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Estimates for the Poisson kernel and Hardy spaces on compact manifolds

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Abstract

We study Hardy spaces on the boundary of a smooth open subset or $\mathbb{R}^n$ and prove that they can be defined either through the intrinsic maximal function or through Poisson integrals, yielding identical spaces. This extends to any smooth open subset of $\mathbb{R}^n$ results already known for the unit ball. As an application, a characterization of the weak boundary values of functions that belong to holomorphic Hardy spaces is given, which implies an F. and M. Riesz type theorem.

0. Introduction

The real Hardy space $H^p(\mathbb{R}^N)$, $0 < p \leqslant \infty$, introduced in 1971 by Stein and Weiss [16], is equal to $L^p(\mathbb{R}^N)$ for $p > 1$, is properly contained in $L^1(\mathbb{R}^N)$ for $p = 1$ and is a space of not necessarily locally integrable distributions for $0 < p < 1$. For $p \leqslant 1$, $H^p(\mathbb{R}^N)$ is an advantageous substitute for $L^p(\mathbb{R}^N)$ [15], as the latter is not a space of distributions and has trivial dual if $p < 1$ while for $p = 1$, $L^1(\mathbb{R}^N)$ is not preserved by singular integrals.

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Let us choose a function $\Phi \in \mathcal{S}(\mathbb{R}^N)$, with $\int \Phi\, dz \neq 0$ and write $\Phi_\varepsilon(z) = \varepsilon^{-N}\Phi(z/\varepsilon)$, $z \in \mathbb{R}^N$, and

$$M_\Phi f(z) = \sup_{0<\varepsilon<\infty} \left|(\Phi_\varepsilon * f)(z)\right|.$$

Then [15]

$$H^p(\mathbb{R}^N) = \left\{ f \in \mathcal{S}'(\mathbb{R}^N):\ M_\Phi f \in L^p(\mathbb{R}^N)\right\}.$$

An obstacle to the localization of the elements of $H^p(\mathbb{R}^N)$, $0 < p \leqslant 1$, is that ψu may not belong to $H^p(\mathbb{R}^N)$ for $\psi \in C_c^\infty(\mathbb{R}^N)$ and $u \in H^p(\mathbb{R}^N)$. In particular, $H^p(\mathbb{R}^N)$, $0 < p \leqslant 1$, is not preserved by pseudo-differential operators and is not well defined on manifolds, a fact that hinders applications to PDE with variable coefficients. On the other hand, $H^p(\mathbb{R}^N)$ is preserved by singular integrals with sufficiently smooth kernels, which implies that it is *locally* preserved by pseudo-differential operators of order zero (and type $\rho = 1$, $\delta = 0$). This fact was used by Strichartz in 1972 [17] who defined $H^1(\Sigma)$ for a compact smooth N-dimensional manifold Σ as the space of all $f \in L^1(\Sigma)$ such that $Tf \in L^1(\Sigma)$ for all pseudo-differential operators T of order zero. Then Peetre [14] proposed in 1975 a more elementary definition of $H^p(\Sigma)$, $p > 0$, in terms of an intrinsic maximal function. More precisely, he set

$$H^p(\Sigma) = \left\{ f \in \mathcal{D}'(\Sigma):\ \mathcal{M}_s f \in L^p(\Sigma)\right\},$$
$$\mathcal{M}_s f(x) = \sup_{\phi \in K_s(x)} \left|\langle f, \phi\rangle\right|,$$

where $K_s(x)$ is the space of smooth functions $\phi \in C^\infty(\Sigma)$ such that there is an $h > 0$ such that $\operatorname{supp}\phi \subset B(x,h)$ and $\sup_{0\leqslant k\leqslant s} h^{N+k}\|\phi\|_k \leqslant 1$. Here $\mathcal{D}'(\Sigma)$ is the space of distributions in Σ, $\operatorname{supp}\phi$ denotes the support of ϕ, $B(x,h)$ is the Riemannian ball centered at x of radius h (assume a Riemannian metric is given on Σ), $\|\phi\|_k$ denotes the norm in $C^k(\Sigma)$ and s is a conveniently large integer that depends on p. It turns out that for $p = 1$ the spaces $H^1(\Sigma)$ defined by Strichartz and Peetre coincide.

A way around the problem that $H^p(\mathbb{R}^N)$ is not localizable for $0 < p \leqslant 1$ is the definition of localizable Hardy spaces $h^p(\mathbb{R}^N)$ [9,15] by means of the truncated maximal function

$$m_\Phi f(z) = \sup_{0<\varepsilon\leqslant 1} \left|(\Phi_\varepsilon * f)(z)\right|,$$
$$h^p(\mathbb{R}^N) = \left\{ f \in \mathcal{S}'(\mathbb{R}^N):\ m_\Phi f \in L^p(\mathbb{R}^N)\right\}.$$

It follows that the space $h^p(\mathbb{R}^N)$ is stable under multiplication by test functions as well as by change of variables that behave well at infinity and also that $h^p(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ for $1 < p \leqslant \infty$. This opens the doorway to a definition of Hardy spaces on smooth manifolds through localization. Namely, if $\{U_\alpha, \Phi_\alpha\}$ is a family of local charts and $\{\varphi_i\}$ a partition of the unity subordinated to the covering U_α then we say that $f \in h^p(\Sigma)$ if, and only if, $f\varphi_i \circ \Phi_\alpha^{-1} \in h^p(\mathbb{R}^n)$. It is known that $h^p(\Sigma) = H^p(\Sigma)$ (see, e.g., [4]).

Consider now a bounded open subset $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega = \Sigma$ and given $f \in \mathcal{D}'(\Sigma)$ let $u \in C^\infty(\Omega)$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{on } \Omega, \\ u|_\Sigma = f. \end{cases} \tag{0.1}$$

Then u gives rise to two maximal functions:

(i) If $\boldsymbol{\nu}_x$ is the outer normal unit vector field defined at $x \in \Sigma$, the normal maximal function is

$$u^{\perp}(x) = \sup_{0<t<t_0} \bigl|u(x - t\boldsymbol{\nu}_x)\bigr|, \quad x \in \Sigma,$$

where t_0 is chosen small so, in particular, $x - t\boldsymbol{\nu}_x \in \Omega$ and $\operatorname{dist}(x - t\boldsymbol{\nu}_x, \Sigma) = t$ whenever $x \in \Sigma$ and $0 < t \leqslant t_0$.

(ii) For fixed $a > 1$, the nontangential maximal function is

$$u_a^*(x) = \sup_{z \in \Gamma_a(x)} \bigl|u(z)\bigr|, \quad x \in \Sigma,$$

where $\Gamma_a(x) = \{z \in \Omega\colon |z - x| < a \operatorname{dist}(z, \Sigma)\}$ is the n-dimensional analogue of a Stolz region, here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\operatorname{dist}(z, \Sigma)$ the distance from z to Σ.

When $\Omega = B \subset \mathbb{R}^n$, $n \geqslant 2$, is the unit ball and $\Sigma = S^{n-1}$, it is known that $f \in H^p(S^{n-1})$, $0 < p \leqslant \infty$, if and only if $u_a^* \in L^p(S^{n-1})$ or, equivalently, if and only if $u^{\perp} \in L^p(S^{n-1})$. This is classic for the unit circle [7] and due to Colzani [5] for $n \geqslant 3$. A relevant fact in the proof is that explicit formulas are known for the Poisson kernel that furnishes the solution of the boundary problem (0.1) when Ω is a ball. In particular, these formulas show that if $P(z, x)\colon \Omega \times \partial\Omega \to \mathbb{R}$ is the Poisson kernel of the domain Ω then there exist constants $C_{\alpha\beta} > 0$ for every multi-indexes $\alpha, \beta \in \mathbb{Z}_+^n$ such that

$$\bigl|D_z^{\alpha} D_x^{\beta} P(z, x)\bigr| \leqslant \frac{C_{\alpha\beta}}{|x - z|^{n-1+|\alpha|+|\beta|}}, \quad (z, x) \in \Omega \times \partial\Omega, \tag{$K_{\alpha\beta}$}$$

at least when $\Omega = B$. For $\alpha = 0$ and $\beta = 0$ estimate (K_0) is well known for general smoothly bounded domains (actually, class C^2 suffices). A proof of this fact was given by Kerzman in an unpublished set of notes [12] and can be found in [13, p. 332]. In this work we prove $(K_{\alpha\beta})$ for all α and β. This is the key to the characterization of the spaces $H^p(\partial\Omega)$, $0 < p \leqslant \infty$, in terms of the maximal functions $u^{\perp}$ and u_a^*. Since this characterization is well known for $p > 1$, we are mainly concerned in this paper with the case $0 < p \leqslant 1$ although the proofs work as well for any p.

The paper is organized as follows. In Section 1 we prove estimates $(K_{\alpha\beta})$ by locally flattening the boundary and constructing a pseudo-differential approximation of the Poisson operator following the method of Treves [19] to construct a parameterization of the heat equation. The pseudo-differential approximation gives a wealth of information about the Poisson kernel and in particular shows the required estimates for its derivatives. In Section 2 we study approximations of the identity that are obtained from the Poisson operator but converge faster to the identity. In Section 3 we prove several technical lemmas about these approximations that are instrumental in the proof of the equivalence of the L^p "norms" of the different maximal functions defined in terms of Poisson integrals—the equivalence of different Poisson's maximal functions is discussed in Section 4—with the intrinsic maximal function, which is the subject of Section 5. Finally, in Section 6, we discuss holomorphic Hardy spaces $\mathcal{H}^p(\Omega)$, $\Omega \subset \mathbb{C}^n$, and prove that every $f \in \mathcal{H}^p(\Omega)$ has a weak boundary value $bf \in H^p(\partial\Omega)$ of which it is its Poisson integral. This establishes an isomorphism of topological vector spaces between $\mathcal{H}^p(\Omega)$ and the subspace of $H^p(\partial\Omega)$ of distributions that are boundary value of some holomorphic function in Ω. We also prove an "F. and M. Riesz theorem," showing that if a measure in $\partial\Omega$ is the boundary value of a holomorphic function defined on Ω it must be absolutely continuous with respect to Lebesgue measure.

We use the standard notation for distributional spaces, so L^p denotes a Lebesgue space, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space, its dual $\mathcal{S}'(\mathbb{R}^n)$ denotes the tempered distributions, $\mathcal{D}'(\Sigma)$ denotes the space of distributions on a manifold Σ, C^r denotes the space of continuous functions with continuous derivatives up to order r if r is a positive integer and the corresponding Hölder space if $r > 0$ is not integral. Different Hardy spaces are denoted by H^p, $\mathcal{H}^p$ and h^p. We also denote by C a positive constant that may change from one line to the next.

1. Pointwise estimates for the Poisson kernel

The following theorem is the main result of this section. It gives estimates that we shall later need to characterize Hardy spaces on the boundary of a smooth domain of $\mathbb{R}^n$.

Theorem 1.1. *Let $P(z,x)$ be the Poisson kernel of a bounded domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary Σ. For every multi-indexes $\alpha \in \mathbb{Z}_+^n$ and $\beta \in \mathbb{Z}_+^{n-1}$ there exist a constant $C_{\alpha\beta} = C_{\alpha\beta}(\Omega) > 0$ such that*

$$\left|D_z^\alpha D_x^\beta P(z,x)\right| \leqslant \frac{C_{\alpha\beta}}{|x-z|^{n-1+|\alpha|+|\beta|}}, \quad (z,x) \in \Omega \times \Sigma, \tag{$K_{\alpha\beta}$}$$

Proof. Fix $a > 1$ and consider the nontangential region inside Ω with vertex at x given by

$$\Gamma_a(x) = \bigl\{z \in \Omega\colon\ |z-x| < a\,\mathrm{dist}(z,\Sigma)\bigr\}.$$

For fixed $r_0 > 0$ consider the set

$$X = \bigl\{(z,x) \in \overline{\Omega} \times \Sigma\colon\ |z-x| \geqslant r_0\bigr\}$$

and observe that $|z-x|^{n+|\alpha|+|\beta|-1} D_z^\alpha D_x^\beta P(z,x)$ is continuous, thus bounded, on the compact set X, because $P(z,x)$ is smooth on $\overline{\Omega} \times \Sigma \setminus \Sigma \times \Sigma$. Therefore, there is no loss of generality if we prove $(K_{\alpha\beta})$ assuming that $|z-x| < r_0$ and we shall do so. The proof is divided into two cases.

Case 1. $z \notin \Gamma_a(x)$

By the compactness of Σ it is enough to prove the estimate when x is in a small neighborhood of an arbitrary point $x_0 \in \Sigma$. Since $|z-x| < r_0$ we may assume that both x and z belong to a small neighborhood of x_0. The initial step is to flatten the boundary in that neighborhood. Thus we consider a diffeomorphism that takes a neighborhood W of x_0 onto a neighborhood of the closure of the cube $Q \subset \mathbb{R}_x^{n-1} \times \mathbb{R}_t$ given by $|x| < 1$, $|t| < 1$ so that x_0 is mapped to $(0,0)$, $\Omega \cap W$ is mapped to $Q^+ = \{(x,t) \in Q\colon\ t > 0\}$ and Σ is flattened to $\{t=0\}$. Using (x,t) as new coordinates the Poisson kernel may be expressed as $\tilde{P}(y,t,x)$ with $z=(y,t)\in Q^+$ and $x=(x_1,\dots,x_{n-1})\in\partial Q^+\cap\{t=0\}$. If $a>0$ is large enough, the condition $z\notin\Gamma_a(x)$ implies $|x-y|>t$. Notice that for $|x-y|>t$, $(|x-y|^2+t^2)^{1/2}$ is comparable to $|x-y|$. Thus, it will be enough to prove that for any $|x|,|y|<1$ and $0<t<|x-y|$

$$\left|D_x^\alpha D_y^\beta D_t^k \tilde{P}(x,t,y)\right| \leqslant \frac{C_{\alpha\beta k}}{|x-y|^{n-1+|\alpha|+|\beta|+k}}. \tag{1.1}$$

Recall that in the original coordinates

$$u(z)=\mathcal{P}\phi(z)=\int_\Sigma P(z,y)\phi(y)\,d\sigma(y), \tag{1.2}$$

where $d\sigma$ indicates the volume element in Σ, solves the Dirichlet problem

$$\begin{cases} \Delta u(z)=0, & z\in\Omega,\\ u(x)=\phi(x), & x\in\Sigma. \end{cases} \tag{1.3}$$

In the new coordinates, (1.3) becomes, with some abuse of notation,

$$\begin{cases} L(t,x,D_x,D_t)u=0,\\ u(x,0)=\phi(x), \end{cases} \tag{1.4}$$

where

$$L(t,x,D_x,D_t)=\frac{\partial^2}{\partial t^2}+2\sum_{j=1}^{n-1} b_j(x,t)\frac{\partial^2}{\partial x_j\partial t}+\sum_{j,k} c_{jk}(x,t)\frac{\partial^2}{\partial x_j\partial x_k}+\cdots \tag{1.5}$$

is an elliptic differential operator with real coefficients and principal symbol

$$\sigma_L(t,x,\tau,\xi)=-\tau^2-2\sum_{j=1}^{n-1}\tau\xi_j b_j(x,t)-\sum_{j,k} c_{j,k}(x,t)\xi_j\xi_k,$$

and the dots in (1.5) denote terms of order one. We now follow the approach of Treves [19, Chapter 3] to construct parameterization of the heat equation. We will apply the machinery of pseudodifferential operators to find a family of pseudodifferential operators $H(t,x,D_x)$, acting on the variable x and depending smoothly on $t>0$ as a parameter, that solves the problem

$$\begin{cases} L\circ H\sim 0 \quad \text{modulo a smooth kernel},\\ H(0,x,D_x)=I. \end{cases} \tag{1.6}$$

The symbol $\sigma_H(t,x,\xi)$ of H is identically equal to 1 for $t=0$ and has order $-\infty$ for $t>0$; furthermore, $\bigcup_{0<t<1}\{\sigma_H(t,x,\xi)\}$ is a bounded subset of $S^0_{1,0}$, the symbol class of order zero and type $\rho=1$, $\delta=0$, defined for $|x|<1$ and $\xi\in\mathbb{R}^{n-1}$. We denote by $\mathcal{L}^m_{1,0}$ the space of operators of order m and type $\rho=1$, $\delta=0$. Since the integral operator $\mathcal{P}$ defined by (1.2), which in the original variables is given by integration against the Poisson kernel, solves (in the new variables) (1.6) exactly, we may regard H as an approximation of $\mathcal{P}$ by pseudo-differential operators. To find H we first construct an operator $D\in\mathcal{L}^1_{1,0}$, such that

$$L\sim\left(\partial_t+\sum_{j=1}^n b_j\frac{\partial}{\partial x_j}-D\right)\left(\partial_t+\sum_{j=1}^n b_j\frac{\partial}{\partial x_j}+D\right) \quad \operatorname{mod}\mathcal{L}^{-\infty}. \tag{1.7}$$

We begin by choosing a pseudo-differential operator induced by the homogeneous function of order one,

$$d_1(t,x,\xi)=\left(\sum_{j,k}c_{jk}(x,t)\xi_j\xi_k-\left(\sum_{j=1}^{n}b_j(x,t)\xi_j\right)^2\right)^{1/2}.$$

The ellipticity of L implies that $d_1(t,x,\xi)\geqslant c|\xi|$ for some $c>0$. Thus, d_1 is an elliptic homogeneous symbol of degree one. Even though d_1 is not a symbol in $S^1_{1,0}$ because it fails to be smooth at the origin, we proceed as usual and after multiplication by a cut-off function that vanishes for $|\xi|<1/2$ and is identically equal to 1 for $|\xi|>1$, we can obtain a symbol in $S^1_{1,0}$ that we still denote by d_1. If $D_1=\mathrm{Op}(d_1)$ then we want to check that

$$L\sim\left(\partial_t+\sum_{j=1}^{n}b_j\frac{\partial}{\partial x_j}-D_1\right)\left(\partial_t+\sum_{j=1}^{n}b_j\frac{\partial}{\partial x_j}+D_1\right)\quad\mathrm{mod}\ \mathcal{L}^1_{1,0}.\tag{1.8}$$

Set $R_1=L-(\partial_t+\sum_{j=1}^n b_j\frac{\partial}{\partial x_j}-D_1)(\partial_t+\sum_{j=1}^n b_j\frac{\partial}{\partial x_j}+D_1)$. Then, the symbolic calculus of pseudo-differential operators shows after a simple computation that the symbol σ_{R_1} of R_1 belongs to $S^1_{1,0}$.

The next step consists in finding a symbol $d_0\in S^0_{1,0}$ such that the operator $D_0=\mathrm{Op}(d_0)$ satisfies

$$L\sim\big(\partial_t+\ell-(D_1+D_0)\big)\big(\partial_t+\ell+(D_1+D_0)\big)\quad\mathrm{mod}\ \mathcal{L}^0_{1,0},$$

where we have written

$$\ell=\sum_{j=1}^{n}b_j\frac{\partial}{\partial x_j}.$$

Now let Q_1 and Q_2 denote respectively $\partial_t+\ell-D_1$ and $\partial_t+\ell+D_1$ and set $R_0=L-(Q_1-D_0)(Q_2+D_0)$. Then, $R_0=L-Q_1Q_2+(Q_2-Q_1)D_0+D_0D_0+[D_0,Q_2]$, and observing that $Q_2-Q_1=2D_1$ and $L\sim Q_1Q_2$ mod $\mathcal{L}^1$ because of (1.8) we have $R_0=D_0D_0+2D_1D_0+R_1$ for some $R_1\in\mathcal{L}^1$. Then, if r_1 is the symbol of R_1 and d_1 is the symbol of D_1, we may take D_0 with symbol

$$d_0(t,x,\tau,\xi)=-\frac{1}{2}\frac{r_1(t,x,\tau,\xi)}{d_1(t,x,\tau,\xi)}$$

and obtain that R_0 has order zero. Keeping up this process we may define a sequence of symbols

$$d_{-j}=-\frac{1}{2}\frac{r_{1-j}}{d_1+d_0+\cdots+d_{1-j}}\in S^{-j}_{1,0}$$

so that their associated operators $D_k=\mathrm{Op}(d_k)$, $k=1,0,\dots,-j$, satisfy

$$L\sim(\partial_t+\ell-D_1-D_0-\cdots-D_{-j})(\partial_t+\ell+D_1+D_0+\cdots+D{-j})\quad\mathrm{mod}\ \mathcal{L}^{-j}_{1,0}.$$

Since the order of d_{-j} goes to $-\infty$ as $j\to\infty$, we may find a symbol $d\in S^1_{1,0}$ such that

$$d(t,x,\xi)\sim\sum_{j=0}^{\infty}d_{1-j}(t,x,\xi)\quad\mathrm{mod}\ S^{-\infty}$$

in the sense that $d - \sum_{j=0}^{k} d_{1-j} \in S_{1,0}^{-k}$ or any $k = 0, 1, 2 \ldots$. Hence, the operator $D = \mathrm{Op}(d)$ satisfies (1.7). If we call

$$A_1 = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} - D \quad \text{and} \quad A = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} + D,$$

we may rewrite (1.7) as

$$L \sim (\partial_t + A_1)(\partial_t + A) \quad \operatorname{mod} \mathcal{L}^{-\infty}.$$

Hence, in order to obtain (1.6) it will suffice to find a family of operators $H(t, x, D_x,)$, $0 \leqslant t < 1$, such that

$$(\partial_t + A) \circ H(t, x, D_x) \sim 0 \quad \text{modulo a smooth kernel,} \tag{1.9}$$

with the additional property $H(0, x, D_x) = \text{identity}$. Note that the symbol $a(t, x, \xi)$ of A has principal symbol $a_1(t, x, \xi) = d_1(t, x, \xi) + i \sum_{j=0}^{n-1} b_j(x, t)\xi_j$. To construct $H(t, x, D_x)$ with symbol $\sigma_H(t, x, \xi) = h(t, x, \xi)$ we propose

$$h(t, x, \xi) \sim e^{-\int_0^t a_1(s,x,\xi)\,ds}\bigl(1 + \kappa_{-1}(t, x, \xi) + \kappa_{-2}(t, x, \xi) + \cdots\bigr) \tag{1.10}$$

with $\kappa_{-j} \in S_{1,0}^{-j}$. An important point here is that, because $d_1(t, x, \xi) \geqslant c > 0$ for $|\xi| > 1$, $e(t, x, \xi) = \exp(-\int_0^t a_1(s, x, \xi)\,ds)$ satisfies the following estimates

$$\bigl|D_x^\alpha D_\xi^\beta D_t^k e(t, x, \xi)\bigr| \leqslant C_{\alpha\beta k}\bigl(1 + |\xi|\bigr)^{k-|\beta|}$$

expressing the fact that $D_t^k e \in S_{1,0}^k$ uniformly in t. The proof of [19, Theorem 1.1] shows that $\kappa_{-1}, \kappa_{-2}, \ldots$ satisfying $\kappa_{-j}(0, x, \xi) = 0$ may be inductively determined by a process similar to the construction of D so that if $h(t, x, \xi)$ is given by (1.10) then $H = \mathrm{Op}(h)$ satisfies (1.9). Furthermore,

$$\bigl|D_x^\alpha D_\xi^\beta D_t^k h(t, x, \xi)\bigr| \leqslant C_{\alpha\beta k}\bigl(1 + |\xi|\bigr)^{k-|\beta|}. \tag{1.11}$$

Consider the kernel of the pseudo-differential $H(t, x, D_x)$

$$\mathfrak{h}(t, x, y) = \frac{1}{(2\pi)^{n-1}} \int e^{i(x-y)\xi} h(t, x, \xi)\,d\xi.$$

It follows from standard estimates for the kernel of pseudo-differential operators (see, e.g., [1,18]) that estimates (1.11) imply the estimates

$$\bigl|D_x^\alpha D_y^\beta D_t^k \mathfrak{h}(t, x, y)\bigr| \leqslant \frac{C_{\alpha\beta k}}{|x - y|^{n-1+k+|\alpha|+|\beta|}}. \tag{1.12}$$

Notice that estimates (1.12) for $\mathfrak{h}$ are analogous to the estimates (1.1) that we wish to prove for $\tilde{P}$. Thus, to obtain (1.1) it will be enough to find smooth functions $\mu(t, x, y)$, $\rho(t, x, y)$ defined for $|x|, |y| < 1$, $0 \leqslant t < 1$ such that $\tilde{P}(x, t, y) = \mu(\mathfrak{h} + \rho)(t, x, y)$. Let $\mathfrak{p}(x, t, y)$ be the kernel of the integral operator $\mathcal{P}$ expressed in the new coordinates: its expression is readily obtained from (1.2) (which gives $\mathcal{P}$ in the original coordinates) by reverting to the new coordinates. We then see that $\mathfrak{p}(x, t, y) = \tilde{P}(x, t, y)/\mu(y)$ where $\mu^{-1}(y)\,dy$ is the expression of the area element $d\sigma$ of Σ in the new coordinates, in particular $\mu > 0$ and is smooth. Therefore, we need only show that $\rho = \mathfrak{p} - \mathfrak{h}$ is smooth up to the boundary. This follows from the fact that the operators $\mathcal{P}$ and H, whose kernels are respectively $\mathfrak{p}$ and $\mathfrak{h}$, satisfy $L \circ (\mathcal{P} - H) \sim 0$ modulo smoothing operators and $(\mathcal{P} - H)|_{t=0} = 0$. That this is so is already a consequence of the "uniqueness" part of [19, Theorem 1.1] but here it seems simpler to give a direct argument. By returning to the original coordinates, let us transfer the operator H to the initial neighborhood $W \ni x_0 \in \Sigma$ obtaining an operator that we still call H. Using a cut-off function χ that is identically 1 in a neighborhood ω of x_0 such $\overline{\omega} \subset W$, it is easy to construct an operator $\tilde{H} = \chi H \chi : C^\infty(\Sigma) \to C^\infty(\Omega)$ such that

(a) $\Delta \tilde{H}$ is regularizing when acting on distributions compactly supported in ω,
(b) $\tilde{H}\phi|_\Sigma = \phi$ if ϕ is supported in $\omega \cap \Sigma$.

If $\phi \in \mathcal{D}'(\Sigma)$, we have that $\Delta(\mathcal{P}\phi - \tilde{H}\phi) = \psi$, where ψ is smooth in $\overline{\Omega} \cap \omega$ in view of (a). Furthermore, (b) implies that $\mathcal{P}\phi - \tilde{H}\phi$ vanishes on $\omega \cap \Sigma$. By boundary elliptic regularity we conclude that $\mathcal{P}\phi - \tilde{H}\phi \in C^\infty(\overline{\Omega} \cap \omega)$ and since this holds for any distribution $\phi \in \mathcal{D}'(\Sigma)$ we conclude that the kernel of $\mathcal{P} - \tilde{H}$ is smooth when restricted to $(\omega \cap \overline{\Omega}) \times (\omega \cap \overline{\Omega})$, which proves, as we wished, that $\mathfrak{p} - \mathfrak{h}$ is smooth up to the boundary. The proof of Case 1 is complete.

Case 2. $|x - y| \leqslant t$

For $|x - y| \leqslant t$, $(|x - y|^2 + t^2)^{1/2}$ is comparable to t. Thus, it will be enough to prove that for any $|x|, |y| < 1$ and $0 < |x - y| \leqslant t < 1$

$$\left|D_x^\alpha D_y^\beta D_t^k \tilde{P}(x, t, y)\right| \leqslant \frac{C_{\alpha\beta k}}{t^{n-1+|\alpha|+|\beta|+k}}. \tag{1.13}$$

As before, it is enough to prove analogous estimates for the kernel $\mathfrak{h}(t, x, y)$ of the pseudo-differential approximation H of the Poisson kernel. This leads us to look more closely to its symbol $h(t, x, y) = \exp(-\int_0^t a_1(s, x, \xi)\, ds)\kappa(t, x, \xi)$ given by (1.10). We recall that a_1 is defined by $a_1(t, x, \xi) = d_1(t, x, \xi) + i \sum_{j=0}^{n-1} b_j(x, t)\xi_j$ where $d_1(t, x, \xi) > c|\xi|$ for $|\xi| > 1$ and $\kappa(t, x, \xi)$ has order zero uniformly in t, furthermore the functions $b_j(x, t)$ are real. Thus

$$\left|h(t, x, y)\right| \leqslant C \exp\bigl(-tc|\xi|\bigr), \quad \xi \in \mathbb{R}^{n-1},\ 0 < t < 1,$$

and

$$\mathfrak{h}(t, x, y) = \frac{1}{(2\pi)^{n-1}} \int e^{i(x-y)\xi} h(t, x, \xi)\, d\xi \tag{1.14}$$

is easily seen to satisfy the estimate

$$\left|\mathfrak{h}(t, x, y)\right| \leqslant C \int\limits_{\mathbb{R}^{n-1}} \exp\bigl(-tc|\xi|\bigr)\, d\xi \leqslant \frac{C'}{t^{n-1}}.$$

Similarly, we see that for $\xi \in \mathbb{R}^{n-1}$ and $0 < t < 1$

$$\left|D_x^\alpha D_t^k h(t, x, \xi)\right| \leqslant C_{\alpha k}\bigl(1 + |\xi|\bigr)^{|\alpha|+k} \exp\bigl(-tc|\xi|\bigr)$$

which implies, after differentiation of (1.14), that

$$\begin{aligned}\left|D_x^\alpha D_y^\beta D_t^k \mathfrak{h}(t,x,y)\right| &\leqslant C_{\alpha\beta k} \int\limits_{\mathbb{R}^{n-1}} \bigl(1+|\xi|\bigr)^{|\alpha|+|\beta|+k} \exp\bigl(-tc|\xi|\bigr)\, d\xi \\ &\leqslant \frac{C_{\alpha\beta k}}{t^{n-1+|\alpha|+|\beta|+k}}. \end{aligned} \tag{1.15}$$

Reasoning as in Case 1, we see that the estimates (1.15) for $D_x^\alpha D_y^\beta D_t^k \mathfrak{h}$ imply the analogous estimates (1.13). The proof of Theorem 1.1 is complete. □

Corollary 1.2. *Let $P(z,x)$ be the Poisson kernel of a bounded domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary Σ. There exist a constant $C = C(\Omega) > 0$ such that for all $(z,x) \in \Omega \times \Sigma$*

$$\left|P(z,x)\right| \leqslant C \operatorname{dist}(z,\Sigma) \min\left(\frac{1}{\operatorname{dist}(z,\Sigma)^n}, \frac{1}{|x-z|^n}\right). \tag{1.16}$$

Proof. It is enough to prove (1.16) when $|z-x|$ is small using local coordinates (x,t). As in the proof of the theorem, we work in a thin tubular neighborhood of Σ. We first point out that if $z=(y,t)$, (1.13) for $\alpha=\beta=k=0$ gives $|P(z,x)| \leqslant Ct^{1-n} \leqslant C' \operatorname{dist}(z,\Sigma)^{1-n}$, since $t \sim \operatorname{dist}(z,\Sigma)$. This gives (1.16) when $z \in \Gamma_a(x)$, in which case $|z-x| \sim \operatorname{dist}(z,\Sigma)$. When $z=(y,t) \notin \Gamma_a(x)$ we use the mean value theorem to get $|P(z,x)| \leqslant \sup|\nabla_z P(z,x)| \operatorname{dist}(z,\Sigma)$, where the supremum is taken along the segment that joins z to the point $\zeta \in \Sigma$ such that $|z-\zeta| = \operatorname{dist}(z,\Sigma)$. Using $(K_{\alpha\beta})$ with $\alpha=0$, $|\beta|=1$ we obtain $|P(z,x)| \leqslant C|x-z|^{-n} \operatorname{dist}(z,\Sigma)$. The corollary easily follows. □

Remark 1.3. The proof of Theorem 1.1 shows that the Laplacian may be replaced for any second order elliptic operator with smooth real coefficients defined in a neighborhood of $\overline{\Omega}$. In this case, the Poisson kernel must be replaced by the kernel of the operator that solves the Dirichlet problem.

Remark 1.4. Estimates for the Poisson kernel can be obtained from estimates on the Green function. However, we point out that classical estimates for the Green function of the Laplace–Beltrami operator on compact manifolds with boundary are interior estimates, in the sense that constants blow up when approaching the boundary [2, p. 112], so they do not seem to imply Theorem 1.1.

2. Approximations of the identity

Assume without loss of generality that Ω is such that $z_t = x - t\boldsymbol{\nu}_x \in \Omega$ and $t = \operatorname{dist}(z_t,\Sigma)$ if $x \in \Sigma$, $0 < t \leqslant 1$. We may then shrink Ω along the normal direction for $0<t<1$ and obtain the open set

$$\Omega_t = \bigl\{z \in \Omega\colon \operatorname{dist}(z,\Sigma) > t\bigr\}$$

with boundary

$$\Sigma_t = \partial\Omega_t = \bigl\{x - t\boldsymbol{\nu}_x(x)\colon x \in \Sigma\bigr\}.$$

Clearly, $\bigcup_{0<t<1} \Omega_t = \Omega$. The map $\Sigma \ni x \mapsto x - t\boldsymbol{\nu}_x$ is a diffeomorphism for fixed $0 < t < 1$ and thus we may identify each Σ_t with Σ. With this identification, the operator

$$\mathcal{D}'(\Sigma) \ni \phi \mapsto \mathcal{P}\phi|_{\Sigma_t} \in C^\infty(\Sigma_t)$$

may be regarded as an operator from $\mathcal{D}'(\Sigma)$ into $C^\infty(\Sigma)$ that we denote by $\mathcal{P}_t$. The fact that $\mathcal{P}\phi(z) \to \phi(x)$ as $\Omega \ni z \to x \in \Sigma$ when $\phi \in C(\Sigma)$ implies that the regularizations $\mathcal{P}_t\phi$ converge to ϕ as $t \to 0$, i.e., $\mathcal{P}_t$ may be regarded as an approximation of the identity. Let $0 < r < 1$. The Hölder space $C^r(\Sigma)$ is defined as

$$C^r(\Sigma) = \{u \in C(\Sigma),\ \|u\|_r < \infty\},$$

where

$$\|u\|_r = |u|_r + |u|_0,$$
$$|u|_0 = \sup_{x\in\Sigma} |u(x)|,$$
$$|u|_r = \sup_{\substack{x,y\in\Sigma\\ x\neq y}} \frac{|u(x)-u(y)|}{d(x,y)^r}$$

and $d(x, y)$ denotes the geodesic distance in Σ. If $k > 0$ is an integer and $r = k + \gamma$, for some $0 < \gamma < 1$, $C^r(\Sigma)$ is defined as

$$C^r(\Sigma) = \{u \in C^k(\Sigma),\ \|Q(x,D)u\|_\gamma < \infty\}$$

for all differential operators $Q(x, D)$ of order $\leqslant k$ with smooth coefficients defined on Σ. Since $u = \mathcal{P}\phi$ solves the Dirichlet problem with boundary value ϕ, it turns out that if $\phi \in C^r(\Sigma)$, $0 < r < 1$, then $\mathcal{P}\phi \in C^r(\overline{\Omega})$ [8]. It follows that $|\phi(x) - \mathcal{P}_t\phi(x)| = O(t^r)$ uniformly in $x \in \Sigma$, so this gives an estimate of approximation speed of $\mathcal{P}_t\phi$ to ϕ for $0 < r < 1$ which increases with r. However, if we take $r > 1$ the exponent will not increase beyond 1. Thus, it is convenient to replace $\mathcal{P}_t$ by another approximation of the identity that yields a faster approximation. This can be obtained by linear combinations of $\mathcal{P}_t$ evaluated at different times t. If $f(s)$, $s \in \mathbb{R}$, we recall that the difference operator with step $t > 0$ is defined as $\Delta_t f(s) = f(s+t) - f(s)$. Then $\Delta_t^2 f(s) = \Delta_t(\Delta_t f)(s) = f(s+2t) - 2f(s+t) + f(s)$ and if $L \geqslant 1$ is an integer

$$\Delta_t^L f(s) = \sum_{j=0}^{L} (-1)^{L-j} \binom{L}{j} f(s+jt).$$

Let $f^{(L)}$ denote the derivative of order L of f. Taylor's formula for $\Delta^L f(s)$ when $f^{(L)}$ is continuous is given by

$$\Delta_t^L f(s) = t^L \int_{[0,1]^L} f^{(L)}\big(s + t(\tau_1 + \cdots + \tau_L)\big)\, d\tau. \tag{2.1}$$

If $f \in C^r(\mathbb{R})$, $r = L + \gamma$, $0 < \gamma < 1$, we have

$$\Delta_t^{L+1} f(s) = t^L \int_{[0,1]^L} \Delta_t f^{(L)}\big(s + t(\tau_1 + \cdots + \tau_L)\big)\, d\tau$$

which gives the estimate

$$\left|\Delta_t^{L+1} f(s)\right| = t^r \|f^{(L)}\|_\gamma \leqslant t^r \|f\|_r. \tag{2.2}$$

Next we define

$$S_t^L = \sum_{j=0}^{L} (-1)^{L-j} \binom{L}{j} \mathcal{P}_{jt}, \quad 0 < t \leqslant 1/L,$$

where it is understood that $\mathcal{P}_0 = I =$ identityoperator. For $t = 0$ we see that $S_0^L = (1-1)^L I = 0$, so

$$\Gamma_t^L = \sum_{j=1}^{L} (-1)^{j+1} \binom{L}{j} \mathcal{P}_{jt} = I - (-1)^L S_t^L$$

is an approximation of the identity. The next lemma shows that $\Gamma_t^L \phi$ approximates ϕ faster than $\mathcal{P}_t \phi$ is ϕ is sufficiently regular.

Lemma 2.1. *Let $0 < r < L$ not be an integer. There exists $C > 0$ such that for every $\phi \in C^L(\Sigma)$ and $\alpha \in \mathbb{Z}_+^{n-1}$, $0 \leqslant |\alpha| < r$,*

$$\sup_{x \in \Sigma_t} \left| D_x^\alpha \left(\Gamma_t^L \phi(x) - \phi(x) \right) \right| = \left\| D_x^\alpha S_t^L \phi \right\|_{L^\infty} \leqslant C t^{r-|\alpha|} \|\phi\|_r. \tag{2.3}$$

Proof. Consider first the case $\alpha = 0$, so we wish to show that $|S_t^L \phi(x)| \leqslant C t^r \|\phi\|_r$. Let $u = \mathcal{P}\phi$, so u is harmonic in Ω and has the boundary value ϕ. Since $\phi \in C^r(\Sigma)$, standard Hölder boundary estimates [8] imply that $u \in C^r(\overline{\Omega})$ and $\|u\|_r \leqslant C\|\phi\|_r$. Then

$$\begin{aligned} S_t^L \phi(x) &= \sum_{j=0}^{L} (-1)^{L-j} \binom{L}{j} u(x - t\boldsymbol{\nu}_x) \\ &= t^{L-1} \int\limits_{[0,1]^{L-1}} \Delta_t D_t^{L-1} u\big(x - t(\tau_1 + \cdots + \tau_{L-1})\boldsymbol{\nu}_x\big)\, d\tau. \end{aligned} \tag{2.4}$$

We may majorize $|\Delta_t D_t^{L-1} u|$ by $t^\gamma \|D_t^{L-1} u\|_\gamma$ where $\gamma = r - L + 1$. Writing

$$\begin{aligned} &D_t^{L-1} u\big(x - t(\tau_1 + \cdots + \tau_{L-1})\boldsymbol{\nu}_x\big) \\ &\quad = \sum_{|\alpha| = L-1} \big(-t(\tau_1 + \cdots + \tau_{L-1})\boldsymbol{\nu}_x\big)^\alpha D_z^\alpha u\big(x - t(\tau_1 + \cdots + \tau_{L-1})\boldsymbol{\nu}_x\big) \end{aligned}$$

as a sum of derivatives of u of order $L-1$ we have $\|D_t^{L-1} u\|_\gamma \leqslant C\|u\|_{L-1+\gamma} = C\|u\|_r \leqslant C'\|\phi\|_r$, so we get $|\Delta_t D_t^{L-1} u| \leqslant C t^\gamma \|\phi\|_r$. Plugging this estimate in (2.4) yields (2.3) for $\alpha = 0$.

For $|\alpha| = 1$ we write $D_x^\alpha S_t^L \phi = S_t^L D_x^\alpha \phi + [D_x^\alpha, S_t^L]\phi$. The estimate already proved for $\alpha = 0$ with $r - 1$ in the place of r gives

$$\left\| S_t^L D_x^\alpha \phi \right\|_{L^\infty} \leqslant C t^{r-1} \|D^\alpha \phi\|_{r-1} \leqslant C t^{r-1} \|\phi\|_r \tag{2.5}$$

and we need only show a similar estimate for the commutator term $[D_x^\alpha, S_t^L]\phi$. We saw in the proof of Theorem 1.1 how to find family of pseudo-differential operators H_t depending on a parameter t differing from $\mathcal{P}$ by a smooth kernel. That was a local construction but using a finite partition of unity in a tubular neighborhood of Σ we can make it global and find a family of pseudo-differential $H_t \in \mathcal{L}_{1,0}^0(\Sigma)$, $0 < t < 1$, such that

(i) for any $k = 0, 1, \ldots$, the set $\{D_t^k H_t\}_{0<t<1}$ is a bounded subset of $\mathcal{L}_{1,0}^k(\Sigma)$;
(ii) $\mathcal{P}_t - H_t$ is regularizing, more precisely, there exists $r(x, y, t) \in C^\infty(\Sigma \times \Sigma \times [0, 1))$ such that

$$(\mathcal{P}_t - H_t)\phi(x) = \int_\Sigma r(x, y, t)\phi(y)\,d\sigma(y), \quad \phi \in D'(\Sigma).$$

Notice that (i) follows from estimates (1.11) and (ii) from the fact that the kernel $\mathfrak{p} - \mathfrak{h}$ is smooth up to the boundary as shown in the proof or Theorem 1.1. Since $H_t\phi \to \phi$ as $t \to 0$ by construction it follows that $r(x, y, 0) \equiv 0$. Consider the pseudo-differential approximation of S_t for $0 < t \leqslant 1/L$,

$$\tilde{S}_t = \sum_{j=0}^{L} (-1)^{L-j} \binom{L}{j} H_{jt} = t^{L-1} \int_{[0,1]^{L-1}} \Delta_t D_t^{L-1} H_{t(\tau_1+\cdots+\tau_{L-1})}\,d\tau.$$

By (i) $D_t^{L-1} H_t \in \mathcal{L}_{1,0}^{L-1}$ which implies that $\Delta_t D_t^{L-1} H_t \in \mathcal{L}_{1,0}^{L-1}$. It follows that $[D_x^\alpha, \Delta_t D_t^{L-1} H_t] \in \mathcal{L}_{1,0}^{L-1}$. Hence, $[D_x^\alpha, \Delta_t D_t^{L-1} H_{t(\tau_1+\cdots+\tau_{L-1})}]$ maps continuously $C^r(\Sigma)$ to $C^{r-L+1}(\Sigma) = C^\gamma(\Sigma) \subset L^\infty(\Sigma)$ (for the continuity of pseudo-differential operators in $\mathbb{R}^{n-1}$ see [15, p. 253]; these results are extended to smooth compact manifolds by localization). We conclude that

$$\sup_{x\in\Sigma_t} \big|\big[D_x^\alpha, \tilde{S}_t\big]\phi(x)\big| \leqslant Ct^{L-1}\|\phi\|_r \leqslant Ct^{r-1}\|\phi\|_r, \quad \phi \in C^r(\Sigma). \tag{2.6}$$

Since the kernel of $\mathcal{P}_t - H_t$ is $r(x, y, t)$ it follows that the kernel of $S_t - \tilde{S}_t$ is

$$\sum_{j=0}^{L} (-1)^{L-j} \binom{L}{j} r(x, y, jt) = t^L \int_{[0,1]^L} D_t^L r\big(x, y, t(\tau_1 + \cdots + \tau_L)\big)\,d\tau,$$

which easily implies

$$\sup_{x\in\Sigma_t} \big|\big[D_x^\alpha, (S_t - \tilde{S}_t)\big]\phi(x)\big| \leqslant Ct^L\|\phi\|_{L^\infty} \leqslant Ct^{r-1}\|\phi\|_r. \tag{2.7}$$

Now (2.6) and (2.7) imply

$$\sup_{x\in\Sigma_t} \big|\big[D_x^\alpha, S_t\big]\phi(x)\big| \leqslant Ct^{r-1}\|\phi\|_r,$$

which together with (2.5) prove (2.3) for $|\alpha| = 1$. Keeping up this process we may prove (2.3) for all $|\alpha| < r$. $\square$

3. Definitions and technical lemmas

In this section we keep the notation of the previous one, in particular, $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary and we denote its boundary $\partial\Omega$ by Σ. We consider in Σ the Riemannian structure inherited from $\mathbb{R}^n$ a denote by $d\sigma$ its volume element. We recall that the Poisson integral $u(z) = \mathcal{P}\phi(z) = \int_\Sigma P(z,x)\phi(x)\,d\sigma(x)$ which is initially defined for $\phi \in C(\Sigma)$ and solves the Dirichlet problem on Ω with boundary value ϕ has a natural extension to $\mathcal{D}'(\Sigma)$. For fixed $z \in \Omega$, $\Sigma \ni x \mapsto P(z,x) \in C^\infty(\Sigma)$ and we set

$$u(z) = \mathcal{P}f(z) = \bigl\langle f, P(z,\cdot)\bigr\rangle, \quad f \in \mathcal{D}'(\Sigma), \tag{3.1}$$

where the brackets denote the pairing between $\mathcal{D}'(\Sigma)$ and $C^\infty(\Sigma)$ that extends the bilinear form

$$C^\infty(\Sigma) \times C^\infty(\Sigma) \ni (\psi,\phi) \mapsto \int_\Sigma \psi(x)\phi(x)\,d\sigma(x).$$

If u is given by (3.1) then u is harmonic in Ω and its weak boundary value bu is f. That means that for any $\phi \in C^\infty(\Sigma)$

$$\lim_{t\searrow 0}\int_\Sigma u(x - t\boldsymbol{\nu}_x)\phi(x)\,d\sigma(x) = \langle f,\phi\rangle,$$

where $\boldsymbol{\nu}_x$ is the outer normal unit vector field. Since our conclusions are invariant under dilations of Euclidean space, we will assume without loss of generality that Ω is such that for any $x \in \Sigma$ and $0 < t \leqslant 1$ $z_t = x - t\boldsymbol{\nu}_x \in \Omega$ and $t = \operatorname{dist}(z_t, \Sigma)$. To simplify the notation we will often write in the sequel

$$d(z,\Sigma) = \operatorname{dist}(z,\Sigma) = \inf_{x\in\Sigma}|z - x|.$$

We consider now three different maximal functions associated to three different ways of approaching the boundary Σ.

(1) The normal maximal function:

$$u^\perp(x) = \sup_{0<\epsilon<1}\bigl|u(x - \epsilon\boldsymbol{\nu}_x)\bigr|,$$

where $\boldsymbol{\nu}_x$ is the outer normal unit vector field.

(2) The nontangential maximal function:

$$u_a^*(x) = \sup_{z\in\Gamma_a(x)}\bigl|u(z)\bigr|,$$

where for a given $a > 1$ the region

$$\Gamma_a(x) = \bigl\{z \in \Omega\colon |z - x| < a d(z,\Sigma),\ d(z,\Sigma) < 1\bigr\}$$

is the n-dimensional analogue of a truncated Stolz angle.

(3) The tangential maximal function:

$$u_m^{**}(x) = \sup_{\substack{z\in\Omega \\ d(z,\Sigma)<1}} \left|u(z)\right| \left(\frac{d(z,\Sigma)}{|z-x|}\right)^m,$$

where m is a positive integer.

We will make use in Σ of both the geodesic distance $d(x, y)$ and the Euclidean distance $|x-y|$. Since $c_1|x-y| \leqslant d(x, y) \leqslant c_2|x-y|$, $x, y \in \Sigma$, for some positive constants c_1, c_2, switching from one distance to the other in an inequality will cause no trouble. We denote by $B(x, r)$ the ball of center x and radius $r > 0$ in $\mathbb{R}^n$ and by $B_\Sigma(x, r)$ the geodesic ball in Σ. We shall also consider a special family of smooth functions defined on Σ.

Definition 3.1. For every $s \in \mathbb{Z}_+$ and $x \in \Sigma$ let $K_s(x)$ denote the set of all $\phi \in C^\infty(\Sigma)$ such that for some $h > 0$ the conditions below are satisfied:

(i) $\operatorname{supp}\phi \subseteq B_\Sigma(x, h)$,
(ii) $\sup_{0\leqslant k\leqslant s} h^{n-1+k}\|\phi\|_k \leqslant 1$.

Definition 3.2. For $f \in \mathcal{D}'(\Sigma)$ we define the grand maximal function by

$$\mathcal{M}_s f(x) = \sup_{\phi\in K_s(x)} \left|\langle f, \phi\rangle\right|.$$

The space $H^p(\Sigma)$, $p > 0$, is the subspace of $\mathcal{D}'(\Sigma)$ of those f such that $\mathcal{M}_s f \in L^p$ for $s \geqslant [(n-1)/p] + 2$.

Although the definition of $H^p(\Sigma)$ seems to depend on s it does not as long as s is sufficiently large ($s > (\dim \Sigma)/p$ suffices). That this is so for $s \geqslant [(n-1)/p] + 2$ follows also from Theorem 5.1 below.

If $T : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is a continuous linear operator, we denote by ${}^tT : \mathcal{D}'(\Sigma) \to \mathcal{D}'(\Sigma)$ the transpose operator, defined by $\langle {}^tTv, \phi\rangle = \langle v, T\phi\rangle$, $v \in \mathcal{D}'(\Sigma)$, $\phi \in C^\infty(\Sigma)$. In particular, we denote by ${}^t\Gamma_h$, $0 < h < 1$, the transpose of the approximation of the identity discussed in the previous section (from now on, in order to alleviate the notation, we often write Γ_h rather than Γ_h^L unless there is a need to stress the role of L). By Lemma 2.1, $\Gamma_h\phi \to \phi$ in $C^\infty(\Sigma)$ if $\phi \in C^\infty(\Sigma)$. Furthermore, ${}^t\Gamma_h$ is bounded in $C^r(\Sigma)$ for every nonintegral $r > 0$. Indeed, this is clearly so if we replace Γ_h by its pseudo-differential approximation $\tilde{\Gamma}_h$, which is a pseudo-differential of order zero, and the conclusion follows because the difference between the two operators has a smooth kernel. Hence, if $\phi \in C^\infty(\Sigma)$, ${}^t\Gamma_h\Gamma_h\phi \to \phi$ in $C^\infty(\Sigma)$ as $h \searrow 0$. Thus, if $\phi \in C^\infty(\Sigma)$ and $0 < h < 1$ we have the following representation:

$$\phi = {}^t\Gamma_h\Gamma_h\phi + \sum_{j=0}^{\infty} \left({}^t\Gamma_{2^{-j-1}h}\Gamma_{2^{-j-1}h} - {}^t\Gamma_{2^{-j}h}\Gamma_{2^{-j}h}\right)\phi. \tag{$*$}$$

Definition 3.3. Let m, L be positive integer numbers and $f \in \mathcal{D}'(\Sigma)$. We define on Σ a tangential maximal function associated to the approximation of identity Γ_t as

$$\Gamma_m^{**} f(x) = \sup_{\substack{y \in \Sigma \\ 0 \leqslant t \leqslant 1/L}} \left|\Gamma_t f(y)\right| \left(1 + \frac{d(x, y)}{t}\right)^{-m},$$

where $d(x, y)$ denotes the geodesic distance in Σ.

Lemma 3.4. *For all $f \in \mathcal{D}'(\Sigma)$, the following pointwise inequality holds*:

$$\Gamma_m^{**} f(x) \leqslant C u_m^{**}(x), \quad x \in \Sigma. \tag{3.2}$$

Proof. We recall that $u_m^{**}(x)$ is given by

$$u_m^{**}(x) = \sup_{z \in \Omega} \left|u(z)\right| \left(\frac{d(z, \Sigma)}{|z - x|}\right)^m,$$

where $u(z) = \langle f, P(z, \cdot)\rangle$. We have

$$\begin{aligned} \Gamma_m^{**} f(x) &= \sup_{\substack{y \in \Sigma \\ 0 \leqslant t \leqslant 1/L}} \left|\Gamma_t f(y)\right| \left(1 + \frac{d(x, y)}{t}\right)^{-m} \\ &\leqslant \sum_{j=1}^{L} \binom{L}{j} \sup_{\substack{y \in \Sigma \\ 0 \leqslant t \leqslant 1/L}} \left|u(y - jt\boldsymbol{\nu}_y)\right| \left(1 + \frac{d(x, y)}{t}\right)^{-m}. \end{aligned}$$

Notice that for $z_j = y - jt\boldsymbol{\nu}_y$ we have, because $jt = d(z_j, \Sigma)$ when $tj \leqslant 1$,

$$\left(1 + \frac{d(x, y)}{t}\right)^{-m} \leqslant \left(1 + \frac{d(x, y)}{jt}\right)^{-m} \leqslant \left(\frac{d(z_j, \Sigma)}{|z_j - x|}\right)^m$$

so that

$$\Gamma_m^{**} f(x) \leqslant \sum_{j=1}^{L} \binom{L}{j} \sup_{z \in \Omega} \left|u(z)\right| \left(\frac{d(z, \Sigma)}{|z - x|}\right)^m \leqslant 2^L u_m^{**}(x)$$

which proves (3.2). $\square$

We now recall the operator

$$S_t = \sum_{j=0}^{L} (-1)^{L-j} \binom{L}{j} \mathcal{P}_{jt}$$

defined in Section 2 and denote by $\sigma_t(x, y)$, $x, y \in \Sigma$, its kernel. The next lemma depends strongly on the estimates $(K_{\alpha\beta})$ proved in Theorem 1.1.

Lemma 3.5. *There exists $C_\alpha > 0$ depending only of L and n such that for all $x \neq y$ in Σ*

$$\left|D_x^\alpha \sigma_t(x, y)\right| \leqslant \frac{C_\alpha t^L}{|x - y|^{n-1+L+|\alpha|}}, \quad 0 \leqslant t \leqslant 1/L.$$

Proof. Notice that the first term in the sum that defines S_t is $\pm I$ so its kernel is supported in the diagonal $x = y$. Hence, in the proof of the lemma we need only look at terms with $j \geqslant 1$. We prove the estimate for $\alpha = 0$, the proof for $|\alpha| > 0$ is similar and will be left to the reader. We consider two cases.

Case 1. Let $x, y \in \Sigma$ be such that $x \neq y$ and $|y - x| < 2Lt$. Then

$$\left(\frac{1}{2}\right)^{n-1+L} \leqslant \frac{t^{n-1+L}}{|x-y|^{n-1+L}} L^{n-1+L} \tag{3.3}$$

since $L \geqslant 1$. Taking account of $(K_{\alpha\beta})$ with $\alpha = \beta = 0$ we get

$$\begin{aligned}\bigl|\sigma_t(x,y)\bigr| &\leqslant \sum_{j=1}^{L} \binom{L}{j} P(x - jt\boldsymbol{v}_x, y) \leqslant \sum_{j=1}^{L} \binom{L}{j} \frac{C}{|x - jt\boldsymbol{v}_x - y|^{n-1}} \\ &\leqslant \sum_{j=1}^{L} \binom{L}{j} \frac{C}{(jt)^{n-1}} \leqslant \frac{1}{t^{n-1}} C 2^L \leqslant C' t^L |x-y|^{-n+1-L},\end{aligned}$$

where the last inequality is a consequence of (3.3).

Case 2. Let $x, y \in \Sigma$ such that $|x - y| > 2Lt$. Then, using Taylor's formula, we get

$$\begin{aligned}\bigl|\sigma_t(x,y)\bigr| &\leqslant \left|(Lt)^L \sum_{|\alpha|=L} \int\limits_{[0,1]^L} \frac{\partial^\alpha}{\partial z^\alpha} P\bigl(x - t(s_1 + \cdots + s_L)\boldsymbol{v}_x, y\bigr)\, ds\right| \\ &\leqslant (Lt)^L \sum_{|\alpha|=L} \int\limits_{[0,1]^L} \left|\frac{\partial^\alpha}{\partial z^\alpha} P\bigl(x - t(s_1 + \cdots + s_L)\boldsymbol{v}_x, y\bigr)\right| ds \\ &\leqslant t^L \int\limits_{[0,1]^L} \frac{C(L,n)}{|x - t(s_1 + \cdots + s_L)\boldsymbol{v}_x - y|^{n-1+L}}\, ds,\end{aligned}$$

where we used $(K_{\alpha\beta})$ to obtain the last inequality. Since, $|x - y| > 2Lt$ and $|t(s_1 + \cdots + s_L)\boldsymbol{v}_x| < Lt$ it follows that

$$\frac{1}{2}|x-y| \leqslant \bigl|x - t(s_1 + \cdots + s_L)\boldsymbol{v}_x - y\bigr|$$

which gives the desired estimate also in this case. □

Lemma 3.6. *Let $1 \leqslant m < L$ be integers and $0 < s < L$. Let $\phi \in C^\infty(\Sigma)$ be such that*

(1) $\operatorname{supp}\phi \subseteq B_\Sigma(x,h)$ *and*
(2) $\|\phi\|_{L^\infty} \leqslant h^{1-n}$ *and* $\|\phi\|_{C^L} \leqslant h^{1-n-L}$.

There exists a positive constant $c = c(n, \Sigma, L, m, s)$ independent of h and ϕ such that if h and $th \in (0, 1/L)$

$$\int\limits_\Sigma \bigl|S_{th}\phi(y)\bigr| \left(1 + \frac{d(x,y)}{h}\right)^m d\sigma(y) \leqslant ct^s \tag{3.4}$$

and

$$\int\limits_{\Sigma} \left|\Gamma_h\phi(y)\right|\left(1+\frac{d(x,y)}{h}\right)^m d\sigma(y) \leqslant c. \tag{3.5}$$

Proof. Since (3.4) for a given value of s implies the same estimate for smaller values of s we may assume that $L-1<s<L$. Interpolating the estimates (2) we have, with $\rho=s/L$,

$$\|\phi\|_s \leqslant C\|\phi\|_{L^\infty}^{1-\rho}\|\phi\|_{C^L}^{\rho} \leqslant Ch^{1-n-s}. \tag{3.6}$$

Note that (2) also implies that $\|\phi\|_{L^1} \leqslant C$. Let

$$B=\left\{y\in\Sigma\colon\ d(x,y)<h\right\},\qquad B^*=\left\{y\in\Sigma\colon\ d(x,y)<2h\right\}.$$

Let us write the integral I on the left hand side of (3.4) as $I=I_1+I_2$ where

$$I_1=\int\limits_{B^*}\left|S_{th}\phi(y)\right|\left(1+\frac{d(x,y)}{h}\right)^m d\sigma(y)$$

and

$$I_2=\int\limits_{\Sigma\setminus B^*}\left|S_{th}\phi(y)\right|\left(1+\frac{d(x,y)}{h}\right)^m d\sigma(y).$$

To estimate I_1 note that for every $y\in B^*$ we have $(1+d(x,y)/h)^m \leqslant 3^m$ so

$$\int\limits_{B^*}\left|S_{th}\phi(y)\right|\left(1+\frac{d(x,y)}{h}\right)^m d\sigma(y) \leqslant 3^m\int\limits_{B^*}\left|S_{th}\phi(y)\right| d\sigma(y). \tag{3.7}$$

Recalling estimate (2.3) in Lemma 2.1 with $|\alpha|=0$ and (3.6) we have

$$\begin{aligned}\int\limits_{B^*}\left|S_{th}\phi(y)\right| d\sigma(y) &\leqslant C\int\limits_{B^*}(th)^s\|\phi\|_s\, d\sigma(y) \leqslant C\int\limits_{B^*}(th)^s h^{1-n-s}\, d\sigma(y)\\ &\leqslant Ct^s h^{1-n}\int\limits_{B^*} d\sigma(y) \leqslant Ct^s,\end{aligned} \tag{3.8}$$

with the constant C depending only of s, L, n, Σ. Thus, (3.7) and (3.8) give $I_1 \leqslant Ct^s$. To estimate I_2 observe that

$$\left(1+\frac{d(x,y)}{h}\right)^m \leqslant 2^m h^{-m} d(x,y)^m,\quad y\in\Sigma\setminus B^*.$$

Hence, using Lemma 3.5 with $\alpha=0$ to estimate the kernel σ_{th} of S_{th}, we get

$$\begin{aligned}I_2 &\leqslant 2^m h^{-m}\int\limits_{\Sigma\setminus B^*}\left(C(th)^L\int\limits_B \frac{|\phi(w)|}{d(y,w)^{n-1+L}}\,dw\right) d(x,y)^m\, d\sigma(y)\\ &\leqslant Ct^L h^{L-m}\|\phi\|_{L^1(\Sigma)}\int\limits_{\Sigma\setminus B^*} d(x,y)^{m-n+1-L}\, d\sigma(y)\end{aligned}$$

$$\leqslant Ct^L h^{L-m} h^{m-L} = c'' t^L \leqslant c'' t^s,$$

which concludes the proof of (3.4).

To prove (3.5) we note that $|\Gamma_h \phi(y)| \leqslant |\mathcal{S}_h \phi(y)| + |\phi(y)|$, so (3.4) with $t = 1$ implies

$$\int_\Sigma \left|\Gamma_h \phi(y)\right| \left(1 + \frac{d(x,y)}{h}\right)^m d\sigma(y) \leqslant c + \int_\Sigma \left|\phi(y)\right| \left(1 + \frac{d(x,y)}{h}\right)^m d\sigma(y)$$

and the last integral is majorized by $|B_\Sigma(x,h)| h^{1-n} 2^m \leqslant c$. □

In the next lemma we recall a standard majorization of $u^\perp(x)$ by the Hardy–Littlewood maximal function of f

$$Mf(x) = \sup_{r>0} \frac{1}{|B_\Sigma(x,r)|} \int_{B_\Sigma(x,r)} \left|f(y)\right| d\sigma(y).$$

Lemma 3.7. *There exists $C > 0$ such that*

$$u^\perp(x) \leqslant CMf(x), \quad f \in L^1(\Sigma). \tag{3.9}$$

Proof. We decompose the integral

$$u(x - t\boldsymbol{\nu}_x) = \int_\Sigma P(x - t\boldsymbol{\nu}_x, y) f(y)\, d\sigma(y)$$

as

$$\int_\Sigma = \int_{B_\Sigma(x,2t)} + \int_{B_\Sigma(x,4t)\setminus B_\Sigma(x,2t)} + \int_{B_\Sigma(x,8t)\setminus B_\Sigma(x,4t)} + \cdots = I_1 + I_2 + I_3 + \cdots.$$

By Corollary 1.2, $|P(x - t\boldsymbol{\nu}_x, y)| \leqslant Ct^{1-n}$ so

$$|I_1| \leqslant Ct^{1-n} \int_{B_\Sigma(x,2t)} \left|f(y)\right| d\sigma(y) \leqslant C' Mf(x).$$

To estimate I_j, $j \geqslant 2$, we recall that

$$\left|P(x - t\boldsymbol{\nu}_x, y)\right| \leqslant C \frac{t}{d(x,y)^n},$$

again by Corollary 1.2, which implies

$$|I_j| \leqslant Ct \int_{B_\Sigma(x,2^j t)} \frac{|f(y)|}{(t2^j)^n} d\sigma(y) \leqslant C' 2^{-j} Mf(x).$$

Hence,

$$\left|u(x - t\boldsymbol{\nu}_x)\right| \leqslant \sum_j |I_j| \leqslant CMf(x) \sum_{j=1}^\infty 2^{-j} \leqslant CMf(x)$$

and taking the supremum in t we get (3.9). □

Since the Hardy–Littlewood maximal operator is bounded in L^2 we have

Corollary 3.8. *If $f \in L^2(\Sigma)$,*

$$\|u^\perp\|_{L^2(\Sigma)} \leqslant C\|f\|_{L^2(\Sigma)}. \tag{3.10}$$

4. Equivalence of Poisson's maximal functions

In this section we show that the three maximal functions defined through the Poisson integral—the normal, nontangential and tangential maximal functions $u^\perp$, u_α^* and u_m^{**}—have equivalent L^p norms for fixed $m > (n-1)/p$ and aperture $\alpha > 1$. As before, $\Omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary $\partial\Omega$ denoted by Σ.

Theorem 4.1. *Let $f \in \mathcal{D}'(\Sigma)$ be a distribution and let u denote its Poisson integral. Let $0 < p \leqslant \infty$, $1 < \alpha < \infty$ and $m > (n-1)/p$ be an integer. The following conditions are equivalent*:

(i) $u^\perp \in L^p(\Sigma)$;
(ii) $u_\alpha^* \in L^p(\Sigma)$;
(iii) $u_m^{**} \in L^p(\Sigma)$.

Moreover, the L^p-norms of the three maximal functions involved are equivalent.

Proof. The proof has three steps.

(i) $\Rightarrow$ (ii) We recall the generalized mean value property for harmonic functions due to Hardy and Littlewood [10]:

Lemma 4.2. *Let $B \subset \mathbb{R}^n$ be a ball centered at z and suppose that u is harmonic on B and continuous on its closure $\overline{B}$. Then for all $q > 0$ there exists a constant $C = C(n,q)$ such that*

$$\left|u(z)\right|^q \leqslant \frac{C}{|B|}\int_B \left|u(w)\right|^q \, dw. \tag{4.1}$$

Following [6], we apply (4.1) to obtain for a fixed $q > 0$ the estimate

$$u_\alpha^*(x) \leqslant CM\left[u^{\perp q}\right](x)^{1/q}, \quad x \in \Sigma,$$

where M denotes Hardy–Littlewood maximal function. It is a classical result that M is bounded in $L^2(\mathbb{R}^n)$. This remains true if $\mathbb{R}^n$ is replaced by a compact Riemannian manifold such as Σ. Choosing $q = p/2$ we have

$$\int_\Sigma \left|u_\alpha^*(x)\right|^p \, d\sigma(x) \leqslant C\int_\Sigma \left\{M\left[u^{\perp q}\right]\right\}^{p/q}(x)\, d\sigma(x) \leqslant C\int_\Sigma \left|u^\perp(x)\right|^p \, d\sigma(x),$$

which shows that (i) implies (ii).

(ii) $\Rightarrow$ (iii) The control of u_m^{**} by u_α^* follows from very general arguments. Consider the sets

$$A_0 = \{z \in \Omega \colon |z - x| \leqslant \alpha d(z, \Sigma)\}$$

and

$$A_j = \{z \in \Omega \colon 2^{j-1}\alpha d(z, \Sigma) \leqslant |z - x| < 2^j \alpha d(z, \Sigma)\}, \quad j = 1, 2, \ldots.$$

Since the collection $\{A_j\}_{j=0}^\infty$ is a covering of Ω we have

$$u_m^{**}(x) \leqslant \sup_{A_0}\left\{|u(z)|\left(\frac{d(z,\Sigma)}{|z-x|}\right)^m\right\} + \sum_{j=1}^\infty \sup_{A_j}\left\{|u(z)|\left(\frac{d(z,\Sigma)}{|z-x|}\right)^m\right\}.$$

It now follows from the definition of u_α^* that

$$u_m^{**}(x)^p \leqslant u_\alpha^*(x)^p + \sum_{j=1}^\infty 2^{(1-j)mp} u_{2^j\alpha}^*(x)^p.$$

Integrating over Σ we obtain

$$\int_\Sigma u_m^{**}(x)^p \, d\sigma(x) \leqslant \int_\Sigma u_\alpha^*(x)^p \, d\sigma(x) + \sum_{j=1}^\infty 2^{(1-j)mp} \int_\Sigma u_{2^j\alpha}^*(x)^p \, d\sigma(x). \tag{4.2}$$

We now use a variation of a useful lemma that relates maximal functions defined with respect to different apertures [15, p. 62]:

Lemma 4.3. *There exists a positive constant C depending on Σ such that if a and b are real numbers satisfying $0 < b \leqslant a$ then the following estimate holds*:

$$\int_\Sigma u_a^*(x)^p \, d\sigma(x) \leqslant C\left(1 + \frac{2a}{b}\right)^{n-1} \int_\Sigma u_{1+b}^*(x)^p \, d\sigma(x).$$

Invoking Lemma 4.3 with $a = 2^j\alpha$ and $b = \alpha - 1$ we get

$$\int_\Sigma u_{2^j\alpha}^*(x)^p \, d\sigma(x) \leqslant C(\Sigma, \alpha)\, 2^{j(n-1)} \int_\Sigma u_\alpha^*(x)^p \, d\sigma(x),$$

which combined with (4.2) yields

$$\begin{aligned}\|u_m^{**}\|_{L^p(\Sigma)}^p &\leqslant \|u_\alpha^*\|_{L^p(\Sigma)} + C\sum_{j=1}^\infty 2^{j(n-1-mp)} \|u_\alpha^*\|_{L^p(\Sigma)}^p \\ &\leqslant C(\Sigma, n, p, m, \alpha)\|u_\alpha^*\|_{L^p(\Sigma)}^p\end{aligned}$$

because $n - 1 - mp < 0$.

(iii) $\Rightarrow$ (i) This implication follows trivially from the obvious pointwise inequality $u^\perp(x) \leqslant u_m^{**}(x)$. $\square$

5. Comparison with the intrinsic maximal function

We now compare Poisson's maximal functions with the intrinsic maximal function. As always, $\Omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary $\partial\Omega$ denoted by Σ with the property that for any $0 < t \leqslant 1$ and $x \in \Sigma$ the point $z_t = x - t\boldsymbol{\nu}_x$ belongs to Ω and $d(z_t, \Sigma) = t$. First we see that the normal maximal function $u^\perp(x)$ is dominated pointwise by $\mathcal{M}_s f(x)$ for any $s \in \mathbb{Z}_+$. We may find a finite partition of unity $\{\rho_k(x)\}_{k=1}^N$ in Σ such that for $j = 1, \ldots, N$, $\operatorname{supp} \rho_j$ is contained in a ball with radius $r_j < r_\Sigma$, where r_Σ is the injectivity radius of Σ. If $f \in \mathcal{D}'(\Sigma)$ we write $f = \sum_{j=1}^N \rho_j f = \sum_{j=1}^N f_j$. Since $\mathcal{M}_s(\rho_j f)(x) \leqslant C\mathcal{M}_s f(x)$, it is enough to prove that $u^\perp(x) \leqslant C\mathcal{M}_s f(x)$ when $\operatorname{supp} f$ is contained in a ball with a radius smaller than r_Σ. That means that in the definition of $\mathcal{M}_s f(x) = \sup_{\phi \in K_s(x)} |\langle f, \phi\rangle|$ we may consider test functions in $K_s(x)$ supported in $B_\Sigma(x, r_\Sigma)$ and we shall do so. To estimate $u^\perp(x)$ we find $z_t = x - t\boldsymbol{\nu}_x \in \Omega$, $0 < t < 1$, such that $u^\perp(x) < 2|u(z_t)|$. Next we split $P(z_t, x)$ as a sum of elements of $K_s(x)$. Consider the finite covering of $B_\Sigma(x, r_\Sigma)$,

$$B_\Sigma(x, 2t) \cup \big(B_\Sigma(x, 4t) \setminus \bar{B}_\Sigma(x, t)\big) \cup \big(B_\Sigma(x, 8t) \setminus \bar{B}_\Sigma(x, 2t)\big) \cup \cdots,$$

and find a partition of unity $\{\psi_j(x)\}_{j=0}^N$ subordinated to this covering that satisfies the estimates $\|D_x^\alpha \psi_j\|_{L^\infty} \leqslant C(t2^j)^{-|\alpha|}$, $\alpha \in \mathbb{Z}_+^{n-1}$, $|\alpha| \leqslant s$. We know from Theorem 1.1 that

$$\big|D_y^\alpha P(z_t, y)\big| \leqslant \frac{C}{|z_t - y|^{n-1+|\alpha|}} \leqslant \frac{C}{(t + |x - y|)^{n-1+|\alpha|}}, \quad |\alpha| \leqslant s.$$

Hence,

$$\big|D_y^\alpha\big(\psi_0(y)P(z_t, y)\big)\big| \leqslant \frac{C}{t^{n-1+|\alpha|}}, \quad |\alpha| \leqslant s,$$

since $|x - y| \leqslant 2t$ on the support of ψ_0, showing that $C^{-1}\psi_0(y)P(z_t, y)$ belongs to $K_s(x)$. For $j \geqslant 1$, $|x - y| \geqslant t2^{j-1}$ on $\operatorname{supp} \psi_j$, which leads to

$$\big|D_y^\alpha\big(\psi_j(y)P(z_t, y)\big)\big| \leqslant \frac{C}{(2^j t)^{n-1+|\alpha|}}, \quad |\alpha| \leqslant s + 1.$$

This already shows that $C^{-1}\psi_j(y)P(z_t, y) \in K_s(x)$ but we need a better estimate to compensate for the possibly large number of terms in the sum. Thus, invoking Corollary 1.2, we have

$$\big|\psi_j(y)P(z_t, y)\big| \leqslant \frac{Ct}{(2^j t)^n} = \frac{C2^{-j}}{(2^j t)^{n-1}}.$$

Summing up, we know that

$$\big\|\psi_j(\cdot)P(z_t, \cdot)\big\|_{L^\infty} \leqslant C2^{-j}(2^j t)^{-n+1}$$

and

$$\big|D_y^\beta\big(\psi_j(y)P(z_t, y)\big)\big| \leqslant C(2^j t)^{-n+1}(2^j t)^{-L-1} \quad \text{for } |\beta| = L + 1.$$

For $0 \leqslant k \leqslant L$ and $|\alpha| = k$ we derive by interpolation

$$\begin{aligned}\left|D_y^\alpha \psi_j(y)P(z_t,y)\right| &\leqslant C\left\|\psi_j(\cdot)P(z_t,\cdot)\right\|_{L^\infty}^{1-\rho_k}\Bigl(\max_{|\beta|=L+1}\left\|D_y^\beta\psi_j P(z_t,y)\right\|_{L^\infty}\Bigr)^{\rho_k}\\ &\leqslant C(2^{-j})^{1-\rho_k}(2^j t)^{-(n-1)}(2^j t)^{-(L+1)\rho_k}\\ &\leqslant C(2^{-j})^{1-\rho_k}(2^j t)^{-(n-1+k)} \leqslant C(2^{-j})^{\delta}(2^j t)^{-(n-1+k)},\end{aligned}$$

where $\rho_k = k/(L+1)$ and $\delta = 1/(L+1)$. The estimates show that on $B_\Sigma(x, r_\Sigma)$ we may write

$$P(z_t,x) = C\sum_{j=0}^{N} 2^{-j\delta}\phi_j(x), \quad \phi_j(x)\in K_s(x).$$

It follows that

$$\left|u(z_t)\right| = \left|\langle f, P(z_t,x)\rangle\right| \leqslant C\sum_{j=0}^{N} 2^{-j\delta}\left|\langle f,\phi_j(x)\rangle\right| \leqslant C'\mathcal{M}_s f(x).$$

Thus

$$u^{\perp}(x) \leqslant C\mathcal{M}_s f(x) \tag{5.1}$$

which may be viewed as a sharper version of Lemma 3.7.

The next step is the control of $\mathcal{M}_s f(x)$ by $u_m^{**}(x)$ when $s > m$. We will need to write the identity

$$\phi = {}^t\Gamma_h\Gamma_h\phi + \sum_{j=0}^{\infty}\bigl({}^t\Gamma_{2^{-j-1}h}\Gamma_{2^{-j-1}h} - {}^t\Gamma_{2^{-j}h}\Gamma_{2^{-j}h}\bigr)\phi, \quad \phi\in C^\infty(\Sigma), \tag{$*$}$$

already discussed in Section 3, in a convenient way. Here $\Gamma_t = \Gamma_t^L$, $0 < t < 1$, is chosen with $L > s$.

Setting

$$\Gamma_t^+ = \Gamma_{t/2} + \Gamma_t, \quad \Gamma_t^- = \Gamma_{t/2} - \Gamma_t, \quad 0 < t < 1,$$

we may write

$$\Gamma_{t/2} = \frac{1}{2}\bigl(\Gamma_t^+ + \Gamma_t^-\bigr), \quad \Gamma_t = \frac{1}{2}\bigl(\Gamma_t^+ - \Gamma_t^-\bigr), \quad 0 < t < 1.$$

Substitution of these formulas in $(*)$ gives after some simplifications

$$\phi = {}^t\Gamma_h\Gamma_h\phi + \frac{1}{2}\sum_{j=1}^{\infty}{}^t\Gamma^-_{2^{-j-1}h}\Gamma^+_{2^{-j-1}h}\phi + {}^t\Gamma^+_{2^{-j-1}h}\Gamma^-_{2^{-j-1}h}\phi \tag{$**$}$$

for any $\phi\in C^\infty(\Sigma)$. Thus, if $f\in\mathcal{D}'(\Sigma)$ we have

$$\langle f,\phi\rangle = \langle\Gamma_h f,\Gamma_h\phi\rangle + \frac{1}{2}\sum_{j=1}^{\infty}\bigl\langle\Gamma^-_{2^{-j-1}h} f, \Gamma^+_{2^{-j-1}h}\phi\bigr\rangle + \bigl\langle\Gamma^+_{2^{-j-1}h} f, \Gamma^-_{2^{-j-1}h}\phi\bigr\rangle.$$

We must estimate each term in the expression above when $\phi\in K_s(x)$. Assuming that $\operatorname{supp}\phi\subset B_\Sigma(x,h)$ and that $0 < h < 1/L$, we have

$$\langle\Gamma_h f,\Gamma_h\phi\rangle = \int_\Sigma \Gamma_h f(y)\Gamma_h\phi(y)\,d\sigma(y).$$

Taking account of Definition 3.3 and Lemma 3.4 we get

$$\left|\Gamma_h f(y)\right| \leqslant Cu_m^{**}(x)\left(1+\frac{d(x,y)}{h}\right)^m$$

so

$$\left|\left\langle \Gamma_h f, \Gamma_h \phi\right\rangle\right| \leqslant Cu_m^{**}(x)\int_\Sigma \left(1+\frac{d(x,y)}{h}\right)^m \left|\Gamma_h\phi(y)\right| d\sigma(y) \leqslant Cu_m^{**}(x), \tag{5.2}$$

in view of (3.5) in Lemma 3.6. To study the terms $\langle \Gamma^+_{2^{-j-1}h} f, \Gamma^-_{2^{-j-1}h}\phi\rangle$ we point out that since $\Gamma_t^L = I - (-1)^L S_t^L$ we have

$$\left|\Gamma^-_{2^{-j-1}h}\phi\right| \leqslant |S_{2^{-j-1}h}\phi| + |S_{2^{-j-2}h}\phi|.$$

Then, (3.4) in Lemma 3.6 gives

$$\int_\Sigma \left(1+\frac{d(x,y)}{h}\right)^m \left|\Gamma^-_{2^{-j-1}h}\phi(y)\right| d\sigma(y) \leqslant C2^{-js}.$$

On the other hand, using once more Lemma 3.4, we have

$$\left|\Gamma^+_{h2^{-j-1}} f(y)\right| \leqslant Cu_m^{**}(x)\left(1+\frac{d(x,y)}{2^{-j-1}h}\right)^m \leqslant Cu_m^{**}(x)2^{jm}\left(1+\frac{d(x,y)}{h}\right)^m$$

so

$$\begin{aligned}\left|\left\langle \Gamma^+_{2^{-j-1}h} f, \Gamma^-_{2^{-j-1}h}\phi\right\rangle\right| &\leqslant C2^{jm}u_m^{**}(x)\int_\Sigma \left(1+\frac{d(x,y)}{h}\right)^m \left|\Gamma^-_{h2^{-j-1}}\phi(y)\right| d\sigma(y)\\ &\leqslant C2^{j(m-s)}u_m^{**}(x) \leqslant C2^{-j}u_m^{**}(x).\end{aligned} \tag{5.3}$$

To handle the terms $\langle \Gamma^-_{2^{-j-1}h} f, \Gamma^+_{2^{-j-1}h}\phi\rangle$ we write $\Gamma^+_{2^{-j-1}h}\phi = 2\phi - (-1)^L(S_{2^{-j-1}h}\phi + S_{2^{-j-2}h}\phi)$ which leads to

$$\left\langle \Gamma^-_{2^{-j-1}h} f, \Gamma^+_{2^{-j-1}h}\phi\right\rangle = 2\left\langle \Gamma^-_{2^{-j-1}h} f, \phi\right\rangle + R_j \tag{5.4}$$

with

$$|R_j| \leqslant \left|\left\langle \Gamma^-_{2^{-j-1}h} f, S_{2^{-j-1}h}\phi\right\rangle\right| + \left|\left\langle \Gamma^-_{2^{-j-1}h} f, S_{2^{-j-2}h}\phi\right\rangle\right|.$$

The terms on the right hand side can be estimated using Lemmas 3.4 and 3.6—in the same fashion used to obtain (5.3)—in order to get

$$|R_j| \leqslant C2^{-j}u_m^{**}(x). \tag{5.5}$$

Since

$$2\sum_{j=1}^{\infty}\left\langle \Gamma^-_{2^{-j-1}h} f, \phi\right\rangle = 2\langle f - \Gamma_{h/2} f, \phi\rangle = 2(-1)^L\langle S_{h/2} f, \phi\rangle,$$

we obtain, in view of (5.2)–(5.5), that

$$\left|\langle f, \phi\rangle\right| \leqslant Cu_m^{**}(x) + \left|\langle S_{h/2} f, \phi\rangle\right|.$$

In the proof of this inequality we assumed for simplicity and ease of notation the small restriction that ϕ was supported in a ball of radius $h < 1/L$, which was useful for technical reasons (the relevant operators in all terms of $(**)$ satisfied the hypotheses of Lemma 3.6). In the general case, if d is the diameter of Σ, we may find $\lambda > 0$ so that $\lambda d < 1/L$ and replace $(**)$ by

$$\phi = {}^t\Gamma_{\lambda h}\Gamma_{\lambda h}\phi + \frac{1}{2}\sum_{j=0}^{\infty} {}^t\Gamma^{-}_{2^{-j-1}\lambda h}\Gamma^{+}_{2^{-j-1}\lambda h}\phi + {}^t\Gamma^{+}_{2^{-j-1}\lambda h}\Gamma^{-}_{2^{-j-1}\lambda h}\phi.$$

Carrying out the proof with this representation we get

$$\left|\langle f, \phi\rangle\right| \leqslant C(\lambda)u_m^{**}(x) + \left|\langle f, {}^tS_{\lambda h/2}\phi\rangle\right|$$

for any $\phi \in K_s(x)$, so taking the supremum in $\phi \in K_s(x)$ we obtain

$$\mathcal{M}_s f(x) \leqslant C u_m^{**}(x) + \left|\langle f, {}^tS_{\lambda h/2}\phi\rangle\right|. \tag{5.6}$$

The operator ${}^tS_h$ has similar properties to S_h. In fact, the latter is related to

$$L = \frac{\partial^2}{\partial t^2} + 2\sum_{j=1}^{n-1} b_j(x,t)\frac{\partial^2}{\partial x_j \partial t} + \sum_{j,k} c_{jk}(x,t)\frac{\partial^2}{\partial x_j \partial x_k} + \cdots,$$

in the same way ${}^tS_h$ is related to

$$\tilde{L} = \frac{\partial^2}{\partial t^2} - 2\sum_{j=1}^{n-1} b_j(x,t)\frac{\partial^2}{\partial x_j \partial t} + \sum_{j,k} c_{jk}(x,t)\frac{\partial^2}{\partial x_j \partial x_k} + \cdots,$$

which is elliptic if and only if L is. From the analog of Lemma 3.5 for the kernel $\tilde{\sigma}_t(x,y)$ of ${}^tS_h$ (notice that $\tilde{\sigma}_t(x,y)$ vanishes for $t = 0$ and $x \neq y$) we get for all $x \neq y$ in Σ

$$\left|D^\alpha \tilde{\sigma}_t(x,y)\right| \leqslant \frac{C\,t^L}{|x-y|^{n-1+L+|\alpha|}}, \quad 0 \leqslant t \leqslant 1/L, \tag{5.7}$$

for some $C > 0$ depending only on L and n. Set $\Phi = {}^tS_{\lambda h/2}\phi$. By Lemma 2.1

$$\|D^\alpha \Phi\|_{L^\infty} \leqslant C(\lambda h)^{r-|\alpha|}\|\phi\|_r \leqslant C'\lambda h^{1-n-|\alpha|} \tag{5.8}$$

for $|\alpha| < r < L$. Assuming without loss of generality that h is smaller that the injectivity radius r_Σ we find a partition of unity $\{\psi_j(x)\}_{j=0}^N$ subordinated to the covering

$$B_\Sigma(x, 2h) \cup \big(B_\Sigma(x, 4h) \setminus \bar{B}_\Sigma(x, h)\big) \cup \big(B_\Sigma(x, 8h) \setminus \bar{B}_\Sigma(x, 2h)\big) \cup \cdots$$

of $B_\Sigma(x, r_\Sigma)$ that satisfies the estimates $\|D_x^\alpha \psi_j\|_{L^\infty} \leqslant C(h2^j)^{-|\alpha|}$, $\alpha \in \mathbb{Z}_+^{n-1}$, $|\alpha| \leqslant s$ and write

$$\Phi_j(y) = \psi_j(y)\Phi(y) = \psi_j(y)\int \tilde{\sigma}_{\lambda h/2}(y, y')\phi(y')\,d\sigma(y')$$

so $\Phi = \sum_{j=0}^N \Phi_j$. Choosing $r > s$, (5.8) and the fact that $\operatorname{supp}\psi_0 \subset B_\Sigma(x, 2h)$ allows us to write $\Phi_0 = C\lambda\Psi_0$ with $\Psi \in K_s(x)$. For $j \geqslant 1$, $y \in \operatorname{supp}\psi_j$ and $y' \in \operatorname{supp}\phi$, $d(y, y') \sim d(y, x) \sim 2^j h$, so for those values of y, y' we have

$$\left|D_y^\alpha \tilde{\sigma}_{\lambda h/2}(y, y')\right| \leqslant C(\lambda 2^{-j})^L (2^{-j}h^{-1})^{n-1+|\alpha|}$$

which shows that $\Phi_j = C\lambda 2^{-j}\Psi_j$ with $\Psi_j \in K_s(x)$. Thus, taking λ sufficiently small we may assume that ${}^t S_{\lambda h/2}\phi = (1/2)\sum_{j=0}^{N} 2^{-j-1}\Psi_j$ with $\Psi_j \in K_s(x)$. Now (5.6) gives

$$\mathcal{M}_s f(x) \leqslant C u_m^{**}(x) + \frac{1}{2}\mathcal{M}_s f(x)$$

which shows that

$$\mathcal{M}_s f(x) \leqslant C u_m^{**}(x) \tag{5.9}$$

if $\mathcal{M}_s f(x) < \infty$. For arbitrary $x \in \Sigma$ we may reason with an approximation of $\mathcal{M}_s f(x)$. Set

$$\mathcal{M}_s^{\varepsilon} f(x) = \sup_{\phi \in K_s^{\varepsilon}(x)} \left|\langle f, \phi\rangle\right|,$$

where $K_s^{\varepsilon}(x)$ is the space of smooth functions $\phi \in C^{\infty}(\Sigma)$ such that there is an $h > \varepsilon$ such that $\operatorname{supp}\phi \subset B(x,h)$ and $\sup_{0 \leqslant k \leqslant s} h^{N+k}\|\phi\|_k \leqslant 1$. Reasoning as before with $\mathcal{M}_s^{\varepsilon} f(x)$ which is always finite in the place of $\mathcal{M}_s f(x)$ we get

$$\mathcal{M}_s^{\varepsilon} f(x) \leqslant C u_m^{**}(x)$$

and letting $\varepsilon \to 0$ we obtain (5.9) in general.

Summing up, (5.1) and (5.6) show that for $s > m$ we have the pointwise inequalities

$$u^{\perp}(x) \leqslant C\mathcal{M}_s f(x) \leqslant C^2 u_m^{**}(x)$$

which trivially imply

$$\|u^{\perp}\|_{L^p(\Sigma)} \leqslant C\|\mathcal{M}_s f\|_{L^p(\Sigma)} \leqslant C^2 \left\|u_m^{**}\right\|_{L^p(\Sigma)}.$$

However, if $m > (n-1)/p$, Theorem 4.1 asserts that the L^p norms of $u^{\perp}$ and u_m^{**} are comparable. We have proved

Theorem 5.1. *Let $f \in \mathcal{D}'(\Sigma)$ be a distribution and let u denote its Poisson integral. Let $0 < p \leqslant \infty$, $1 < \alpha < \infty$, and assume $s > m > (n-1)/p$ are integers. The following conditions are equivalent*:

(i) $f \in H^p(\Sigma)$;
(ii) $\mathcal{M}_s f \in L^p(\Sigma)$;
(iii) $u^{\perp} \in L^p(\Sigma)$;
(iv) $u_{\alpha}^{*} \in L^p(\Sigma)$;
(v) $u_m^{**} \in L^p(\Sigma)$.

Moreover, the L^p-norms of all maximal functions involved are comparable.

6. Complex Hardy spaces

In this section Ω will denote a bounded open subset of complex space $\mathbb{C}^n$ with smooth boundary $\partial\Omega = \Sigma$. Denote by ρ a smooth real function that vanishes precisely on Σ

such that $d\rho \neq 0$ on Σ and $\rho > 0$ on Ω. For $\epsilon > 0$ sufficiently small the set $\Sigma_\epsilon^\rho = \{z \in \Omega\colon \rho(z) = \epsilon\}$ is a smooth embedded orientable hypersurface with a Riemannian structure inherited from $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

For $0 < p < \infty$, the complex Hardy space $\mathcal{H}^p(\Omega)$ is defined as the space of all holomorphic functions f defined on Ω such that

$$\sup_{0<\epsilon<\epsilon_0} \int_{\Sigma_\epsilon^\rho} \left|f(z)\right|^p \, d\sigma_\epsilon^\rho(z) < \infty. \tag{6.1}$$

Since $|f|^p$ is subharmonic, the independence of condition (6.1) from the particular defining function ρ follows from the following lemma of Stein [13].

Lemma 6.1. *Let ρ and ρ' be two defining functions for Ω and let u be a positive subharmonic function on Ω. Then*

$$\sup_{0<\epsilon<\epsilon_0} \int_{\Sigma_\epsilon^\rho} u(z)\, d\sigma_\epsilon^\rho(z) < \infty$$

if and only if

$$\sup_{0<\epsilon<\epsilon_0} \int_{\Sigma_\epsilon^{\rho'}} u(z)\, d\sigma_\epsilon^{\rho'}(z) < \infty.$$

We may take as ρ the function $\Omega \ni x - t\boldsymbol{\nu}_x \mapsto t$, defined for $x \in \Sigma$ and $0 < t < t_0$, and set

$$\|f\|_{\mathcal{H}^p}^p = \sup_{0<t<t_0} \int_{\Sigma_t} \left|f(z)\right|^p \, d\sigma_t(z) < \infty$$

for $f \in \mathcal{H}^p(\Omega)$. With this choice of ρ, the level sets $t = \text{const} > 0$ are the submanifolds Σ_t already considered in Section 2.

We recall that if $f(z)$ is holomorphic on Ω and has tempered growth at the boundary, i.e., $|f(z)| \leqslant C\,\mathrm{dist}(z, \Sigma)^{-N}$ for some positive constants C and N, then $f(z)$ has a weak boundary value $bf \in \mathcal{D}'(\Sigma)$ [11, p. 66]. This means that if we regard the restrictions $f_t = f|_{\Sigma_t}$ as distributions defined on Σ via the identification $\Sigma_t \ni x - t\boldsymbol{\nu}_x \mapsto x \in \Sigma$, then $\langle f_t, \phi\rangle \to \langle bf, \phi\rangle$ for any $\phi \in C^\infty(\Sigma)$ as $t \to 0$. Conversely, if bf exists, f must have tempered growth at the boundary. We denote by $\mathcal{H}_b(\Omega)$ the space of holomorphic functions on Ω with tempered growth at the boundary.

Theorem 6.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded open subset with smooth boundary $\partial\Omega = \Sigma$. Let $0 < p \leqslant \infty$ and let $f(z)$ be a holomorphic function on Ω. The following properties are equivalent*:

(i) $f \in \mathcal{H}^p(\Omega)$;
(ii) $|f|^p$ *has a harmonic majorant on* Ω;
(iii) $f \in \mathcal{H}_b(\Omega)$ *and* $bf \in H^p(\Sigma)$;

(iv) *f is the Poisson integral of some $f_0 \in H^p(\Sigma)$.*

Proof. When $p = \infty$ (ii) has to be understood as "$\|f\|_{L^\infty}$ has a harmonic majorant on Ω"; the proof of this case is simpler and will be left to the reader, so from now on we assume that $0 < p < \infty$ and prove the theorem in four steps.

(i) $\Rightarrow$ (ii) Let $G(z, w)$ be the Green function of Ω. Fix a point $z_0 \in \Omega$ and consider the function $w \mapsto G(z_0, w)$. Since the normal derivative $-\boldsymbol{\nu}_w(D)G(z_0, w) = P(z_0, w) > 0$ for $w \in \Sigma$ and $G(z_0, w) > 0$ for $w \in \Omega$, we may use a defining function $\rho(w)$ for Ω that coincides with $G(z_0, w)$ when $w \in \Omega$ is close to Σ. For $\varepsilon > 0$ and small set $\Omega_\varepsilon = \{z \in \Omega\colon \rho(z) > \varepsilon\}$. Then the Green function $G_\varepsilon(z, w)$ of Ω_ε is given by $G_\varepsilon(z, w) = G(z, w) - \varepsilon$ and writing the Poisson kernel $P_\varepsilon(z_0, w)$ of Ω_ε, $w \in \partial\Omega_\varepsilon$, as the normal derivative of the Green function it follows that $P_\varepsilon(z_0, w) \to P(z_0, w)$ uniformly on $w \in \Sigma$ as $\varepsilon \to 0$ if we identify points in $\partial\Omega_\varepsilon$ with their normal projections onto Σ. Let $f \in \mathcal{H}^p(\Omega)$. Then the restriction of $|f|^p$ to $\partial\Omega_\varepsilon$ belongs to $L^1(\partial\Omega_\varepsilon)$ uniformly in $\varepsilon \searrow 0$. By our identification we may think of $|f|^p_{\partial\Omega_\varepsilon}$ as bounded subset of $L^1(\Sigma)$ and find a sequence ε_j such that $v_j = |f|^p_{\partial\Omega_{\varepsilon_j}}$ converges weakly to a positive Radon measure $\mu \in \mathcal{M}(\Sigma)$. Let u_j be harmonic function on Ω_{ε_j} with boundary value $v_j = |f|^p_{\partial\Omega_{\varepsilon_j}}$. Since $|f|^p$ is subharmonic, $v_j \leqslant u_j$ on Ω_{ε_j}, so for large j we have $|f(z_0)|^p \leqslant u_j(z_0)$. We may write

$$u_j(z_0) = \int\limits_{\partial\Omega_{\varepsilon_j}} P_{\varepsilon_j}(z_0, y) v_j(y)\, d\sigma_{\varepsilon_j}(y)$$

and let $\varepsilon_j \to 0$ to get

$$\big|f(z_0)\big|^p \leqslant \big\langle \mu, P(z_0, \cdot)\big\rangle \doteq u(z_0)$$

so $u(z)$, the Poisson integral of μ, is the required harmonic majorant.

(ii) $\Rightarrow$ (iii) Since $|f|^p$ has a harmonic majorant, $|f|^{p/2} \leqslant 1 + |f|^p$ also does. Reasoning as before, we may find a function $v \in L^2(\Sigma)$ which is the weak limit of the restrictions of $|f|^{p/2}$ to $\partial\Omega_{\varepsilon_j}$. Hence, if $u(z)$ is the Poisson integral of v we have $|f|^{p/2}(z) \leqslant u(z)$, $z \in \Omega$. Then

$$f^\perp(x) \doteq \sup_{0<t<t_0} \big|f(x - t\boldsymbol{\nu}_x)\big| \leqslant u^\perp(x)^{2/p}. \tag{6.2}$$

By Corollary 3.8, $u^\perp(x) \in L^2(\Sigma)$ because u is the Poisson integral of $v \in L^2(\Sigma)$, which implies that

$$\int\limits_\Sigma f^\perp(y)^p\, d\sigma(y) \leqslant C.$$

Consider $z = x - t\boldsymbol{\nu}_x \in \Omega$ lying at a small distance $t = d(z, \Sigma) < \varepsilon_0/2$ from the boundary and consider the ball $B(z, t/2) \subset \Omega$. For $w = y - s\boldsymbol{\nu}_y \in B(z, t/2)$ we have $|f(w)| \leqslant f^\perp(y)$. Hence,

$$\begin{aligned}
|f(z)|^p &\leqslant \frac{1}{|B(z,t/2)|} \int\limits_{B(z,t/2)} |f(w)|^p \, dV(w) \\
&\leqslant Ct^{-n} \int\limits_{t/2}^{3t/2} ds \int\limits_{\Sigma} f^{\perp}(y)^p \, d\sigma(y) \leqslant Ct^{1-n} = C \operatorname{dist}(z,\Sigma)^{1-n}.
\end{aligned}$$

Thus $f(z)$ has tempered growth as $z \to \partial\Omega$ and possesses a weak boundary value $bf = f_0 \in \mathcal{D}'(\Sigma)$. The Poisson integral U of f_0 and the holomorphic function f are both harmonic and have the same boundary value so f is the Poisson integral of f_0. Hence, (6.2) may be rewritten as

$$U^{\perp}(x) \leqslant u^{\perp}(x)^{2/p}$$

showing that $U^{\perp} \in L^p(\Sigma)$ so $f_0 \in H^p(\Sigma)$ by Theorem 5.1.

(iii) $\Rightarrow$ (iv) It is enough to write f as the Poisson integral of bf.

(iv) $\Rightarrow$ (i) Let $f = U$ be the Poisson integral of f_0. For $t > 0$ small we have $|f(x - t\boldsymbol{v}_x)| = |U(x - t\boldsymbol{v}_x)| \leqslant U^{\perp}(x)$ so

$$\int\limits_{\Sigma_t} |f(z)|^p \, d\sigma_t(z) \leqslant C \int\limits_{\Sigma} U^{\perp}(x)^p \, d\sigma(x) < \infty$$

which shows that $f \in \mathcal{H}^p(\Omega)$. $\square$

Corollary 6.3. *If $f \in \mathcal{H}^p(\Omega)$, the "norms" $\|f\|_{\mathcal{H}^p}$, $\|f^{\perp}\|_{L^p}$, $\|f_a^*\|_{L^p}$ are all comparable.*

Although the usual product of functions cannot be extended to distributions preserving the associative property, it is possible to define the product of two distributions $f_0, g_0 \in \mathcal{H}_b(\Omega)$ as $f_0 g_0 = b(fg)$ where $bf = f_0$ and $bg = g_0$, turning $\mathcal{H}_b(\Omega)$ into an associative algebra. A more precise version of this fact is given by

Corollary 6.4. *If $f_0, g_0 \in H^p(\Sigma)$, the product $f_0 g_0 \in H^{p/2}(\Sigma)$.*

Proof. Let f and g be respectively the Poisson integrals of f_0 and g_0, so $f, g \in \mathcal{H}^p(\Omega)$ by Theorem 6.2. Hence, Schwarz inequality gives

$$\int\limits_{\Sigma_t} |fg(z)|^{p/2} \, d\sigma_t(z) \leqslant \Bigg(\int\limits_{\Sigma_t} |f(z)|^p \, d\sigma_t(z)\Bigg)^{1/2} \Bigg(\int\limits_{\Sigma_t} |g(z)|^p \, d\sigma_t(z)\Bigg)^{1/2} \leqslant C$$

showing that $fg \in \mathcal{H}^{p/2}(\Omega)$. Thus, $f_0 g_0 = b(fg) \in H^{p/2}(\Sigma)$. $\square$

A simple consequence of the Poisson representation for $\mathcal{H}^1(\Omega)$ functions is the following version of the F. and M. Riesz theorem.

Theorem 6.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded open subset with smooth boundary $\partial\Omega = \Sigma$ and assume that $f(z)$ is holomorphic in Ω with tempered growth at the boundary and has a measure $\mu \in \mathcal{M}(\Sigma)$ as weak boundary value. Then μ is absolutely continuous with respect to $d\sigma$.*

Proof. Since $\mathcal{M}(\Sigma) \subset H^p(\Sigma)$ for $p < 1$, Theorem 6.2 shows that f is the Poisson integral of its boundary value μ. Moreover, the Poisson representation $f(z) = \langle \mu(y), P(z, y)\rangle$ shows that $|f(z)| \leqslant \langle |\mu|(y), P(z, y)\rangle$ where $|\mu|$ is the variation of μ. Interchanging the order of the integration we see that

$$\int_{\Sigma_t} \bigl|f(z)\bigr|\,d\sigma_t(z) \leqslant \left\langle |\mu|(y), \int_{\Sigma_t} P(z, y)\,d\sigma_t(z)\right\rangle \leqslant C|\mu|(\Sigma).$$

Thus $f \in \mathcal{H}^1(\Omega)$ and Theorem 6.2 implies that $\mu = bf \in H^1(\Sigma) \subset L^1(\Sigma)$, as we wished to prove. □

Remark 6.6. Theorem 6.5 also follows from an analogous and stronger local result due to Brummelhuis [3] according to which if a measure μ is defined on an open subset V of Σ and is the boundary value of a holomorphic function f defined on one side of Σ then μ is absolutely continuous with respect to $d\sigma$ on V.

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