{\lambda_n} \sum_{\substack{i=1 \\ i \neq n}}^{\infty} \frac{\lambda_i \langle u, \theta_i \rangle}{(\lambda_i - \lambda_n)} \theta_i(X).$$

It then follows that

$$\| Tu \|_0 \leqslant \| G \|_0 + \frac{1}{\lambda_n} \| u \|_0 \leqslant \left[C' \sum_{i=1}^{\infty} \lambda_i + \frac{1}{\lambda_n} \right] \| u \|_0 = C \| u \|_0 .$$

[3] If we require only that $u \in C^0[0, \ell]$, then it is possible for $S_j(X)$ to diverge at some points (cf. [14]). It therefore follows that T is not bounded on $C^0[0, \ell]$.

Next we consider

$$\sup_{X,Y\in[0,\ell]} \frac{|Tu(X) - Tu(Y)|}{|X - Y|^{\alpha}} \leqslant \sup_{X,Y\in[0,\ell]} \left[\frac{|G(X) - G(Y)| + \lambda_n^{-1} |u(X) - u(Y)|}{|X - Y|^{\alpha}}\right].$$

To estimate this last expression, we will utilize

$$\sup_{X,Y\in[0,\ell]} \frac{|u(X) - u(Y)|}{|X - Y|^{\alpha}} \leqslant C \| u \|_0,$$

which follows since $u(X)/\| u \|_0 \in C^{0+\alpha}[0, l]$. We also find

$$\sup_{X,Y\in[0,\ell]} \frac{|G(X) - G(Y)|}{|X - Y|^{\alpha}} \leqslant C \| u \|_0 \sup_{X,Y\in[0,\ell]} \sum_{i=1}^{\infty} \lambda_i \frac{|\theta_i(X) - \theta_i(Y)|}{|X - Y|^{\alpha}}$$

and

$$\frac{|\theta_i(X) - \theta_i(Y)|}{|X - Y|^{\alpha}} \leqslant C \frac{\left|\sin\left(\frac{X - Y}{2(\lambda_i)^{1/2}}\right)\right|}{|X - Y|^{\alpha}} \leqslant C\lambda_i^{-\alpha/2} \frac{|\sin \tau|}{|\tau|^{\alpha}} \leqslant C\lambda_i^{-\alpha/2}.$$

Combining these gives

$$\sup_{X,Y\in[0,\ell]} \frac{|G(X) - G(Y)|}{|X - Y|^{\alpha}} \leqslant C \| u \|_0 \sum_{i=1}^{\infty} \lambda_i^{1-\alpha/2} \leqslant C \| u \|_0, \qquad 0 < \alpha < 1.$$

Thus we find

$$\| Tu \|_{0+\alpha} \leqslant C \| u \|_0. \tag{4.23}$$

Next we treat the derivative of Tu. For $u' \in C^{0+\alpha}[0, \ell]$, we have

$$\langle \theta_i, u \rangle = \lambda_i \langle \theta_i', u' \rangle, \qquad (\lambda_i)^{1/2} \theta_i' = -\sin[X/(\lambda_i)^{1/2}],$$

and

$$(Tu)'(X) = \sum_{\substack{i=1 \\ i\neq n}}^{\infty} \frac{\langle (\lambda_i)^{1/2} \theta_i', u' \rangle}{\lambda_i - \lambda_n} (\lambda_i)^{1/2} \theta_i'(X).$$

Now the procedure that was used to bound Tu, $u \in C^{0+\alpha}[0, \ell]$, can be repeated to yield

$$\| (Tu)' \|_{0+\alpha} \leqslant C' \| u' \|_0. \tag{4.24}$$

Thus we find that, if $u \in C^{1+\alpha}[0, \ell]$, then

$$\| Tu \|_{1+\alpha} = \| Tu \|_{0+\alpha} + \|(Tu)'\|_{0+\alpha} \leqslant \max(C, C') \| u \|_1 = \| T \| \| u \|_1 . \quad (4.25)$$

The other estimates that we will need in our proof of Theorems 3 and 4 can be stated in terms of the norm of $C^1[0, \ell]$. First, we have

LEMMA 3. *The remainder operator R has the property that*

$$\| R(y, y', P) - R(z, z', P)\|_1 \leqslant K(y, y', z, z', P) \| y - z \|_1 , \quad (4.26)$$

where $K(y, y', z, z', P) \to 0$ as $\| y \|_1 \to 0$ and $\| z \|_1 \to 0$.

Proof. We have R defined by (4.12), and dR/dX is found by replacing $g(X \mid \xi)$ with $(dg/dX)(X \mid \xi)$. Since $g(X \mid \xi)$ is continuous on $[0, \ell]$ and $(dg/dX)(X \mid \xi)$ is piecewise continuous on $[0, \ell]$, there exists a constant $C < \infty$ such that

$$\| R(y, y', P) - R(z, z', P)\|_1 \leqslant C \| F(y, y', P) - F(z, z', P)\|_0 . \quad (4.27)$$

Using a mean value theorem, we find

$$\begin{aligned} &\| F(y, y', P) - F(z, z', P)\|_0 \\ &\quad \leqslant K_1 \| y - z \|_0 + K_2 \| y' - z' \|_0 + K_3 \| y - z \|_0 \| y' - z' \|_0 , \end{aligned}$$

where $K_1 \to 0$, $K_2 \to 0$, and $K_3 \to 1$ as $\| y \|_1 \to 0$ and $\| z \|_1 \to 0$.

Since $\| y - z \|_0 \leqslant \| y - z \|_1$ and $\| y' - z' \|_0 \leqslant \| y - z \|_1$, we have

$$\| R(y, y', P) - R(z, z', P)\|_1 \leqslant C(K_1 + K_2 + K_3 \| y - z \|_1) \| y - z \|_1 , \quad (4.28)$$

which implies (4.26).

Next we consider a decomposition of the remainder R which will single out the "leading term" of the nonlinearity.

Let

$$R(y, y', P) = D_0(y, y', P) + D(y, y', P), \quad (4.29)$$

where the "leading term" operator D_0 is defined as

$$D_0(y, y', P) = \int_0^{\ell} g(X \mid \xi) F_0[y(\xi), y'(\xi), P] \, d\xi,$$

with

$$F_0(y, y', P) = - \frac{\hat{\delta}(P) + 3P\hat{\delta}'(P)}{6\hat{\delta}(P)} y^3 - \frac{\hat{M}'''(0)}{2\hat{M}'(0)} y(y')^2. \quad (4.30)$$

It therefore follows that D is given by

$$D(y, y', P) = \int_0^{\ell} g(X \mid \xi) \{F[y(\xi), y'(\xi), P] - F_0[y(\xi), y'(\xi), P]\} \, d\xi. \quad (4.31)$$

The nature of this decomposition is exposed in

LEMMA 4. *The operators D_0 and D have the properties that*

$$\| D_0(y, y', P) - D_0(z, z', P)\|_1 \leqslant C(\| y \|_1^2 + \| z \|_1^2) \| y - z \|_1 , \quad (4.32)$$

$$\| D(y, y', P) - D(z, z', P)\|_1 \leqslant C(\| y \|_1^3 + \| z \|_1^3) \| y - z \|_1 . \quad (4.33)$$

Proof. We have D_0 given by (4.30), and dD_0/dX is found by replacing $g(X \mid \xi)$ with $(dg/dX)(X \mid \xi)$. Then, as in the proof of Lemma 3, there is a constant C such that

$$\| D_0(y, y', P) - D_0(z, z', P)\|_1 \leqslant C[\| y^3 - z^3 \|_0 + \| y(y')^2 - z(z')^2 \|_0]. \quad (4.34)$$

We also have that

$$\| y^3 - z^3 \|_0 \leqslant 2(\| y \|_0^2 + \| z \|_0^2) \| y - z \|_0 ,$$

as well as

$$\| y(y')^2 - z(z')^2 \|_0 \leqslant 2(\| y \|_1^2 + \| z \|_1^2) \| y - z \|_1 .$$

These, together with $\| y \|_0 \leqslant \| y \|_1$, bring (4.34) into the form of (4.32).

To establish (4.33), we begin, as in the proof of Lemma 3, with the estimate

$$\begin{aligned} &\| D(y, y', P) - D(z, z', P)\|_1 \\ &\quad \leqslant C \| F(y, y', P) - F_0(y, y', P) - F(z, z', P) + F_0(z, z', P)\|_0 . \end{aligned} \quad (4.35)$$

Then, using a mean value theorem and some tedious but straightforward estimation, we are led to the desired result.

In the proofs of Theorems 3 and 4 we keep in mind what is required to show branching. We first need to know that some critical value P_c does satisfy (4.10) and that $\hat{\nu}(P)$ is invertible in some neighborhood of P_c. Then we must prove the existence of a nontrivial solution for $\mid P - P_c \mid$ or $\mid \nu - \nu_n \mid$ sufficiently small. We begin with

Proof of Theorem 3. Hypothesis 1 provides that there is a P_1 which satisfies (4.10) with $n = 1$, and in particular it is the smallest such value. Also, (3.6) is sufficient to insure that $\hat{\nu}(P)$ is invertible in some neighborhood of P_1. The existence of a nontrivial solution of (3.3) or, equivalently, of (4.7), follows as a special case of Theorem 4.

Proof of Theorem 4. Hypothesis 2 provides that $\hat{\nu}(P) \in C^2[-P, P]$ and is strictly increasing on $[0, \bar{N}]$ for some $\bar{N} > 0$. Hence, on $[0, \bar{N}]$ we have $\max \hat{\nu}(P) = \hat{\nu}(\bar{N})$. Therefore, if $\hat{\nu}(\bar{N}) \geqslant n^2\pi^2/\ell^2$ for some values of n, say $n = 1, 2,..., k$, then there must be k distinct values of P which satisfy (4.10); we call them P_n, $n = 1, 2,..., k$. Also, the monotonicity of $\hat{\nu}(P)$ provides that $P_1 < P_2 < \cdots < P_k$. Furthermore, the strictly increasing property insures that $\hat{\nu}(P)$ is invertible in some neighborhood of each P_n. This establishes the first part of the theorem regarding the branch points.

We now turn our attention to proving that nontrivial solutions of (4.7) exist in the neighborhood of the branch points. We have followed the Lyapunov–Schmidt method in converting (4.7) into a pair of Equations (4.20) and (4.21) for $w(X)$ and $a(\epsilon)$.

For $w \in C^{1+\alpha}[0, \ell]$, we have from Lemmas 2 and 3

$$\| Bw \|_{1+\alpha} \leqslant \| T \| \{ | \epsilon | \, \| w \|_1 + K \| v + w \|_1 \}. \tag{4.36}$$

Take $| \epsilon | \leqslant (4 \| T \|)^{-1}$ and pick an ϵ_1 such that $\| v \|_{1+\alpha} \leqslant \epsilon_1$ and $\| w \|_{1+\alpha} \leqslant \epsilon_1$ imply $K \leqslant (4 \| T \|)^{-1}$. Then (4.36) gives $\| Bw \|_{1+\alpha} \leqslant \epsilon_1$, which shows that the ball $\| w \|_{1+\alpha} \leqslant \epsilon_1$ is mapped into itself.

To show that B is a contraction on $C^{1+\alpha}[0, \ell]$, we consider $\| Bw_1 - Bw_2 \|_{1+\alpha}$. Again, from Lemmas 2 and 3, we have

$$\begin{aligned} \| Bw_1 - Bw_2 \|_{1+\alpha} &\leqslant \| T \| \, (| \epsilon | + K) \| w_1 - w_2 \|_1 \\ &\leqslant \| T \| \, (| \epsilon | + K) \| w_1 - w_2 \|_{1+\alpha}, \end{aligned} \tag{4.37}$$

where $K \to 0$ as $\| v \|_1 \to 0$, $\| w_1 \|_1 \to 0$, and $\| w_2 \|_1 \to 0$. Take $| \epsilon | < (2 \| T \|)^{-1}$ and pick an ϵ_2 such that $\| v \|_{1+\alpha} \leqslant \epsilon_2$, $\| w_1 \|_{1+\alpha} \leqslant \epsilon_2$, and $\| w_2 \|_{1+\alpha} \leqslant \epsilon_2$ imply $K < (2 \| T \|)^{-1}$. Then $\| T \| \, (| \epsilon | + K) < 1$, and we have a contraction implied by (4.37).

Therefore, if we set $\epsilon_0 = \min(\epsilon_1, \epsilon_2)$ we find that B is a contraction mapping of the ball $\| w \|_{1+\alpha} \leqslant \epsilon_0$ into itself. So there exists a unique $w \in C^{1+\alpha}[0, \ell]$ which satisfies (4.20). Of course, this means that for a given ν_n there is a unique solution, which gives bifurcation at each of the branch points.

Now that the existence of a solution w to (4.20) has been established, let us derive a bound on that solution. As a preliminary bound, we have from (4.20) and (4.36) that

$$\| w \|_{1+\alpha} = \| Bw \|_{1+\alpha} \leqslant \| T \| \{ (| \epsilon | + K) \| w \|_1 + K \| v \|_1 \}. \tag{4.38}$$

For $|\epsilon| \leqslant (4\|T\|)^{-1}$ and $K \leqslant (4\|T\|)^{-1}$, we find

$$\|w\|_{1+\alpha} \leqslant \tfrac{1}{2}\|v\|_1 = \tfrac{1}{2}|a(\epsilon)|\,\|\theta_n\|_1 . \tag{4.39}$$

To improve this estimate, we make a more careful examination of (4.20) and find

$$\begin{aligned}\|w\|_{1+\alpha} \leqslant \|T\|\{|\epsilon|\,\|w\|_1 + \|D_0(v+w, v'+w', P)\|_1 \\ + \|D(v+w, v'+w', P)\|_1\}.\end{aligned} \tag{4.40}$$

Employing Lemma 4 gives

$$\|w\|_{1+\alpha} \leqslant \|T\|\{|\epsilon|\,\|w\|_1 + C\|v+w\|_1^3 + C\|v+w\|_1^4\}. \tag{4.41}$$

In view of (4.39), we see $\|w\|_1 \leqslant \|w\|_{1+\alpha} \leqslant \frac{1}{2}\|v\|_1$ and

$$\|w\|_{1+\alpha} \leqslant \|T\|\{|\epsilon|\,\|w\|_1 + C'\|v\|_1^3 + C''\|v\|_1^4\}. \tag{4.42}$$

Consequently, if $|\epsilon| \leqslant (2\|T\|)^{-1}$, then

$$\|w\|_{1+\alpha} \leqslant C\|v\|_1^3 = C|a|^3\|\theta_n\|_1^3 . \tag{4.43}$$

The remaining part of the proof is to determine $a(\epsilon)$. We consider (4.21) expressed as

$$\begin{aligned}\epsilon a = \langle\theta_n, D_0(v, v', P)\rangle + \langle\theta_n, D_0(v+w, v'+w', P) - D_0(v, v', P)\rangle \\ + \langle\theta_n, D(v+w, v'+w', P)\rangle.\end{aligned} \tag{4.44}$$

From (4.30) we see that

$$D_0(v, v', P) = D_0(a\theta_n, a\theta_n', P) = a^3 D_0(\theta_n, \theta_n', P).$$

Moreover, from Lemma 4 and our estimate (4.43), we have

$$\begin{aligned}&|\langle\theta_n, D_0(v+w, v'+w', P) - D_0(v, v', P)\rangle| \\ &\qquad\leqslant C(\|v+w\|_1^2 + \|v\|_1^2)\|w\|_1 \leqslant C|a|^5,\end{aligned}$$

and

$$|\langle\theta_n, D(v+w, v'+w', P)\rangle| \leqslant C\|v+w\|_1^4 \leqslant Ca^4. \tag{4.45}$$

Thus, (4.44) becomes

$$\epsilon a = a^3\langle\theta_n, D_0(\theta_n, \theta_n', P)\rangle + Q_0(a), \qquad |Q_0(a)| \leqslant Ca^4. \tag{4.46}$$

After the calculation of $\langle \theta_n, D_0(\theta_n, \theta_n', P)\rangle$, we find

$$a^2 = \left(1 - \frac{\nu_n}{\nu}\right) \frac{4\ell \hat{\delta}(P)}{\hat{\delta}(P) + 3P\hat{\delta}'(P) + \nu_n \hat{\delta}(P) \hat{M}'''(0) [\hat{M}'(0)]^{-1}} + Q_1(a),$$

$$|Q_1(a)| \leqslant C |a|^3. \tag{4.47}$$

A Taylor series expansion about P_n yields the form indicated in the theorem. Higher-order terms in $P - P_n$ can be included with w. The positive square root was used since $\varphi(0) = \varphi_0 > 0$.

There is another method used in bifurcation theory which is an alternative to that of Lyapunov–Schmidt. The Poincaré–Keller method[4] can be applied directly to the boundary value problem (2.24). We have carried out this procedure to obtain the results of Theorem 4 (except for the estimates on w). This alternative proof is omitted.

Turning to the proofs of the uniqueness theorems, we consider

Proof of Theorem 5. Our goal will be show that, under the given conditions, $\varphi_0 = 0$, and hence, by (2.23), $\varphi \equiv 0$. We achieve this goal in two stages. First, we show that $\varphi_0 \leqslant \pi/2$; then we show that $\varphi_0 = 0$.

From (4.2) we have $\varphi(\ell) = \pm\varphi_0$. Thus, for any solution of (3.3) it follows that

$$\pm\varphi_0 = P \int_0^\ell \left(\frac{\xi^2}{2\ell} - \frac{\ell}{6}\right) \frac{\hat{\delta}[P \cos \varphi(\xi)]}{\hat{M}'[\varphi'(\xi)]} \sin \varphi(\xi)\, d\xi. \tag{4.48}$$

Utilizing (3.1), we eliminate one of the terms and find

$$\varphi_0 \leqslant \frac{P\ell^2}{6} \left\| \frac{\hat{\delta}(P \cos \varphi)}{\hat{M}'[\varphi']} \sin \varphi \right\|_0 \leqslant \frac{Pm\ell^2}{6} \| \hat{\delta}(P \cos \varphi) \sin \varphi \|_0, \tag{4.49}$$

where (3.19) has been used in the latter inequality. Since $\hat{\delta}$ is strictly decreasing and $\cos \varphi \geqslant \cos \varphi_0$, we have

$$\varphi_0 \leqslant \frac{Pm\ell^2}{6} \hat{\delta}(P \cos \varphi_0) \| \sin \varphi \|_0 . \tag{4.50}$$

If $\varphi_0 \leqslant \pi/2$, then $|\sin \varphi| \leqslant \sin \varphi_0$. On the other hand, if $\varphi_0 > \pi/2$, then $\varphi = \pi/2$ at some point on $(0, \ell)$ and $|\sin \varphi| \leqslant 1$. Thus (4.50) becomes

$$\varphi_0 \leqslant \frac{Pm\ell^2}{6} \hat{\delta}(P \cos \varphi_0) \left[\sin \varphi_0 + (1 - \sin \varphi_0) H\left(\varphi_0 - \frac{\pi}{2}\right)\right]. \tag{4.51}$$

[4] This method has been so named by Stakgold [13] owing to the elegant application by Keller [5] of the Poincaré technique.

To first show that $\varphi_0 \leqslant \pi/2$, we assume the contrary. If $\varphi_0 > \pi/2$, then $\cos\varphi_0 < 0$ and (3.20) applies. Thus,

$$\hat{\delta}(P\cos\varphi_0) \leqslant 1 + \int_{P\cos\varphi_0}^{0} |\hat{\delta}'(\tilde{N})|\, d\tilde{N} \leqslant 1 + \frac{Pm\ell^2}{6} |\cos\varphi_0| . \tag{4.52}$$

So (4.51) becomes

$$\varphi_0 \leqslant \frac{Pm\ell^2}{6}\left(1 + \frac{Pm\ell^2}{6} |\cos\varphi_0|\right), \qquad \frac{\pi}{2} < \varphi_0 < \pi. \tag{4.53}$$

For $\pi/2 < \varphi_0 < \pi$, $|\cos\varphi_0| \leqslant \varphi_0 - \pi/2$, so that we have

$$|\cos\varphi_0| + \frac{\pi}{2} \leqslant \frac{Pm\ell^2}{6} + \left(\frac{Pm\ell^2}{6}\right)^2 |\cos\varphi_0|, \qquad \frac{\pi}{2} < \varphi_0 < \pi. \tag{4.54}$$

But if $Pm\ell^2/6 < 1$, this is clearly a contradiction. Hence, we conclude that, under the hypothesis of the theorem, $\varphi_0 \leqslant \pi/2$.

For $\varphi_0 \leqslant \pi/2$, we have that $\hat{\delta}(P\cos\varphi_0) \leqslant 1$, and (4.51) yields

$$\varphi_0 \leqslant \frac{Pm\ell^2}{6} \sin\varphi_0 , \tag{4.55}$$

which can only be satisfied by $\varphi_0 = 0$ if $Pm\ell^2/6 < 1$.

Finally, we come to

Proof of Theorem 6. Suppose there is a solution $\varphi \in C^{1+\alpha}[0, \ell]$ that satisfies (3.3), which we express in the form

$$\varphi = PL\left[\frac{\hat{\delta}(P\cos\varphi)}{\hat{M}'(\varphi')} \sin\varphi\right], \tag{4.56}$$

where the linear integral operator L is defined by (4.8).

We wish to analyze (4.56) in the space of square integrable functions $L_2[0, \ell]$ with norm

$$\|\Psi\|_{L_2} = \left[\int_0^{\ell} \Psi^2(X)\, dx\right]^{1/2}.$$

Since $C^{1+\alpha}[0, \ell] \subset L_2[0, \ell]$ and due to the smoothness of $\hat{\delta}$ and $\hat{M}$, we are justified in considering

$$\|\varphi\|_{L_2} = P\left\| L\left[\frac{\hat{\delta}(P\cos\varphi)}{\hat{M}'(\varphi')} \sin\varphi\right]\right\|_{L_2} . \tag{4.57}$$

From the Hilbert–Schmidt theory of integral operators (cf. Stakgold [12]), we have that

$$\| L\Psi \|_{L_2} \leqslant \| L \|_{L_2} \| \Psi \|_{L_2}, \qquad \| L \|_{L_2} = \frac{1}{\nu_1} = \frac{\ell^2}{\pi^2}. \tag{4.58}$$

By Hypothesis 1, there exists a P_1 such that $P_1\hat{\delta}(P_1) = \nu_1 \hat{M}'(0)$. Thus, (4.57) yields the inequality

$$\| \varphi \|_{L_2} \leqslant \frac{P\hat{M}'(0)}{P_1\hat{\delta}(P_1)} \left\| \frac{\hat{\delta}(P\cos\varphi)\sin\varphi}{\hat{M}'(\varphi')\,\varphi} \right\|_0 \| \varphi \|_{L_2}. \tag{4.59}$$

Now the properties of $\hat{\delta}$ and $\hat{M}$ admit the representation

$$\frac{\hat{\delta}(P\cos\varphi)\sin\varphi}{\hat{M}'(\varphi')\,\varphi} = [\hat{\delta}(P) - q(P\cos\beta\varphi)\,\varphi\sin\beta\varphi]\left[(\hat{M}'(0))^{-1} - r(\eta\varphi')\frac{(\varphi')^2}{2}\right]$$

for

$$0 < \beta < 1 \qquad \text{and} \qquad 0 < \eta < 1, \tag{4.60}$$

with

$$q(N) = \hat{\delta}(N) + 3N\hat{\delta}'(N) - (P^2 - N^2)\,\hat{\delta}''(N)$$

and

$$r(\mu) = \frac{\hat{M}'(\mu)\,\hat{M}'''(\mu) + [\hat{M}''(\mu)]^2}{[\hat{M}'(\mu)]^3}. \tag{4.61}$$

The conditions of Hypothesis 4 give $q(N) > 0$ and $r(\mu) \geqslant 0$, so that

$$\frac{\hat{\delta}(P\cos\varphi)\sin\varphi}{\hat{M}'(\varphi')\,\varphi} \leqslant \frac{\hat{\delta}(P)}{\hat{M}'(0)}, \tag{4.62}$$

and hence (4.59) becomes

$$\| \varphi \|_{L_2} \leqslant \frac{P\hat{\delta}(P)}{P_1\hat{\delta}(P_1)} \| \varphi \|_{L_2}. \tag{4.63}$$

If $P\hat{\delta}(P) < P_1\hat{\delta}(P_1)$, then (4.63) can only be satisfied by $\| \varphi \|_{L_2} = 0$, which gives us uniqueness for the trivial solution. Moreover, $P\hat{\delta}(P) < P_1\hat{\delta}(P_1)$ implies that $P < P_1$.

As an alternative approach to the proof of Theorem 6, one could apply variational principles to the boundary value problem (2.24) while utilizing (3.2). A proof of this type was used in a simpler buckling problem considered by Reiss [9].

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REFERENCES

1. S. Antman, General solutions for plane extensible elasticae having nonlinear stress-strain laws, *Quart. Appl. Math.* **26** (1968), 35–47.
2. S. Antman, Equilibrium states of nonlinearly elastic rods, *J. Math. Anal. Appl.* **13** (1968), 476–501.
3. P. B. Bailey, L. F. Shampine, and P. E. Waltman, "Nonlinear Two Point Boundary Value Problems," Academic Press, New York, 1968.
4. J. M. Greenberg, On the equilibrium configurations of compressible slender bars, *Arch. Rational Mech. Anal.* **27** (1967), 181–194.
5. J. B. Keller, Bifurcation theory for ordinary differential equations, *in* "Bifurcation Theory and Nonlinear Eigenvalue Problems" (J. B. Keller and S. Antman, Eds.), Benjamin, New York, 1969.
6. A. E. H. Love, "A Treatise on the Mathematical Theory of Elasticity" Cambridge University Press, Cambridge, England, 1927.
7. W. E. Olmstead, On the preclusion of buckling of a nonlinearly elastic rod, Report No. 70-15, Series in Applied Mathematics, Northwestern University, Evanston, Ill., 1970.
8. P. H. Rabinowitz, A note on a nonlinear elliptic equation, *Indiana U. Math. J.* **22** (1972), 43–49.
9. E. Reiss, Column buckling — an elementary example of bifurcation, *in* "Bifurcation Theory and Nonlinear Eigenvalue Problems (J. B. Keller and S. Antman, Eds.), Benjamin, New York, 1969.
10. D. H. Sattinger, Topics in stability and bifurcation theory, *in* "Lecture Notes in Mathematics," No. 309, Springer–Verlag, New York, 1973.
11. D. H. Sattinger, Bifurcation of Periodic Solutions of the Navier–Stokes Equations, *Arch. Rational Mech. Anal.* **41** (1971), 66–80.
12. I. Stakgold, "Boundary Value Problems of Mathematical Physics I," Macmillan, New York, 1967.
13. I. Stakgold, Branching of solutions of nonlinear equations, *SIAM Rev.* **13** (1971), 289–332.
14. E. C. Titchmarsh, "The Theory of Functions," Oxford University Press, Oxford, England, 1939.