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# Pricing general insurance in a reactive and competitive market

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ABSTRACT

A simple parameterisation is introduced which represents the insurance market's response to an insurer adopting a pricing strategy determined via optimal control theory. Claims are modelled using a lognormally distributed mean claim size rate, and the market average premium is determined via the expected value principle. If the insurer maximises its expected wealth then the resulting Bellman equation has a moving boundary in state space that determines when it is optimal to stop selling insurance. This stochastic optimisation problem is simplified by the introduction of a stopping time that prevents an insurer leaving and then re-entering the insurance market. Three finite difference schemes are used to verify the existence of a solution to the resulting Bellman equation when there is market reaction. All of the schemes use a front-fixing transformation. If the market reacts, then it is found that the optimal strategy is altered, in that premiums are raised if the strategy is of loss-leading type and lowered if it is optimal for the insurer to set a relatively high premium and sell little insurance.



## 1. Introduction

In actuarial science, a premium principle [1] equates the cost of a general insurance[1] policy to the moments of the corresponding claim arrival and severity distributions. Insurers add a loading to this cost price in order to make a profit and cover their expenses. Many lines of insurance are highly competitive, and so in practice the loading depends critically on the price other insurers charge for comparable policies.

Emms and Haberman [2] use optimal control theory in order to calculate the optimal pricing strategy for an insurer, given the average price process of the rest of the insurance market, building on the earlier work of Taylor [3]. Taylor's paper constructs a deterministic, discrete, demand model for pricing insurance policies, which means that the future average price of the market must be known with certainty. If one adopts a continuous stochastic process for the market average price, then the generalisation to a stochastic, continuous time model is not entirely straightforward.

For example, there is some freedom of choice in how one sets up a continuous time model: one can either model the *premium* rate charged by the insurer for a unit of insurance cover, or charge a premium up front for a finite period of cover thereafter.

In the former case, policyholders pay a premium $p(t)$ continuously over the course of their policies [2]. Thus, for each interval of length $\mathrm{d}t$ of insurance cover, the policyholder pays a premium $p(t)\,\mathrm{d}t$. There are two natural choices for the premium rate that the insurer should charge a policyholder: either the rate currently set by the insurer $p(t)$, or the rate at the start of the policy held constant over the duration of cover. If we adopt the rate currently in force, then the insurer can force arbitrarily high premium increases on the policyholder, and the optimal control is bang–bang. If we adopt the rate at the start of the policy, then the insurer receives an infinite number of premium rates at time $t$ from each policyholder, and it

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[1] The term "general insurance" is used by actuaries to refer to short-term policies (such as car and house insurance) as opposed to longer-term life insurance.

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becomes necessary to introduce an accrued premium rate. Emms and Haberman [2] discuss these complications in greater detail.

If one adopts the convention of an up-front premium, Emms [4] finds an analytical expression for the optimal premium strategy for a linear demand model. The availability of analytical results and the simplicity of the modelling means that, here, we construct the continuous pricing model so the policyholder pays a premium up front for an insurance policy.

One feature of unconstrained pricing models is that under certain conditions it can be optimal to set a negative premium [5]. This reflects the strategy of initially pricing low relative to the market in order to generate sales, which for an insurer increases its reputation provided that it settles on the resulting claims. By using this increased reputation, the insurer can then set a higher relative price, which still generates sales and thus makes a profit. A further limitation of the modelling is addressed by Emms and Haberman [6], who suppose that the market for policies is finite, and so there comes a point after which sales no longer significantly contribute to the reputation of the insurer. Such an assumption leads to a nonlinear state model for the insurer, and this complicates the dynamics of the optimal control. Nevertheless, one can categorise the type of optimal premium strategy according to the number of policies currently in force, and the corresponding adjoint variable.

In all of these papers, it is assumed that there is a single optimising insurer, whose price does not affect the premiums charged by other insurers in the market for comparable policies. This seems a reasonable assumption for a small insurer in a large market. In mathematical finance, there is a similar assumption for the optimal asset allocation problem [7], which posits that the allocation the investor makes in the stock does not influence the price of that stock. However, many lines of insurance are dominated by a few large insurers who monitor each other insurer's premiums and regularly update their prices. In such a market, existing competitive pricing models for general insurance policies are not applicable, and one must consider how insurers react to each other insurer's prices.

Time delays are an inherent part of the actuarial process, since claims are often settled some time after a policy has lapsed. Thus, it becomes difficult to model the precise financial state of a competitor, and so the reaction of the market to pricing changes is unlikely to be instantaneous. In this paper, we introduce a parameterisation into a general insurance model in order to investigate how market reaction might affect the optimising insurer's pricing strategy. Specifically, we suppose that the market average premium drifts towards the premium of the optimising insurer over time, which models the case of a large insurer whose price is a benchmark for the industry.

One might formulate this class of problem as a Stackelberg (or leader–follower) game [8], that is, try and find a pricing strategy which is optimal for each insurer, assuming that they do not cooperate, but where one insurer takes the lead. However, usually this requires a parameterisation of the control (i.e., a prescribed relationship between the leading insurer's premium and state) in order that one can find time-consistent Markovian equilibria. The technical difficulties in solving such games are considerable, because the leader's optimisation problem contains that of the follower, and if one seeks a control in feedback form then the optimal control problem is not in a standard form. Consequently, rather than parameterise the control, we consider an explicit parameterisation of the reaction of the market, since this leads to a hard but still tractable optimisation problem.

Competitive models for pricing general insurance policies [2] employ a demand parameterisation, which is a function of the insurer's premium *relative* to the market average premium. If the insurer sets a premium below the market average, then the demand law says policies will be sold. However, demand cannot be solely a function of relative price, because that can lead to price collusion. Here, the entire insurance market sets a high premium, which, according to the demand law, generates significant sales for an optimising insurer. In reality, customers will not pay a premium above the value of what they are insuring, and may not take out insurance if they have that option. This leads us to a generalisation of the demand law such that demand is a function not only of the relative premium, but also of the claims [9], which are assumed to be distributed identically for each insurer. In this way, the demand for policies is related to their cost price (through expected claims), and if all sellers of policies price far above cost, our demand law dictates that very few policies are sold.

In the following section, we present the motivation behind the modelling and introduce the notation. Section 3 contains the insurance pricing problem formulated using optimal control theory. In Section 3.1, we discuss the difficulties in solving such a problem without further simplification of the model. Section 3.2 introduces a stopping time that prevents an insurer leaving and then re-entering the insurance market. An analytical optimal pricing strategy is found if there is no market reaction. The subsequent numerical algorithms must reproduce this solution if we are to be confident in their predictions when there is market reaction. Section 3.3 contains the solution of the deterministic, reactive problem. Three numerical schemes are given in Section 4 in order to solve the stochastic case, and a discussion and interpretation of the sensitivity of the model to its parameters is contained in Section 5. Conclusions are drawn in Section 6.

## 2. Motivation and modelling

Following Taylor [3], we specify all prices (and claims) per unit of exposure, where the exposure is the unit of risk for an insurer, and it varies according to the line of insurance under consideration. This means that we can ignore the precise details of a policy, but we require that each insurer ascribes a certain number of units of exposure to each policy sold.

Let us consider one general insurer who sets a continuous premium, $\mathbf{p}_t,\ (t \geq 0)$ per unit exposure (hereafter we omit this qualification) for a general insurance policy of fixed duration $l$. Let us suppose that the market average premium on offer from competing insurers for a policy of the same duration is $\bar{\mathbf{p}}_t$ per unit exposure. We shall call $\bar{\mathbf{p}}_t$ the market average premium, and take it as understood that this average *excludes* the price of the single insurer under consideration. The exposure, $\mathbf{q}_t$, of the insurance company is a measure of the potential liabilities of the insurer, since it reflects the number of insurance policies currently in force and the potential size of the claims on these policies. We suppose that the current exposure evolves according to a demand law $G$, which is a function of the current premiums $\mathbf{p}_t$, $\bar{\mathbf{p}}_t$ and the policy loading.

The reserve, $\mathbf{w}_t$, denotes the amount of capital currently held by the insurer, and increases as policies are sold and decreases as claims are settled. Although the reserve is an indication of the financial state of the insurer, it does not describe the insurer's solvency or wealth, since insurance policies are contracts, and a proportion of those currently in force will likely lead to future claims made on the insurer. Usually, there is a constraint set by government on the minimum amount of reserve that the insurer must hold, so that it might respect future claims, although that is not the focus of the modelling here.

We define the optimal premium strategy, $\mathbf{p}_t$, as one which maximises the expected wealth of the insurer at the end of the planning horizon $T$, by explicitly taking into account the expected future claims on policies in force at time $t \le T$.

We develop an insurance model which uses the mean claim size rate $\mathbf{u}_t$ (per unit exposure) to model claims, and we suppose that this process is lognormally distributed:

$$\mathrm{d}\mathbf{u}_t = \mathbf{u}_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}\mathbf{W}_t), \tag{1}$$

where $\mu$ and $\sigma$ are constants and $\{\mathbf{W}_t\}$ is a standard Brownian motion. Consequently, $\mathbf{u}_t$ is always positive (provided that $\mathbf{u}_0$ is positive), and it incorporates the intensity as well as the magnitude of the claims. In the terminology of control theory, the state vector for this model is $\mathbf{X}_t = (\mathbf{u}_t, \bar{\mathbf{p}}_t, \mathbf{q}_t, \mathbf{w}_t)^T$, whereas the control is $\mathbf{p}_t$.

We suppose that a policyholder pays the premium, $\mathbf{p}_t$, at the start of a policy of length $l = \kappa^{-1}$, and that if the policyholder renews their insurance they are charged the same premium as a new customer. Following the exposure model of Emms [4], we suppose that the change in exposure over a time interval of length $\mathrm{d}t$ is given by

$$\mathrm{d}\mathbf{q}_t = \mathbf{q}_t(G - \kappa)\,\mathrm{d}t. \tag{2}$$

Here, we model the size of the insurer by its current exposure, and suppose that large insurers tend to gain greater exposure than small insurers with comparable premiums, since large insurers are often perceived as less likely to fail. Consequently, the gain in exposure is linearly proportional to current exposure multiplied by a demand function $G$, which models competitive pricing. The loss of exposure at rate $\kappa$ models the expiry of policies. Emms and Haberman [6] have generalised this demand model by looking at a nonlinear dependency on the current exposure.

With our model, the increase in reserve, $\mathbf{w}_t$, from selling insurance at time $t$ is the increase in exposure from selling policies times the current premium, $\mathbf{q}_t G\,\mathrm{d}t \times \mathbf{p}_t$, and this increase arises from both renewals and new business. The loss of reserve due to claims paid out over a time interval of length $\mathrm{d}t$ is $\mathbf{u}_t\mathbf{q}_t\mathrm{d}t$, because $\mathbf{u}_t$ is the mean claim size rate per unit of exposure. Thus, the insurer's reserve evolves according to

$$\mathrm{d}\mathbf{w}_t = -\alpha\mathbf{w}_t\,\mathrm{d}t + \mathbf{q}_t(G\mathbf{p}_t - \mathbf{u}_t)\,\mathrm{d}t,$$

where the constant $\alpha$ determines the loss of wealth due to returns to shareholders.

The optimal premium strategy for the insurer in a demand model is often loss leading [2]. This form of optimal control relies on the assumption that the market average premium is independent of the insurer's premium strategy. If the insurer sets a premium below the market average and gains significant exposure, then the market as a whole may react. An insurer who initially tries to gain exposure faces the possibility that the market will follow the same course of action by setting a comparable premium. In this scenario, the insurer will not gain significant exposure, but it will suffer reduced income.

Consequently, we split up the drift in the market average premium into that due to claims and that due to the reaction of the market to an optimising insurer. If there is *no market reaction*, then we adopt the expected value principle [1], and suppose that the market average premium is perfectly correlated with the claims: $\bar{\mathbf{p}}_t = \gamma^{-1}\mathbf{u}_t$, where $\gamma$ is a constant and represents a fixed loss ratio per unit time. We incorporate the change in the market average premium due to market reaction using a mean-reversion term $\lambda(\mathbf{p}_t - \bar{\mathbf{p}}_t)\,\mathrm{d}t$, where the constant $\lambda \ge 0$ is a measure of the rate at which the market reacts to the optimising insurer's price. The evolution equation for the market average premium is

$$\mathrm{d}\bar{\mathbf{p}}_t = \gamma^{-1}\mathrm{d}\mathbf{u}_t + \lambda(\mathbf{p}_t - \bar{\mathbf{p}}_t)\,\mathrm{d}t.$$

If the optimising insurer's premium $\mathbf{p}_t$ is smaller/larger than $\bar{\mathbf{p}}_t$ then the market average premium decreases/increases in value in order to follow the optimising insurer. In this paper, we assume that $\lambda$ is independent of the insurer's exposure and so the reaction is solely based on the premium difference. If $\lambda$ is linearly related to the exposure, then the market reacts to the insurer's premium *income*, and we may consider this case in future work.

## 3. The optimisation problem

Summarising the modelling in the previous section, we suppose that the state process $\{\mathbf{X}_t\}$ evolves according to

$$\begin{aligned}
\mathrm{d}\mathbf{u}_t &= \mathbf{u}_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}\mathbf{W}_t),\\
\mathrm{d}\bar{\mathbf{p}}_t &= \gamma^{-1}\mathrm{d}\mathbf{u}_t + \lambda(\mathbf{p}_t - \bar{\mathbf{p}}_t)\,\mathrm{d}t,\\
\mathrm{d}\mathbf{q}_t &= \mathbf{q}_t(G - \kappa)\,\mathrm{d}t,\\
\mathrm{d}\mathbf{w}_t &= -\alpha\mathbf{w}_t\,\mathrm{d}t + \mathbf{q}_t(G\mathbf{p}_t - \mathbf{u}_t)\,\mathrm{d}t,
\end{aligned} \tag{3}$$

where the demand law $G = G(\mathbf{p}_t, \bar{\mathbf{p}}_t, \mathbf{u}_t)$.

Taylor [3] assumes that the demand for policies is a function of the premium relative to the market average premium

$$\mathbf{k}_t := \mathbf{p}_t/\bar{\mathbf{p}}_t.$$

If this ratio is small, then the insurer has a very competitive premium, and consequently gains significant exposure. The expected cost of selling an insurance policy of duration $l$ at time $t$ is

$$\mathbb{E}\left(\int_t^{t+l} \mathbf{u}_s \,\mathrm{d}s \middle| \mathbf{X}_t = x\right) = \left(\frac{\mathrm{e}^{\mu l} - 1}{\mu}\right) u_t, \tag{4}$$

where the current state of the insurer is $x = (u_t, \bar{p}_t, q_t, w_t)^T$. The premium to claims rate ratio $\mathbf{p}_t/\mathbf{u}_t$ is proportional to the loading applied by the insurer, and it is a measure of the "value" of the insurance to the policyholder. If the insurer sets a very large loading, then buying insurance becomes an unattractive means to minimise risk for the policyholder. We set the demand law

$$G(\mathbf{p}_t, \bar{\mathbf{p}}_t, \mathbf{u}_t) = \begin{cases} a - \mathbf{p}_t \,\mathrm{d}(\bar{\mathbf{p}}_t, \mathbf{u}_t) & \text{if } 0 < \mathbf{p}_t \le \dfrac{a}{\mathrm{d}(\bar{\mathbf{p}}_t, \mathbf{u}_t)}, \\ 0 & \text{if } \mathbf{p}_t > \dfrac{a}{\mathrm{d}(\bar{\mathbf{p}}_t, \mathbf{u}_t)}, \end{cases} \tag{5}$$

where $\mathrm{d}(\bar{\mathbf{p}}_t, \mathbf{u}_t) \ge 0$. The simplest plausible choice for d yields a demand function which is a linear decreasing function of the two ratios $\mathbf{k}_t$ and $\mathbf{p}_t/\mathbf{u}_t$; that is,

$$\mathrm{d}(\bar{\mathbf{p}}_t, \mathbf{u}_t) = \frac{b}{\bar{\mathbf{p}}_t} + \frac{c}{\mathbf{u}_t}, \tag{6}$$

with constants $b, c > 0$. With this parameterisation, the demand for insurance policies depends on both the competitiveness of the premium and its perceived value for money. The larger the parameter $c$ then the lower the demand for policies with a given loading. Therefore, large values of $c$ imply an insurance market which is very sensitive to the loading that the insurer applies to its policies.

The reserve process, $\{\mathbf{w}_t\}$, does not reflect the outstanding liabilities of the insurer. If the insurer sells no more insurance at time $t$ then the exposure decays exponentially (at least according to the model), and so the expected liability from claims on policies currently in force is

$$\mathbb{E}\left[\int_t^{\infty} q_s \mathbf{u}_s \,\mathrm{d}s \middle| \mathbf{X}_t = x\right] = \frac{u_t q_t}{\kappa - \mu},$$

provided that $\kappa > \mu$. Thus, a measure of the *wealth* of the insurer is its current reserve, $w_t$, minus its current expected liability for holding policies with exposure $q_t$.

We suppose that the insurer aims to maximise this wealth, so an appropriate objective for the insurer is

$$\max_p \mathbb{E}\left[\mathbf{w}_T - \frac{\mathbf{u}_T \mathbf{q}_T}{\kappa - \mu}\right]. \tag{7}$$

Thus, the optimising insurer aims to maximise its expected final wealth at some fixed planning horizon $T$.

Let us define the value function

$$V(x, t) = \max_p \mathbb{E}\left[\mathbf{w}_T - \frac{\mathbf{u}_T \mathbf{q}_T}{\kappa - \mu} \middle| \mathbf{X}_t = x\right].$$

The Bellman equation corresponding to system (3) is

$$\begin{aligned} J_t + \max_p \Big\{ & \mu u J_u + \frac{1}{2}\sigma^2 u^2 J_{uu} + (\gamma^{-1} u\mu + \lambda(p - \bar{p}))J_{\bar{p}} + \frac{1}{2}(\gamma^{-1} u\sigma)^2 J_{\bar{p}\bar{p}} + \gamma^{-1}(u\sigma)^2 J_{\bar{p}u} \\ & + q(G - \kappa)J_q + (-\alpha w + q(Gp - u))J_w \Big\} = 0, \end{aligned} \tag{8}$$

where $J = J(t, u, \bar{p}, q, w)$ is the candidate value function [10]. The feedback premium strategy $p^f$ is given by the first-order condition for a maximum in the Bellman equation:

$$p^f = \frac{1}{2}\left(\frac{a + \frac{\lambda J_{\bar{p}}}{q J_w}}{\mathrm{d}(\bar{p}, u)} - \frac{J_q}{J_w}\right). \tag{9}$$

We denote the feedback demand function by

$$G^f = \frac{1}{2}\left(a - \frac{\lambda J_{\bar{p}}}{qJ_w} + \frac{\mathrm{d}(\bar{p}, u)J_q}{J_w}\right), \tag{10}$$

which denotes the demand corresponding to the feedback premium $p^f$ provided that this demand is non-zero. Thus, if $G(p^f, \bar{p}, u) \neq 0$, then $G(p^f, \bar{p}, u) = G^f(t, u, \bar{p}, q, w)$, and we say that the feedback premium is interior to the demand function.

Substituting (9) and (10) into the Bellman equation leads to

$$\begin{aligned} &J_t + \mu u J_u + \frac{1}{2}\sigma^2 u^2 J_{uu} + \left(\gamma^{-1}u\mu - \lambda\bar{p} + \frac{a\lambda}{2\mathrm{d}(\bar{p}, u)}\right)J_{\bar{p}} \\ &+ \frac{1}{2}(\gamma^{-1}u\sigma)^2 J_{\bar{p}\bar{p}} + \gamma^{-1}(u\sigma)^2 J_{\bar{p}u} + q\left(\frac{1}{2}a - \kappa\right)J_q + \left(-\alpha w + q\left(\frac{a^2}{4\mathrm{d}(\bar{p}, u)} - u\right)\right)J_w \\ &+ \frac{q\mathrm{d}(\bar{p}, u)}{4J_w}\left(\frac{\lambda J_{\bar{p}}}{q\mathrm{d}(\bar{p}, u)} - J_q\right)^2 = 0, \end{aligned} \tag{11}$$

where we have collected the nonlinear terms at the end of the equation. The boundary condition at termination is

$$J(t = T) = w - \frac{qu}{\kappa - \mu}, \tag{12}$$

so the terminal feedback premium and demand are

$$p^f(T) = \frac{1}{2}\left(\frac{a}{\mathrm{d}(\bar{p}, u)} + \frac{u}{\kappa - \mu}\right), \qquad G^f(T) = \frac{1}{2}\left(a - \frac{\mathrm{d}(\bar{p}, u)u}{\kappa - \mu}\right).$$

The argument of the maximum operator in (8) is quadratic in $p$, and the quadratic term has coefficient $-q\mathrm{d}(\bar{p}, u)J_w$. It seems reasonable to suppose that $J$ increases with the current reserve, so this coefficient is negative, and therefore the first-order condition does yield a maximum. Consequently, the feedback premium $p^f$ attains the maximum in the Bellman equation as long as it is interior.

The behaviour of the optimal premium at termination helps to determine the range of validity of the interior control. Using the linear demand function (6), as $\bar{p} \to 0$, then $d \to \infty$, $p^f(T) \to \frac{1}{2}u/(\kappa - \mu)$, $G^f(T) \to -\infty$. These values are outside of the domain of definition of the demand function, so $\bar{p}$ is constrained by the bound $G \geq 0$ which gives

$$\frac{u}{\bar{p}} \leq \frac{a(\kappa - \mu) - c}{b}.$$

As $\bar{p} \to \infty$, $d \to c/u$, $p^f(T) \to \frac{1}{2}u(a/c + 1/(\kappa - \mu))$, and $G^f(T) \to \frac{1}{2}(a - c/(\kappa - \mu))$. As long as $a \geq c/(\kappa - \mu)$, then there is no upper bound on $\bar{p}$ for the terminal feedback premium to remain interior.

### 3.1. *Optimisation problem without an exit set*

Let us look for a solution to the Bellman equation (11) of the form

$$J = \mathrm{e}^{\alpha(t-T)}(w + quf(R, t)), \tag{13}$$

where

$$R = \frac{u}{\bar{p}}, \tag{14}$$

and we suppose that

$$\mathrm{d}(\bar{p}, u)u = D(R).$$

For this candidate value function $J_w > 0$, so the first-order condition does yield a maximum in the Bellman equation. The linear demand function (6) gives $D$ linear in $R$:

$$D(R) = bR + c. \tag{15}$$

The candidate value function $J$ is the value function $V$ if it is sufficiently smooth, the feedback premium attains the supremum in the Bellman equation, and the corresponding optimal state is well defined [11].

Substituting (13) into the Bellman equation (11) yields

$$f_t + R\left(\mu(1-\gamma^{-1}R) - \lambda\left(\frac{aR}{2D(R)} - 1\right) + \sigma^2(1-\gamma^{-1}R)^2\right) f_R + \frac{1}{2}\sigma^2 R^2 (1-\gamma^{-1}R)^2 f_{RR}$$
$$+ \frac{a^2}{4D(R)} - 1 + \left(\frac{1}{2}a - \kappa + \mu + \alpha\right) f + \frac{1}{4}D(R)\left(\frac{\lambda R^2 f_R}{D(R)} + f\right)^2 = 0, \tag{16}$$

with terminal boundary condition

$$f(R,T) = -\frac{1}{\kappa - \mu}. \tag{17}$$

The feedback premium and demand function are now

$$p^f = \frac{u}{2}\left(\frac{a}{D(R)} - f - \frac{\lambda R^2 f_R}{D(R)}\right), \tag{18}$$

$$G^f = \frac{1}{2}\left(a + fD(R) + \lambda R^2 f_R\right). \tag{19}$$

If the insurer's premium $\mathbf{p}_t = a/\mathrm{d}(\bar{\mathbf{p}}_t, \mathbf{u}_t)$, then according to the demand law (5) there is no demand for the insurer's policies. If $G_f < 0$, then the candidate value function does not yield the optimal control because the first-order condition does not yield the maximum in (8). However, even if $G_f \geq 0$, it is not clear that the optimal strategy for this problem is smooth. It may be optimal over the planning horizon to set a discontinuous premium with periods of high policy sales followed abruptly by periods of no sales at all. Thus, it is believed that the problem as it stands is intractable.

In order to rule out such strategies, and to be consistent with observed premium strategies of insurers, we modify the model so that the premium strategy must be a smooth function of time.

### 3.2. Optimisation problem with an exit set

Let us introduce the stopping time

$$\tau := \inf\{t \geq 0 : \mathcal{F}(\bar{\mathbf{p}}_t, \mathbf{u}_t, t) = 0\}, \tag{20}$$

where $\mathcal{F}$ is a function of the state variables. We require the insurer to leave the insurance market if the state function $\mathcal{F}(\bar{\mathbf{p}}_t, \mathbf{u}_t, t) = 0$. Consequently, the objective function of the insurer (7) becomes

$$\max_p \mathbb{E}\left[\mathbf{w}_{\tau\wedge T} - \frac{\mathbf{u}_{\tau\wedge T}\mathbf{q}_{\tau\wedge T}}{\kappa - \mu}\right], \tag{21}$$

and the associated value function is

$$V(x,t) = \max_p \mathbb{E}\left[\mathbf{w}_{\tau\wedge T} - \frac{\mathbf{u}_{\tau\wedge T}\mathbf{q}_{\tau\wedge T}}{\kappa - \mu}\,\middle|\, \mathbf{X}_t = x\right].$$

If $\mathcal{F}$ is a prescribed function of the state variables, then this modification of the objective leads to a fixed time horizon optimal control problem with an exit set [11]. Following Fleming and Soner [11, p. 156], the Hamilton Jacobi Bellman (HJB) equation and terminal boundary condition for this problem are still given by (8) and (12). However, the exit set introduces an additional state boundary condition for the HJB equation:

$$V(x,t) = w - \frac{uq}{\kappa - \mu}, \tag{22}$$

if $\mathcal{F}(\bar{p}, u, t) = 0$.

If $\mathcal{F} \equiv G$, where $G$ is given by (5), then this stopping time prevents the insurer leaving and then re-entering the insurance market. However, the first-order condition may still not yield the maximum in the HJB equation, which means that the candidate value function is not necessarily the value function. Consequently, we choose the state function, $\mathcal{F}$, so that the optimal control is determined by the first-order condition when the candidate value function takes the form (13). In effect, we are imposing a form for the value function guided by the optimisation problem without an exit set and the boundary conditions.

Thus, we set

$$\mathcal{F}(\bar{p}, u, t) = G^f(\bar{p}, u, t), \tag{23}$$

where $G^f$ is given by (10). Thus, if the feedback premium is not interior, then the insurer must stop selling insurance and leave the insurance market for the entire planning horizon. If we can solve this problem numerically, then in principle we can write down an explicit form for $G^f(R,t)$. Returning to the optimisation problem afresh, with the prescribed exit set determined by (20), we can be confident that we have found the optimal control for this particular state condition provided that the solution to the HJB equation is sufficiently smooth. We must verify *a posteriori* that such a state function leads to a reasonable exit set and optimal control.

Let us look for a solution of the HJB equation (11) in the form of (13). The HJB equation becomes (16), with terminal boundary condition (17) and spatial condition

$$f(R,t) = -\frac{1}{\kappa-\mu} \quad \text{at } \mathcal{F}=0, \tag{24}$$

from (22). Choosing the exit set on the basis of a feedback demand function $G^f$ couples the exit set to the optimisation problem. Using (19), we obtain the *implicit moving boundary condition* to determine the upper boundary of the state variable $R = R_f(t)$:

$$\frac{D(R_f(t))}{\kappa-\mu} = a + \lambda R_f(t)^2 f_R(R_f(t),t). \tag{25}$$

If there is no market reaction ($\lambda = 0$), then, from (3), the process defined by $\mathbf{R}_t = \mathbf{u}_t/\bar{\mathbf{p}}_t = \gamma$ is constant. In this case, we write $f = f_0(t;\gamma)$, and Eq. (16) reduces to the ordinary differential equation (ODE)

$$\frac{\mathrm{d}f_0}{\mathrm{d}t} + \frac{a^2}{4D(\gamma)} - 1 + \left(\frac{1}{2}a - \kappa + \mu + \alpha\right) f_0 + \frac{1}{4}D(\gamma) f_0^2 = 0, \tag{26}$$

which can be integrated to yield

$$f_0(t;\gamma) = \begin{cases} \dfrac{1}{D(\gamma)}\left(\Delta_+\left(\dfrac{1+E(t)}{1-E(t)}\right) - 2\phi\right) & \text{if } \zeta > 0, \\ \dfrac{1}{D(\gamma)}\left(\Delta_- \tan\left(\dfrac{1}{4}\Delta_-(T-t) + \tan^{-1}\left(\dfrac{2\phi(\kappa-\mu) - D(\gamma)}{\Delta_-(\kappa-\mu)}\right)\right) - 2\phi\right) & \text{if } \zeta < 0, \\ \dfrac{2}{D(\gamma)}\left(\dfrac{2}{t - T + \frac{4(\kappa-\mu)}{2\phi(\kappa-\mu) - D(\gamma)}} - \phi\right) & \text{if } \zeta = 0, \end{cases} \tag{27}$$

where we introduce the following notation for convenience:

$$\phi = \frac{1}{2}a - \kappa + \mu + \alpha, \tag{28}$$

$$\zeta = 4\phi^2 + 4D(\gamma) - a^2, \tag{29}$$

$$\Delta_\pm = \sqrt{\pm\zeta}, \tag{30}$$

$$E(t) = \left(\frac{(\kappa-\mu)(2\phi - \Delta_+) - D(\gamma)}{(\kappa-\mu)(2\phi + \Delta_+) - D(\gamma)}\right) \mathrm{e}^{\frac{1}{2}(T-t)\Delta_+}. \tag{31}$$

Notice that we cannot have $c = D(0) = 0$, otherwise Eq. (26) is singular when $\gamma = 0$. In addition, $f_0$ is independent of $\sigma$, the claims rate volatility, so the optimal premium strategy is identical to the deterministic problem. It was found in [5] that the optimal strategy derived from (27) can be characterised as either equilibrating ($\zeta > 0$) or singular ($\zeta < 0$) as the planning horizon $T$ gets large, where $\zeta$ is defined by (29).

If there is no reaction, $f_0$ yields the relative to market feedback premium $k_0^f$, the corresponding feedback demand function $G_0^f$, and the change in reserve from selling policies and settling claims per unit of market average premium $\delta w = (Gp - u)/\bar{p}$. Using this notation, we have

$$k_0^f(t;\gamma) = \frac{\gamma}{2}\left(\frac{a}{D(\gamma)} - f_0(t;\gamma)\right), \tag{32}$$

$$G_0^f(t;\gamma) = \frac{1}{2}(a + f_0(t;\gamma)D(\gamma))^+, \tag{33}$$

$$\delta w(t;\gamma) = k_0^f(a - k_0^f(t;\gamma)D(\gamma)/\gamma)^+ - \gamma. \tag{34}$$

If $D$ is given by (15), then for $\gamma > (a(\kappa-\mu) - c)/b$ it is optimal not to sell insurance, while if $\delta w < 0$ then the insurer's reserve decreases, because claims exceed premium income over $\mathrm{d}t$. If the insurer does not sell insurance, then the reserve must decrease in order to pay off the expected claims of existing policy holders.

**Table 1**
Sample data set.

| | |
|---|---|
| Time horizon $T$ | 2.0 yr |
| Depreciation of wealth $\alpha$ | 0.05 p.a. |
| Demand parameterisation $a$ | 3 p.a. |
| Demand parameterisation $b$ | 1 p.a. |
| Demand parameterisation $c$ | 1 (p.a.)$^2$ |
| Length of policy $l = \kappa^{-1}$ | 1 yr |
| Mean claim size rate growth $\mu$ | 0.1 p.a. |
| Mean claim size rate volatility $\sigma$ | 0.3 (p.a.)$^{1/2}$ |
| Loss ratio $\gamma$ | 0.9 p.a. |
| Rate of market reaction $\lambda$ | 0.1 p.a. |
| Number of time steps $N_t$ | 10,000 |
| Number of spatial steps $N_\rho$ | 100 |

If the market does react, then $\mathbf{R_t} = \mathbf{u}_t/\bar{\mathbf{p}}_t$ is stochastic. If the mean claim size rate $u = 0$ at time $t$, then $\mathbf{u}_s \equiv 0$ for $s \in [t, T]$, since $\mathbf{u}_t$ is lognormally distributed from (3). Consequently, provided that the market average premium is non-zero, $\mathbf{R}_s \equiv 0$, and the optimisation problem is deterministic because stochasticity only occurs through variability in claims. This provides the final boundary condition for the moving boundary problem:

$$f(0, t) = f_0(t; 0). \tag{35}$$

Here, we assume that the limiting behaviour of the stochastic problem as stochasticity tends to zero is given by the deterministic problem. For premium pricing models, this is the case, because the Itô terms in (8) do not explicitly contain the control $p$.

Eq. (16) and the boundary conditions (17), (24), (25) and (35) are sufficient to determine $f$ and its domain of definition.

### 3.3. Deterministic problem

If the claims rate is deterministic ($\sigma = 0$), then the state equation for $R$ is

$$\frac{\mathrm{d}R}{\mathrm{d}t} = R[\mu(1 - \gamma^{-1}R) - \lambda(k - 1)], \tag{36}$$

with $R(0) = \gamma$ using (14) and (3). It is clear that $R \equiv \gamma$ is a solution of this equation if there is no reaction $\lambda = 0$. Again, we search for a smooth premium strategy that maximises (7) for the deterministic version of the problem in Section 3.1. Primarily, we solve this problem in order to verify numerically that the analytical solution $f_0$ is the optimal premium strategy (for a restricted range of parameters). In addition, we can see how market reaction changes the deterministic optimal premium strategy.

Rather than solve the boundary value problem arising from Pontryagin's Maximum Principle, we solve the deterministic problem using control parameterisation. Emms [5] solves the pricing problem with constraints using this method, and we adopt a similar approach here. Thus, the state equations for $q$ and $w$ given by (3) are supplemented by (36). However, the demand function $G$ is not smooth at $p = a/d$, and numerical instabilities arise if one tries to integrate the state equations. Instead, we redefine the demand function using the first expression in (5) over the entire domain, and pose the constraint that $G > 0$. This is a state constraint, which under the constraint transcription requires smoothing near the jump in derivative, in order to remove numerical oscillations. The objective is now the deterministic version of (7).

Representative results are shown in Figs. 1 and 2 by varying the parameters $a$, which determines the growth in exposure for given relative premium, and $\gamma$, which indicates whether the market is over-/under-pricing insurance. Fig. 1 uses the parameter set given in Table 1, except that we set $a = 2$ so that $\zeta < 0$ and the equilibrating form of $f_0$ is optimal. There are two graphs in Fig. 1: the first shows the case that $\gamma = 0.3$ so the market is over-pricing insurance, while the second shows $\gamma = 0.7$ and the market is setting a fairer premium. Both graphs show that the numerical method reproduces the analytical premium strategy if $\lambda = 0$ and that market reaction raises the optimal relative premium.

Fig. 2(a) shows the optimal strategy when $a = 3$ and $\gamma = 0.3$ so that $\zeta > 0$ and the non-equilibrating form of $f_0$ is optimal. Again, the non-reacting case is verified, and the optimal relative premium is raised as market reaction increases. If we set $\gamma = 1.5$, then the market under-prices insurance, and it is optimal to lower the premium if the market reacts. We discuss the interpretation of these results in relation to the stochastic problem in Section 5 where the findings are similar.

## 4. Finite difference schemes

Crank [12] and Javierre et al. [13] compare various numerical techniques that solve the canonical moving boundary problem: the one-dimensional Stefan problem. In finance, Nielsen et al. [14] compares the front-fixing and penalty methods for the solution of the American option pricing problem. Here, the numerical solution of (16) is complicated by the implicit moving boundary condition (25). We describe three finite difference schemes, which yield the numerical solution based on a front-fixing transformation, in order to have confidence in the numerical results.

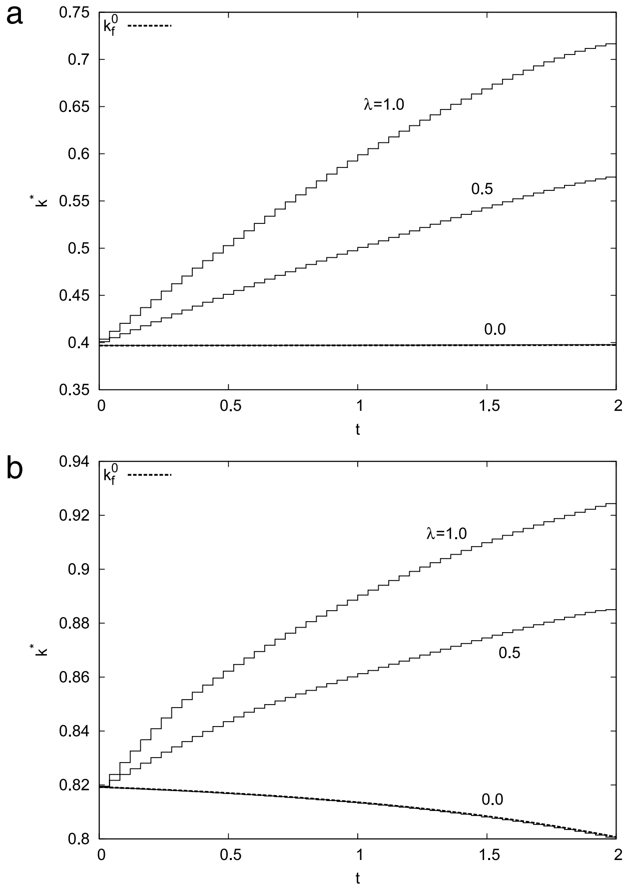


**Fig. 1.** Equilibrating optimal relative premium strategies for a deterministic and reactive market. The parameters are taken from Table 1 except that $a = 2.0$ and (a) $\gamma = 0.3$ or (b) $\gamma = 0.7$.

Let us introduce the new state variable

$$\rho = \frac{R}{R_f(t)} \tag{37}$$

so that the moving boundary is fixed on $\rho = 1$. In terms of this new variable, $f = f(\rho(R, t), t)$, and the (reduced) Bellman equation becomes

$$\begin{aligned} &f_t + \rho\left(\mu(1-\gamma^{-1}\rho R_f(t)) - \frac{\dot{R}_f(t)}{R_f(t)} - \lambda\left(\frac{a\rho R_f(t)}{4D(\rho R_f(t))} - 1\right) + \sigma^2(1-\gamma^{-1}\rho R_f(t))^2\right) f_\rho \\ &\quad + \frac{1}{2}\sigma^2\rho^2(1-\gamma^{-1}\rho R_f(t))^2 f_{\rho\rho} + \frac{a^2}{4D(\rho R_f(t))} - 1 + \left(\frac{1}{2}a - \kappa + \mu + \alpha\right) f \\ &\quad + \frac{D(\rho R_f(t))}{4}\left(\frac{\lambda R_f(t)\rho^2 f_\rho}{D(\rho R_f(t))} + f\right)^2 = 0, \end{aligned} \tag{38}$$

with boundary conditions

$$f(1, t) = f(\rho, T) = -\frac{1}{\kappa - \mu} \quad \text{and} \quad f(0, t) = f_0(t).$$

If we evaluate this equation on $\rho = 1$, then we obtain an equation for the evolution of the moving boundary $R_f(t)$:

$$\begin{aligned} \frac{\mathrm{d}R_f}{\mathrm{d}t} &= \frac{R_f}{f_\rho}\Bigg[\left(\mu(1-\gamma^{-1}R_f) - \lambda\left(\frac{aR_f}{4D(R_f)} - 1\right) + \sigma^2(1-\gamma^{-1}R_f)^2\right) f_\rho \\ &\quad + \frac{1}{2}\sigma^2(1-\gamma^{-1}R_f)^2 f_{\rho\rho} + \frac{a^2}{2D(R_f(t))} - \frac{\frac{1}{2}a + \alpha}{\kappa - \mu}\Bigg], \end{aligned} \tag{39}$$

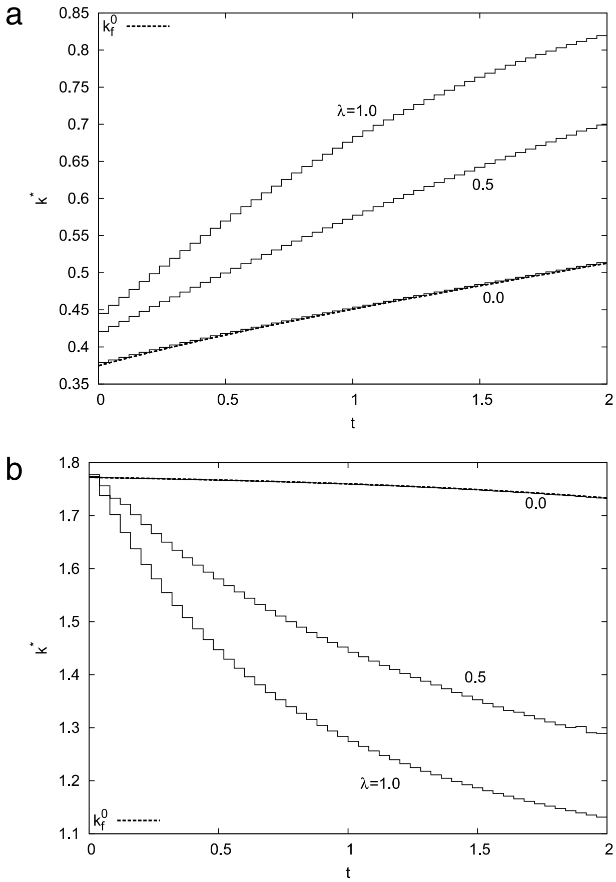


**Fig. 2.** Step functions showing the optimal relative strategy as the reaction parameter $\lambda$ is varied. The parameters are taken from Table 1 except that (a) $\gamma = 0.3$ or (b) $\gamma = 1.5$.

with boundary condition

$$R_f(T) = D^{-1}(a(\kappa - \mu)),$$

from (25).

The derivatives in (39) can be calculated using a set of fictitious grid points, and the moving boundary condition in the new state variable is

$$f_\rho = \frac{1}{\lambda R_f}\left(\frac{D(R_f)}{\kappa - \mu} - a\right). \tag{40}$$

One can also use this expression to simplify (39) and yield

$$\frac{1}{R_f}\frac{\mathrm{d}R_f}{\mathrm{d}t} = \mu(1 - \gamma^{-1}R_f) + \sigma^2(1 - \gamma^{-1}R_f)^2\left(1 + \frac{f_{\rho\rho}}{2f_\rho}\right) + \lambda\left(1 - \frac{3aR_f}{4D(R_f)} - \frac{\alpha R_f}{D(R_f) - a(\kappa - \mu)}\right). \tag{41}$$

In terms of the new state variables, the relative feedback premium is

$$k^f = \frac{\rho R_f(t)}{2}\left(\frac{a}{D(\rho R_f(t))} - f(\rho, t) - \frac{\lambda \rho^2 R_f(t) f_\rho(\rho, t)}{D(\rho R_f(t))}\right),$$

so, on $\rho = 0$, $k_f = 0$, and, on $\rho = 1$,

$$k^f = \frac{aR_f(t)}{D(R_f(t))},$$

from (40).

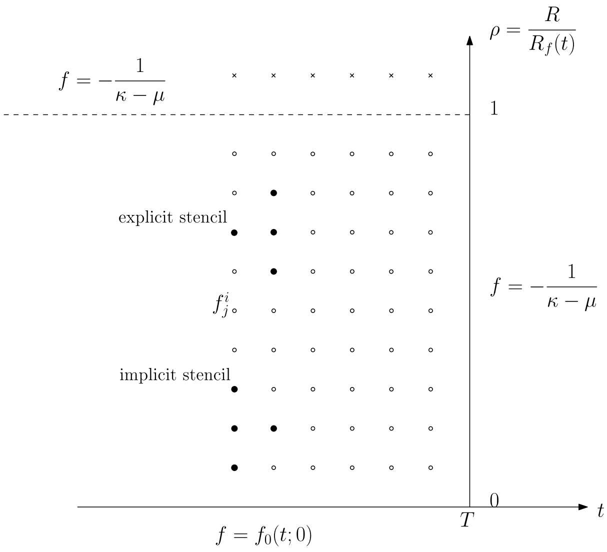


**Fig. 3.** Finite difference grid for the numerical solution of (16). The moving boundary is on $\rho = 1$, and the crosses in $\rho > 1$ denote fictitious grid points.

Consider the uniform grid shown in Fig. 3. There are $N_t$ time steps and $N_\rho$ spatial steps, so the time step size is $\Delta t = T/N_t$ and the spatial step size is $\Delta\rho = 1/N_\rho$. Let us denote the approximate value of $f(j\Delta\rho, i\Delta t)$ and $R_f(i\Delta t)$ on this grid by $f_j^i$ and $R_f^i$, respectively, for $i = 0, 1, \ldots, N_t$ and $j = 0, 1, \ldots, N_\rho + 1$. In all of the subsequent numerical schemes, the solution is calculated stepping backwards in time from the end of the planning horizon $t = T$.

*Explicit scheme*

At $t = T, f = -1/(\kappa - \mu)$, so $f_\rho = 0$, which means that $\mathrm{d}R_f/\mathrm{d}t$ is infinite, from (39). A straightforward explicit finite difference scheme [15] with a first-order time difference and second-order spatial differences does not cope with this singularity. Moreover, there are two time derivatives in (38): one for $f$ and one for the moving boundary $R_f$, and it is not clear what time stepping scheme one should adopt. The simplest approach is to lag the evolution of the moving boundary, so in (38), we evaluate $\dot{R}_f$ at the old time step. In order to avoid the singularity, we modify the moving boundary condition (40) in the discrete approximation:

$$\frac{f_{N_\rho+1}^{i-1} - f_{N_\rho-1}^{i}}{2\Delta\rho} = \frac{1}{\lambda R_f^i}\left(\frac{D(R_f^i)}{\kappa - \mu} - a\right),$$

where the current time level is $i - 1$ and $f_{N_\rho+1}^{i-1}$ is a fictitious grid point. This is called the explicit finite difference scheme. There is no longer a problem with a singularity at $t = T$, since now $f_\rho$ is now small and non-zero. The second-order difference for the first derivative has been replaced by a difference across time levels:

$$\frac{f_{N_\rho+1}^{i-1} - f_{N_\rho-1}^{i}}{2\Delta\rho} = \frac{f_{N_\rho+1}^{i-1} - f_{N_\rho-1}^{i-1} + f_{N_\rho-1}^{i-1} - f_{N_\rho-1}^{i}}{2\Delta\rho}$$

$$\approx f_\rho(j\Delta\rho, (i-1)\Delta t) + \frac{\Delta t}{2\Delta\rho} f_t((j-1)\Delta\rho, i\Delta t).$$

Provided that $\Delta t \ll \Delta\rho$ and $f_t$ is finite, then this gives an approximation to $f_\rho$. In any case, the stability restriction of an explicit scheme requires the time step to be much smaller than the spatial step. Furthermore, the accuracy of this approximation will be tested with reference to more complicated numerical schemes.

*Hybrid scheme*

The hybrid numerical scheme is an implicit scheme, and so requires an iterative solver at each time step. In this scheme, we use a forward first-order time step for $f_t$ in (38), lag $\dot{R}_f$, and use a backward first-order time step for $\dot{R}_f$ in (39). The backward step results in a right-hand side which must be evaluated at $t = (i - 1)\Delta t$, that is, the current time step, so the discretised equation is nonlinear. A secant method is used to solve the discretised form of (39) for $R_f^{i-1}$.

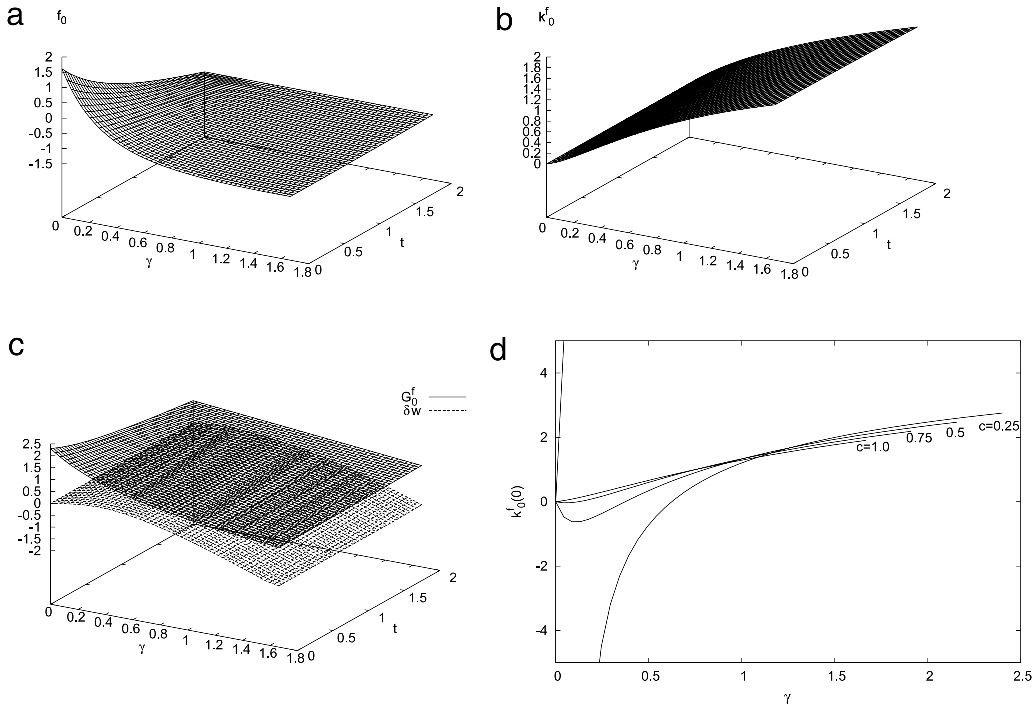


**Fig. 4.** Analytical solution of (16) when there is no market reaction for the sample data set given in Table 1. Graph (a) shows $f_0$ as a surrogate for the value function, graph (b) shows the relative to market feedback premium $k_0^f = p_0^f/\bar{p}$, and graph (c) shows the feedback demand function $G_0^f$ and the change in reserve $\delta w$. Graph (d) shows how the relative premium at $t = 0$ changes as the parameter $c$ of the demand parameterisation is varied. All these quantities are shown as a function of the *constant* loss ratio $0 \le \gamma \le (a(\kappa-\mu)-c)/b$ which generates positive demand.

*Implicit scheme*

The implicit scheme uses backward time steps for $f_t$ and $\dot{R}_f$ in (38), and $\dot{R}_f$ in (39). The reduced Bellman equation (16) becomes, at step $i-1$,

$$\frac{f_j^i - f_j^{i-1}}{\Delta t} + \rho_j^{i-1}\Bigg(\mu(1-\gamma^{-1}\rho_j^{i-1}R_f^{i-1}) - \frac{R_f^i - R_f^{i-1}}{R_f^{i-1}\Delta t} - \lambda\left(\frac{a\rho_j^{i-1}R_f^{i-1}}{4D(\rho_j^{i-1}R_f^{i-1})} - 1\right) + \sigma^2(1-\gamma^{-1}\rho_j^{i-1}R_f^{i-1})^2\Bigg)\left(\frac{f_{j+1}^{i-1}-f_{j-1}^{i-1}}{2\Delta\rho}\right) + \frac{1}{2}(\sigma\rho_j^{i-1})^2(1-\gamma^{-1}\rho_j^{i-1}R_f^{i-1})^2\left(\frac{f_{j+1}^{i-1}-2f_j^{i-1}+f_{j-1}^{i-1}}{\Delta\rho^2}\right) + \frac{a^2}{4D(\rho_j^{i-1}R_f^{i-1})} - 1 + \left(\frac{1}{2}a - \kappa + \mu + \alpha\right) f_j^{i-1} + \frac{D(\rho_j^{i-1}R_f^{i-1})}{4}\left(\frac{\lambda R_f \rho_j^{i-1^2}}{D(\rho_j^{i-1}R_f^{i-1})}\left(\frac{f_{j+1}^{i-1}-f_{j-1}^{i-1}}{2\Delta\rho}\right) + f_j^{i-1}\right)^2 = 0, \tag{42}$$

for $j = 1, 2, \ldots, N_\rho - 1$.

On $j = 0, f_0^i = f_0(i\Delta t; 0)$, while $f_{N_\rho+1}^{i-1}$ is given by the differenced form of the moving boundary condition:

$$\frac{f_{N_\rho+1}^{i-1} - f_{N_\rho-1}^{i-1}}{2\Delta\rho} = \frac{1}{\lambda R_f^{i-1}}\left(\frac{D(R_f^{i-1})}{\kappa-\mu} - a\right).$$

Substituting this expression into the differenced form of (41) gives

$$\frac{R_f^i - R_f^{i-1}}{R_f^{i-1}\Delta t} = \mu(1-\gamma^{-1}R_f^{i-1}) + \sigma^2(1-\gamma^{-1}R_f^{i-1})^2\left(1 + \frac{1}{\Delta\rho} + \frac{\lambda R_f^{i-1}(f_{N_\rho-1}^{i-1}(\kappa-\mu)+1)}{\Delta\rho^2(D(R_f^{i-1}) - a(\kappa-\mu))}\right) + \lambda\left(1 - \frac{3aR_f^{i-1}}{4D(R_f^{i-1})} - \frac{\alpha R_f^{i-1}}{D(R_f^{i-1}) - a(\kappa-\mu)}\right). \tag{43}$$

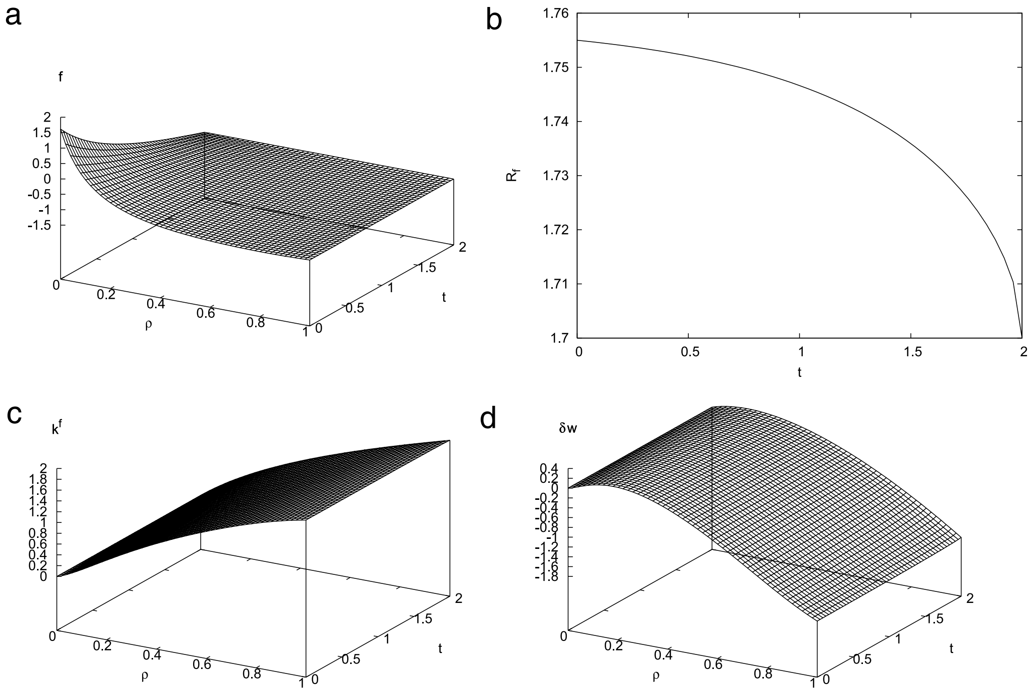


**Fig. 5.** Results from the hybrid finite difference scheme with market reaction using the sample data in Table 1. The explicit and the implicit schemes lead to quantitatively similar results, and are not shown.

Therefore, Eq. (42) for $j = 1, 2, \ldots, N_\rho - 1$ and (43) are $N_\rho$ equations for the $N_\rho$ unknown variables $f_1^{i-1}, f_2^{i-1}, \ldots, f_{N_\rho-1}^{i-1}, R_f^{i-1}$ at time $t = (i-1)\Delta t$. Thus, we have a nonlinear system of equations to solve at each step. The root of this system can be calculated using Broyden's method [16], which is a secant method, and so avoids the evaluation of the Jacobian of the system.

The requirement that we must find the root of a system of nonlinear equations at each time step considerably increases the amount of computation over an explicit scheme. However, as for other implicit schemes for parabolic equations, one can use an increased time step $\Delta t$ for given $\Delta\rho$ without introducing numerical instabilities [15]. Our ability to find a root of these equations as we step back from the time horizon ensures that the derivatives of the candidate value function exist at least in the finite difference approximation. It is not clear that the explicit and hybrid schemes avoid any singularity as a result of the choice of approximation. If all three schemes agree on the solution as we refine the finite difference grid, then it gives us confidence that there is a solution of the Bellman equation for a given parameter set.

## 5. Discussion of results

Again we fix our discussion on the linear demand law (15) and the parameter set in Table 1. These values broadly correspond to those given in [2], although, since we have changed the demand function, $G$, the increase in exposure for a given relative premium is smaller. From (2), the insurer generates exposure when

$$k < \frac{a - \kappa}{bR + c},$$

which for the base parameter set gives $k < 2R/(R+1)$. If the market average price is comparable to the claim size rate, then $R \sim 1$, so the insurer must set its premium below the market average premium in order to generate exposure. A higher premium value does generate policy sales, but this is insufficient to overcome the loss of exposure due to non-renewals. If we increase the value of $a$ or decrease $b$ and $c$, then more policies are sold for a given relative premium. However, the optimal premium is very sensitive to these values, and infinite wealth-generating strategies are possible. Thus, we set the base parameter set shown in Table 1, and subsequently explore the numerical sensitivity of the optimal premium strategy.

First, we focus on the non-reactive case for which $\lambda = 0$. The equations are then the one-factor model mentioned in the Appendix of [4]. Plots of $f_0$, the feedback premium $k_0^f$, the feedback demand function $G_0^f$, and the change in reserve $\delta w$ are shown in Fig. 4 for $\lambda = 0$, using the sample data. These graphs are calculated using the explicit analytical expression (27). The reactive case leads to similar graphs, since the stochastic variable $R$ takes over the role of fixed parameter $\gamma$.

Fig. 4(a) shows $f_0$, which is a surrogate for $J$, the candidate value function, as a function of the parameter $\gamma$ and time $t$. The largest value for $f_0$ is at the origin, since this represents the case that there are no claims and the wealth of the insurer is its reserve. The corresponding feedback premium in Fig. 4(b) is zero, however, because the demand law (5) says that, when

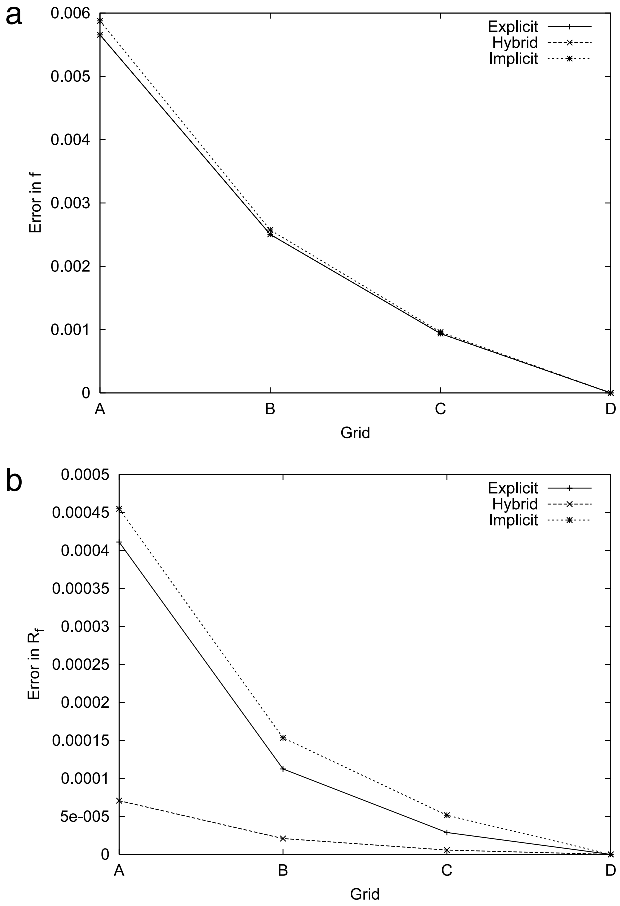


**Fig. 6.** Demonstration of the convergence of the explicit, hybrid, and implicit finite difference schemes using grid refinement. The explicit and hybrid schemes use the difference grids: A ($N_t = 2000, N_\rho = 40$), B ($N_t = 4000, N_\rho = 60$), C ($N_t = 8000, N_\rho = 80$), D ($N_t = 10,000, N_\rho = 100$), while the implicit scheme uses the following grids: A ($N_t = 100, N_\rho = 40$), B ($N_t = 200, N_\rho = 60$), C ($N_t = 300, N_\rho = 80$), D ($N_t = 400, N_\rho = 100$). Graph (a) shows the error in the value function $\|f\| - \|f\|(D)$ while graph (b) shows the error in the moving boundary $\|R_f\| - \|R_f\|(D)$, where $\|.\|$ denotes the average value over the grid. The other parameters were chosen using the sample data.

there are no claims, there is no demand for insurance. Insurance has no value to a policyholder when there can be no claims on that insurance. Notice that this is a qualitative difference from previous competitive pricing models in the literature, e.g. [2], and arises from the modified demand law (5).

The feedback relative premium in Fig. 4(b) increases with $\gamma$, which reflects the increase in the claims rate that the insurer expects to receive. Fig. 4(c) shows the change in reserve $\delta w$ per unit of market average premium, and the feedback demand function $G_0^f$. Since $G_0^f > 0$, the candidate control is interior over the domain. The reserve increases ($\delta w > 0$) over the time horizon if the $\gamma$ is sufficiently small. Thus, if the market overprices insurance, the insurer can undercut the market and still generate demand for its policies. The greater the claim size rate, the greater the loss of reserve. As the demand parameter $c$ is decreased, then $\zeta$ decreases, so, for a given value of $\gamma$, $\zeta < 0$, and the strategy shifts to a singular form of (27). The corresponding feedback premium is shown in Fig. 4(d) for a number of different values of $c$. For the given time horizon, the feedback premium becomes infinite when $c \approx 0.25$, which indicates that infinite wealth can be generated. Similar behaviour occurs in [4] for an unmodified demand law.

In the base parameter set, we have specified $\gamma = 0.9$, which yields $\zeta = 0.29$, so the strategy equilibrates as the time horizon $T$ increases. In addition, $\gamma = 0.9 < (a(\kappa - \mu) - c)/b = 1.7$, so the control is interior, and it is optimal to sell insurance over the planning horizon.

We have demonstrated in the non-reactive case that the candidate value function is sufficiently smooth for at least part of the parameter space. Consequently, $J = V$, and the feedback law gives the optimal control should it satisfy the further conditions of the verification theorem in [10]. Essentially, these conditions require that the first-order condition does yield the maximum in the Bellman equation and that the feedback premium leads to a well-defined optimal state trajectory. The first-order condition does yield the maximum in the Bellman equation provided that the feedback demand $G^f$ is positive, and this has been demonstrated for the parameter sets shown. The control is admissible because the relative feedback premium

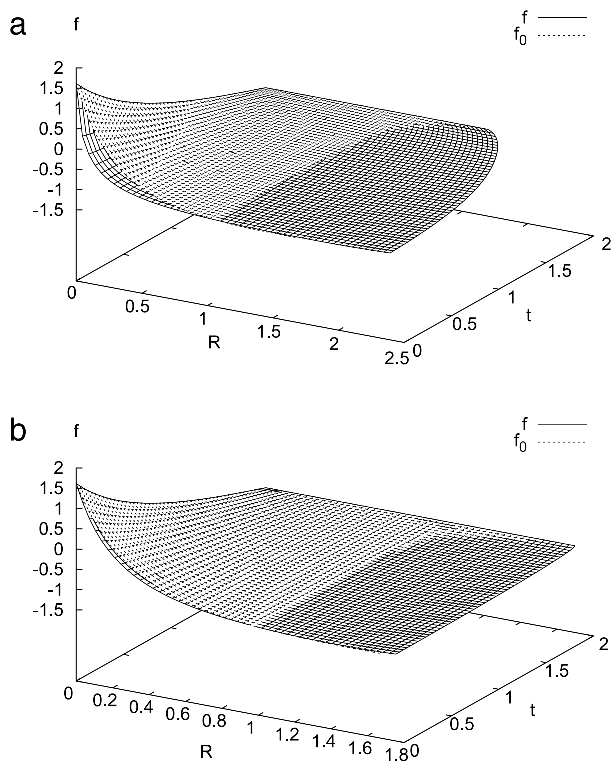


**Fig. 7.** Difference in the value function when the market reaction is (a) rapid, $\lambda = 1$, and (b) slow, $\lambda = 0.1$. The numerical scheme is implicit with mesh $N_\rho = 100$, $N_t = 100$.

is deterministic, and so the state equations become linear in the state variables. Therefore, $k^f$ is the optimal premium strategy under parameter restrictions.

When there is market reaction ($\lambda \neq 0$), the problem must be solved numerically, and one cannot easily apply a verification theorem. We must proceed carefully. The numerical solution of (38) and (39) for the sample data set is shown in Fig. 5 using the hybrid finite difference scheme. Both the explicit scheme and the implicit scheme yield virtually identical results, and so are not shown. The advantage of the explicit and hybrid schemes is that computation is much faster than the implicit method, since this requires the numerical solution of a system of $N$ equations at each time step.

On comparing Figs. 4(a) and 5(a) for a non-reactive and a reactive market, one should remember that $\gamma$ is a parameter in the model, and so Fig. 4(a) yields the value function as this parameter is varied. In reality, the model is imperfect, so at each instant $R$ will not be constant. If one sets the parameter $\gamma$ at each instant equal to the current value of $\mathbf{u}_t/\bar{\mathbf{p}}_t$, then one obtains the premium strategy in Fig. 4(b). In Fig. 5(a), $\rho = R/R_f$ is the rescaled current value of the $\mathbf{R}_t = \mathbf{u}_t/\bar{\mathbf{p}}_t$, and the parameter $\gamma$ is fixed. For the base parameter set, if there is no market reaction, then it is optimal to sell insurance over the entire time horizon. If the market does react, we can see in Fig. 5(b) that $R_f(0) > R_f(T)$, so market reaction increases the region of state space over which it is optimal to sell insurance. In this enlarged region, the premium is increased (Figs. 4(b) and 5(c)), and the reserve is lowered (Figs. 4(c) and 5(d)). However, it is easier to explain the features of the numerical solution in $(R, t)$ space, since $R = u/\bar{p}$ is directly related to the state variables of the insurer.

We support the existence of a solution by confirming in Fig. 6 that convergence is achieved as the finite difference grid is refined. In all three numerical schemes, both the error in the value function $f$ in graph 6(a) and the error in the moving boundary position $R_f$ in graph 6(b) are independent of the grid spacing as the grid is refined. The high-resolution grid D is assumed to represent the true solution in these graphs. Thus, the computed solution is smooth and well behaved, so it appears that it does represent the value function $V$. We suppose that this feedback premium is admissible and leads to a well-defined optimal state. Given the existence of an optimal control for the non-reactive problem, the consistency of the reactive, deterministic case, and the robustness of the numerical solution, this does appear to be a reasonable assumption.

Figs. 7 and 8 show the change in the value function and the optimal relative premium respectively as one changes the reaction of the market to the insurer's strategy in $(R, t)$ space. If the market reacts within the year, then $\lambda$ is $O(1)$, and it is optimal for the insurer to raise its optimal premium especially when $R$ is small. If the reaction is slow, then there is little difference in the computed optimal premium strategy and the analytical solution $f_0$. However, it is still optimal to raise the insurer's premium over a large range of $R$. Notice that these results are presented in $(R, t)$ space, and therefore show the variation in the moving boundary position $R_f$. If the reaction is fast, then the region of state space in which it is optimal to sell insurance is enlarged.

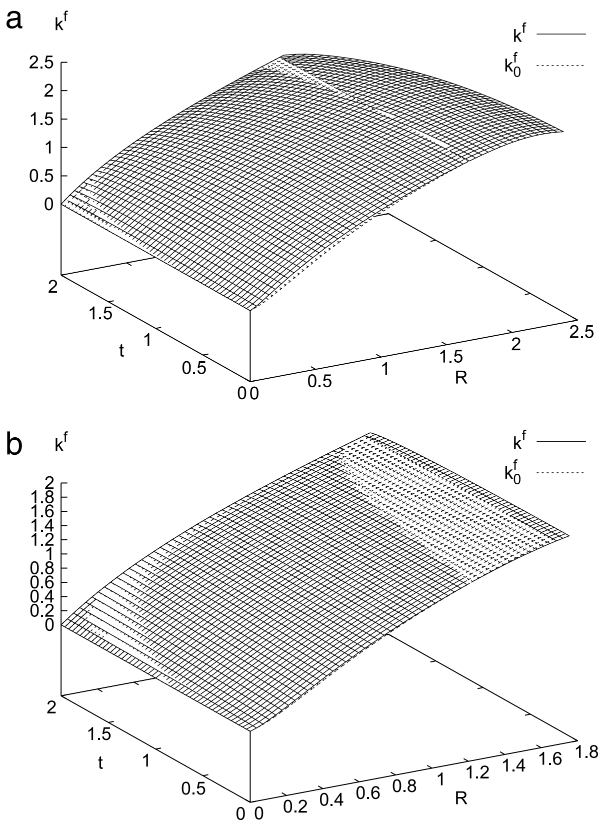


**Fig. 8.** Difference in the feedback premium when the market reaction is (a) rapid, $\lambda = 1$, and (b) slow, $\lambda = 0.1$. The numerical scheme is implicit with mesh $N_\rho = 100$, $N_t = 100$.

The variation in the relative premium at $t = 0$ is made clearer in Fig. 9. From Fig. 9(a), we see that, as the market reaction parameter $\lambda$ decreases, so the deviation of the value function from the non-reacting case decreases. In graph (b), we have highlighted this behaviour by plotting the moving boundary for $\lambda = 0.05$. The moving boundary position deviates only slightly from the non-reacting case, which for the parameter set in Table 1 is $R_f = 1.7$. Consequently, we have demonstrated that the hybrid scheme reproduces the analytical non-reacting solution in the limit as $\lambda \to 0$. The explicit and implicit schemes also reproduce this solution, and are not shown.

The behaviour of the optimal control for a reacting market can be understood as follows. Remember that the cost of selling a policy is assumed to be given by (4) for all insurers. For large values of $R = u/\bar{p}$, the market is underpricing insurance, and, without market reaction, it is optimal for the insurer to withdraw from the insurance market by setting a high premium. If the market reacts to this strategy, then the market also sets a higher premium. Now, there is a greater possibility for the insurer to generate wealth and sell insurance, so the state space is enlarged as the rate of market reaction increases. Moreover, wealth generating strategies involve building up exposure, and that means that the insurer should lower its relative premium (Fig. 9(a)). In Fig. 7(a), the lower premium value raises $f$ for large $R$ above the value for no market reaction because the insurer can now sell more insurance and generate greater profits.

Conversely, if $R$ is small, then the insurance market is overpricing insurance, and an insurer can generate wealth by setting a premium below market average. If the market reacts, then it lowers its premium to match the insurer, and so it becomes more difficult for the insurer to generate demand for its policies. Thus, the insurer raises its premium to withdraw from the market.

Rather than perform an exhaustive sensitivity analysis of all the parameters in the model, we focus on the parameters in the demand function, since that is a significant modification over previous models, and the parameters governing the mean claim size rate, since that process introduces stochasticity. Fig. 10 shows the initial relative premium $k^f(t = 0)$ when we vary each of $a$, $b$, and $c$ in turn, with the remaining parameters held fixed. Making $a$ larger, or $b$ or $c$ smaller, increases demand for a fixed premium, and so extends the region over which it is optimal to sell insurance. This increased demand also causes an increase in the optimal relative premium for small values of $R$ and a decrease in the optimal relative premium near the moving boundary $R_f$.

Fig. 11 shows how the optimal premium strategy changes as we vary the parameters of the mean claim size rate process (1). If the mean claim size rate is expected to rise, then for most values of $R$ the optimal strategy is to set a higher relative premium than if there were no market reaction. If the mean claim size rate is expected to increase but the market

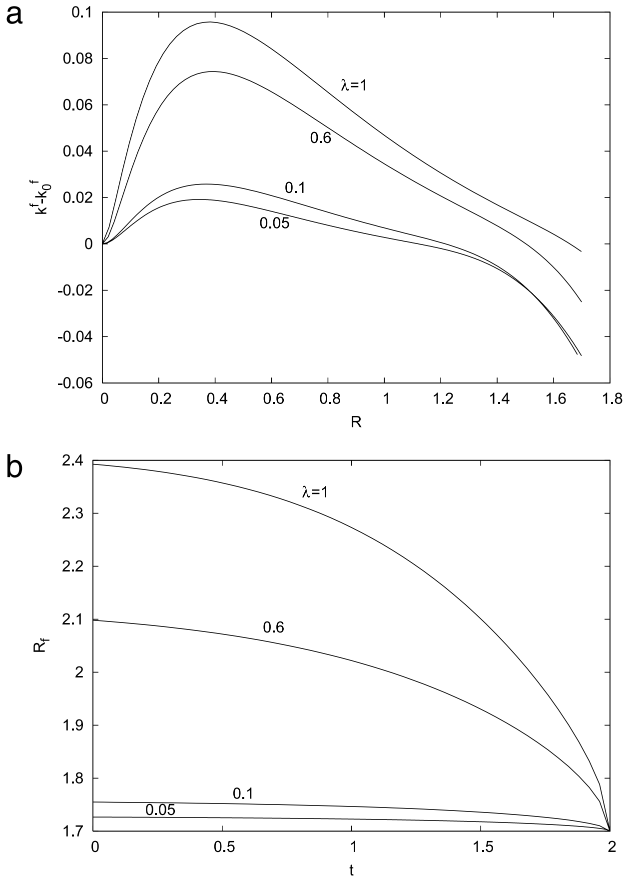


**Fig. 9.** The difference in (a) the initial relative premium strategy, and (b) the moving boundary $R_f$, as we vary the market reaction parameter $\lambda$. In graph (a), $R$ is shown up to $(a(\kappa - \mu) - c)/b$, since this is the extent of the no reaction optimal premium strategy.

average premium remains fixed, then the tendency is for the market to under-price insurance. Following the same argument as before, this leads the insurer to lower its premium as the drift increases. The size of the state space for which it is optimal to sell insurance also diminishes, because it is more difficult for the insurer to generate a profit from selling insurance. If the volatility of the mean claim size rate increases, then it is optimal to set a larger premium than without reaction if $R$ is small, and a smaller relative premium if $R$ is large. Again, the region of state space for which it is optimal to sell insurance is smaller as the volatility of the mean claim size rate is increased, since it is more profitable not to sell insurance if large claims are more likely.

## 6. Conclusions

The generalisation of the demand function (5) impacts significantly on the optimal premium strategy for an insurer. If there is no reaction in the market, then we have found an analytical expression for the optimal relative premium, and if there are no insurance claims, then the optimal relative premium is zero, since there is no need for insurance. Even though the optimal premium strategy is given explicitly, it is not immediately apparent from the analytical solution how the demand function affects the optimal strategy. Consequently, we have introduced a set of parameters and considered the deviation of the optimal strategy corresponding to changes in the parameter set. As the sensitivity of the market to the value of insurance is decreased, demand for insurance increases, and the optimal strategy can lead to negative premium values if the market overprices insurance.

If the market reacts to an insurer who uses optimal control theory in order to calculate premium values, then only a numerical solution can be found for the optimal control problem, with an exit set determined as part of the optimisation problem. In addition, the numerical solution is not entirely straightforward, because the state space is separated into two regions: one where it is optimal for the insurer to leave the market, and the other where the Bellman equation yields the optimal premium strategy. We have fixed the boundary of these two regions by introducing a front-fixing coordinate transformation, which makes the Bellman equation more complicated.

Three numerical schemes have been implemented, and they agree on the computed value function as the mesh is refined. This gives us confidence that we have a robust solution to the Bellman equation, and that the feedback law does

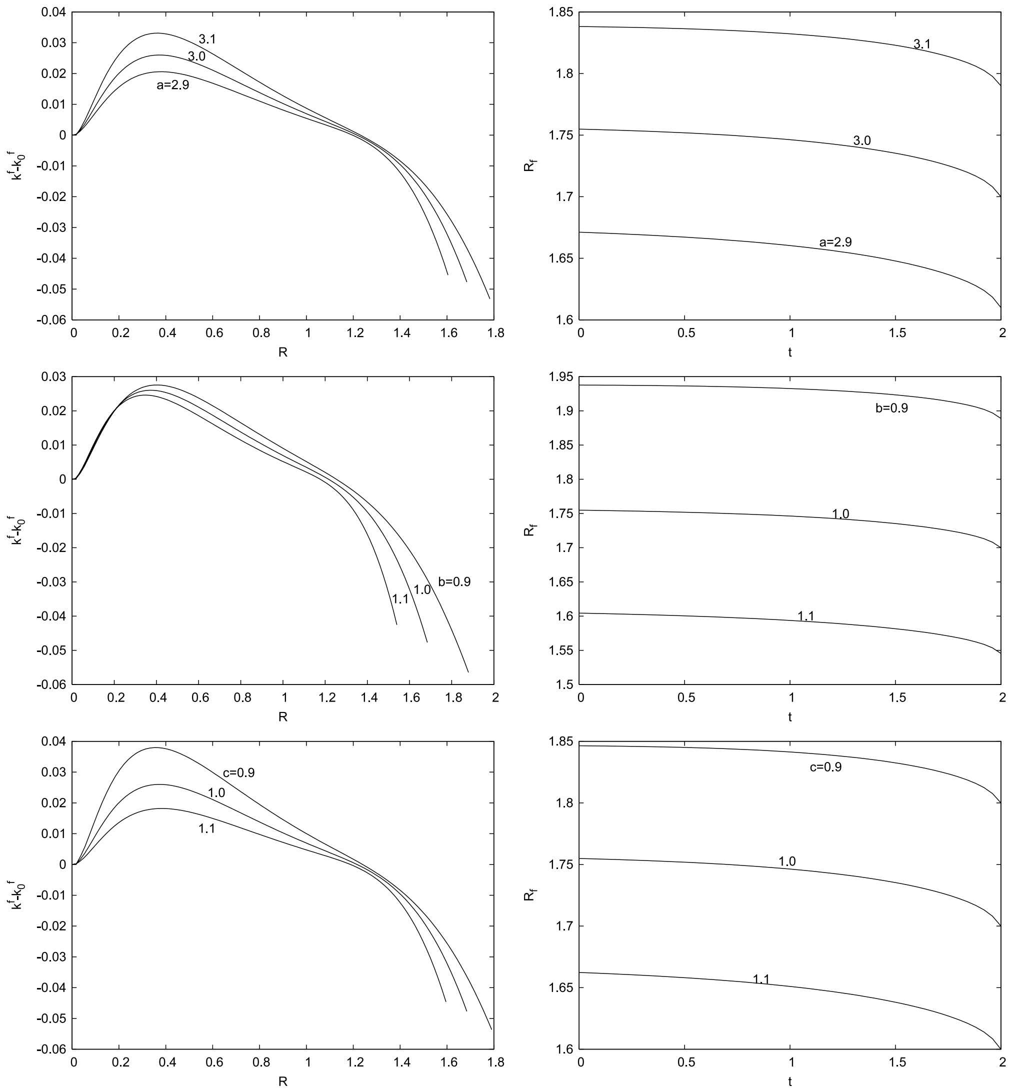


**Fig. 10.** The initial relative premium $k^f - k_0^f$ and moving boundary position $R_f$ as the parameters of the demand function $a$, $b$, $c$ are varied. These results use the implicit scheme, and in the graphs of $k^f - k_0^f$ the optimal strategy is shown up until $(a(\kappa - \mu) - c)/b$, which varies with the parameters under consideration.

yield an optimal control under parametric restrictions. The more implicit the numerical scheme, the more computational effort is required to solve the numerical problem. However, the fully implicit scheme allows a larger time step without introducing numerical instability. For the problems here, the reduction in time step required by the explicit scheme is more than compensated for by its faster overall execution time. All the numerical schemes reproduce the non-reacting optimal premium strategy as $\lambda \to 0$ for the base parameter set.

If one changes the parameters far from the base set, all the numerical schemes show numerical instability. It is clear from the analytical solution that the value function is singular as the sensitivity of the demand function to the market loading is decreased (parameter $c$), which means that infinite wealth can be generated. With market reaction, we must change the demand function in order to prevent collusion. We speculate that the numerical instability indicates that the value function is singular over part of the domain. Since the model is stochastic, there is a finite probability that any value of $0 \le R \le R_f$ can occur. If $R$ is small, then the market is considerably over-pricing insurance, and it seems likely that there are premium strategies that generate infinite wealth.

Infinite wealth generation is a consequence of the form of the exposure Eq. (2). If the insurer sets its premium sufficiently below the market average, then its exposure grows exponentially, and this growth continues indefinitely. In reality, there is a lower-bound on the insurer's reserve and a finite market for insurance policies. The constrained stochastic optimisation problem is formidable, since it is likely that there are no smooth solutions to the Bellman equation [5]. Calculation of the maximum in the Bellman equation at each time step might lead to a more robust numerical scheme for the constrained

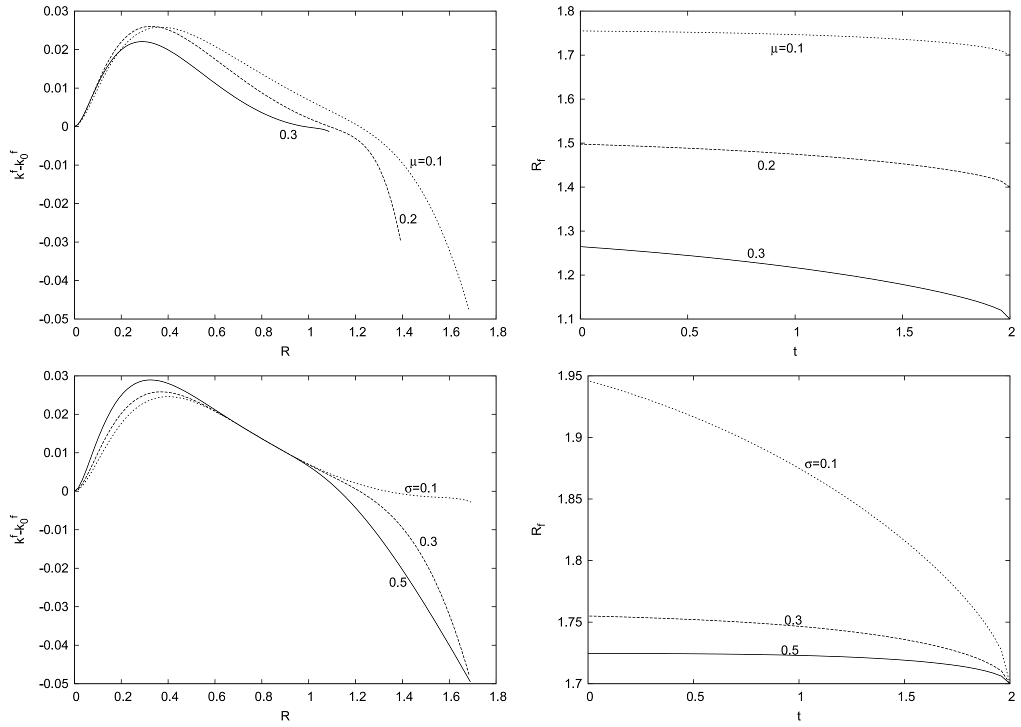


**Fig. 11.** The initial relative premium $k^f - k_0^f$ and moving boundary position $R_f$ as the parameters of the mean claim size rate process $\mu$ and $\sigma$ are varied. These results use the hybrid scheme with $N_t = 100,000$ time steps and $N_\rho = 100$ spatial steps.

problem. The exponential growth in exposure can be removed by introducing a saturation exposure [6], and this may also decrease the numerical sensitivity of the optimisation problem.

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