NORTH-HOLLAND

Theorems of Perron-Frobenius type for Matrices without Sign Restrictions

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ABSTRACT

The paper attempts to solve a problem which was called a "challenge for the future" in *Linear Algebra Appl.* We define and investigate a new quantity for real matrices, the *sign-real spectral radius*, and derive various characterizations, bounds, and properties of it. In certain aspects our quantity shows similar behavior to the Perron root of a nonnegative matrix. It is shown that our quantity also has intimate connections to the componentwise distance to the nearest singular matrix. Relations to the Perron root of the (entrywise) absolute value of the matrix and to the μ-number are given as well. © 1997 Elsevier Science Inc.

1. NOTATION AND INTRODUCTION

We use standard notation from matrix theory; cf. [16, 18]. In particular, Q_{kn} denotes the set of k-tuples of strictly increasing integers out of $\{1, \dots, n\}$. For $A \in M_n(\mathbb{R})$ and $\mu \in Q_{kn}$, $A[\mu] \in M_k(\mathbb{R})$ denotes the principal submatrix of A consisting of rows $i \in \mu$ and columns $j \in \mu$. The adjoint is denoted by adj A.

For $\mu \in Q_{kn}$ we denote the number of elements of μ by $|\mu|$. *For vectors and matrices we use comparison and absolute value always entrywise.* In the following, I denotes the identity matrix of proper dimension.

A *signature matrix* S is a diagonal matrix with diagonal entries $+1$ or -1; in notation, $|S| = I$. The set of signature matrices with n rows and

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columns is denoted by $\mathcal{S}_n$. If the dimension is evident from the context, we simply write $\mathcal{S}$.

Componentwise distances, perturbation bounds, and error bounds have received quite some attention in recent years (see, for example, [6], [15], and the many references cited there). The componentwise distance $\sigma(A, E)$ of $A \in M_n(\mathbb{R})$ to the nearest singular matrix weighted by some $0 \leqslant E \in M_n(\mathbb{R})$ is defined by (cf. [23, 9, 25, 24])

$$\sigma(A, E) := \min\{\alpha \in \mathbb{R} | \text{there is a singular } A' \text{ with} |A' - A| \leqslant \alpha |E|\}. \tag{1.1}$$

If no such α exists, we define $\sigma(A, E) := \infty$. Remember that comparison and absolute value for vectors and matrices is always used entrywise. An explicit formula for this number uses the *real spectral radius* [23, Chapter 5]

$$\rho_0(A) := \max\{|\lambda| | \lambda \text{ a real eigenvalue of } A\}, \tag{1.2}$$

where $\rho_0(A) := 0$ if A has no real eigenvalues. Rohn [23, Theorem 5.1] has shown

$$\sigma(A, E) = \frac{1}{\max_{S_1, S_2 \in \mathcal{S}} \rho_0(S_1 A^{-1} S_2 E)}. \tag{1.3}$$

The computation of $\sigma(A, E)$ is NP-hard [22], which is reflected in the exponential number of signature matrices in (1.3).

In [9], J. Demmel and N. J. Higham conjecture an upper bound for $\sigma(A, E)$. In the course of extending and proving this conjecture [25, Proposition 7.3; 24, Proposition 2.6], the *sign-real spectral radius* $\rho_0^S(A)$ occurs.

DEFINITION 1.1. For $A \in M_n(\mathbb{R})$ the sign-real spectral radius is defined by

$$\rho_0^S(A) := \max_{S \in \mathcal{S}} \rho_0(SA). \tag{1.4}$$

With (1.3) we have

$$\begin{aligned} \sigma(A, E) &= \left(\max_{S_1, S_2 \in \mathcal{S}} \rho_0(S_1 A^{-1} S_2 E) \right)^{-1} \\ &= \left(\max_{S \in \mathcal{S}} \rho_0^S(A^{-1} S E) \right)^{-1} \leqslant \rho_0^S(A^{-1} E)^{-1}, \end{aligned}$$

which means any lower bound on the sign-real spectral radius implies an upper bound on the componentwise distance to the nearest singular matrix. This was the original motivation for defining and investigating the sign-real spectral radius $\rho_0^S(A)$.

It turns out that the sign-real spectral radius is interesting in itself and, in certain aspects, shows similar behavior to the Perron root of a nonnegative matrix: for example, the inheritance property on going to principal submatrices (Corollary 2.4), the identical characterization of $\rho_0^S(A) = 0$ and $\rho(|A|) = 0$ (Theorem 2.7), and especially the max-min characterizations (Theorem 3.1 and following). The relation between $\rho_0^S(A)$ and $\rho(|A|)$ is characterized in Theorem 5.7, and the sign-real spectral radius is proved to be continuous (Corollary 2.5). Moreover, $\rho_0^S(A)$ is proved to be always a *simple* eigenvalue of some SA unless A is permutationally similar to a strictly upper triangular matrix, in which case $\rho_0^S(A) = 0$ and *all* SA have an n-fold eigenvalue zero (Theorem 3.9). The case that $\rho_0^S(A)$ is a multiple eigenvalue of some SA is characterized in Theorem 3.8.

Furthermore, bounds are derived, such as the determinant bound given in Theorem 4.2. This bound is sharp, and it holds similarly for nonnegative matrices (Corollary 4.3). The well-known lower and upper bounds by Collatz for the Perron root of nonnegative matrices are generalized (Lemma 3.3). This gives a simple sufficient condition for the fact that some orthant does not contain an eigenvector of a real matrix (Corollary 3.4). We will prove

$$\sigma(A, E) = \left[\rho_0^S\begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix}\right]^{-2} \tag{1.5}$$

(Theorem 2.8), displaying the intimate connection between $\sigma(A, E)$ and $\rho_0^S(A)$. We also prove that computation of ρ_0^S in NP-hard (Corollary 2.9), and state a relation to the so-called μ-problem[1] (see [8], [5], and the references cited there). A special μ-problem with only diagonal and no block perturbations is

$$\mu_D(A) := \begin{cases} 0 & \text{if } \det(I - A\tilde{D}) \neq 0 \\ & \quad \text{for all } |\tilde{D}| \leqslant D \\ \left(\min\limits_{|\tilde{D}| \leqslant D} \left\{ \|\tilde{D}\|_2 \,\middle|\, \det(I - A\tilde{D}) = 0 \right\}\right)^{-1} & \text{otherwise,} \end{cases} \tag{1.6}$$

where $A, D \in M_n(\mathbb{R})$ and $0 \leqslant D$ diagonal.

[1]The author would like to thank Paul van Dooren for pointing out the μ-problem to him.

Although $\rho_0^S(A^2)$ need not be equal to $\rho_0^S(A)^2$, Theorem 2.16 shows with $\lim_{k\to\infty}[\rho_0^S(A^k)]^{1/k} = \rho(A)$ another normlike behavior of the sign-real spectral radius. Finally, in Section 5 we give bounds for the sign-real spectral radius by means of the geometric mean of cycles. These lead to almost sharp bounds for the componentwise distance to the nearest singular matrix [24].

2. BASIC PROPERTIES OF $\rho_0^S(A)$

We start with some basic properties of the sign-real spectral radius $\rho_0^S(A)$ as defined in (1.4).

LEMMA 2.1. *Let $A, B \in M_n(\mathbb{R})$, signature matrices $S_1, S_2 \in \mathcal{S}$, a permutation matrix P, and a regular diagonal matrix D be given. Then*

$$\begin{aligned}
&\rho_0^S(A) = \rho_0^S(S_1 A S_2) = \rho_0^S(A^T) = \rho_0^S(P^T A P) = \rho_0^S(D^{-1}AD),\\
&\rho_0^S(DA) = \rho_0^S(AD),\\
&\rho_0^S(\alpha A) = |\alpha|\,\rho_0^S(A) \qquad \text{for} \quad \alpha \in \mathbb{R}.
\end{aligned}$$

If there exists a matrix $C \in M_n(\mathbb{R})$, $\operatorname{rank} C = 1$ with $C_{ij} := \operatorname{sign}(A_{ij})$ for $A_{ij} \neq 0$ and $C_{ij} \in \{-1, +1\}$ for $A_{ij} = 0$, then $\rho_0^S(A) = \rho(|A|)$.

For the Kronecker product $\otimes$, we have $\rho_0^S(A)\rho_0^S(B) \leqslant \rho_0^S(A \otimes B)$.

For lower or upper triangular A,

$$\rho_0^S(A) = \max_i |A_{ii}|.$$

If the permutational similarity transformation putting $|A|$ into its irreducible normal form is applied to A, and $A_{(i,i)}$ are the diagonal blocks, then $\rho_0^S(A) = \max_i \rho_0^S(A_{(i,i)})$.

Proof. For $S \in \mathcal{S}$ we have $S^{-1} = S$; thus A has the same eigenvalues as SAS; hence $\rho_0^S(A) = \rho_0^S(AS_2) = \rho_0^S(S_1 A S_2)$. The eigenvalues of $S_1 A S_2$ and $S_2 A^T S_1$ are the same; therefore $\rho_0^S(A) = \rho_0^S(A^T)$. The eigenvalues of $S \cdot P^T A P$ and $PSP^T \cdot A$ are the same, and PSP^T is a signature matrix. The set of eigenvalues of SA and $D^{-1}SAD = S \cdot D^{-1}AD$ are identical, as are those of $S \cdot DA$ and $D^{-1}SDAD = S \cdot AD$. By definition, $C = xy^T$ for $x, y \in \mathbb{R}^n$, $|x| = |y| = (1)$. Hence, $S_x := \operatorname{diag}(x) \in \mathcal{S}$, $S_y := \operatorname{diag}(y) \in \mathcal{S}$, and $S_x A S_y = |A|$ yield $\rho_0^S(A) = \rho_0^S(S_x A S_y) = \rho(|A|)$. The eigenvalues of $(S_1 A) \otimes (S_2 B) = (S_1 \otimes S_2) \cdot (A \otimes B)$ are the products of the eigenvalues of $S_1 A$ and $S_2 B$. The other statements are obvious. ∎

For orthogonal $Q \in M_n(\mathbb{R})$, in general $\rho_0^S(A) \neq \rho_0^S(Q^TAQ)$, and also, in general, $\rho_0^S(AB) \neq \rho_0^S(BA)$. For $\circ$ denoting the Hadamard product, the quantities $\rho_0^S(A^2)$, $\rho_0^S(A \circ A)$ may be less than, equal to, or larger than $\rho_0^S(A)^2$. However, Theorem 2.15 implies, $\rho_0^S(A^TA) \geqslant \rho_0^S(A)^2$.

LEMMA 2.2. *For every orthant there exists some signature matrix S such that SA has an eigenvector in that orthant corresponding to a real nonnegative eigenvalue, i.e.,*

$$\forall T \in \mathcal{S} \ \exists S \in \mathcal{S} \ \exists 0 \neq x \in \mathbb{R}^n :$$

$$x \geqslant 0 \text{ and } SA \cdot Tx = \lambda \cdot Tx \text{ for some } 0 \leqslant \lambda \in \mathbb{R}.$$

For regular A one has $\rho_0^S(A) > 0$.

Proof. Let $T \in \mathcal{S}$ be given. If there exists $0 \neq x \in \mathbb{R}^n$, $x \geqslant 0$, with $ATx = 0$, the proof is finished. Suppose $ATx \neq 0$ for all nonzero $x \geqslant 0$. Define $E := \{x \in \mathbb{R}^n \mid x \geqslant 0 \text{ and } \|x\|_1 = 1\}$, which is a nonempty, compact, and convex set. Then $f(x) := |ATx| / \|ATx\|_1$ is well defined on E, f is continuous, and f maps E into itself. Due to Brouwer's fixed-point theorem, there is some $x \in E$ with $f(x) = x$, and for suitable $S \in \mathcal{S}$ we have

$$SA \cdot Tx = T \cdot |ATx| = \|ATx\|_1 \cdot Tx.$$

For regular A we have $ATx \neq 0$ for all $T \in \mathcal{S}$ and therefore $\|ATx\|_1 > 0$ for all $x \in E$. ■

Lemma 2.2 shows in particular that there is always some $S \in \mathcal{S}$ such that SA has a real eigenvalue, which means that $\rho_0^S(A)$ is always equal to a real eigenvalue of some SA. Shortly, we will characterize the set of matrices with $\rho_0^S(A) = 0$.

The following theorem establishes connections between the sign-real spectral radius and P-matrices. Moreover, it shows the inheritance property of $\rho_0^S(A)$ on going to principal submatrices. In the proof and later on we use (cf., for example, [25, Lemma 4.1])

$$A \in M_n(\mathbb{R}), \quad u, v \in \mathbb{R}^n \quad \Rightarrow \quad \det(A + uv^T) = \det A + v^T(\operatorname{adj} A)u,$$

$$\text{and for regular } A, \quad \det(A + uv^T) = (\det A)(1 + v^TA^{-1}u). \quad (2.1)$$

THEOREM 2.3. *Let $A \in M_n(\mathbb{R})$ and $0 < b \in \mathbb{R}$. Then the following are equivalent*:

(i) $\rho_0^S(A) < b$.
(ii) *For all signature matrices S there holds* $\det(bI - SA) > 0$.
(iii) *For all signature matrices S, the matrix $bI - SA$ is a P-matrix, i.e., for all $\mu \in Q_{kn}$, $1 \leqslant k \leqslant n$, one has* $\det\{(bI - SA)[\mu]\} > 0$.
(iv) *For all diagonal matrices D with $|D| \leqslant I$, one has* $\det(bI - DA) > 0$.

Proof. (i) ⇒ (ii): Assume $\det(bI - SA) \leqslant 0$ for some signature matrix S. That means the characteristic polynomial $P_{SA}(x)$ of SA at b is less than or equal to zero. Now $P_{SA}(x) \to +\infty$ for $x \to +\infty$ implies that $P_{SA}(x)$ intersects with the real axis at some $x \geqslant b$, which means $\rho_0^S(A) \geqslant b$.

(ii) ⇒ (iii): Let $\mu := (1, \dots, n-1)$, and an arbitrary signature matrix $S \in \mathscr{S}$ be given. Define $T := S[\mu] \in \mathscr{S}$, and $S', S'' \in \mathscr{S}_n$ to be the signature matrices with $S'_{ii} = S''_{ii} = T_{ii}$ for $1 \leqslant i \leqslant n-1$ and with $S'_{nn} = -1$, $S''_{nn} = +1$. By assumption,

$$\det(bI - S'A) > 0 \quad \text{and} \quad \det(bI - S''A) > 0.$$

Furthermore, $S'' - S' = 2e_n e_n^T$, and $(bI - S'A) - (bI - S''A) = 2e_n e_n^T A$. Together with (2.1), the linearity of determinants for *rank*-1 updates, this implies

$$\det(bI - S'''A) > 0 \qquad \text{for} \quad S''' := \tfrac{1}{2}(S' + S'').$$

But $S'''_{nn} = 0$, and the last row of $bI - S'''A$ is be_n^T. Hence, by definition of S', S'', we have $0 < \det(bI - S'''A) = b\det\{(bI - SA)[\mu]\}$. This is true for all $S \in \mathscr{S}$ and $\mu = (1, \dots, n-1)$. Renumbering and an induction argument finishes this part of the proof.

(iii) ⇒ (iv): follows by applying a similar argument to that in (ii) ⇒ (iii).

(iv) ⇒ (i): Choosing appropriate D proves that for all $S \in \mathscr{S}$, every principal minor of $bI - SA$ is positive, which means $bI - SA$ is a P-matrix for all $S \in \mathscr{S}$. Hence, for all $\tilde{b} \geqslant b$ and all $S \in \mathscr{S}$, we have $\det(\tilde{b}I - SA) > 0$ (cf. [13, Theorem 5.22]). If λ is an eigenvalue of SA for some $S \in \mathscr{S}$, then $\det(\lambda I - SA) = 0 = \det[(-\lambda)I - (-S)A]$, and therefore $|\lambda| < b$. ■

From the proof we see that Theorem 2.3 remains valid on replacing $bI - SA$ by $bI - AS$, and $bI - DA$ by $bI - AD$.

COROLLARY 2.4. *The sign-real spectral radius has the inheritance property, that is, it cannot increase on going to a principal submatrix*:

$$\mu \in Q_{kn} \quad \text{for } 1 \leqslant k \leqslant n \quad \Rightarrow \quad \rho_0^S(A[\mu]) \leqslant \rho_0^S(A).$$

In particular,

$$\max |A_{ii}| \leqslant \rho_0^S(A).$$

The *real* spectral radius ρ_0 of a matrix is, in general, not continuous in the components of A, because the maximal real eigenvalue may be multiple and may become complex for arbitrarily small perturbations. An example is

$$A(\epsilon) = \begin{pmatrix} 1 & -\epsilon \\ 1 & 1 \end{pmatrix} \qquad \text{with} \quad \epsilon \geqslant 0.$$

Here $\rho_0(A(\epsilon)) = 0$ for $\epsilon > 0$, whereas $\rho_0(A(0)) = 1$.

Interestingly enough, the sign-real spectral radius is continuous in the components of A, although SA may have a real eigenvalue $|\lambda| = \rho_0^S(A)$ of multiplicity greater than one. Nevertheless, ρ_0^S depends continuously on the components of A. The reason for the continuity of $\rho_0^S(A)$ will be explained again after Theorem 3.8.

COROLLARY 2.5. *The sign-real spectral radius $\rho_0^S(A)$ depends continuously on the components of the matrix A.*

Proof. The quantity $\beta := \inf\{0 < b \in \mathbb{R} \mid \det(bI - SA) > 0$ for all signature matrices $S \in \mathcal{S}\}$ is well defined, it depends continuously on the components of A, and Theorem 2.3 implies $\rho_0^S(A) \leqslant \beta$. On the other hand, the continuity of the determinant implies the existence of some $S \in \mathcal{S}$ with $\det(\beta I - SA) = 0$. This includes the case $\beta = 0$. Hence, $\rho_0^S(A) \geqslant \beta$, and therefore $\rho_0^S(A) = \beta$, and the continuity of the determinant proves the corollary. ■

The characterization in Theorem 2.3 shows

$$\rho_0^S(A) = \min\{0 \leqslant b \in \mathbb{R} | \det(bI - SA) \geqslant 0 \text{ for all } S \in \mathcal{S}\}.$$

Using a bisection scheme, this offers a possibility to calculate $\rho_0^S(A)$ to any desired accuracy without eigenvalue computation. The exponential behavior is reflected by the fact that computation of the sign-real spectral radius is NP-hard (see Corollary 2.9). Following are more properties of $\rho_0^S(A)$ showing similarities to Perron-Frobenius theory.

LEMMA 2.6. *For $A \in M_n(\mathbb{R})$, there exist signature matrices S_1, S_2 and $0 \neq x \in \mathbb{R}^n$ with $x \geqslant 0$ and*

$$S_1 A S_2 \cdot x = \rho_0^S(A) \cdot x. \tag{2.2}$$

For $B(\lambda) := \operatorname{adj}(\lambda I - S_1 A S_2)$ one has $B(\rho_0^S(A)) \geqslant 0$.

Proof. According to Lemma 2.2, $\rho_0^S(A)$ is an eigenvalue of SA for some signature matrix S. Hence $SAx = \lambda x$ with $0 \neq x \in \mathbb{R}^n$, $|\lambda| = \rho_0^S(A)$. Proper choice of S_1 and S_2 yields (2.2).

Set $B := B(\rho_0^S(A))$. Then $B \operatorname{adj} B = (\det B)I = 0$, and any column of $\operatorname{adj} B$ is a (possibly zero) multiple of the nonnegative eigenvector x. But Theorem 2.3(iii) with a limit argument implies $\det B[\mu] \geqslant 0$ for all $\mu \in Q_{kn}$, and therefore in particular $\operatorname{diag} \operatorname{adj} B \geqslant 0$. Hence $x \geqslant 0$ yields $\operatorname{adj} B \geqslant 0$. ■

Note that in general we do not have $\operatorname{adj} B(\lambda) \geqslant 0$ for $\lambda > \rho_0^S(A)$ as for nonnegative matrices (cf. [3, Theorem 3.1]). Unlike Perron-Frobenius theory, the eigenvector x may consist of zero components, even if A has no zero entry. Consider

$$A = \begin{pmatrix} 3 & 3 & 4 \\ 4 & -1 & 3 \\ 0.8 & -1.2 & 1 \end{pmatrix} \quad \text{with } \rho_0^S(A) = 5 \quad \text{and} \quad x = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}. \tag{2.3}$$

There is no $S \in \mathscr{S}$ such that SA has an eigenvector corresponding to an eigenvalue $+5$ or -5 without zero component, and the eigenvalue $+5$ or -5 is always simple. However, $\rho_0^S(A) = \rho_0^S(A[\mu])$ for $\mu = (1,2)$. We will come to this again in Lemma 3.7.

For $b > \rho_0^S(A)$ and any $S \in \mathscr{S}$, $bI - SA$ is a P-matrix. However, this does not necessarily imply positive stability. Consider

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -0.01 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

Then $\rho_0^S(A) < 1.07$, but A has eigenvalues $1.1077 \pm 0.187i$ with real part greater than $\rho_0^S(A)$; thus $\rho_0^S(A) \cdot I - A$ is not positive stable.

Theorem 2.3 allows us to characterize the set of matrices with sign-real spectral radius zero.

THEOREM 2.7. *Let $A \in M_n(\mathbb{R})$ be given. Then the following are equivalent*:

(i) $\rho_0^S(A) = 0$.

(ii) *The matrix A is permutationally similar to a strictly upper triangular matrix.*

Proof. By Corollary 2.4 and Lemma 2.2, $\rho_0^S(A) = 0$ is equivalent to $\rho_0^S(A[\mu]) = 0 = \det A[\mu]$ for all $1 \leqslant k \leqslant n$ and all $\mu \in Q_{kn}$. If A had two distinct nonzero full cycles, this would imply existence of a nonzero nonfull cycle (cf. [12, Lemma 2.1]). Thus, an induction argument shows that A has at most one nonzero full cycle, and $\det A = 0$ proves A to be acyclic. ■

Theorem 2.3 allows us to prove the relation (1.5) between the componentwise distance to the nearest singular matrix as defined in (1.1) and the sign-real spectral radius. Together with Corollary 2.5 this gives a simple proof of the continuity of the componentwise distance to the nearest singular matrix $\sigma(A, E)$ (see also [25, Lemma 6.13]).

THEOREM 2.8. *Let $A, E \in M_n(\mathbb{R})$, A regular and $E \geqslant 0$, be given. Then*

$$\sigma(A, E) = \left[\rho_0^S\begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix}\right]^{-2}.$$

This includes $\sigma(A, E) = \infty \Leftrightarrow \rho_0^S\begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix} = 0$.

Proof. Applying a Schur-complement argument shows

$$\det\begin{pmatrix} xI & -S_2 E \\ -S_1 A^{-1} & xI \end{pmatrix} = \det\left(x^2 I - S_1 A^{-1} S_2 E\right).$$

That means the eigenvalues of $S_1 A^{-1} S_2 E$ are exactly $\pm\sqrt{\lambda}$ for λ an eigenvalue of

$$\begin{pmatrix} 0 & S_2 E \\ S_1 A^{-1} & 0 \end{pmatrix}.$$

Using $\mathcal{S} = -\mathcal{S}$ and Rohn's characterization (1.3) proves the theorem. ■

Note that $\rho_0^S\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^2 = \max_S \rho_0^S(ASB) = \max_S \rho_0^S(BSA)$, but, in general, $\rho_0^S(AB) \neq \rho_0^S(BA)$.

As a corollary we note that computation of $\rho_0^S(A)$ for $A \in M_n(\mathbb{Q})$ is NP-hard. For $e \in \mathbb{R}^n$ denoting the vector of all ones and $E = ee^T$, Poljak and Rohn showed in [22] that for rational A computation of $\sigma(A, E)$ is NP-hard. By (1.3) and Theorem 2.8,

$$\sigma(A, E)^{-1} = \max_{S_1, S_2 \in \mathcal{S}} \rho_0(S_1 A^{-1} S_2 E)$$

$$= \max_{S_1, S_2 \in \mathcal{S}} e^T S_1 A^{-1} S_2 e = \rho_0^S \begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix}^2,$$

which is a rational number for rational A and $E = ee^T$. Since matrix inversion is polynomially bounded, we have the following corollary.

COROLLARY 2.9. *For $A \in M_n(\mathbb{Q})$ computation of the sign-real spectral radius $\rho_0^S(A)$ is NP-hard. More precisely, if for $A \in M_n(\mathbb{Q})$, E being the matrix of all ones, there exists a polynomial-time algorithm for calculating the rational number*

$$\rho_0^S \begin{pmatrix} 0 & E \\ A & 0 \end{pmatrix}^2,$$

then $P = NP$.

Moreover, Theorem 2.3 allows to state the connection with the μ-number.

COROLLARY 2.10. *Let $A, D \in M_n(\mathbb{R})$ with $0 \leqslant D$ diagonal. Then for $\mu_D(A)$ defined in (1.6) we have*

$$\mu_D(A) = 0 \quad \Leftrightarrow \quad \rho_0^S(AD) < 1,$$

and

$$\rho_0^S(AD) \cdot \|D\|^{-1} \leqslant \mu_D(A) \leqslant \rho_0^S(A) \qquad \text{for} \quad \rho_0^S(AD) \geqslant 1.$$

Proof. By Theorem 2.3(iv) and the remark after Theorem 2.3, we have

$$\rho_0^S(AD) < 1 \quad \Leftrightarrow \quad \forall |\Delta| \leqslant D : \det(I - A\Delta) > 0,$$

which is equivalent to $\mu_D(A) = 0$. For $b := \rho_0^S(AD) \geqslant 1$ we have $\rho_0^S(b^{-1}AD) = 1$, and we have $\det(I - S \cdot b^{-1}AD) = \det(I - A \cdot b^{-1}DS) = 0$ for some $S \in \mathscr{S}$. From $\rho_0^S(AD) \geqslant 1$ we get $\|D\| \neq 0$, and by the definition (1.6), it follows $\mu_D(A) \geqslant \|b^{-1}DS\|^{-1} = \rho_0^S(AD) \cdot \|D\|^{-1}$. Let $\mu_D(A) = \|\tilde{D}\|^{-1}$ for some diagonal $|\tilde{D}| \leqslant D$. Then $\det(I - A\tilde{D}) = 0 = \det[\rho_0^S(A) \cdot I - A \cdot \rho_0^S(A) \cdot \tilde{D}]$, and Theorem 2.3(iv) implies $\rho_0^S(A) \cdot |\tilde{D}_{ii}| \geqslant 1$ for some i. Hence $\mu_D(A) = \|\tilde{D}\|^{-1} \leqslant \rho_0^S(A)$. ■

The inverse μ-number minimizes $\|\Delta\| = \sigma_1(\Delta)$ for $|\Delta| \leqslant D$ and $I - A\Delta$ singular. This explains the factor $\|D\|^{-1}$. In connection with the μ-number the following observation may also be useful.

LEMMA 2.11. *For $A, D \in M_n(\mathbb{R})$, A regular, $0 \leqslant D$ diagonal, one has*

$$\sigma(A, D) = \rho_0^S(A^{-1}D)^{-1}.$$

This includes

$$\sigma(A, D) = \infty \quad \Leftrightarrow \quad \rho_0^S(A^{-1}D) = 0.$$

In particular,

$$\rho_0^S(A) = \sigma(A^{-1}, I)^{-1}.$$

Proof. Rohn's formula (1.3) and Lemma 2.1 yield

$$\sigma(A, D)^{-1} = \max_{S \in \mathscr{S}} \rho_0^S(A^{-1}SD) = \max_{S \in \mathscr{S}} \rho_0^S(A^{-1}DS) = \rho_0^S(A^{-1}D). \quad ■$$

Another view of the componentwise distance to the nearest singular matrix is the following. For $A, \Delta \in M_n(\mathbb{R})$, $\Delta \geqslant 0$, the set $[A - \Delta, A + \Delta] := \{\tilde{A} \in M_n(\mathbb{R}) \mid A - \Delta \leqslant \tilde{A} \leqslant A + \Delta\}$ is called an *interval matrix*. An interval matrix is called regular if all $\tilde{A} \in [A - \Delta, A + \Delta]$ are regular. Then $\sigma(A, E) = \sup\{\alpha \mid [A - \alpha E, A + \alpha E] \text{ regular}\}$. In [23, Theorem 5.1], Rohn gave fourteen equivalent formulations for $[A - \Delta, A + \Delta]$ being regular, among them [condition (C4)] in our notation

$$[A - \Delta, A + \Delta] \text{ regular}$$

$$\Leftrightarrow \quad \forall S_1, S_2 \in \mathscr{S}\ \forall k : \left[A \cdot (A - S_1 \Delta S_2)^{-1}\right]_{kk} > \tfrac{1}{2} \qquad (2.5)$$

(to simplify the formulation, here and in the following Theorem 2.12, inverse matrices are always assumed to exist when spoken of). If $\rho_0^S(A) < b$, then we know by Theorem 2.3 that all $bI - DA$ are regular for $|D| \leqslant I$. Following we give a characterization of $\rho_0^S(A)$, which is similar to (2.5).

THEOREM 2.12. *For* $A \in M_n(\mathbb{R})$, $0 < b \in \mathbb{R}$ *we have*

$$\rho_0^S(A) < b \quad \Leftrightarrow \quad \forall S \in \mathcal{S}\ \forall k : b \cdot \left[(bI - SA)^{-1}\right]_{kk} > \tfrac{1}{2}. \tag{2.6}$$

Proof. We start with a general statement. Let $S, T \in \mathcal{S}$ differ exactly in the kth component, that is, $T = S - 2S_{kk}e_k e_k^T$, and assume $\det(bI - SA) > 0$. Then by (2.1),

$$\det(bI - TA) = \det(bI - SA) \cdot \left[1 + 2S_{kk} \cdot e_k^T A \cdot (bI - SA)^{-1} \cdot e_k\right]. \tag{2.7}$$

Furthermore

$$\begin{aligned} &S_{kk} \cdot e_k^T A \cdot (bI - SA)^{-1} \cdot e_k \\ &\quad = S_{kk} \cdot e_k^T \cdot \{S(SA - bI) + bS\} \cdot (bI - SA)^{-1} \cdot e_k \\ &\quad = S_{kk}\left\{-S_{kk} + bS_{kk} \cdot \left[(bI - SA)^{-1}\right]_{kk}\right\} = b \cdot \left[(bI - SA)^{-1}\right]_{kk} - 1. \end{aligned}$$

Together with (2.7), this implies

$$\det(bI - TA) = \det(bI - SA) \cdot \left\{2b\left[(bI - SA)^{-1}\right]_{kk} - 1\right\}.$$

Summarizing, for $S, T \in \mathcal{S}$, $T = S - 2S_{kk}e_k e_k^T$, and $\det(I - SA) > 0$, we have

$$\det(bI - TA) > 0 \quad \Leftrightarrow \quad b \cdot \left[(bI - SA)^{-1}\right]_{kk} > \tfrac{1}{2}. \tag{2.8}$$

Now assume $\rho_0^S(A) < b$. Then $\det(bI - SA) > 0$ for all $S \in \mathcal{S}$, and (2.8) proves " $\Rightarrow$." To prove " $\Leftarrow$," we assume $\rho_0^S(A) \geqslant b$. By Theorem 2.3, there is some $S' \in \mathcal{S}$ with $\det(bI - S'A) \leqslant 0$. Furthermore,

$$\sum_{S \in \mathcal{S}} \det(bI - SA) = 2^n b^n,$$

and $b > 0$ shows that there exists some $S'' \in \mathcal{S}$ with $\det(bI - S''A) > 0$. Hence, there are $S, T \in \mathcal{S}$ with $\det(bI - SA) > 0$ and $\det(bI - TA) \leqslant 0$, such that S and T differ in exactly one component. Then (2.8) implies that the right-hand side of (2.6) is not valid, and this finishes the proof. ■

Another formulation of Theorem 2.12 is

$$\rho_0^S(A) < b \quad \Leftrightarrow$$

$$\forall \mu \in Q_{n-1,n}\ \forall S \in \mathcal{S}: \det(bI - SA) < 2b \det\{(bI - SA)[\mu]\}. \quad (2.9)$$

Note that $\rho_0^S(A) < b$ implies $0 < \det\{(bI - SA)[\mu]\}$ for all $S \in \mathcal{S}$ and all $\mu \in Q_{kn}$, $1 \leqslant k \leqslant n$.

There is another characterization of the sign-real spectral radius without the use of signature matrices. The proof uses geometrical properties of the system of cones spanned by the columns of two matrices [26, 17].[2]

THEOREM 2.13. *For $A \in M_n(\mathbb{R})$ and $0 < b \in \mathbb{R}$ the following are equivalent*:

(*i*) $(bI - A)^{-1}(bI + A)$ *is a P-matrix.*

(ii) $\rho_0^S(A) < b$.

REMARK. For simplicity of notation, inverse matrices are assumed to exist when spoken of.

Proof. We have

$$(bI - A)^{-1}(bI + A) = -U^{-1}V \qquad \text{with} \quad U := A - bI, \quad V := A + bI.$$

Denote the columns of U, V by u_i, v_i. According to [26] (see also [17, Theorem 6.6]), consider the system of cones $\mathscr{C}(U, V)$ spanned by all n-tuples of column vectors $c_1, \dots, c_n$ with $c_i \in \{u_i, v_i\}$. Then $-U^{-1}V$ is a P-matrix if and only if for any choice of $c_i \in \{u_i, v_i\}$ the vectors $c_1, \dots, c_n$ are linearly independent, and for $1 \leqslant j \leqslant n$ the hyperplane spanned by 0 and c_i, $i \neq j$, separates u_j from v_j. This means

$$\text{(i)} \quad \Leftrightarrow \quad \det(A + bS')\det(A + bS'') < 0$$
$$\text{for all } S', S'' \in \mathcal{S} \text{ differing in exactly one component.}$$

[2] The author wishes to thank L. Elsner for pointing out to him the paper by Kuhn and Löwen.

The linearity of the determinant for rank-1 updates, $b > 0$, and $\det S' \det S'' = -1$ imply (similarly to the argument used in the proof of Theorem 2.3)

$$\text{(i)} \quad \Leftrightarrow \quad \det(bI + SA) > 0 \quad \text{for all } S \in \mathcal{S}.$$

Now Theorem 2.3 finishes the proof. ■

For nonnegative A, Theorem 2.13 implies that $\rho(A) < b$ iff $(bI - A)^{1}(bI + A)$ is a P-matrix. Note that A is (negative) stable if and only if $(I - A)^{-1}(I + A)$ is convergent (see, for example, Theorem 7.21 in [13]).

Theorem 2.13 also shows similarities to Theorem 2.12. For $\rho_0^S(A) < b$ it implies for all $S \in \mathcal{S}$ that $(bI - SA)^{-1}(bI - SA) = (bI - SA)^{-1}(-bI + SA + 2bI) = 2b(bI - SA)^{-1} - I$ is a P-matrix. This implies in particular the right-hand side of (2.6).

Theorems 2.3, 2.12, and 2.13 together with the continuity of ρ_0^S (Corollary 2.5) yield different characterizations of the sign-real spectral radius.

COROLLARY 2.14. *For $A \in M_n(\mathbb{R})$,*

$$\begin{aligned}
\rho_0^S(A) &= \inf\{0 < b \in \mathbb{R} \,|\, \det(bI - SA) \\
&\qquad\qquad > 0 \text{ for all } S \in \mathcal{S}\} \\
&= \inf\left\{0 < b \in \mathbb{R} \,\middle|\, b\left[(bI - SA)^{-1}\right]_{kk} \right. \\
&\qquad\qquad \left. > \tfrac{1}{2} \text{ for all } 1 \leqslant k \leqslant n \text{ and all } S \in \mathcal{S} \right\} \\
&= \inf\left\{0 < b \in \mathbb{R} \,\middle|\, (bI - A)^{-1}(bI + A) \text{ is a } P\text{-matrix}\right\} \\
&= \max\left\{\rho_0(DA) \,\middle|\, |D| \leqslant I\right\}.
\end{aligned}$$

The first characterization uses determinants of certain n-by-n matrices for all signature matrices, the second the diagonal elements of the inverses of certain n-by-n matrices for all signature matrices, and the third the minors of one matrix.

Next we give the sign-real spectral radius for symmetric matrices together with norm bounds, and give a maximum characterization.

THEOREM 2.15. *For $A \in M_n(\mathbb{R})$,*

$$\rho_0^S(A) \leqslant \|A\|_p \qquad \textit{for} \quad 1 \leqslant p \leqslant \infty. \tag{2.10}$$

Furthermore,

$$A = A^T \quad \Rightarrow \quad \rho_0^S(A) = \|A\|_2,$$

and

$$\max_{Q^T = Q^{-1}} \rho_0^S(QA) = \|A\|_2.$$

Proof. For $\rho_0^S(A)$ an eigenvalue of SA we have

$$\rho_0^S(A) = \rho_0(SA) \leqslant \rho(SA) \leqslant \|SA\|_p \leqslant \|S\|_p \cdot \|A\|_p = \|A\|_p.$$

This implies for $A = A^T$

$$\|A\|_2 = \rho_0(A) \leqslant \rho_0^S(A) \leqslant \|A\|_2.$$

Finally, for the singular-value decomposition $A = U\Sigma V^T$ and $Q := VU^T$, we have $(QA)^T = QA$ and therefore $\rho_0^S(QA) = \|QA\|_2 = \|A\|_2$. ■

The bound (2.10) can be arbitrarily weak, as is seen by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad \rho_0^S(A) = 0 \text{ and } \|A\|_2 = 1.$$

However, in this case also $\rho_0^S(|A|) = 0$. In Section 5 we will see that in any case $\rho_0^S(A)$ and $\rho(|A|)$ must not be too far apart.

Note that neither $\rho_0^S(A) \leqslant \rho(A)$ nor $\rho_0^S(A) \leqslant r(A)$ (the numerical radius) need to be true in general. For example, for

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

one has $\rho_0^S(A) = 2$, whereas $\rho(A) = 0$, $r(A) = 1$. However, $\rho_0^S(A) \leqslant 2r(A)$ is always true, because $\|A\|_2 \leqslant 2r(A)$.

Theorem 2.15 implies $\rho_0^S(A)^2 \leqslant \rho_0^S(A^TA)$, whereas $\rho_0^S(A)^2$ can be less than, equal to or greater than $\rho_0^S(A^2)$. Nevertheless, the following theorem holds, which has a well-known counterpart for norms. We state the result and defer the proof to Section 5.

THEOREM 2.16. $\lim_{k \to \infty}[\rho_0^S(A^k)]^{1/k} = \rho(A)$.

3. MAX-MIN AND FURTHER CHARACTERIZATIONS OF $\rho_0^S(A)$

We start with the following max-min characterization of $\rho_0^S(A)$, which is almost identical to the well known formula by Collatz for nonnegative matrices (cf. [7] or [16, Corollary 8.3.3]).

THEOREM 3.1. *Let $A \in M_n(\mathbb{R})$. Then*

$$\rho_0^S(A) = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \min_{x_i \neq 0} \left| \frac{(Ax)_i}{x_i} \right|. \tag{3.1}$$

Proof. Define

$$\varphi_A(x) := \min_{x_i \neq 0} \left| \frac{(Ax)_i}{x_i} \right|. \tag{3.2}$$

By Lemma 2.6, there is a signature matrix S with $SAz = \rho_0^S(A)z$ for some $0 \neq z \in \mathbb{R}^n$, and hence $\rho_0^S(A) = \varphi_A(z) \leqslant \sup_{x \in \mathbb{R}^n} \varphi_A(x)$. Let $0 \neq x \in \mathbb{R}^n$ be given, and for the purpose of establishing a contradiction suppose $b := \varphi_A(x) > \rho_0^S(A)$. Then $|Ax| \geqslant b|x|$, and with suitable $S_1, S_2 \in \mathcal{S}$ we have $S_1 A S_2 \cdot y \geqslant b \cdot y$ for $y := S_2 x = |x|$. Hence for all k, $[(bI - S_1 A S_2)y]_k y_k \leqslant 0$, and the well-known characterization by Fiedler and Pták [14, Theorem 1.1] implies that $bI - S_1 AS_2$ is not a P-matrix. But this contradicts Theorem 2.3, because $b > \rho_0^S(A)$. Hence, $\rho_0^S(A) = \varphi_A(z) = \max_{x \in \mathbb{R}^n} \varphi_A(x)$. The theorem is proved. ■

The function $\varphi_A(x)$ in (3.2) is basically the Collatz-Wielandt function (cf. [20, Chapter 1.3]). Theorem 3.1 gives a convenient tool to obtain lower bounds for the sign-real spectral radius. As for (reducible) nonnegative matrices, the corresponding min-max equality does not hold. In fact, it cannot, because $\rho_0^S(A) = \rho(A)$ for nonnegative A. But even $\rho_0^S(A) \leqslant \max_{x_i \neq 0} |(Ax)_i / x_i|$ is not true, as every singular matrix which is not permutationally similar to a strictly upper triangular matrix shows (cf. Theorem 2.7).

If we just exchange max and min in (3.1), we arrive at $\min_{x \in \mathbb{R}^n} \max_{x_i \neq 0} |(Ax)_i / x_i| =: M$. Without proof we mention that M is equal to the minimum absolute value of the real eigenvalues of $(SA)[\mu]$, where the minimum runs over all $S \in \mathcal{S}$ and all $\mu \in Q_{kn}$, $1 \leqslant k \leqslant n$. In particular, $M = 0$ iff some $A[\mu]$ is singular.

For nonnegative matrices, Perron-Frobenius theory offers a min-max characterization for irreducible matrices complementing the max-min characterization. Our next aim is to derive a similar characterization for the sign-real spectral radius, which is complementary to Theorem 3.1. For this purpose we need two preparatory lemmas, the second one being of interest by itself.

LEMMA 3.2. $\max_{S \in \mathcal{S}} \inf_{x>0} \max_i \left| \frac{(ASx)_i}{x_i} \right| \leqslant \rho_0^S(A)$.

Proof. Let fixed but arbitrary $S \in \mathcal{S}$ be given. We show that for $\varepsilon > 0$ there exists some $x > 0$ with $\max|(ASx)_i/x_i| \leqslant \rho_0^S(A) + O(\varepsilon)$, which proves the lemma. By Lemma 2.2, there exists $0 \leqslant z^{(1)} \in \mathbb{R}^n$, $z^{(1)} \neq 0$ with $|ASz^{(1)}| = \lambda_1 z^{(1)}$ with $0 \leqslant \lambda_1 \in \mathbb{R}$. We have $\lambda_1 \leqslant \rho_0^S(A)$. If $z^{(1)} > 0$, the proof is finished. Denote the index set of nonzero components of $z^{(1)}$ by α_1, i.e., $i \in \alpha_1 \Leftrightarrow z_i^{(1)} \neq 0$. Applying the same argument to $(AS)[\mu]$, $\mu := \{1, \dots, n\} \setminus \alpha_1$, and filling the eigenvector with zeros yields existence of some $0 \leqslant z^{(2)} \in \mathbb{R}^n$, $z^{(2)} \neq 0$ with $|(ASz^{(2)})[\mu]| = \lambda_2 z^{(2)}[\mu]$, $0 \leqslant \lambda_2 \in \mathbb{R}$. By Corollary 2.4, $\lambda_2 \leqslant \rho_0^S(A)$. Denoting the index set of nonzero components of $z^{(2)}$ by α_2 and continuing with this process we obtain a splitting $\{1, \dots, n\} = \alpha_1 \cup \cdots \cup \alpha_m$ with the following properties.

$$
\begin{aligned}
z_i^{(k)} \neq 0 \quad &\Leftrightarrow \quad i \in \alpha_k && \text{for} \quad 1 \leqslant k \leqslant m, \\
|(ASz^{(k)})[\mu]| &= \lambda_k z^{(k)}[\mu] && \text{for} \quad \mu := \{1, \dots, n\} \setminus \bigcup_{\nu=1}^{k-1} \alpha_\nu, \qquad 1 \leqslant k \leqslant m, \\
(ASz^{(\nu)})[\alpha_k] &= 0 && \text{for} \quad \nu < k, \\
|(ASz^{(k)})[\alpha_k]| &= \lambda_k z^{(k)}[\alpha_k].
\end{aligned}
$$

Define

$$
x := \sum_{\nu=1}^{m} \varepsilon^\nu z^{(\nu)} \qquad \text{for} \quad \varepsilon > 0. \tag{3.3}
$$

Then $x > 0$, and for $1 \leqslant k \leqslant m$,

$$
|(ASx)[\alpha_k]| = \left| \sum_{\nu=1}^{m} (ASz^{(\nu)})[\alpha_k] \cdot \varepsilon^\nu \right| \leqslant \lambda_k z^{(k)}[\alpha_k] \cdot \varepsilon^k + O(\varepsilon^{k+1}).
$$

For $i \in \alpha_k$ we have $x_i = z_i^{(k)}\varepsilon^k$, and hence for all $i \in \alpha_k$,

$$\left|\frac{(ASx)_i}{x_i}\right| = \left|\frac{\lambda_k z_i^{(k)}\varepsilon^k + O(\varepsilon^{k+1})}{z_i^{(k)}\varepsilon^k}\right| = \lambda_k + O(\varepsilon) \leqslant \rho_0^S(A) + O(\varepsilon).$$

Applying this for $1 \leqslant k \leqslant m$ and using $\lambda_k \leqslant \rho_0^S(A)$ shows $\max_i|(ASx)_i/x_i| \leqslant \rho_0^S(A) + O(\varepsilon)$ for the x defined in (3.3), and proves the lemma. ∎

The next lemma gives lower and upper bounds for a real eigenvalue of $A \in M_n(\mathbb{R})$, A not sign-restricted, provided the sign pattern of the corresponding left eigenvector is known.

LEMMA 3.3. *Let $A \in M_n(\mathbb{R})$ be given with real eigenvalue $\lambda \in \mathbb{R}$ and left eigenvector $0 \neq z \in \mathbb{R}^n$, i.e. $A^T z = \lambda z$. Then for any vector $x \in \mathbb{R}^n$, $x_i \neq 0$ and $z_i x_i \geqslant 0$ for $1 \leqslant i \leqslant n$, we have*

$$\min_i \frac{(Ax)_i}{x_i} \leqslant \lambda \leqslant \max_i \frac{(Ax)_i}{x_i}.$$

Proof. Set $\mu_i := (Ax)_i/x_i$; then

$$\sum_i (\lambda - \mu_i) x_i z_i = \sum_i \left[(A^T z)_i x_i - (Ax)_i z_i\right] = x^T A^T z - z^T A x = 0.$$

By assumption, not all $x_i z_i$ can be zero. Hence $x_i z_i \geqslant 0$ shows that not all μ_i can be strictly less than or stictly greater than λ. ∎

The proof is almost the same as the one by Collatz for his Satz in Section 2 of [7]. We mention that Lemma 3.3 can be used to prove that a specific orthant $\{x \in \mathbb{R}^n \mid Sx \geqslant 0\}$, $S \in \mathcal{S}$ does not contain an eigenvector (to a real eigenvalue).

COROLLARY 3.4. *Let $A \in M_n(\mathbb{R})$, $S \in \mathcal{S}$, and $x, y \in \mathbb{R}^n$ be given with $Sx > 0$, $Sy > 0$, and*

$$\min_i \frac{(A^T x)_i}{x_i} > \max_i \frac{(A^T y)_i}{y_i}.$$

Then there is no eigenvector of A in the orthant $\{x \in \mathbb{R}^n \mid Sx \geqslant 0\}$.

In particular, if A^T has two eigenvectors with no zero components corresponding to distinct eigenvalues in the same orthant, then this orthant contains no eigenvector of A.

We will apply Lemma 3.3 to the eigenvalue $\lambda = \rho_0^S(A)$ of $S_1 A S_2$, where $S_1, S_2 \in \mathscr{S}$ are chosen according to Lemma 2.6. If λ is a simple eigenvalue of $S_1 A S_2$, then according to Lemma 2.6 $S_1 A S_2$ has a left and right eigenvector corresponding to λ where none of the corresponding components are of opposite sign.

We are now ready to prove the following theorem, a duality theorem very much in the spirit of the corresponding assertion in Perron-Frobenius theory.

THEOREM 3.5. *For $A \in M_n(\mathbb{R})$,*

$$\max_{S \in \mathscr{S}} \max_{\substack{x > 0 \\ x \neq 0}} \min_{x_i \neq 0} \left| \frac{(ASx)_i}{x_i} \right| = \rho_0^S(A) = \max_{S \in \mathscr{S}} \inf_{x > 0} \max_i \left| \frac{(ASx)_i}{x_i} \right|. \quad (3.4)$$

Proof. The left equality follows by Theorem 3.1. Using Lemma 3.2, it remains to show max inf max $\geqslant \rho_0^S(A)$. Using Lemma 2.6, let $S_1 A^T S_2 \cdot z = \rho_0^S(A^T) \cdot z = \rho_0^S(A) \cdot z$ with $z \geqslant 0$, $z \neq 0$. Then for all $x > 0$, Lemma 3.3 implies

$$\max_i \frac{(S_2 A S_1 x)_i}{x_i} \geqslant \rho_0^S(A).$$

Hence

$$\max_{S \in \mathscr{S}} \inf_{x > 0} \max_i \left| \frac{(ASx)_i}{x_i} \right| \geqslant \inf_{x > 0} \max_i \left| \frac{(AS_1 x)_i}{x_i} \right|$$

$$\geqslant \inf_{x > 0} \max_i \frac{(S_2 A S_1 x)_i}{x_i} \geqslant \rho_0^S(A).$$

■

The matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and $x = (1, 0)^T$ show that in the right-hand side of (3.4) the $\inf_{x > 0} \max_i$ cannot be replaced by a $\min_{x \geqslant 0} \max_{x_i \neq 0}$.

The sign-real spectral radius and the componentwise distance to the nearest singular matrix are in a sense inversely proportional (see Theorem 2.8 and Lemma 2.11). The validity of an *upper* bound b for $\sigma(A, E)$ is verified if a matrix $\tilde{A}$ is known with $|\tilde{A} - A| \leqslant bE$ and $\det A \det \tilde{A} \leqslant 0$. Similarly, any nonzero vector x together with Theorem 3.1 provides an easy-to-calculate *lower* bound for $\rho_0^S(A)$. Conversely, it is comparatively difficult to prove b to be a *lower* bound of $\sigma(A, E)$, because this includes the proof of regularity of *all* $\tilde{A}$ with $|\tilde{A} - A| \leqslant bE$. Similarly, according to Theorem 3.5, we have, for some vector x, to calculate the ratio $|(ASx)_i/x_i|$ for *all* $S \in \mathcal{S}$ to obtain a valid *upper* bound for $\rho_0^S(A)$.

We note the similarity to the corresponding result for nonnegative matrices. For any $A \in M_n(\mathbb{R})$, $A \geqslant 0$, we have

$$\max_{\substack{x \geqslant 0 \\ x \neq 0}} \min_{x_i \neq 0} \frac{(Ax)_i}{x_i} = \rho(A) = \inf_{x > 0} \max_i \frac{(Ax)_i}{x_i}. \tag{3.5}$$

As in Perron-Frobenius theory, replacing inf by min in (3.5), we find that

$$\min_{\substack{x \geqslant 0 \\ x \neq 0}} \max_{x_i \neq 0} \frac{(Ax)_i}{x_i}$$

is, in general, even not an upper bound for $\rho(A)$.

The following bounds for $\rho_0^S(A)$ under operations with diagonal matrices are straightforward implications of Theorem 3.1 and $\rho_0^S(AD) = \rho_0^S(DA)$ (cf. Lemma 2.1).

LEMMA 3.6. *Let $A \in M_n(\mathbb{R})$ and D be a diagonal matrix. Then*

$$\rho_0^S(A) - \max_i |D_{ii}| \leqslant \rho_0^S(A + D) \leqslant \rho_0^S(A) + \max_i |D_{ii}|$$

and

$$\rho_0^S(A) \min_i |D_{ii}| \leqslant \rho_0^S(AD) \leqslant \rho_0^S(A) \max_i |D_{ii}|.$$

Some properties of the spectral radius of nonnegative irreducible matrices do not (immediately) carry over to the sign-real spectral radius, even when A has no zero component. For example, if $\lambda = \rho_0^S(A)$ is an eigenvalue of some

SA, then

$$\text{it may hold that } \lambda < \rho(SA), \tag{3.6α}$$

$$\text{it may hold that } \lambda = \rho_0^S(A[\mu]) \text{ for some } \mu \in Q_{kn},\ k < n, \tag{3.6β}$$

$$\lambda \text{ may be a multiple eigenvalue of } SA. \tag{3.6γ}$$

The matrix in (2.4) is an example for (3.6α). However, Theorem 5.7 will show that the ratio $\rho(SA)/\rho_0^S(A)$ is bounded for every $S \in \mathcal{S}$, namely, $\rho(SA)/\rho_0^S(A) \leqslant \rho(|A|)/\rho_0^S(A) \leqslant (3 + 2\sqrt{2})n$. If (3.6$\beta$) or (3.6$\gamma$) holds true, the matrix A has special properties. If (3.6β) is true, then by Corollary 2.4 one has $\rho_0^S(A[\mu']) = \rho_0^S(A)$ for every index set $\mu' \supseteq \mu$.

If an eigenvector x of some SA, corresponding to the eigenvalue $\rho_0^S(A)$, has zero components, then $\rho_0^S(A[\mu]) = \rho_0^S(A)$, where μ is the index set of nonzero entries in x. The converse is not true: If $\rho_0^S(A[\mu]) = \rho_0^S(A)$, there need not exist an eigenvector corresponding to $\rho_0^S(A)$ of some SA with a zero entry. An example is

$$A = \begin{pmatrix} 51 & 60 & 2 \\ 84 & 75 & -2 \\ 38 & 76 & 47 \end{pmatrix}, \tag{3.7}$$

for which $\rho_0^S(A) = \rho_0^S(A[\mu]) = 135$ for $\mu = (1,2)$, but $\rho_0^S(A)$ is a simple eigenvalue of SA for all $S \in \mathcal{S}$, and no eigenvector corresponding to $\rho_0^S(A)$ has a zero component. However, every left eigenvector of SA to a real eigenvalue λ with $|\lambda| = \rho_0^S(A)$ has a zero component.

Following we will see that this is true in general. The conditions for (3.6β) and (3.6γ) can be characterized as follows.

LEMMA 3.7. *For $A \in M_n(\mathbb{R})$ the following are equivalent:*

(i) *There exists $\mu \in Q_{n-1,n}$ with $\rho_0^S(A[\mu]) = \rho_0^S(A)$.*

(ii) *There exists some $S \in \mathcal{S}$ such that SA has a left or a right eigenvector to the eigenvalue $\lambda = \rho_0^S(A)$ with a zero component.*

Proof. (i) $\Rightarrow$ (ii): There is $S \in \mathcal{S}$ with $\det\{(\rho_0^S(A) \cdot I - SA)[\mu]\} = 0$. The linearity of the determinant (2.1) subject to rank-1 perturbations, the inheritance of ρ_0^S, and Theorem 2.3 imply $\det\{\rho_0^S(A) \cdot I - SA\} = 0$. Now, every matrix $C \in M_n(\mathbb{R})$ such that C and $C[\mu]$, $\mu \in Q_{n-1,n}$, have an eigenvalue λ in common has a left or right eigenvector corresponding to λ

with a zero component. This is seen as follows. For $B := \operatorname{adj}(\lambda I - C)$ we have $B_{kk} = 0$ for $k \in \{1, \dots, n\}$ with $k \notin \mu$. If $B \equiv 0$, then all $(n-1) \times (n-1)$ minors of $\lambda I - C$ are zero, and the kernel of $\lambda I - C$ is at least of dimension 2. If $B \not\equiv 0$, there is a nonzero row or column of B that is a left or right eigenvector of C, respectively, and it has a zero component, because $B_{kk} = 0$ and B is of rank 1.

(ii) $\Rightarrow$ (i) follows by Theorem 3.1, Corollary 2.4, and $\rho_0^S(A) = \rho_0^S(A^T)$. ■

THEOREM 3.8. *For $A \in M_n(\mathbb{R})$ the following are equivalent:*

(i) *For some $S \in \mathcal{S}$, $\rho_0^S(A)$ is an eigenvalue of SA of multiplicity greater or equal to m.*

(ii) *For all $\mu \in Q_{kn}$, $0 \leqslant k \leqslant m-1$ it is $\rho_0^S(A(\mu)) = \rho_0^S(A)$, where $A(\mu) := A[\mu']$ with $\mu' := \{1, \dots, n\} \setminus \mu$.*

Proof. Let $S \in \mathcal{S}$, set $B := \rho_0^S(A) \cdot I - SA$, and denote the characteristic polynomial of B by $\sum_{i=0}^n p_i x^i$. Then $|p_i| = |\sum_{|\mu|=i} \det B(\mu)|$; see for example [27, (6.2.9)]. By Theorem 2.3(iii), $\det B(\mu) \geqslant 0$ for all μ. Therefore, we have the following equivalences: (i) $\Leftrightarrow$ [B has an eigenvalue 0 of multiplicity greater or equal to m] $\Leftrightarrow$ [$p_i = 0$ for $0 \leqslant i \leqslant m-1$] $\Leftrightarrow$ [$\det B(\mu) = 0$ for all $0 \leqslant |\mu| \leqslant m-1$]. By the inheritance property of the sign-real spectral radius (Corollary 2.4) this is equivalent to (ii). ■

In classical Perron-Frobenius theory the spectral radius is a simple eigenvalue for irreducible matrices. We may ask whether something similar is true for the sign-real spectral radius. The answer will yield another way to explain its continuity. It may happen that $\lambda := \rho_0^S(A)$ is a multiple eigenvalue of some SA, and ε-perturbations move λ into the complex plane. However, the following theorem proves that $\rho_0^S(A)$ is always a *simple* eigenvalue of some SA, unless A is permutationally similar to a strictly upper triangular matrix, in which case *all* SA have the n-fold eigenvalue zero.

THEOREM 3.9. *For $A \in M_n(\mathbb{R})$, $n \geqslant 2$, exactly one of the two following statements is true.*

(i) *$\rho_0^S(A)$ is a simple eigenvalue of SA for some $S \in \mathcal{S}$.*

(ii) *A is permutationally similar to a strictly upper triangular matrix (and therefore $\rho_0^S(A) = 0$).*

Proof. Let λ be an eigenvalue of SA with $|\lambda| = \rho_0^S(A)$. If λ is simple, we use SA or $-SA$ and we are finished. Suppose the multiplicity of λ is greater than one. By the proof of Theorem 3.8 we know $\det(\lambda I - SA) = 0 = \det\{(\lambda I - SA)[\mu]\}$ for all $\mu \in Q_{n-1,n}$. Define diagonal S', S'' with $S_{ii} = S'_{ii} = S''_{ii}$ for $1 \leqslant i \leqslant n-1$, and $S'_{nn} := 0$, $S''_{nn} := -S_{nn}$. Then for $\mu := \{1, \dots, n-1\}$ we have $\det(\lambda I - S'A) = \lambda \det\{(\lambda I - SA)[\mu]\} = 0$, and the linearity of determinants subject to rank-1 perturbations (2.1) implies $\det(\lambda I - S''A) = 0$. The signature matrix S'' differs from S in exactly one component, and λ is an eigenvalue of SA and of $S''A$. Repeating this argument, we either arrive at some $\tilde{S} \in \mathcal{S}$ with a simple eigenvalue λ, or else $\det(\lambda I - SA) = 0$ for all $S \in \mathcal{S}$. In the latter case, (2.1) implies $\det\{(\lambda I - SA)[\mu]\} = 0$ for all $\mu \in Q_{kn}$, $1 \leqslant k \leqslant n$, and therefore $\lambda = 0$. For $\rho_0^S(A) = |\lambda|$, Theorem 2.7 finishes the proof. ■

Set $r := \rho_0^S(A)$. Note that it may happen that $\det(rI - S_1 A) = \det(rI - S_2 A) = 0$, but r is simple eigenvalue of $S_1 A$ and $S_2 A$, and $\rho_0^S(A[\mu]) < \rho_0^S(A)$ for all $\mu \in Q_{n-1,n}$. An example is

$$A = \begin{pmatrix} 2 & 8 & 2 & 2 \\ 1 & 4 & 5 & 2 \\ 5 & 1 & -7 & 1 \\ 1 & 2 & 5 & 4 \end{pmatrix},$$

for which $\rho_0^S(A) = 10$, and $S_1 = I$, $S_2 = \operatorname{diag}(-1, 1, -1, 1)$.

Corollary 2.4 tells us that $\rho_0^S(A)$ cannot increase on going to a principal submatrix. The corresponding property for individual elements of nonnegative matrices does not carry over. That means there are matrices $A \in M_n(\mathbb{R})$ such that increasing the absolute value and preserving the sign of an individual element A_{ij} may decrease the signature spectral radius. This is because in Perron-Frobenius theory the "direction" is evident from the fact that A is nonnegative; for the sign-real spectral radius there is no such generic "direction."

However, we have the following property of $\rho_0^S(A)$, which shows its behavior in certain "directions."

THEOREM 3.10. *Let $A \in M_n(\mathbb{R})$ and $J \subseteq \{1, \dots, n\} \times \{1, \dots, n\}$ be an arbitrary set of index pairs. Then there exists a matrix $\Sigma \in M_n(\mathbb{R})$ with the following properties.*

(i) $|\Sigma_{ij}| = 1$ *for* $(i,j) \in J$, *and* $\Sigma_{ij} = 0$ *for* $(i,j) \notin J$.

(ii) *For every matrix* $B \in M_n(\mathbb{R})$ *with* $\operatorname{sign}(B_{ij}) = \Sigma_{ij}$ *for* $1 \leqslant i, j \leqslant n$,

$$\rho_0^S(A) \leqslant \rho_0^S(A + B).$$

In words: For every set J of index pairs there exists a "direction" Σ in which the sign-real spectral radius cannot decrease below $\rho_0^S(A)$.

Proof. According to Lemma 2.6, there exist signature matrices S, T with

$$SAT \cdot x = \rho_0^S(A) \cdot x \qquad \text{for some} \quad 0 \neq x \in \mathbb{R}^n \quad \text{with} \quad x \geqslant 0.$$

Define $\Sigma_{ij} := S_{ii}T_{jj}$ for $(i, j) \in J$, and $\Sigma_{ij} := 0$ otherwise. Any $B \in M_n(\mathbb{R})$ with $\operatorname{sign}(B_{ij}) = \Sigma_{ij}$ satisfies $SBT \geqslant 0$, and hence

$$S(A + B)Tx = \rho_0^S(A) \cdot x + SBT \cdot x \geqslant \rho_0^S(A) \cdot x \geqslant 0.$$

Using Theorem 3.1 and Lemma 2.1, this yields $\rho_0^S(A + B) = \rho_0^S(S(A + B)T \geqslant \rho^S(A)$. ■

Regarding the perturbation of an individual component, this means that towards $+\infty$ or $-\infty$, the sign-real spectral radius can never fall below the current value. Using (2.1), this observation can be generalized to the following corollary.

COROLLARY 3.11. *For $A \in M_n(\mathbb{R})$ and $u, v \in \mathbb{R}^n$ the following is true: There exists $s \in \{-1, +1\}$ such that for all $0 \leqslant \alpha \in \mathbb{R}$,*

$$\rho_0^S(A + s\alpha \cdot uv^T) \geqslant \rho_0^S(A).$$

Proof. For $r := \rho_0^S(A)$ and $\det(rI - SA) = 0$, (2.1) implies $\det[rI - S(A + \alpha \cdot uv^T)] = -\alpha v^T \operatorname{adj}(rI - SA)Su$, which is nonpositive for all $\alpha \geqslant 0$ or for all $\alpha \leqslant 0$. The assertion follows. ■

Theorem 3.10 and Corollary 3.11 offer possibilities to find, for a given matrix A, another matrix $\tilde{A}$ with $\rho_0^S(\tilde{A}) \leqslant \rho_0^S(A)$. The following theorem shows that with a certain Gauss transformation we may construct a matrix $\tilde{A}$ with $\rho_0^S(\tilde{A}) \leqslant \rho_0^S(A)$.

THEOREM 3.12. *Let $A \in M_n(\mathbb{R})$ and $1 \leqslant k \leqslant n$, $\mu := \{1, \ldots, n\} \setminus \{k\}$. For $\varphi \in \mathbb{R}$ with $\varphi A_{kk} \geqslant 0$ and $|\varphi| \rho_0^S(A) \leqslant 1$, define $B := (I + \varphi \cdot Ae_k e_k^T)A$. Then*

$$\rho_0^S(B[\mu]) \leqslant \rho_0^S(A).$$

Proof. For $\rho_0^S(A) = 0$, by Theorem 2.7 the matrix A, and by definition also B are permutationally similar to a strictly upper diagonal matrix. Therefore, we may assume $r := \rho_0^S(A) > 0$, and by Corollary 2.4 we may assume $\varphi \neq 0$. We will show $\det(rI - \tilde{S}B[\mu]) \geqslant 0$ for all $\tilde{S} \in \mathcal{S}_{n-1}$, from which the assertion follows by Theorem 2.3. Let $\tilde{S} \in \mathcal{S}_{n-1}$ be given, define diagonal $S' \in M_n(\mathbb{R})$ by

$$S'_{ii} := \begin{cases} S_{ii} & \text{for} \quad i < k, \\ 0 & \text{for} \quad i = k, \\ S_{i-1,i-1} & \text{for} \quad i > k, \end{cases}$$

and set $D := S' + r\varphi \cdot e_k e_k^T$. Then $D_{kk} = r\varphi$, $|D| \leqslant I$, and D is regular. In the following we use $e_k^T DD = r^2\varphi^2 \cdot e_k^T$, (2.1), and $(\operatorname{adj} C)C = (\det C)I$, and we set $d := \det(rI - DA)$. Then

$$\begin{aligned}
r \det\left(rI - \tilde{S}B[\mu]\right)
&= \det(rI - S'B) \\
&= \det\left[rI - \left(D - r\varphi \cdot e_k e_k^T\right)\left(I + \varphi \cdot A e_k e_k^T\right)A\right] \\
&= \det\left[rI - DA + \varphi \cdot (rI - DA) e_k e_k^T A + r\varphi^2 \cdot A_{kk} \cdot e_k e_k^T A\right] \\
&= \det(rI - DA) + e_k^T A \operatorname{adj}(rI - DA) \cdot \left[\varphi \cdot (rI - DA) e_k + r\varphi^2 \cdot A_{kk} e_k\right] \\
&= d + \varphi \cdot e_k^T A \operatorname{adj}(rI - DA) \cdot (rI - DA) e_k \\
&\quad + r^{-1} A_{kk} \cdot e_k^T D \cdot DA \cdot \operatorname{adj}(rI - DA) \cdot e_k \\
&= d + \varphi d \cdot A_{kk} + r^{-1} A_{kk} \cdot e_k^T D[(DA - rI)\operatorname{adj}(rI - DA) \\
&\qquad + r \operatorname{adj}(rI - DA)] e_k \\
&= d + \varphi d \cdot A_{kk} - r^{-1} d \cdot A_{kk} D_{kk} + A_{kk} D_{kk} \det\{(rI - DA)[\mu]\} \\
&= d + r\varphi \cdot A_{kk} \det\{(rI - DA)[\mu]\} \geqslant 0.
\end{aligned}$$

The last term is nonnegative because by Theorem 2.3 and $|D| \leqslant I$ we have $d \geqslant 0$ and $\det\{(rI - DA)[\mu]\} \geqslant 0$, and by assumption $\varphi A_{kk} \geqslant 0$. ■

For $A_{kk} = 0$, the sign of φ is not fixed. From the last line of the proof it follows that for $A_{kk} = 0$ we have $\rho_0^S(B[\mu]) = \rho_0^S(A)$ for $B := [I + s \cdot$

$\rho_0^S(A)^{-1} \cdot Ae_k e_k^T]A$ and $s = +1$ or $s = -1$. For any A and any φ with $\varphi A_{kk} \geqslant 0$, $\rho_0^S(B[\mu]) \geqslant \varphi^{-1}$ implies $\rho_0^S(A) \geqslant \varphi^{-1}$.

4. LOWER AND UPPER BOUNDS USING DETERMINANTS

The absolute value of the determinant of a matrix is, in general, neither a lower nor an upper bound for the sign-real spectral radius. However, using the determinants of all principal submatrices, we can derive a lower and an upper bound on $\rho_0^S(A)$. In a way this generalizes Corollary 2.4. The new bounds are shown to be sharp. Corresponding bounds for the Perron root of a nonnegative matrix, which are sharp as well, follow as a corollary.

DEFINITION 4.1. Define

$$\delta(A) := \max_{\mu} |\det A[\mu]|^{1/|\mu|}, \tag{4.1}$$

where the maximum is taken over all $\mu \in Q_{kn}$, $1 \leqslant k \leqslant n$.

THEOREM 4.2. *For $A \in M_n(\mathbb{R})$ we have*

$$\delta(A) \leqslant \rho_0^S(A) \leqslant \delta(A) \cdot (2^{1/n} - 1)^{-1}. \tag{4.2}$$

The inequalities are sharp in the sense that for each $n \in \mathbb{N}$, left equality and right equality can be achieved. One has

$$(2^{1/n} - 1)^{-1} < 1.443n. \tag{4.3}$$

Proof. Denote $\mathcal{S}^+ := \{S \in \mathcal{S} \mid \det S = +1\}$ and $\mathcal{S}^- := \mathcal{S} \setminus \mathcal{S}^+$. The characteristic polynomial P_A of A satisfies (cf. [18, 2.15])

$$P_A(x) = \det(xI - A) = \sum_{k=0}^{n} x^{n-k}(-1)^k \operatorname{trace} C_k(A). \tag{4.4}$$

For $1 \leqslant k \leqslant n - 1$ there holds

$$\sum_{S \in \mathscr{S}^+} \operatorname{trace} C_k(SA) = \sum_{S \in \mathscr{S}^+} \sum_{|\mu| = k} \det(SA)[\mu]$$

$$= \sum_{|\mu|=k} \left(\sum_{S \in \mathscr{S}^+} \det S[\mu] \right) \det A[\mu] = 0,$$

and therefore

$$\sum_{S \in \mathscr{S}^+} \det(xI - SA) = 2^{n-1}\left[x^n + (-1)^n \det A\right], \tag{4.5}$$

and similarly

$$\sum_{S \in \mathscr{S}^-} \det(xI - SA) = 2^{n-1}\left[x^n - (-1)^n \det A\right]. \tag{4.6}$$

For $x := |\det A|^{1/n}$, the value of at least one of the two sums (4.5) and (4.6) is zero. Hence there exists a signature matrix $S \in \mathscr{S}$ with $\det(xI - SA) \leqslant 0$. Using $\det(xI - SA) \to +\infty$ for $x \to +\infty$ implies the existence of a real eigenvalue λ of SA with $\rho_0^S(A) \geqslant \lambda \geqslant |\det A|^{1/n}$. The inheritance property (Corollary 2.4) implies the left inequality in (4.2).

Another way of writing (4.4) for $S \in \mathscr{S}$ is

$$\det(xI - SA) = x^n + \sum_{|\mu| \geqslant 1} (-1)^{|\mu|} \det\{(SA)[\mu]\} \cdot x^{n-|\mu|}, \tag{4.7}$$

where the sum is taken over all $\mu \in Q_{kn}$, $1 \leqslant k \leqslant n$. Setting $\alpha := \delta(A) = \delta(SA)$ implies

$$\left| \sum_{|\mu| \geqslant 1} (-1)^{|\mu|} \cdot \det\{(SA)[\mu]\} \cdot x^{n-|\mu|} \right| \leqslant \sum_{|\mu| \geqslant 1} |\det A[\mu]| \cdot |x|^{n-|\mu|}$$

$$\leqslant \sum_{k=1}^{n} \binom{n}{k} \alpha^k |x|^{n-k}$$

$$= (|x| + \alpha)^n - |x|^n.$$

For $|x| \geqslant \alpha(2^{1/n}-1)^{-1}$ it follows that

$$(|x|+\alpha)^n - |x|^n \leqslant \left(|x| + (2^{1/n}-1)|x|\right)^n - |x|^n = |x|^n.$$

Together with (4.7) this shows that the value of the characteristic polynomial of SA is nonzero for all real x with $|x| > \alpha(2^{1/n}-1)^{-1}$. This proves (4.2).

The left inequality in (4.2) is an equality for the identity matrix. Consider the circulant $A \in M_n(\mathbb{R})$ with

$$A = \begin{pmatrix} 1 & a & a^2 & \cdots & \cdots \\ a^{n-1} & 1 & a & a^2 & \cdots \\ a^{n-2} & a^{n-1} & 1 & a & \ddots \\ \vdots & \ddots & \ddots & 1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad a := 2^{1/n}. \tag{4.8}$$

Define $D := \operatorname{diag}(a^{n-1}, \ldots, a^2, a, 1)$. Then

$$D^{-1}AD = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 2 & 1 & 1 & \cdots \\ 2 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

a matrix with 1's on and above the diagonal and 2's below. Subtracting the second row from the first row of $D^{-1}AD$ and using an induction argument, it follows that all principal minors of $D^{-1}AD$ and hence of A are of absolute value 1. Therefore $\delta(D^{-1}AD) = \delta(A) = 1$. All row sums of the positive matrix A are equal, and therefore they are equal to its Perron root, which is

$$\rho(A) = \rho_0^S(A) = \sum_{k=0}^{n-1} a^k = \frac{a^n - 1}{a-1} = (2^{1/n}-1)^{-1}.$$

Hence the right inequality of (4.2) is sharp. Finally,

$$2^{1/n} - 1 = e^{(1/n)\ln 2} - 1 > \frac{1}{n}\ln 2$$

implies (4.3). The theorem is proved. ∎

We mention that for polynomials an inequality in the spirit of (4.2) has been given by Birkhoff [4]; see also Marden [19].

COROLLARY 4.3. *For nonnegative* $A \in M_n(\mathbb{R})$ *we have*

$$\delta(A) \leqslant \rho(A) \leqslant \delta(A) \cdot (2^{1/n} - 1)^{-1}.$$

The inequalities are sharp in the sense that for each $n \in \mathbb{N}$, *left and right equality can be achieved.*

COROLLARY 4.4. *For orthogonal* $Q \in M_n(\mathbb{R})$ *we have*

$$\rho_0^S(Q) = 1.$$

Proof. Theorem 4.2 implies $\rho_0^S(Q) \geqslant 1$, and Theorem 2.15 yields $\rho_0^S(Q) \leqslant \|Q\|_2 = 1$. ■

Orthogonal matrices map the unit ball into itself. With Q also SQ is orthogonal, and Corollary 4.4 states that there exists a signature matrix S and real $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$ and $SQx = x$.

COROLLARY 4.5. *For* $A \in M_n(\mathbb{R})$ *with singular values* $\sigma_1(A) \geqslant \cdots \geqslant \sigma_n(A)$,

$$\left(\prod_{i=1}^{n} \sigma_i(A)\right)^{1/n} \leqslant \rho_0^S(A) \leqslant \sigma_1(A). \tag{4.9}$$

Proof. Follows by Theorem 2.15, Theorem 4.2, and $|\det A| = \prod \sigma_i(A)$. ■

The geometric mean on the left of (4.9) cannot be replaced by an arithmetic mean, as any nonzero strictly upper triangular matrix shows.

Definition 4.1 of $\delta(A)$ together with Theorem 4.2 yields a number of interesting properties of the δ-number. For example, for signature matrices

$S_1, S_2 \in \mathcal{S}$, regular diagonal matrix D, and permutation matrix P,

$$\begin{aligned} \delta(A) &= \delta(A^T) = \delta(S_1 A S_2) = \delta(D^{-1}AD) = \delta(P^T A P), \\ \delta(AD) &= \delta(DA), \qquad \delta(A) \leqslant \|A\|_p, \\ \delta(A) &= 0 \quad \Leftrightarrow A \\ &\quad \text{permutationally similar to a strictly upper triangular matrix,} \end{aligned}$$

$$Q^T = Q^{-1} \quad \Rightarrow \quad \delta(Q) = 1.$$

In general, $\delta(A^2) \neq \delta(A)^2$, $\delta(A \circ A) \neq \delta(A)^2$, and $\delta(Q^T A Q) \neq \delta(A)$ for $Q^T = Q^{-1}$, but, of course, $\delta(A)$ has the inheritance property. A geometrical interpretation of $|\det A|^{1/n}$ is the length of an edge of a cube of the same volume as the parallelepiped spanned by the rows of A.

5. LOWER AND UPPER BOUNDS USING CYCLIC PRODUCTS

A set $\omega = (\omega_1, \dots, \omega_k)$ of $|\omega| := k \geqslant 1$ mutually distinct integers out of $\{1, \dots, n\}$ is called a cycle and defines the cyclic product

$$\prod A_\omega := \prod_{i=1}^{|\omega|} A_{\omega_i \omega_{i+1}}, \qquad \text{where} \quad \omega_{|\omega|+1} := \omega_1.$$

Note that, in contrast with [10] or [12], by our definition each diagonal element A_{ii} is a cyclic product (of length one). Next, we derive bounds for $\rho_0^S(A)$ using the geometric mean of the absolute value of cyclic products.

LEMMA 5.1. *For $A \in M_n(\mathbb{R})$ and a cycle ω with $|\omega| \leqslant 2$ we have*

$$\left|\prod A_\omega\right|^{1/|\omega|} \leqslant \rho_0^S(A). \tag{5.1}$$

Proof. For $|\omega| = 1$ this follows by Corollary 2.4. For $|\omega| = 2$ it suffices to show $\rho_0^S(A) \geqslant \sqrt{|A_{ij} A_{ji}|}$, $i \neq j$, and by the inheritance of ρ_0^S it suffices to show

$$\rho_0^S(A) \geqslant \sqrt{|bc|} \qquad \text{for} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

According to Lemma 2.1 we know $\rho_0^S(A) = \rho_0^S(S_1 A S_2)$ for $S_1, S_2 \in \mathcal{S}$, and therefore we may assume $a, b, c \geqslant 0$, and either $d \geqslant 0$ or $d < 0$. In either case, the eigenvalues $\frac{1}{2}[a + d \pm \sqrt{(a-d)^2 + 4bc}\,]$ are real, and at least one of them is larger in absolute value than $\sqrt{bc}$. ■

We tried to prove (5.1) for $|\omega| > 2$, because this would imply

$$\rho(|A^{-1}|E)^{-1} \leqslant \sigma(A, E) \leqslant n\rho(|A^{-1}|E)^{-1}$$

with sharp lower and upper bounds (cf. [25]). An attempt to prove (5.1) for $|\omega| > 2$ could proceed as follows.

If A consists of a zero row, then deleting this row and the corresponding column does not change $\rho_0^S(A)$. Assume A has no zero row, so that $\max_j |A_{ij}| > 0$ for all $1 \leqslant i \leqslant n$. Then suitable renumbering and an induction argument together with Corollary 2.4 assures that we may assume w.l.o.g. $|A_{ii'}| = \max_j |A_{ij}|$, where $i' := i + 1$ for $1 \leqslant i < n$ and $i' := 1$ for $i = n$. Furthermore, define $b := (\prod_i |A_{ii'}|)^{1/n}$. Then a similarity transformation with the diagonal matrix D with

$$D_{\nu\nu} := \prod_{i=1}^{\nu-1} \frac{b}{A_{ii'}}$$

yields $|(D^{-1}AD)_{ii'}| = b$ for $1 \leqslant i \leqslant n$. That means, with a proper scaling, we can restrict our attention to matrices of the form

$$A = P + A^* \qquad \text{with} \quad A^*_{ii'} = 0 \text{ and}$$

$$P \text{ the permutation matrix with } P_{ii'} = 1. \tag{5.2}$$

P is a cyclic shift. Obviously, $\rho_0^S(P) = 1$ (which is also a consequence of Corollary 4.3). In the following, we will prove $\rho_0^S(P + \epsilon A^*) > 1$ for small $\epsilon > 0$.

We use a formula for the determinant of the sum of two matrices. First, we need the following definition. For $A \in M_n(\mathbb{R})$ and $\alpha, \beta \in Q_{kn}$, $A[\alpha|\beta] \in M_k(\mathbb{R})$ denotes the matrix with rows $i \in \alpha$ and columns $j \in \beta$, whereas $A(\alpha|\beta) \in M_{n-k}(\mathbb{R})$ is obtained by deleting rows $i \in \alpha$ and columns $j \in \beta$. Similar to the kth compound matrix $C_k(A) \in M_{\binom{n}{k}}(\mathbb{R})$ of $A \in M_n(\mathbb{R})$, the $\alpha\beta$ entry of which is defined to be $\det A[\alpha|\beta]$ for $\alpha, \beta \in Q_{kn}$, we define the kth *adjoint matrix* $\operatorname{adj}_k(A) \in M_{\binom{n}{k}}(\mathbb{R})$, the $\alpha\beta$ entry of which is defined by

$$\operatorname{adj}_k(A) := (-1)^{\Sigma\alpha_i + \Sigma\beta_i} A(\beta|\alpha).$$

We define

$$C_0(A) := 1 \quad \text{and} \quad \mathrm{adj}_n(A) := 1.$$

The kth adjoint seems to be not so common in the recent English-language literature; we found it in [21]. However, Peschl uses the transpose of our definition. One easily verifies

$$C_1(A) = A, \qquad C_n(A) = \det A,$$

$$\mathrm{adj}_0(A) = \det A, \qquad \mathrm{adj}(A) := \mathrm{adj}_1(A) = A^{-1} \det A$$

provided A is regular,

$$C_k(AB) = C_k(A) C_k(B),$$

and

$$\mathrm{adj}_k(AB) = \mathrm{adj}_k(B)\,\mathrm{adj}_k(A).$$

Furthermore, using our definition, Laplace's expansion theorem reads (cf. [21, Satz 42])

$$\mathrm{adj}_k(A)\, C_k(A) = (\det A) I_{\binom{n}{k}}.$$

Using this yields the following expansion for the determinant of a sum of two matrices. Hans Schneider pointed out to us that the result can be found in [2, No. 5, p. 101], but without proof. Therefore, we state the short proof.

LEMMA 5.2. *For $A, B \in M_n(\mathbb{R})$ and $\delta, \epsilon \in \mathbb{R}$,*

$$\det(\delta A + \epsilon B) = \sum_{k=0}^{n} \mathrm{trace}[\mathrm{adj}_k(A)\, C_k(B)] \cdot \delta^{n-k} \epsilon^k. \tag{5.3}$$

REMARK. The apparent asymmetry in (5.3) is resolved by the trace operation, because $\mathrm{trace}[\mathrm{adj}_k(A)\, C_k(B)] = \mathrm{trace}[\mathrm{adj}_{n-k}(B)\, C_{n-k}(A)]$.

Proof. For the characteristic polynomial P_A of A we have

$$P_A(\lambda) = \det(\lambda I - A) = \sum_{k=0}^{n} \lambda^{n-k} (-1)^k \,\mathrm{trace}\, C_k(A).$$

Therefore,

$$\det(A + I) = \sum_{k=0}^{n} \operatorname{trace} C_k(A).$$

For regular A, using $C_k(A^{-1}) = C_k(A)^{-1}$, this implies

$$\begin{aligned}
\det(A + B) &= \det A \det(I + A^{-1}B) \\
&= (\det A) \sum_{k=0}^{n} \operatorname{trace}[C_k(A^{-1}B)] \\
&= (\det A) \sum_{k=0}^{n} \operatorname{trace}[C_k(A^{-1})C_k(B)] \\
&= (\det A) \sum_{k=0}^{n} \operatorname{trace}[\det A^{-1} \operatorname{adj}_k(A)\, C_k(B)] \\
&= \sum_{k=0}^{n} \operatorname{trace}[\operatorname{adj}_k(A)\, C_k(B)],
\end{aligned}$$

For singular A, the same formula follows by a continuity argument. Observing $\operatorname{adj}_k(\delta A) = \delta^{n-k} \operatorname{adj}_k(A)$ and $C_k(\epsilon B) = \epsilon^k C_k(B)$ proves the lemma. ∎

This allows us to prove that P as defined in (5.2) is a strict local minimum of $\rho_0^S(P + \epsilon A^*)$ for small $\epsilon \neq 0$ and $P \circ A^* = 0_{n \times n}$, $\circ$ denoting the Hadamard product.

THEOREM 5.3. *Let $P \in M_n(\mathbb{R})$ be the permutation matrix (cyclic shift) with $P_{ii'} = 1$ where*

$$i' := \begin{cases} i + 1 & \text{for} \quad 1 \leqslant i < n, \\ 1 & \text{for} \quad i = n, \end{cases}$$

and let $A^* \in M_n(\mathbb{R})$, not identical to the zero matrix, be such that $A_{ii'} = 0$ for $1 \leqslant i \leqslant n$. Then for small enough $0 < \epsilon \in \mathbb{R}$,

$$\rho_0^S(P + \epsilon A^*) > \rho_0^S(P) = 1. \tag{5.4}$$

Proof. Using $\rho_0^S(P) = 1$ and Theorem 2.3, it suffices to show

$$\det[I - S(P + \epsilon A^*)] < 0 \tag{5.5}$$

for some $|S| = I$ and sufficiently small $\epsilon > 0$. Using $\det(-P) = -1$ and Lemma 5.2, it follows for $\det S = 1$,

$$\det(I - SP - \epsilon SA^*) = \det(-SP)\det(I - P^TS + \epsilon P^TA^*)$$

$$= -\sum_{k=0}^{n} \operatorname{trace}\left[\operatorname{adj}_k(I - P^TS)\, C_k(P^TA^*)\right] \cdot \epsilon^k. \tag{5.6}$$

Furthermore, $\operatorname{adj}_0(I - P^TS) = \det(I - P^TS) = 0$ for $\det S = 1$, that is, the summand for $k = 0$ in (5.6) vanishes. Regarding $C_1(P^TA^*) = P^TA^*$, (5.5) is satisfied for small enough $\epsilon > 0$, and therefore the theorem is proved, if we can show

$$\operatorname{trace}\left[\operatorname{adj}(I - P^TS)\, P^TA^*\right] > 0 \tag{5.7}$$

for some signature matrix S with $\det S = 1$. We have $\operatorname{rank}\operatorname{adj}(I - P^TS) = 1$, and defining $\mathscr{S}^+ := \{|S| = I \mid \det(S) = 1\}$ we have

$$\left\{\operatorname{adj}(I - P^TS) \mid S \in \mathscr{S}^+\right\} = \left\{x \cdot x^T \,\middle|\, |x_i| = 1 \text{ for } 1 \leqslant i \leqslant n \text{ and } x_1 = 1\right\}. \tag{5.8}$$

For $\mu \in Q_{kn}$, $1 \leqslant k \leqslant n$, define

$$X_\mu := \left\{x \in \mathbb{R}^n \,\middle|\, |x| = (1) \text{ and } x_\nu = 1 \text{ for } \nu \in \mu\right\}.$$

An induction argument shows for every $B \in M_n(\mathbb{R})$,

$$\sum_{x \in X_\mu} x^TBx = 2^{n-|\mu|}\left(\sum_{\nu \notin \mu} B_{\nu\nu} + \sum_{i,j \in \mu} B_{ij}\right). \tag{5.9}$$

By (5.8) we have $\{\mathrm{adj}(I - P^T S) \mid S \in \mathcal{S}^+\} = \{xx^T \mid x \in X_{\{1\}}\}$, and (5.9) implies for $\mu = \{1\}$

$$\sum_{S \in \mathcal{S}^+} \mathrm{trace}\big[\mathrm{adj}(I - P^T S)\, P^T A^*\big] = \sum_{x \in X_{\{1\}}} \mathrm{trace}(xx^T \cdot P^T A^*)$$

$$= \sum_{x \in X_{\{1\}}} x^T \cdot P^T A^* \cdot x = 2^{n-1}\, \mathrm{trace}(P^T A^*).$$

From the definition of P and A^* we know $(P^T A^*)_{ii} = 0$ for $1 \leqslant i \leqslant n$, and therefore $\mathrm{trace}(P^T A^*) = 0$. Hence, either (5.7) is satisfied and the theorem is proved, or $\mathrm{trace}(\mathrm{adj}(I - P^T S)\, P^T A^*) = 0$ for all $S \in \mathcal{S}^+$.

By (5.8) this leaves us with proving our result for the case $x^T \cdot P^T A^* \cdot x = 0$ for all $x \in X_{\{1\}}$. In this case, $x^T \cdot P^T A^* \cdot x = 0$ for all $x \in X_\mu$ with $1 \in \mu$. For $\mu \in \{1, i\}$, (5.9) and $(P^T A^*)_{ii} = 0$ imply

$$(P^T A^*)_{1i} + (P^T A^*)_{i1} = 0 \qquad \text{for} \quad 2 \leqslant i \leqslant n.$$

For $\mu = \{1, 2, i\}$, again using (5.9), we obtain

$$(P^T A^*)_{2i} + (P^T A^*)_{i2} = 0 \qquad \text{for} \quad 3 \leqslant i \leqslant n.$$

Following these lines, we arrive at

$$P^T A^* = -(P^T A^*)^T. \tag{5.10}$$

That means the theorem is proved if for P, A^* with (5.10), A^* not identically zero, there exists some $S \in \mathcal{S}^+$ such that (5.5) is satisfied for small enough $\epsilon > 0$. The zeros of the characteristic polynomial of a skew-symmetric matrix have real part zero. Therefore, the characteristic polynomial $P_{P^T A^*}(x) = \sum_{k=0}^{n} c_k x^{n-k}$ of $P^T A^*$ has the form

$$c_{2k+1} = -\mathrm{trace}\, C_{2k+1}(P^T A^*) = 0 \quad \text{and} \quad c_{2k} = \mathrm{trace}\, C_{2k}(P^T A^*) \geqslant 0. \tag{5.11}$$

It is easy to see that

$$\sum_{S \in \mathcal{S}^+} \mathrm{adj}_k(I - P^T S) = 2^{n-1} I_{\binom{n}{k}} \qquad \text{for} \quad 1 \leqslant k \leqslant n.$$

Hence, (5.6) implies

$$\sum_{S\in\mathcal{S}^+} \det(I - SP - \epsilon\cdot SA^*)$$

$$= -\sum_{k=0}^{n}\sum_{S\in\mathcal{S}^+} \operatorname{trace}\left[\operatorname{adj}_k(I - P^TS)\, C_k(P^TA^*)\right]\cdot\epsilon^k$$

$$= -2^{n-1}\sum_{k=2}^{n}\left[\operatorname{trace} C_k(P^TA^*)\right]\cdot\epsilon^k,$$

because the terms for $k = 0$ and $k = 1$ vanish. Therefore, (5.11) yields successively for $k = 2,\dots,n$ that either there exists a nonzero, and hence also a positive and a negative, coefficient of ϵ^k, or the coefficients of ϵ^k in (5.6) are zero for all signature matrices S with $\det S = 1$. The first case implies (5.5) for small enough $\epsilon > 0$ and proves the theorem; the latter case is not possible for all $k \leqslant n$, because of (5.10) and because, by assumption, A^* is not identical to the zero matrix. Therefore not all coefficients of ϵ^k can be zero. The theorem is proved. ■

Theorem 5.3 shows that $\rho_0^S(P) = 1$ is a strict *local* minimum of $\rho_0^S(P)$ with respect to perturbations A^* with $P\circ A^* = 0$. The discussion at the beginning of this section showed that (5.1) is a valid lower bound for all ω if and only if $\rho_0^S(P)$ is a *global* minimum in the prescribed sense. However, that is not true. Consider

$$A^* = -\frac{1}{10}\begin{pmatrix} 3 & 0 & 8\\ 8 & 3 & 0\\ 0 & 8 & 3\end{pmatrix} \quad\text{and}\quad A = P + A^* = \begin{pmatrix} -0.3 & 1 & -0.8\\ -0.8 & -0.3 & 1\\ 1 & -0.8 & -0.3\end{pmatrix}. \tag{5.12}$$

Then $\rho_0^S(A) = \rho_0^S(P + A^*) < 0.95$. The graph of $\rho_0^S(P + tA^*)$ for $-0.5 \leqslant t \leqslant 1.5$ is shown in Figure 1. It also displays the strict local minimum at $t = 0$. Nevertheless, the lower bound (5.1) becomes valid for $|\omega| > 2$ when multiplied by a *constant* factor less than one.

DEFINITION 5.4. For $A \in M_n(\mathbb{R})$ define $\zeta(A)$ to be the maximum geometric mean of the cyclic products of $|A|$, that is,

$$\zeta(A) := \max_{\omega}\left|\prod A_\omega\right|^{1/|\omega|},$$

where the maximum is taken over all cycles ω.

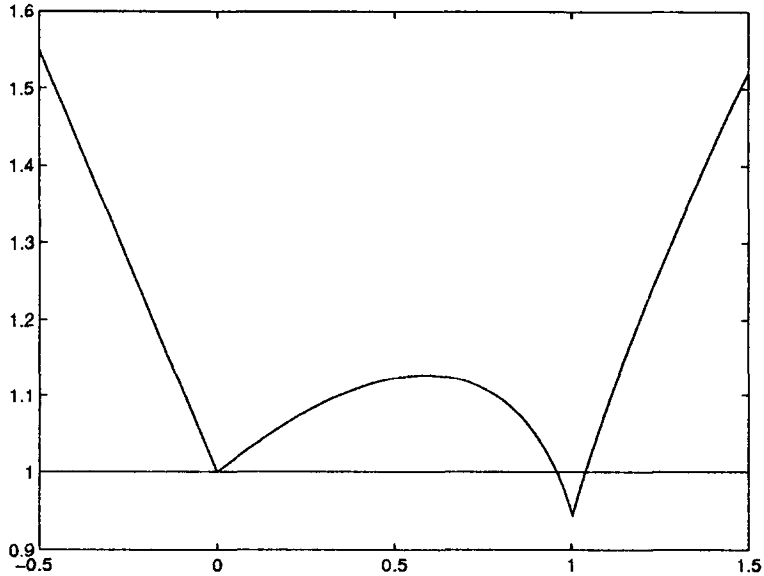

FIG. 1. $\rho_0^S(P + tA^*)$ for $-0.5 \leqslant t \leqslant 1.5$.

Remember that the cyclic products of $|A|$ include the diagonal elements $|A_{ii}|$. The following Theorem shows that indeed $\zeta(A)$ and $\rho_0^S(A)$ cannot be very far apart. The proof of the lower bound of the following theorem 5.5 is lengthy. It uses $\rho_0^S(S_1 A S_2) = \rho_0^S(A)$ essentially and constructs a suitable vector x for the use of Theorem 3.1. The theorem allows us to prove an almost sharp upper bound for the componentwise distance to the nearest singular matrix, thus extending and improving a conjecture by J. Demmel and N. J. Higham [9]. The theorem is included in [24].

THEOREM 5.5. *For* $A \in M_n(\mathbb{R})$,

$$(3 + 2\sqrt{2})^{-1}\zeta(A) \leqslant \rho_0^S(A) \leqslant n\zeta(A). \tag{5.13}$$

The right inequality is sharp, and the left inequality is sharp up to a constant factor.

Proof. The lower bound in (5.13) has been proven in [24, Lemma 2.1]. The upper bound follows from (see [11, Theorem 7.2 and Remark 7.3])

$$\zeta(A) = \inf\{\|D^{-1}AD\| \mid D \text{ nonsingular diagonal}\}, \tag{5.14}$$

$\|A\| \max|A_{ij}|$, and $\rho_0^S(A) \leqslant \rho(|A|) \leqslant n \max|A_{ij}|$. For $A = (1)$, the right inequality in (5.13) becomes an equality; for $A = I$ the left inequality is sharp up to a constant factor. ■

We mention that the factor $(3 + 2\sqrt{2})^{-1}$ in Theorem 5.5 can be improved depending on n (see [24, Theorem 2.4]). For nonnegative A, the result corresponding to Theorem 5.5 reads

$$\zeta(A) \leqslant \rho(A) = \rho_0(A) = \rho_0^S(A) \leqslant n\zeta(A) \qquad \text{for} \quad A \geqslant 0. \quad (5.15)$$

In that case both inequalities are sharp, as is seen by $A = I$ and $A = (1)$.

Next we show that the sign-real spectral radius of A and the Perron root of $|A|$ cannot be too far apart. The corresponding bounds are also sharp up to a constant factor, which will result from the following lemma.

LEMMA 5.6. *For $A \in M_n(\mathbb{R})$, $n \geqslant 2$, with*

$$A_{ij} = \begin{cases} 1 & \textit{for} \quad i < j, \\ 0 & \textit{for} \quad i = j, \\ -1 & \textit{for} \quad i > j, \end{cases} \qquad \text{i.e.,} \qquad A = \begin{pmatrix} 0 & & +1 \\ & \ddots & \\ -1 & & 0 \end{pmatrix},$$

one has $\rho_0^S(A) = 1$.

Proof. By Theorem 2.3, $\rho_0^S(A) = 1$ is equivalent to $\det(I - SA) \geqslant 0$ for all $S \in \mathcal{S}$ and $\det(I - SA) = 0$ for some $S \in \mathcal{S}$. We prove this by induction. The statement is true for $n = 2$. For $n > 2$, let $S \in \mathcal{S}$ be given. If $S_{11} = -1$ and $S_{nn} = +1$, then the first and last rows of $I - SA$ are identical and $\det(I - SA) = 0$. Adding the first to the last column of $I - SA$ yields 0 in components 2 to $n - 1$. For $S_{11} = +1$ and $S_{nn} = -1$ we obtain a zero column, and for $S_{11} = S_{nn} = +1$, the induction hypothesis shows $\det(I - SA) \geqslant 0$. For $S_{11} = S_{nn} = -1$, adding the last to the first column of $I - SA$ and the induction hypothesis finish the proof. ■

THEOREM 5.7. *For $A \in M_n(\mathbb{R})$,*

$$\rho_0^S(A) \leqslant \rho(|A|) \leqslant n \cdot (3 + 2\sqrt{2})\rho_0^S(A). \qquad (5.16)$$

The left inequality is sharp; the right inequality is sharp up to the constant factor $3 + 2\sqrt{2}$.

Proof. Theorem 5.5 and (5.15) yield

$$\rho_0^S(A) \leqslant \rho(|A|) \leqslant n\zeta(A) \leqslant n \cdot (3 + 2\sqrt{2})\rho_0^S(A).$$

The left inequality in (5.16) is sharp for any nonnegative matrix; the right inequality is sharp up to the constant factor $3 + 2\sqrt{2}$ for the matrix defined in Lemma 5.6. ■

Finally, we can prove Theorem 2.16.

Proof of Theorem 2.16

Theorem 2.15 implies $\rho_0^S(A^k) \leqslant \|A^k\|_2$, and for $\varphi := 3 + 2\sqrt{2}$,

$$\rho_0^S(A^k) \geqslant n^{-1}\varphi^{-1}\rho(|A|^k) \geqslant n^{-1}\varphi^{-1}\rho(A)^k.$$

Hence, $\lim_{k\to\infty}\|A^k\|_2^{1/k} = \rho(A)$ proves the theorem. ■

6. FURTHER REMARKS AND OPEN PROBLEMS

By Theorem 5.5, the ratio $\rho_0^S(A)/\zeta(A)$ is bounded below by $(3 + 2\sqrt{2})^{-1}$, and the matrix in (5.12) shows that the ratio can be less than one. What is the minimum ratio (depending on n)? What are properties of a matrix achieving this minimum ratio?

For example, is it true that there is always a matrix A achieving the minimum ratio such that $|A|$ is a circulant [like the matrix (5.12)]?

The matrix (5.12) showed that (5.1) need not to be true for $|\omega| > 2$. However, there is evidence that the estimation (5.1) is true for matrices with zero diagonal. This is equivalent to the following conjecture, easy to formulate in simple terms.

CONJECTURE 6.1. For $A \in M_n(\mathbb{R})$ of the form

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & * \\ & & \ddots & \ddots & \\ & * & & & 1 \\ 1 & & & & 0 \end{pmatrix} \qquad (6.1)$$

there exists a nonzero vector $x \in \mathbb{R}^n$ with $|Ax| \geq |x|$.

To see that the conjecture is equivalent to

$$\rho_0^S(A) \geqslant \zeta(A) \qquad \text{for } A \text{ with zero diagonal,} \tag{6.2}$$

we use the sequence of arguments preceding (5.2), and Theorem 3.1 proves the equivalence.

It is not difficult to prove the conjecture for $n = 3$, and therefore (5.1) for matrices with zero diagonal and $|\omega| = 3$. The conjecture has a number of implications. If Conjecture 6.1 is true, then Theorem 5.5 improves to

$$\rho_0^S(A) \geqslant \tfrac{1}{2}\zeta(A) \qquad \text{for any} \quad A \in M_n(\mathbb{R}). \tag{6.3}$$

This is seen as follows. If $|A_{ii}| \leqslant \frac{1}{2}\zeta(A)$ for all $1 \leqslant i \leqslant n$, define A^0 to be the matrix A with zero diagonal instead. Then $\zeta(A^0) = \zeta(A)$, and Lemma 3.6 and (6.2) imply

$$\rho_0^S(A) \geqslant \rho_0^S(A^0) - \tfrac{1}{2}\zeta(A) \geqslant \zeta(A^0) - \tfrac{1}{2}\zeta(A) = \tfrac{1}{2}\zeta(A).$$

If $|A_{ii}| \geqslant \frac{1}{2}\zeta(A)$ for some $1 \leqslant i \leqslant n$, define A^1 to be the matrix A with $A^1_{ii} := \operatorname{sign}(A_{ii})\,\zeta(A)$. Then $\rho_0^S(A^1) \geqslant \zeta(A)$ by Corollary 2.4, and Lemma 3.6 implies

$$\rho_0^S(A) \geqslant \rho_0^S(A^1) - [\zeta(A) - |A_{ii}|] \geqslant |A_{ii}| \geqslant \tfrac{1}{2}\zeta(A).$$

Conjecture 6.1 also implies

$$\rho(|A|) \leqslant (n-1)\rho_0^S(A) \qquad \text{for} \quad A \in M_n(\mathbb{R}) \text{ with zero diagonal.} \tag{6.4}$$

To see this, we may assume by (5.14) that for $\varepsilon > 0$, $\max|A_{ij}| \leqslant \zeta(A) + \varepsilon$. The zero diagonal and (6.2) imply $\rho(|A|) \leqslant (n-1)[\zeta(A) + \varepsilon] \leqslant (n-1)\rho_0^S(A) + O(\varepsilon)$. The bound (6.4) is sharp as by Lemma 5.6.

For an arbitrary matrix A, define A^0 to be that matrix with zero diagonal. Then $\rho(|A|) \leqslant \rho(|A^0|) + \max|A_{ii}| \leqslant (n-1)\rho_0^S(A) + \rho_0^S(A)$. Therefore, if Conjecture 6.1 is true,

$$\rho(|A|) \leqslant n\rho_0^S(A) \qquad \text{for any} \quad A \in M_n(\mathbb{R}). \tag{6.5}$$

Finally, we mention that if Conjecture 6.1 is true, the results in [24] imply

$$\frac{1}{\rho(|A^{-1}|E)} \leqslant \sigma(A, E) \leqslant \frac{2n-1}{\rho(|A^{-1}|E)}. \tag{6.6}$$

In [25] a general $n \times n$ example has been given with $\sigma(A, E)\rho(|A^{-1}|E) = n$.

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